HECKE ALGEBRAS OF NORMALIZERS OF PARABOLIC SUBGROUPS

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Abstract. In the context of Hecke algebras of complex reflection groups, we prove that the generalized Hecke algebras of normalizers of parabolic subgroups are semidirect products, under suitable conditions on the parameters involved in their definition.

Contents

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1. Introduction

Let W be a complex reflection group, that is, a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by complex (pseudo-)reflections. Let W_0 be a parabolic subgroup of W, that is, the pointwise stabilizer of a subset of $\mathbb{C}^n$, which is also a complex reflection group. In [20], the second author defined a generalized Hecke algebra $\widetilde{H}_0$ attached to the *normalizer* $N_0 = N_W(W_0)$, which is a natural extension of the Hecke algebra H_0 of W_0 by the group algebra of $\overline{N_0} = N_0/W_0 = N_W(W_0)/W_0$. This algebra turns out to be particularly useful for understanding (up to Morita equivalence) the 'braid subalgebra' of the Yokonuma-Hecke algebras introduced in [19].

It was proved in [20] that $\widetilde{H}_0$ is a free module over its ring of definition, with a direct sum decomposition $\widetilde{H}_0 \simeq \bigoplus_{g \in \overline{N_0}} [gH_0]$ as a free H_0-module of rank $|\overline{N_0}|$. Since it has been proven by Muraleedaran and Taylor in [23] that the extension

$$1 \to W_0 \to N_0 \to \overline{N_0} \to 1 \tag{1.1}$$

is always split, it is expected that $\widetilde{H}_0$ is isomorphic to a semidirect product $\overline{N_0} \ltimes H_0$.

In [13], Henderson and the authors positively answered this question when W is a *real* reflection group (and actually in this case the reflection subgroup W_0 does not even need to be parabolic), regardless of the ring of definition K and the defining parameters of $\widetilde{H}_0$. In the present paper, we explore the general case, for which conditions need to be added. We assume that K is a domain and denote by $K^\times$ its group of invertible elements.

Our first main result is the following Theorem (see Theorem 2.7 below for a more precise statement):

Theorem 1.1. *Let W_0 be a parabolic subgroup of W. If the defining parameters of H_0 are generic, and K is a sufficiently large field of characteristic 0, then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$.*

However, this does not apply in general to the non-generic case. In this paper we find explicit, sufficient algebraic conditions to ensure such a semidirect product decomposition, using the classification of irreducible complex reflection groups. Indeed, it is not difficult to see that for this problem we can assume that W is irreducible.

In the case of the general series $G(de, e, n)$ of complex reflection groups, we will prove that these problems can be reduced to the case of a parabolic subgroup of the form

$$W_0 = G(de, e, n_0) \times \prod_{k=1}^{n} G(1, 1, k)^{b_k}$$

(see Section 4 for precise definitions). The Hecke algebra of the group $G(1, 1, k)$ is the Hecke algebra of type A_{k-1} associated to the symmetric group $\mathfrak{S}_k$ (considered as a Coxeter group). Let $\Delta(k)$ denote the element of its standard basis associated to the element of maximal length of $\mathfrak{S}_k$ – which is the image of Garside's fundamental element of the usual braid group on k strands. Our main result for the general series is the following one, proved in Sections 3 and 4:

Theorem 1.2. *Let $W = G(de, e, n)$ and $W_0 = G(de, e, n_0) \times \prod_{k=1}^{n} G(1, 1, k)^{b_k}$. Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ as soon as, whenever $b_k \neq 0$,*

- *there exists $T_k \in K[X]$ such that $T_k(\Delta(k))^{-de} = \Delta(k)^2$, where the equality holds inside the Iwahori-Hecke algebra of type A_{k-1}, and*

- *if moreover* $e \neq 1$ *and* $n_0 \geq 1$*, there exists* $T_{0,k} \in K[X]$ *such that* $T_{0,k}(\sigma)^{de} = \sigma^{kd}$ *whenever* σ *is a braided reflection associated to the hyperplane* $z_1 = 0$*.*

In particular the second condition is void when $d = 1$*, as* $z_1 = 0$ *is not a reflecting hyperplane in that case.*

In most cases, the above conditions on the existence of polynomials have a natural translation in terms of the parameters of the Hecke algebra of W_0 (see Lemmas 2.8 and 2.9 below).

In exceptional types, we determine semidirect product decompositions for all parabolic subgroups of maximal rank, as well as for some parabolic subgroups of rank 1. In rank 3, we do this for all the groups except G_{27}. Since, in rank 3, proper parabolic subgroups either have rank 1 or are maximal, this solves our problem for these groups (that is, for G_{24}, G_{25} and G_{26}). We also solve it for the rank 4 group G_{32}, for which we also have to consider parabolic subgroups of rank 2. In particular, we get the following general result (see Theorem 5.1 for more details), where B_0 is the braid group of W_0 and the element z_{B_0} is defined in Section 2.

Theorem 1.3. *Let* W *be an irreducible complex reflection group of exceptional type, and* W_0 *a parabolic subgroup of maximal rank. Let* z_{B_0} *be the canonical positive central element of* B_0*. Except for two exceptions for ranks* 3 *and* 5*, if there exists* $T \in K[X]$ *such that the equality* $T(z_{B_0})^{|Z(W)|} = z_{B_0}^{-|Z(W_0)|}$ *holds inside* H_0*, then* $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$*.*

In the last Section 6, we explore the remaining exceptional cases. There, we explain in particular why a systematic exploration failed for the largest cases, and we nevertheless manage to solve the problem for some of them, including all the (Shephard) groups whose braid group is an Artin group.

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2. General results

2.1. **The Hecke algebra of a normalizer.** Let W_0 be a reflection subgroup of the finite complex reflection group $W < \mathrm{GL}_n(\mathbb{C})$. We let $\mathcal{A}_0, \mathcal{A}$ be the collection of reflecting hyperplanes of W_0 and W, respectively, and H_0, H the corresponding Hecke algebras. The hyperplane complement $X = \mathbb{C}^n \setminus \bigcup \mathcal{A}$ is acted upon by W, and the braid group of W is defined by $B = \pi_1(X/W)$. We denote $\pi : B \twoheadrightarrow W$ the natural projection.

The group B contains an important central element, which we denote z_B. When W is irreducible, its center $Z(W)$ is cyclic of some order m, generated by $\zeta_m \mathrm{Id}$ for $\zeta_m = \exp(2\pi \mathrm{i}/m) \in \mathbb{C}^\times$. In this setting, z_B is the homotopy class inside X/W of the path $t \mapsto \exp(2\pi \mathrm{i}t/m).*$, where $* \in X$ is the chosen base-point. In the general case, the ambient space $\mathbb{C}^n$ admits a canonical direct sum decomposition $\mathbb{C}^n \simeq \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_r}$ yielding a decomposition $W \simeq W_1 \times \cdots \times W_r$, where $W_i < \mathrm{GL}_{n_i}(\mathbb{C})$ is an irreducible reflection group. Letting $m_i = |Z(W_i)|$, then z_B is the homotopy class inside X/W of the path $t \mapsto (\exp(2\pi \mathrm{i}t/m_1).*_1, \dots, \exp(2\pi \mathrm{i}t/m_r).*_r)$, where $* = (*_1, \dots, *_r) \in X \subset \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_r}$ is the chosen base-point.

We recall from [9] the construction of the Hecke algebra of W over some ring K. It is defined using parameters $u_{i,s} \in K^\times$ for s running among the distinguished reflections of W, where $0 \leq i < o(s)$ and $u_{i,s} = u_{i,t}$ when s, t belong to the same conjugacy class. Then H is the quotient of KB by the relations $\prod_i (\sigma - u_{i,s}) = 0$ for every braided reflection σ associated to s – so that its most general definition ring is the ring of Laurent polynomials $\mathbb{Z}[u_{i,s}^{\pm 1}]$. Its basic structural property is the now proven BMR freeness conjecture, as a combination of [1, 2, 7, 16, 17, 21, 10, 18, 24].

Theorem 2.1. *The algebra H is a free K-module of rank $|W|$.*

We consider the normalizer $N_W(W_0) = N_0$ of W_0 inside W. By definition, the *Hecke algebra* $\widetilde{H}_0$ *of* N_0 as defined in [20] is a quotient of the group algebra $K\hat{B}_0$ of $\hat{B}_0 = \pi^{-1}(N_0) = \pi_1(X/N_0)$, by two types of relations:

- The relations $\sigma^{m_L} = 1$, for every braided reflection σ associated to a hyperplane $L \in \mathcal{A} \setminus \mathcal{A}_0$. Here m_L is the order of the pointwise stabilizer of L,
- The defining relations of the Hecke algebra H_0 on the braided reflections with respect to hyperplanes in $\mathcal{A}_0$.

We have the following generalization of Theorem 2.1, proven in [20].

Theorem 2.2. *The algebra $\widetilde{H}_0$ is a free H_0-module of rank $|\overline{N_0}| = |N_0/W_0|$, with a natural direct sum decomposition*

$$\widetilde{H}_0 = \bigoplus_{g \in \overline{N_0}} [gH_0]$$

with $[gH_0]$ a free right H_0-module of rank 1. As a consequence it is a free K-module of rank $|N_0|$.

An equivalent definition of $\widetilde{H}_0$ can be given as follows. We introduce the normal subgroup Q_0 of $\hat{B}_0$ generated by all the σ^{m_L}, for σ a braided reflection associated as above to some hyperplane $L \in \mathcal{A} \setminus \mathcal{A}_0$. Let $\widetilde{B}_0 = \hat{B}_0/Q_0$. We define $\widetilde{H}_0$ as the quotient of $K\widetilde{B}_0$ by the Hecke relations of W_0, which makes sense as all the braided reflections of B with respect to a hyperplane in $\mathcal{A}_0$ belong to $\hat{B}_0$. These elements of the form σ^{m_L} are exactly the *meridians* around L, in the terminology of [3] (also called *generators-of-the-monodromy* in [9]). We set $X_0 = \mathbb{C}^n \setminus \bigcup \mathcal{A}_0$. Letting $B_0 = \pi_1(X_0/W_0)$ denote the braid group of W_0, we have a short exact sequence of groups (see [20, Section 2.2])

$$1 \to B_0 \to \widetilde{B}_0 \to \overline{N_0} \to 1 \tag{2.1}$$

and the direct sum decomposition of Theorem 2.2 is such that $[gH_0] \subset \widetilde{H}_0$ is equal to $bH_0 = H_0 b$ for $b \in \widetilde{B}_0$ having $g \in \overline{N_0}$ for image. We have $bH_0b^{-1} = H_0$ for $b \in \widetilde{B}_0$.

Lemma 2.3. *Assume that there exists a group homomorphism $\psi : \overline{N_0} \to \widetilde{H}_0^{\times}$ such that, for every $g \in \overline{N_0}$, there exists $b \in \widetilde{B}_0$ with the following property*

- *b maps to g under $\widetilde{B}_0 \twoheadrightarrow \overline{N_0}$*
- *$\psi(g)$ belongs to the image $bH_0 \subset \widetilde{H}_0$ of $b(KB_0)$ under $K\widetilde{B}_0 \to \widetilde{H}_0$.*

Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$. In particular, if the short exact sequence (2.1) splits, then $\widetilde{H}_0$ is a semidirect product $\overline{N_0} \ltimes H_0$.

Proof. Because of $\psi(g) \in \widetilde{H}_0^{\times}$ and these conditions, we have $\psi(g)H_0 = bH_0 = [gH_0]$. Writing $\psi(g) = bm$ for $m \in H_0$, since b and $\psi(g)$ are invertible we get that m is also invertible, and $\psi(g)H_0\psi(g)^{-1} = bmH_0m^{-1}b^{-1} = bH_0b^{-1} = H_0$. It follows that there is an algebra morphism $\overline{N_0} \ltimes H_0 \to \widetilde{H}_0$ mapping $g \otimes x$ for $g \in \overline{N_0}$, $x \in H_0$, to $\psi(g)x$. Since it maps each $g \otimes H_0$ to $[gH_0]$ isomorphically, this is an isomorphism $\overline{N_0} \ltimes H_0 \simeq \widetilde{H}_0$. □

The following has been proven in [13, Theorems 3.15 and 3.19]:

Theorem 2.4. *If W is a finite real reflection group and W_0 is an arbitrary reflection subgroup of W, then the short exact sequence (2.1) splits and $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$.*

In [13, Proposition 5.1], it is moreover shown that the short exact sequence (2.1) also splits in the case where W is the complex reflection group $G(r,1,n)$ and W_0 is a standard parabolic subgroup of type $G(d,1,k)$, $k \leq n$. However, this conclusion cannot be expected for arbitrary finite complex reflection groups. Indeed, the splitting of the short exact sequence (2.1) implies the splitting of the short exact sequence (1.1), but there are pairs (W, W_0) where W is a reflection group and W_0 a reflection subgroup of W such that the short exact sequence above does *not* split (see [13, Section 6]). Nevertheless, we will show in Subsection 2.2 below that, generically in characteristic 0, this is the only obstruction for a semidirect product decomposition of the Hecke algebra $\widetilde{H}_0$ of N_0.

For later use, we prove the following result:

Lemma 2.5. *Let W_1, W_2 be two reflection subgroups of W which are conjugate, let $G_i = N_W(W_i)$, and $\hat{B}_i, \widetilde{B}_i, B_i, \widetilde{H}_i$ be the groups and algebras $\hat{B}_0, \widetilde{B}_0, B_0, \widetilde{H}_0$ attached to $W_0 = W_i$ as above, $i = 1,2$.*

Then there is a group isomorphism $\widetilde{B}_1 \to \widetilde{B}_2$ mapping B_1 to B_2 and an algebra isomorphism $\widetilde{H}_1 \to \widetilde{H}_2$ mapping H_1 to H_2.

Proof. Let $w \in W$ such that $W_2 = wW_1w^{-1}$ and $b \in \pi^{-1}(\{w\}) \in B$. Setting $G_i = N_W(W_i)$, we have $G_i = wG_1w^{-1}$, hence $\hat{B}_2 = \pi^{-1}(G_2) = \pi^{-1}(wG_2w^{-1}) = b\pi^{-1}(G_1)b^{-1} = b\hat{B}_1b^{-1}$. Let $Q_i = \mathrm{Ker}(\hat{B}_i \twoheadrightarrow B_i)$. By definition, Q_i is generated as a group by the set of all the meridians around the reflecting hyperplanes of W which are not reflecting hyperplanes for W_i. Now, $x \mapsto bxb^{-1}$ realizes a bijection between the meridians, mapping the generating ones for Q_1 to the generating ones for Q_2, hence $bQ_1b^{-1} = Q_2$. It follows that $x \mapsto bxb^{-1}$ restricts to an isomorphism $\hat{B}_1 \to \hat{B}_2$ which maps Q_1 to Q_2, therefore induces an isomorphism $\widetilde{B}_1 \to \widetilde{B}_2$. Since Q_1, Q_2 are subgroups of the pure braid group P, this isomorphism fits into a commutative diagram of the form

$$\begin{array}{ccc} \widetilde{B}_1 & \longrightarrow & \widetilde{B}_2 \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ G_1 & \longrightarrow & G_2 \end{array}$$

where $G_1 \to G_2$ is $x \mapsto wxw^{-1}$. Since $W_2 = wW_1w^{-1}$ this implies that it maps $B_1 = \mathrm{Ker}(\widetilde{B}_1 \twoheadrightarrow N_W(W_1)/W_1)$ to B_2. Finally, since $x \mapsto bxb^{-1}$ maps braided reflections around reflecting hyperplanes of W_1 to braided reflections around reflecting hyperplanes of W_2, the defining ideal of $\widetilde{H}_1$ inside $K\widetilde{B}_1$ is mapped to the defining ideal of $\widetilde{H}_2$ inside $K\widetilde{B}_2$, and this induces an isomorphism $\widetilde{H}_1 \to \widetilde{H}_2$, which maps the image of KB_1 inside $\widetilde{H}_2$ to the image of KB_2 inside $\widetilde{H}_2$, namely H_1 to H_2, and this proves the claim.

□

2.2. **The generic Hecke algebra of the normalizer.** Let $W < \mathrm{GL}_n(\mathbb{C})$ be a complex reflection group, and let $\mathcal{R}^* \subset \mathcal{R}$ be the collection of its distinguished (pseudo-)reflections. Let $W_0 < W$ be a full reflection subgroup of W, and $\mathcal{R}_0 \subset \mathcal{R}$, $\mathcal{R}_0^* \subset \mathcal{R}^*$ its collection of (distinguished) reflections. Let $\Bbbk$ be the ring of Laurent polynomials $\mathbb{Z}[u_{s,i}^{\pm 1}]$, where $s \in \mathcal{R}^*$

and $i \in \{0, \dots, o(s)-1\}$, with the convention that $u_{s,i} = u_{wsw^{-1},i}$ for all $w \in N_0$. This is the most general ring over which $\widetilde{H}_0$ is defined. In this section we consider the generic case, that is, the case where K is a field containing $\Bbbk$. In particular K has characteristic 0.

2.2.1. *An isomorphism à la Cherednik.* There is a natural bijection $\mathcal{R}^* \to \mathcal{A}$ given by $s \mapsto \mathrm{Ker}(s-1)$. We denote $L \mapsto s_L$ its inverse. For any choice of elements $\varphi_L \in \mathbb{C}W_L$ with $W_L = \langle s_L \rangle$, $L \in \mathcal{A}$, with the condition that $\varphi_{w(L)} = w\varphi_L w^{-1}$ for every $w \in W$, it is well-known (see for instance [9]) that the 1-form $\sum_{L\in\mathcal{A}} h\varphi_L\omega_L \in \Omega^1(X) \otimes \mathbb{C}[[h]]W$, with ω_L the logarithmic 1-form over X associated to L (that is, $\mathrm{d}\alpha_L/\alpha_L$ for α_L any linear form defining L), is integrable and W-equivariant and provides an algebra isomorphism $H \to KW$, as a consequence of Theorem 2.1, where $K = \mathbb{C}((h))$ is the field of Laurent series, and the $u_{s_L,k} \in K^\times$ depend on φ_L.

More precisely, we have $\mathbb{C}\langle s_L\rangle = \prod_{k=0}^{m_H-1} \mathrm{Ker}(s_L - \zeta_L^k) \simeq \mathbb{C}^{m_L}$, where we identified s_L with its image under the multiplication operator map $\mathbb{C}\langle s_L\rangle \hookrightarrow \mathrm{End}(\mathbb{C}\langle s_L\rangle)$, and $\zeta_L = \exp(2\mathrm{i}\pi/m_L)$. We denote $\varepsilon_{L,k}$ the primitive idempotent associated with $\mathrm{Ker}(s_L - \zeta_L^k)$. Then letting $\varphi_L = \sum_{k=0}^{m_H-1} = \lambda_{L,k}\varepsilon_{L,k} \in \mathbb{C}\langle s_H\rangle$ with scalars such that $\varphi_{w(L)} = \varphi_L$ for all $L \in \mathcal{A}, w \in W$, we get a morphism $H \to KW$ with $u_{s_L,k} = \exp(2\mathrm{i}\pi\lambda_{L,k}h/m_L)$, which we call the parameters of L associated to the collection of $\varphi_L, L \in \mathcal{A}$. This morphism is an isomorphism as soon as the $u_{s_L,k}$ ($L \in \mathcal{A}, 0 \le k \le m_H - 1$) are algebraically independent over $\mathbb{C}$, which holds as soon as the $\lambda_{L,k}$ are linearly independent over $\mathbb{Q}$. We call such a choice of parameters a *generic choice* for φ_L.

We denote $\widetilde{H}_0^!$ and $H_0^!$ the Hecke algebras of N_0 and W_0 defined over $\mathbb{C}[[h]] \subset \mathbb{C}((h)) = K$. We prove the following.

Proposition 2.6. *For any choice of elements $\varphi_L \in \mathbb{C}W_L$, $L \in \mathcal{A}_0$, such that $\varphi_{g(L)} = \varphi_L$ for all $g \in N_0$, the 1-form*

$$\omega_0 = \sum_{L\in\mathcal{A}_0} \varphi_L\omega_L \in \Omega^1(X) \otimes \mathbb{C}W_0$$

is integrable over X, and N_0-equivariant. The monodromy of $h\omega_0$ over X/N_0 provides an algebra homomorphism $\mathbb{C}[[h]]\widetilde{B}_0 \to \mathbb{C}[[h]]N_0$ mapping $\mathbb{C}[[h]]B_0$ to $\mathbb{C}[[h]]W_0$. The latter morphism factorizes through $\widetilde{H}_0^!$ and induces a $\mathbb{C}((h))$-algebra morphism $\widetilde{H}_0 \to \mathbb{C}((h))N_0$ mapping H_0 to KW_0 for the parameters of H_0 associated to the chosen values of $\varphi_L, L \in \mathcal{A}_0$. If this choice is generic in the above sense, then this algebra morphism factorizes through $\widetilde{H}_0^!$ and induces isomorphisms $\widetilde{H}_0 \to KN_0$ and $H_0 \to KW_0$.

Proof. The 1-form ω_0 is the restriction to X of the usual 1-form on $\mathbb{C}^n \setminus \bigcup \mathcal{A}_0$ attached to W_0, therefore it is integrable as well. The N_0-equivariance is clear. Thus, by e.g. Chen's iterated integrals, one gets that the monodromy of $h\omega_0$ provides a morphism $\mathbb{C}[[h]]\widehat{B}_0 = \mathbb{C}[[h]]\pi_1(X/N_0) \to \mathbb{C}[[h]]N_0$ which restricts to the usual monodromy morphism $\mathbb{C}[[h]]B_0 \to \mathbb{C}[[h]]W_0 \subset \mathbb{C}[[h]]N_0$. Since the map $\mathbb{C}\widehat{B}_0 \to \mathbb{C}N_0$ induced by $\pi : \widehat{B}_0 \to N_0$ is surjective and coincides with reduction modulo h of $\mathbb{C}[[h]]\widehat{B}_0 \to \mathbb{C}[[h]]N_0$, by Nakayama's lemma one gets that the latter algebra homomorphism is surjective. Since the monodromy of ω_0 along meridians around $L \notin \mathcal{A}_0$ is trivial, this morphism induces an algebra morphism $\Phi : \mathbb{C}[[h]]\widetilde{B}_0 \to \mathbb{C}[[h]]N_0$ which is still surjective, and still extends $\mathbb{C}[[h]]B_0 \to \mathbb{C}[[h]]W_0$.

The latter is known to factorize through $H_0^!$, hence we get that Φ induces an algebra morphism $\mathbb{C}[[h]]\widetilde{B}_0 \to \mathbb{C}[[h]]N_0$ which is still surjective, and still extends $\mathbb{C}[[h]]B_0$ to $\mathbb{C}[[h]]W_0$.

Since the latter is known to factorize through $\widetilde{H}_0'$, we get that Φ induces a surjective algebra morphism $\widetilde{H}_0 \to KN_0$. As a consequence of Theorem 2.1, we have equality of dimensions, therefore this provides an isomorphism $\widetilde{H}_0 \to KN_0$ mapping H_0 to KW_0.

□

2.2.2. *Consequences in the generic case.* We now assume that W_0 admits a complement inside N_0, that is, we assume that $G = U_0 \ltimes W_0$ for some $U_0 < N_0$. We also also assume that the parameters of H_0 are generic in characteristic 0, that is, that K is a field containing the generic ring $\Bbbk = \mathbb{Z}[u_{s,i}^{\pm}]$.

For $K \supset \Bbbk$ large enough, by Proposition 2.6 we know that there exists an algebra isomorphism $\Phi : \widetilde{H}_0 \to KG$ mapping H_0 to KW_0. As a consequence, setting $M := \Phi^{-1}(U_0)$, we get $\widetilde{H}_0 = \bigoplus_{g \in M} gH_0$ and $\widetilde{H}_0 = M \ltimes H_0 \simeq \overline{N_0} \ltimes H_0$. When W_0 is a parabolic subgroup of W, the existence of such a complement U_0 is proven in all cases in [23], therefore a consequence of the above argument is the following:

Theorem 2.7. *Let W_0 be a parabolic subgroup of W. Then, for K a sufficiently large field containing $\Bbbk = \mathbb{Z}[u_{s,i}^{\pm}]$, we have $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$.*

2.3. **Genericity conditions.** Thanks to Theorem 2.7, the question is therefore to determine algebraic criteria on the domain K and on the parameters $u_{s,i} \in K$ so that the extension $\widetilde{H}_0$ is a semidirect product.

To this end, we will often need the following result of commutative algebra.

Lemma 2.8. *Let K be an arbitrary domain, let $a, b \in \mathbb{Z} \setminus \{0\}$, and let $\chi(X) \in K[X]$ be a monic and split polynomial with roots $v_1, \dots, v_r \in K^{\times}$ such that $i \neq j \Rightarrow v_i - v_j \in K^{\times}$.*

We call such a split polynomial square-free. *Then the following are equivalent:*

- *There exists $T \in K[X]$ such that $T(X)^a \equiv X^b \mod \chi(X)$ inside $K[X, X^{-1}]$,*
- *The domain K contains a a-th root of each $v_i^b, 1 \leq i \leq r$.*

Proof. By assumption, we can write $\chi(X) = (X - v_1) \dots (X - v_r)$. Because of the conditions $v_i \in K^{\times}$ and $v_i - v_j \in K^{\times}$, applying the Chinese Remainder Theorem we have

$$K[X, X^{-1}]/(\chi) \simeq K[X]/(\chi) \simeq \prod_i K[X]/(X - v_i) \simeq K^r.$$

Then the equation is equivalent to the equations $T(v_i)^a = v_i^b$. There is no solution to such an equation if v_i^b has no a-th root in K. If it has such an a-th root $v_i^{b/a}$, we are looking for a polynomial $T \in K[X]$ such that $T(v_i) = v_i^{b/a}$ for $1 \leq i \leq r$. This is then a linear equation in the coefficients of T, whose determinant is the Vandermonde determinant attached to the v_i's. This determinant is invertible in K because of our assumptions, and this proves the claim.

□

One specific element whose minimal polynomial will play a major role is the following one.

Lemma 2.9. *The image of Garside's fundamental element in the braid group on $n+1$ strands inside the Hecke algebra of type A_n is annihilated by the polynomial*

$$\prod_{\substack{i,j \geq 0 \\ i+j = n(n+1)}} (X^2 - (-1)^i u_0^i u_1^j) \tag{2.2}$$

$n+1$	i
2	$\{2,0\}$
3	$\{6,3,0\}$
4	$\{12,8,6,4,0\}$
5	$\{20,15,12,10,8,5,0\}$
6	$\{30,24,20,18,15,12,10,6,0\}$

TABLE 1. Eigenvalues of Garside's element in type A

Proof. The image in the statement is equal to the element T_{w_0} of the standard basis of the Hecke algebra, associated to the longest element w_0 of the symmetric group, which has length $n(n+1)/2$. Setting $q = -u_1 u_0^{-1}$ and renormalizing each braided reflection σ as $-u_0^{-1}\sigma$, we get the equivalent formulation that, inside the Hecke algebra defined over $\mathbb{Z}[q^{\pm 1}]$ by the equation $(\sigma - q)(\sigma + 1)$ as in [14], the element T_{w_0} is annihilated by the polynomial $\prod_{0 \le i \le n(n+1)}(X^2 - q^i)$. This Hecke algebra is semisimple over the algebraic closure $\overline{\mathbb{Q}(q)}$ of the field of fractions of $\mathbb{Z}[q^{\pm 1}]$, and the central element $T_{w_0}^2$ acts by the scalar $z_\lambda \in \overline{\mathbb{Q}(q)}$ on the irreducible representation attached to a partition $\lambda \vdash n+1$.

Therefore T_{w_0} is annihilated by the polynomial $\prod_{a \in A}(X^2 - a)$, where A denotes the set of all values $v_\lambda, \lambda \vdash n+1$, inside the Hecke algebra over $\overline{\mathbb{Q}(q)}$. This happens already inside the original Hecke algebra defined over $\mathbb{Z}[q^{-1}]$, because it is a free module over it. We thus only need to prove that each of the z_λ is of the form q^i, $0 \le i \le n(n+1)$.

By a result of Springer (see [14] 9.2.2), we have $z_\lambda = q^{v_\lambda \frac{n(n+1)}{2}}$ with $v_\lambda \frac{n(n+1)}{2} \in \mathbb{Z}$, where $v_\lambda = \gamma(\lambda)/\dim(\lambda)$, with $\dim(\lambda)$ of the irreducible representation of the symmetric group $\mathfrak{S}_n$ attached to λ, and $\gamma(\lambda)$ is the trace of $1 + (1\ 2)$. We need to prove that v_λ always belongs to the real interval $[0,2]$.

Notice that $v_{[1^{n+1}]} = 0$ and $v_{[n+1]} = 2$, and in particular the statement on the $z_\lambda' s$ is true for $n+1 = 2$. We proceed to prove it by induction on n. For $\mu \vdash n$, let us denote $[\lambda : \mu]$ the multiplicity of (the representation of $\mathfrak{S}_n$ attached to) μ in the restriction of λ. Then $\dim(\lambda) = \sum[\lambda : \mu]\dim(\mu)$ and $\gamma(\lambda) = \sum[\lambda : \mu]\gamma(\mu)$. As a consequence,

$$v_\lambda = \gamma(\lambda)/\dim(\lambda) = \frac{1}{\dim(\lambda)}\sum[\lambda : \mu]\gamma(\mu) = \frac{\sum[\lambda : \mu]\dim(\mu)v_\mu}{\sum[\lambda : \mu]\dim(\mu)}$$

hence v_λ belongs to the minimal interval containing all the $v_\mu, \mu \vdash n$. By the induction assumption this interval is included in (and even equal to) $[0,2]$, and this concludes the induction step and the proof. □

Actually, for any given n, the proof provides a more specific polynomial, as every z_λ is explicitely computable. For the small values of n, the index i in the formula (2.2) belongs to the sets given in Table 1.

In particular, determining whether the polynomial (2.2) splits or not highly depends on n, as it is related to the appearance of odd integers among the possible values of i.

3. Groups of type $G(r,1,n)$

In this section, we prove Theorem 1.2 in the case where W is the complex reflection group $G(r,1,n)$, that is, we find explicit conditions on K to ensure that we have an isomorphism $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ in this case.

In [13, Proposition 5.1], Henderson and the authors showed that if W_0 is a parabolic subgroup of type $G(r,1,k)$ of W ($k \leq n$), then the group-theoretic short exact sequence (2.1) splits, and we even have a direct product decomposition $\widetilde{B}_0 = B_0 \times \overline{N_0}$ in that case, yielding a direct tensor product decomposition $\widetilde{H}_0 = H_0 \otimes K\overline{N_0}$ (as a K-algebra).

In particular, $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ without any condition on the parameters. However, this does *not* hold in general, as shown in [13, Example 6.6].

We begin by a few observations on normalizers of parabolic subgroups and the existence of complements (proven in [23]).

3.1. Special elements and parabolic subgroups.

Let $n \geq 1, r \geq 2$. Recall that $W = G(r,1,n)$ has a Coxeter-like presentation with n generators $s_0, s_1, \dots, s_{n-1}$, with relations given by the type B_n braid relations (with $s_0s_1s_0s_1 = s_1s_0s_1s_0$), together with the relations $s_0^r = 1$, $s_i^2 = 1$ for all $1 \leq i \leq n-1$. This is exactly the Coxeter presentation of the Weyl group of type B_n, except that s_0 has order r (for $r = 2$ we recover the Weyl group of type B_n).

It is shown in [22, Theorem 3.9] that every parabolic subgroup of W is conjugate to a parabolic subgroup (which we will call *standard*) generated by a subset S_0 of the set $S := \{s_0, s_1, \dots, s_{n-1}\}$ of Coxeter-like generators of W. By Lemma 2.5, it is enough to deal with the standard parabolic subgroups in order to prove $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ – which, in certain cases, may be achieved thanks to Lemma 2.3 by showing the existence of a splitting of the short exact sequence (2.1).

Recall that W is the group of $(n \times n)$-monomial matrices whose entries are r-th roots of unity. In this description, s_0 is the diagonal matrix having $\zeta = \exp(2\pi i/r)$ as first entry and 1 everywhere else, with $s_1, s_2, \dots, s_{n-1}$ acting on the basis vectors by permuting them in the obvious way. The action of W on $\mathbb{C}^n$ is irreducible.

Let $S' = \{s_1, s_2, \dots, s_{n-1}\}$, $\mathfrak{S}_n = \langle S' \rangle$. We have $G(r,1,n) = (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$,

where the generator t_i in the i-th factor of $(\mathbb{Z}/r\mathbb{Z})^n$ is given by the diagonal matrix having $\zeta = \det(s_0)$ as i-th entry and 1 everywhere else. It is a reflection, and can be expressed as a product of elements of S as s_0 if $i = 1$ and $s_{i-1} \cdots s_1 s_0 s_1 \cdots s_{i-1}$ if $i \geq 2$. Note that $(\mathbb{Z}/r\mathbb{Z})^n$ can also be defined as the subgroup generated by the $\mathfrak{S}_n$-conjugates of those elements which lie in $S \setminus S' = \{s_0\}$, generalizing in this specific case the semi-direct product decomposition of Coxeter groups given in [5].

The braid group $B = B(r,1,n)$ of $G(r,1,n)$ is isomorphic to the Artin group of type B_n. We denote its standard Artin generators by $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$.

In [6], it is shown that every element $w \in W$ can be written uniquely in the form

$$t_{0,k_0} t_{1,k_1} \cdots t_{n-1,k_{n-1}} p(w),$$

where $p(w) \in \mathfrak{S}_n$ and $0 \leq k_i \leq r-1$. Here $t_{i,k}$ is defined as $s_i s_{i-1} \cdots s_0^k$ if $k \neq 0$, and as 1 if $k = 0$.

Let $W(r)$ denote the set that consists of those $w \in W$ such that in the above normal form, we have $k_i \in \{0,1\}$ for all $i = 0, \dots, n-1$. Note that it is exactly the set of monomial matrices in $\{1, \zeta\}$.

Replacing every $t_{i,k}$ by $s_i s_{i-1} \cdots s_0^k$ if $k \neq 0$ and by 1 if $k = 0$ in the above normal form and $p(w)$ by a reduced expression of it in $\mathfrak{S}_n$, by [6, Lemma 1.5] one obtains a reduced expression in the generating set S of W. Moreover, the same lemma shows that, if we restrict to $W(r)$, then taking any reduced expression of $w \in W(r)$ in S and replacing every s_i by σ_i yields a well-defined element $\mathbf{w}$ of B. Indeed, here B is an Artin group of type B_n, and [6, Lemma 1.5] restricted to $W(r)$ shows that any two reduced expressions of $w \in W(r)$ can be related by the defining relations of B. Let $\mathcal{D} := \{\mathbf{w} \mid w \in W(r)\}$. Note that $\mathcal{D}$ is independent of r.

In fact, the set $\mathcal{D}$ is the set of (left or right) divisors of the Garside element $(\sigma_0\sigma_1 \ldots \sigma_{n-1})^n$ of the braid monoid $B^+ \subseteq B$, which is a Garside monoid. In other words, the morphism of monoids $B^+ \longrightarrow W$ induces by restriction an injective set-theoretic map $\mathcal{D} \to W$, with image equal to $W(r)$. Note that for $r = 2$ we have $W = W(r)$ and $\mathcal{D}$ is just the set of positive lifts of elements of W, which is a Coxeter group of type B_n in that case; for $r \neq 2$ the subset $W(r)$ is *not* a subgroup of W.

Denoting by ℓ the length function on W with respect to S, what we just recalled implies the following important property, which will be used repeatedly below:

Lemma 3.1. *Let $u, v \in W(r)$ such that $w := uv$ lies in $W(r)$ and $\ell(u) + \ell(v) = \ell(w)$. Then* $\mathbf{uv} = \mathbf{w}$.

3.2. **Description of the complements.** Howlett [15] found a description of complements of parabolic subgroups of Coxeter groups inside their normalizers; it is not hard to see that his construction does not generalize to the $G(r,1,n)$ case, even though these groups are close to being Coxeter groups. In this section, we therefore recall a different way of finding complements of parabolic subgroups of W inside their normalizers; this construction is due to Taylor and Muraleedaran [23] and, up to slight variations, will be relevant to construct suitable lifts of these complements inside $\widetilde{H}_0$.

By the above remarks, we can assume that $S_0 \subseteq S$ has at least one irreducible component which generates a Coxeter group of type A, since we have already a group-theoretic splitting otherwise, according to [13, Proposition 5.1].

We first explain how to define a subgroup U_0 which is complementary to $W_0 = \langle S_0 \rangle$ inside N_0 by recalling results from [23]. We introduce some technical notation, illustrated in Example 1 below. As in [23], we associate a partition of n to W_0 by writing W_0 as a direct product of irreducible standard factors, also counting the trivial group, i.e., we write

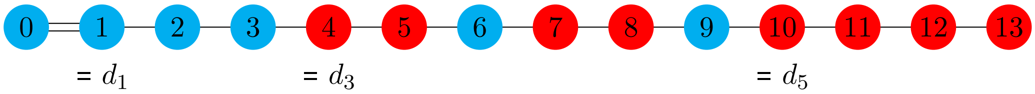
$$W_0 = G(d,1,n_0) \times \prod_{k=1}^{n} G(1,1,k)^{b_k}$$

in such a way that $1^{b_1}2^{b_2}\cdots n^{b_n}$ is a partition of $n - n_0$. Let $b_{i_1}, b_{i_2}, \cdots, b_{i_\ell}$ be the sequence of those b_k such that $b_k \neq 0$, where $i_1 < i_2 < \cdots < i_\ell$. If $b_k \neq 0$, we write $R_k := G(1,1,k)^{b_k} \cong \mathfrak{S}_k^{b_k}$. Let $d_{i_1} = n_0 + 1$, $d_{i_{j+1}} = d_{i_j} + b_{i_j} i_j$ for $1 \leq j \leq \ell - 1$. Up to replacing W_0 by a $\mathfrak{S}_n$-conjugate, if $k = i_j$, then we can assume that the b_k irreducible factors of Coxeter type A_{k-1} have generating sets given by

$$\begin{aligned} J_k^1 &:= \{s_{d_k}, s_{d_k+1}, \ldots, s_{d_k+k-2}\}, \\ J_k^2 &:= \{s_{d_k+k}, \ldots, s_{d_k+2k-2}\}, \\ &\ldots, \\ J_k^{b_k} &:= \{s_{d_k+(b_k-1)k}, \ldots, s_{\underbrace{d_k+b_k k}_{=d_{i_{j+1}}}-2}\}. \end{aligned}$$

We set $J_k = J_k^1 \cup J_k^2 \cup \cdots \cup J_k^{b_k}$ and we have $R_k = \langle J_k \rangle$ (note that $J_1^m = \varnothing$ for all $1 \le m \le b_1$, as J_1 is the trivial group). With this choice of labeling, we get that d_k indexes the leftmost node of J_k in the Dynkin-like diagram associated to W (see Example 1 below), and the irreducible factors of type A of W_0 are arranged from the left to the right, always separated by a single node, with their size increasing from left to right.

Example 1. *Let $W = G(r,1,14)$ and $W_0 = G(1,1,1)^3 \times G(1,1,3)^2 \times G(1,1,5)$. We have $n_0 = 0$ and $b_1 = 3$, $b_3 = 2$, $b_5 = 1$; $i_1 = 1, i_2 = 3, i_3 = 5$, and $d_1 = 1, d_3 = 4, d_5 = 10$. The nodes that are elements of S_0 are the red ones in the picture below. We have $J_3^1 = \{s_4, s_5\}$, $J_3^2 = \{s_7, s_8\}$, $J_5^1 = J_5 = \{s_{10}, s_{11}, s_{12}, s_{13}\}$.*



We now assume that the conditions given in Theorem 1.2 is satisfied, providing suitable polynomials $T_k, T_{0,k} \in K[X]$. As shown in [23], for every k such that $b_k \neq 0$, there is a subgroup $N_k \subseteq N_0$, which normalizes R_k (and centralizes every R_j with $j \neq k$) and acts as the reflection group $G(r,1,b_k)$ on a suitable subspace of the natural module $V = \mathbb{C}^n$ (see [23, Theorem 3.12 and its proof]). Moreover, $U_0 = \prod_{k, b_k \neq 0} N_k$ is a complement to W_0 inside N_0. We give explicit generators of the subgroups N_k, corresponding to the Coxeter-like generators of $G(r,1,b_k)$.

Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{C}^n$, where $W < \mathrm{GL}_n(\mathbb{C})$. The direct product decomposition of W_0 corresponds to a tensor product decomposition $\mathbb{C}^n = \mathbb{C}^{n_0} \oplus \bigoplus_{k=1}^n \mathbb{C}^k \otimes \mathbb{C}^{b_k}$ of the space, where the element $a_p \otimes b_q$ of the canonical basis of $\mathbb{C}^k \otimes \mathbb{C}^{b_k}$ is mapped to $e_{d_k+(q-1)k+p-1}$, when $b_k \neq 0$. Then $G(r,1,b_k)$ acts on each $\mathbb{C}^k \otimes \mathbb{C}^{b_k}$ as $f_k = \mathbb{1} \otimes \rho$, where ρ is the natural representation of $G(r,1,b_k)$ on $\mathbb{C}^{b_k}$ and $\mathbb{1}$ is the trivial representation on $\mathbb{C}^k$. In particular, $s_k^{(0)} := f_k(s_0)$ acts by multiplication by ζ on each $a_p \otimes b_1$ and by the identity on the other basis vectors, while $s_k^{(i)} := f_k(s_i)$ acts through $a_r \otimes b_p \mapsto a_r \otimes b_{s_i(p)}$.

In fact, we have $s_k^{(1)}, \dots, s_k^{(b_k-1)} \in \langle S' \rangle \subseteq \mathfrak{S}_n$, while $s_k^{(0)} = t_{d_k} t_{d_k+1} \cdots t_{d_k+k-1} \in (\mathbb{Z}/r\mathbb{Z})^n$.

Hence the semidirect product decomposition of N_k (obtained by viewing it as the reflection group $G(r,1,b_k)$) is compatible with the semidirect product decomposition $(\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$ of W, in the sense that it is the same as the intersection of the semidirect product decomposition of W with N_k. In fact, as $s_k^{(1)}, \dots, s_k^{(b_k-1)} \in \mathfrak{S}_n \cap N_W(R_k)$, we have $s_k^{(1)}, \dots, s_k^{(b_k-1)} \in N_{\mathfrak{S}_n}(R_k)$, and they turn out to be among the generators of the complement to R_k inside $\mathfrak{S}_n$ as described by Howlett [15]: more precisely, for all $i = 1, \dots, b_k - 1$, we have

$$s_k^{(i)} = w_0(K_i) w_0(J_k^{i+1} \cup J_k^i), \tag{3.1}$$

where $K_i = J_k^{i+1} \cup J_k^i \cup \{s_{d_k - 1 + ik}\}$ is the connected closure of J_k^i and J_k^{i+1} (obtained by just adding the unique simple reflection between them in the Dynkin diagram) and $w_0(X)$ denotes the longest element in the Coxeter group of type A generated by $X \subseteq S'$.

Also note that $s_k^{(0)}, s_k^{(1)}, \dots, s_k^{(b_k-1)}$ all lie in $W(r)$.

Example 2. *Let $W = G(r,1,4)$, $S_0 = \{s_1, s_3\}$. Then $W_0 = G(1,1,2)^2$, hence 2 is the only integer k such that $b_k \neq 0$ and we have $S_0 = J_2$, $b_2 = 2$. Hence $U_0 = N_2 \cong G(r,1,2)$, $E_2^1 = \{e_1, e_2\}$, $E_2^2 = \{e_3, e_4\}$, and the Coxeter-like generators $s_2^{(0)}$, $s_2^{(1)}$ of N_2 are given by the* *matrices*

$$s_2^{(0)} = \left(\begin{array}{cc|cc} \zeta & 0 & & \\ 0 & \zeta & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right), \quad s_2^{(1)} = \left(\begin{array}{cc|cc} & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \end{array}\right).$$

3.3. **Lifting the complement.** Let $\tau_1 = \sigma_0 \in B$ and for all $2 \leq i \leq n$, let

$$\tau_i := \sigma_{i-1}\sigma_{i-2}\cdots\sigma_1\sigma_0\sigma_1\cdots\sigma_{i-1} \in B.$$

Note that τ_i lies in $\mathcal{D}$, and is the simple attached to t_i. We now construct lifts of the generators of N_k in $\widetilde{B}_0$. As $s_k^{(i)} \in W(r)$ for all $i = 0, \ldots, b_k - 1$ we can take their lifts in $\mathcal{D} \subseteq B^+ \subseteq B$ (obtained by lifting any reduced expression of them in B), which we denote by $\sigma_k^{(i)}$. As $s_k^{(i)} = \pi(\sigma_k^{(i)}) \in U_0 \subseteq N_0$, we have $\sigma_k^{(i)} \in \hat{B}_0$ for all $i = 0, \ldots, b_k - 1$, and we can take their image in $\widetilde{B}_0$, which we still denote by $\sigma_k^{(i)}$.

Note that $s_k^{(0)} = t_{d_k}t_{d_k+1}\cdots t_{d_k+k-1}$ and we have $\ell(s_k^{(0)}) = \sum_{i=0}^{k-1}\ell(t_{d_k+i})$, hence by Lemma 3.1 we have $\sigma_k^{(0)} = \tau_{d_k}\tau_{d_k+1}\cdots\tau_{d_k+k-1}$.

Lemma 3.2. *Let $W = G(r,1,n)$ and W_0 a standard parabolic subgroup as above. Let k such that $b_k \neq 0$. In $\widetilde{B}_0$, we have*

$$(\sigma_k^{(0)})^r = \Delta_{J_k^1}^2,$$

where $\Delta_{J_k^1}$ is the image in $\widetilde{B}_0$ of the simple attached to the longest element $w_0(J_k^1)$ in the type A parabolic subgroup of W generated by J_k^1. In particular, if $k = 1$, then $(\sigma_k^{(0)})^r = 1$.

Proof. We claim that in $\widetilde{B}_0$, for all $d_k \leq i \leq d_k + k - 1$, we have $\tau_{d_k}^d = 1$ and

$$(\tau_i)^d = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_{d_k+1}\sigma_{d_k}^2\sigma_{d_k+1}\cdots\sigma_{i-1}$$

for $i > d_k$. Note that

$$\tau_i = \sigma_{i-1}\cdots\sigma_1\sigma_0\sigma_1^{-1}\cdots\sigma_{i-1}^{-1}(\sigma_{i-1}\cdots\sigma_2\sigma_1^2\sigma_2^{-1}\cdots\sigma_{i-1}^{-1})(\sigma_{i-1}\cdots\sigma_3\sigma_2^2\sigma_3^{-1}\cdots\sigma_{i-1}^{-1})\cdots\sigma_{i-1}^2.$$

For all $j \leq d_k - 1$, we have $\sigma_{i-1}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\cdots\sigma_{i-1}^{-1} \in Q_0$ (since the reflection $s_{i-1}\cdots s_j\cdots s_{i-1}$ is not in W_0 in that case, as the reflection s_{d_k-1} appears in the reduced expression $s_{i-1}\cdots s_j\cdots s_{i-1}$). Hence deleting these factors, we get that

$$\begin{aligned} \tau_i &= \sigma_{i-1}\cdots\sigma_1\sigma_0\sigma_1^{-1}\cdots\sigma_{i-1}^{-1}(\sigma_{i-1}\cdots\sigma_2\sigma_{d_k}^2\sigma_2^{-1}\cdots\sigma_{i-1}^{-1})(\sigma_{i-1}\cdots\sigma_3\sigma_{d_k+1}^2\sigma_3^{-1}\cdots\sigma_{i-1}^{-1})\cdots\sigma_{i-1}^2 \\ &= \sigma_{i-1}\cdots\sigma_1\sigma_0\sigma_1^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}\cdots\sigma_{i-1}. \end{aligned}$$

Raising to the power d we obtain

$$(\tau_i)^d = \sigma_{i-1}\cdots\sigma_1\sigma_0(\underbrace{\sigma_1^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}\cdots\sigma_{i-1}^2\sigma_{i-2}\cdots\sigma_1}_{=:\beta}\sigma_0)^{d-1}\sigma_1^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}\cdots\sigma_{i-1}.$$

We claim that $\beta \in Q_0$. Indeed, we have

$$\begin{aligned} \beta &= \sigma_1^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}\cdots\sigma_{i-1}^2\sigma_{i-2}\cdots\sigma_1 \\ &= (\sigma_1^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}\cdots\sigma_{i-1}^2\sigma_{i-2}^{-1}\cdots\sigma_{d_k+1}^{-1}\sigma_{d_k}\cdots\sigma_1)\cdots(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}^2\sigma_{d_k-1}\cdots\sigma_1), \end{aligned}$$

and all the factors are in Q_0 (since $s_{d_k-1} \notin W_0$). Hence this gives

$$(\tau_i)^d = \sigma_{i-1}\cdots\sigma_1 \underbrace{\sigma_0^d}_{\in Q_0} \sigma_1^{-1}\cdots\sigma_{d_k-1}^{-1}\sigma_{d_k}\cdots\sigma_{i-1}$$

which lies in Q_0 if $i = d_k$ and is equal to $\sigma_{i-1}\cdots\sigma_{d_k}^2\sigma_{d_k+1}\cdots\sigma_{i-1}$ otherwise. This shows the claim. Now using the fact that the τ_i's commute with each other, we get that

$$(\sigma_k^{(0)})^d = \sigma_{d_k}^2(\sigma_{d_k+1}\sigma_{d_k}^2\sigma_{d_k+1})\cdots(\sigma_{d_k+k-2}\sigma_{d_k+k-3}\cdots\sigma_{d_k}^2\sigma_{d_k+1}\cdots\sigma_{d_k+k-2}) = \Delta_{J_k^1}^2,$$

where the last equality follows from the fact that in a type A_m braid group with standard generators $\sigma_1, \sigma_2, \dots, \sigma_m$ (and corresponding Coxeter generators $s_1, s_2, \dots, s_m$), we have

$$\begin{aligned} \Delta_{m-1}^2\sigma_m\sigma_{m-1}\cdots\sigma_1^2\sigma_2\cdots\sigma_m &= \Delta_{m-1}(\Delta_{m-1}\sigma_m\sigma_{m-1}\cdots\sigma_1)\sigma_1\sigma_2\cdots\sigma_m \\ &= \Delta_{m-1}\Delta_m\sigma_1\sigma_2\cdots\sigma_m = (\Delta_{m-1}\sigma_m\sigma_{m-1}\cdots\sigma_1)\Delta_m = \Delta_m^2, \end{aligned}$$

where Δ_m is the lift of the longest element in $\langle s_1, s_2, \dots, s_m\rangle$ and Δ_{m-1} is the lift of the longest element of the parabolic subgroup $\langle s_1, s_2, \dots, s_{m-1}\rangle$.

□

We would like to show that our lifts $\sigma_k^{(i)}$ satisfy the defining relations of the Coxeter-like presentation of U_0. Unfortunately, this is not true, as it would give a splitting of the short exact sequence (2.1). But we shall prove that the braid relations between these generators are satisfied already in B; the problem will come from the fact that the generators $\sigma_k^{(0)}$ fail to be of order d inside $\widetilde{B}_0$, as we have just shown in Lemma 3.2. We will need to twist our lift by a suitable element of H_0 to get the order relation in that case.

3.3.1. *Braid relations.* The fact that the lifts $\sigma_k^{(i)}$ satisfy the same braid relations as their images in W, that is, that

- $\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)} = \sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)}$ for all k such that $1 < b_k$,
- $\sigma_k^{(i)}\sigma_k^{(i+1)}\sigma_k^{(i)} = \sigma_k^{(i+1)}\sigma_k^{(i)}\sigma_k^{(i+1)}$ for all $1 \leq i < b_k - 1$,
- $\sigma_k^{(i)}\sigma_k^{(j)} = \sigma_k^{(j)}\sigma_k^{(i)}$ for all $0 \leq i < j - 1 \leq b_k - 2$,
- $\sigma_k^{(i)}\sigma_\ell^{(j)} = \sigma_\ell^{(j)}\sigma_k^{(i)}$ for all $k \neq \ell$,

follows from the fact that if we consider these relations inside U_0 (that is, if we replace every $\sigma_k^{(i)}$ by $s_k^{(i)}$), then each side of each relation is an element of $W(r)$, and moreover, it is readily checked using [6, Lemma 1.5] that the length of each side in terms of the generating set S is the sum of the lengths of the various factors. For instance, we have

$$\ell(s_k^{(0)}s_k^{(1)}s_k^{(0)}s_k^{(1)}) = \ell(s_k^{(0)}) + \ell(s_k^{(1)}) + \ell(s_k^{(0)}) + \ell(s_k^{(1)}).$$

Hence the fact that the relations above are satisfied in B is a consequence of Lemma 3.1.

Remark 3.3. *In fact, as U_0 is isomorphic to a complex reflection group which is a direct product of groups of type $G(r, 1, b_k)$, we can consider its braid group $\mathcal{B}(U_0)$, which is a direct product of Artin groups of type B_{b_k}. What we did above is nothing but constructing a group morphism $\psi : \mathcal{B}(U_0) \to \widetilde{B}_0$.*

3.3.2. *Order relations.* Consider $s_k^{(i)}$ with $1 \leq i \leq b_k - 1$. As $i \geq 1$ we have that $s_k^{(i)} \in \mathfrak{S}_n \cap N_W(G_0)$, where $G_0 = W_0 \cap \mathfrak{S}_n$. Hence we have $s_k^{(i)} \in N_{\mathfrak{S}_n}(G_0)$. The group G_0 is a (standard) parabolic subgroup of the type A Coxeter group $\mathfrak{S}_n$ and $s_k^{(i)}$ is equal to $w_0(K_i)w_0(J_k^{i+1} \cup J_k^i)$ (see (3.1)), which is a generator of the Howlett complement U_0^G to G_0 inside $N_{\mathfrak{S}_n}(G_0)$.

In [13], the definition of the Howlett complement U_0^G was generalized to any reflection subgroup G_0 of a (possibly infinite) Coxeter group G. Let π denote the quotient map $\mathcal{B}(G) \twoheadrightarrow G$. In the case where G is finite, it was shown by Henderson and the authors (see [13, Corollary 3.12]) that the set-theoretic map $U_0^G \to \pi^{-1}(N_G(G_0))$ given by taking positive lifts of elements in $U_0^G \subseteq G$ in the Artin group $\mathcal{B}(G)$ of G becomes an injective group homomorphism $U_0^G \to \pi^{-1}(N_G(G_0))/Q_0^G$ when passing to the quotient on the right side, where Q_0^G is the subgroup of the pure braid group of G generated by the squares of the braided reflections around hyperplanes attached to reflections which are not in G_0.

Applied to our case with $G = \mathfrak{S}_n$, where $\mathcal{B}(G)$ is the usual braid group $\mathcal{B}_n$ on n strands, this result implies that the squares of the $\sigma_k^{(i)}$'s inside $\mathcal{B}_n$ belong to Q_0^G. Indeed, they are images of $1 = (s_k^{(i)})^2$ under the above injective morphism. Now, the morphism $\mathcal{B}_n \to B$ mapping the j-th Artin generator σ_j for $1 \leq i \leq n-1$ to $\sigma_j \in B$ is injective – for instance because its composition with $B \to \mathcal{B}_{n+1}$, $\sigma_0 \mapsto \sigma_1^2$, $\sigma_j \mapsto \sigma_{j+1}$ for $j \geq 1$ is the injective morphism $\mathcal{B}_n \to \mathcal{B}_{n+1}$, $\sigma_i \mapsto \sigma_{i+1}$ of adding one strand on the left. Moreover, it maps any meridian around some reflecting hyperplane in $\mathfrak{S}_n$ (that is, a conjugate of some squared generator of $\mathcal{B}_n$) to a meridian around the same reflecting hyperplane for W, hence embeds Q_0^G inside Q_0. Therefore $(\sigma_k^{(i)})^2 \in Q_0$ and, inside $\widetilde{B}_0$, we have $(\sigma_k^{(i)})^2 = 1$.

It remains to treat the case of the generators $s_k^{(0)}$, for all k such that $b_k \neq 0$. We claim that for every such k, $\sigma_k^{(0)} T_k(\Delta_{J_k^1})$ is a lift of $s_k^{(0)}$ inside $\widetilde{H}_0$, where $T_k \in K[X]$ is a polynomials satisfying the assumptions of Theorem 1.2. Here, we abuse notation and write $\Delta_{J_k^1}$ for the image of the Garside element (i.e., the lift of the longest element of $\langle J_k^1 \rangle$) of the type A Artin group $\mathcal{B}(\langle J_k^1 \rangle) \subseteq B$ attached to $\langle J_k^1 \rangle$ inside the Hecke algebra H_0 of W_0. As $\widetilde{H}_0$ is a quotient of the group algebra of $\widetilde{B}_0$ which contains the Hecke algebra H_0 of the parabolic subgroup W_0 (which contains $\langle J_k^1 \rangle$), this gives a well-defined element of $\widetilde{H}_0$.

Note that $\sigma_k^{(0)}$ commutes with every Artin generator σ_i of $\mathcal{B}(\langle J_k^1 \rangle)$, hence $\sigma_k^{(0)} \Delta_{J_k^1} = \Delta_{J_k^1} \sigma_k^{(0)}$. Let T_k be a polynomial as in the assumptions of Theorem 1.2. By Lemma 3.2, we have

$$(\sigma_k^{(0)} T_k(\Delta_{J_k^1}))^d = (\sigma_k^{(0)})^d T_k(\Delta_{J_k^1})^d = (\Delta_{J_k^1}^2) T_k(\Delta_{J_k^1})^d = 1$$

Note that $\sigma_k^{(i)}$ commutes with $\Delta_{J_k^1}$ whenever $2 \geq i$, hence the commutation relation between $T(\Delta_{J_k^1})\sigma_k^{(0)}$ and $\sigma_k^{(i)}$ is satisfied inside $H_0 \subseteq \widetilde{H}_0$ in that case. Also note that $\sigma_k^{(1)} \Delta_{J_k^j} = \Delta_{J_k^\ell} \sigma_k^{(1)}$, $\{j, \ell\} = \{1, 2\}$ (as the images in $\mathfrak{S}_n$ of the various factors satisfy the same relation, with the length of the various factors adding). As $\Delta_{J_k^1}$ and $\Delta_{J_k^2}$ commute with each other and

$\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)} = \sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)}$ as seen in Subsection 3.3.1, we deduce that

$$\begin{aligned}\sigma_k^{(0)}T(\Delta_{J_k^1})\sigma_k^{(1)}\sigma_k^{(0)}T(\Delta_{J_k^1})\sigma_k^{(1)} &= T(\Delta_{J_k^1})T(\Delta_{J_k^2})\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)} \\ &= T(\Delta_{J_k^1})T(\Delta_{J_k^2})\sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)} \\ &= T(\Delta_{J_k^2})T(\Delta_{J_k^1})\sigma_k^{(1)}\sigma_k^{(0)}\sigma_k^{(1)}\sigma_k^{(0)} \\ &= \sigma_k^{(1)}\sigma_k^{(0)}T(\Delta_{J_k^1})\sigma_k^{(1)}\sigma_k^{(0)}T(\Delta_{J_k^1}),\end{aligned}$$

hence the braid relation between $\sigma_k^{(0)}T(\Delta_{J_k^1})$ and $\sigma_k^{(1)}$ is still satisfied. This shows that one can lift N_k, for all k such that $b_k \neq 0$.

To conclude this case, note that the defined lifts of two generators belonging to different factors N_k and N_ℓ of U_0 have to commute with each other. Indeed, on one hand we have that $\Delta_{J_k^1}$ and $\Delta_{J_\ell^1}$ commute with each other for all k, ℓ. On the other hand, $\Delta_{J_k^1}$ (resp. $\Delta_{J_\ell^1}$) also commutes with all $\sigma_\ell^{(i)}$ (resp. $\sigma_k^{(i)}$), and $s_k^{(i)}$, $s_\ell^{(j)}$ are elements in $W(r)$ with product also in $W(r)$ and such that $s_k^{(i)} \cdot s_\ell^{(j)} = s_\ell^{(j)} \cdot s_k^{(i)}$ and $\ell(s_k^{(i)} \cdot s_\ell^{(j)}) = \ell(s_k^{(i)}) + \ell(s_\ell^{(j)})$ for all i, j and $k \neq \ell$, hence these observations together with Lemma 3.1 allow us to conclude that generators from distinct components N_k, N_ℓ commute with each other.

Hence, we just showed that, modifying the group morphim ψ from Remark 3.3 by twisting $\sigma_k^{(0)}$ by $\sigma_k^{(0)}T_k(\Delta_{J_k^1})$ for all k such that $b_k \neq 0$, we get a group morphism ψ' from $U_0 \cong \overline{N_0}$ to $\widetilde{H}_0^\times$, which satisfies the assumptions of Lemma 2.3. This concludes the proof of Theorem 1.2 in this case.

4. Groups of type $G(de, e, n)$

In this section, we complete the proof of Theorem 1.2, using the results of the preceeding section for $G(r, 1, n)$ by setting $r = de$.

4.1. General considerations. Let $W = G(de, e, n)$, and $W^! = G(r, 1, n)$ with $r = de$. We denote by $\mathcal{A}$, $\mathcal{A}^!$ the corresponding hyperplane arrangements and, for $L_0 \subset \mathbb{C}^n$, we denote by W_0 and $W_0^!$ the parabolic subgroups defined as the pointwise stabilizers of L_0 in W and $W^!$, respectively. We have $W_0 = W \cap W_0^!$.

Let $\mathcal{E}$ (resp. $\mathcal{E}^!$) denote the collection of all hyperplanes in $\mathcal{A}$ (resp. $\mathcal{A}^!$) containing L_0. We can and will assume that L_0 is the intersection of some collection of hyperplanes of $\mathcal{A}$, which implies that L_0 is equal to the intersection $\bigcap \mathcal{E}$ of the hyperplanes inside $\mathcal{E}$ (and hence also that $L_0 = \bigcap \mathcal{E}'$).

Since W_0 (resp. $W_0^!$) is generated by its reflections, we have

$$g \in N_0 := N_W(W_0) \Leftrightarrow g(\mathcal{E}) = \mathcal{E} \Leftrightarrow g(L_0) = L_0$$

and

$$g \in N_0^! := N_{W^!}(W_0^!) \Leftrightarrow g(\mathcal{E}^!) = \mathcal{E}^!.$$

Hence, as $g(L_0) = L_0$ implies that $g(\mathcal{E}^!) = \mathcal{E}^!$, we get $N_0 \subset N_0^! \cap W$. Conversely, if $g \in N_0^! \cap W$ we get $g(\mathcal{E}^!) = \mathcal{E}^!$ hence $g(\bigcap \mathcal{E}^!) = \bigcap \mathcal{E}^!$. But $\bigcap \mathcal{E} = L_0 = \bigcap \mathcal{E}^!$ hence $\bigcap \mathcal{E}^! = L_0$ and $g(L_0) = L_0$, whence $g \in N_0$. This proves the following:

Lemma 4.1. *We have $N_0 = N_0^! \cap W$.*

At the level of braid groups, we denote by B the braid group of W, $B^!$ the braid group of $W^!$. Let $\pi : B \twoheadrightarrow W$ and $\pi^! : B^! \to W^!$ be the natural projections, with kernels $P, P^!$, respectively. Recall that W is a normal subgroup of $W^!$ with cyclic quotient C.

We set $M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup \mathcal{A}$, $M(\mathcal{A}^!) = \mathbb{C}^n \setminus \bigcup \mathcal{A}^!$. The inclusion map $M(\mathcal{A}^!) \subset M(\mathcal{A})$ induces a surjective homomorphism $\pi_1(M(\mathcal{A}^!)) \twoheadrightarrow \pi_1(M(\mathcal{A}))$. Recall that $\hat{B}_0^! = \pi_1(M(\mathcal{A}^!)/N_0^!)$, $\hat{B}_0 = \pi_1(M(\mathcal{A})/N_0)$.

We introduce $B^\sharp = \pi_1(M(\mathcal{A}^!)/W)$. It is a normal subgroup of $B^! = \pi_1(M(\mathcal{A}^!)/W^!)$, and there is a natural projection map $B^\sharp \twoheadrightarrow B$ with kernel $\mathrm{Ker}(M(\mathcal{A}^!) \twoheadrightarrow M(\mathcal{A}))$.

In particular we have $B^\sharp = B$ when $\mathcal{A}^! = \mathcal{A}$, that is, as soon as $d > 1$. We denote by $\pi^\sharp : B^\sharp \twoheadrightarrow W$ the natural projection map.

We denote by $p_W : W^! \to C = W^!/W$ the canonical projection and set $p_B = p_W \circ \pi^!$. We have $B^\sharp = \mathrm{Ker} p_B$ (see [9]). This yields the following commutative diagram:

(4.1)
$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & P^! & = & P^! & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & B^\sharp & \longrightarrow & B^! & \xrightarrow{p_B} & C & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi^\sharp} & & \downarrow{\scriptstyle \pi^!} & & \| & & \\
1 & \longrightarrow & W & \longrightarrow & W^! & \xrightarrow{p_W} & C & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The usefulness of introducing this 'fake braid group' $B^\sharp$ is that it can be used in general, regardless of the value of d, to define $\widetilde{B}_0$ as a quotient of $\hat{B}_0^\sharp := (\pi^\sharp)^{-1}(N_0)$. Recall that $\widetilde{B}_0 = \hat{B}_0/Q_0$ where Q_0 is the kernel of the natural map $P = \pi_1(M(\mathcal{A})) \twoheadrightarrow \pi_1(M(\mathcal{E}))$. Let us denote $q : P^! \twoheadrightarrow P$ the projection map orginating from the inclusion $M(\mathcal{A}^!) \subset M(\mathcal{A})$, and set $Q_0^\sharp = q^{-1}(Q_0)$. It is the kernel of the projection map $P^! = \pi_1(M(\mathcal{A}^!)) \twoheadrightarrow \pi_1(M(\mathcal{E}))$. We have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P^! & \longrightarrow & \hat{B}_0^\sharp & \longrightarrow & N_0 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle q} & & \downarrow & & \| & & \\
1 & \longrightarrow & P & \longrightarrow & \hat{B}_0 & \longrightarrow & N_0 & \longrightarrow & 1
\end{array}$$

which implies that the middle map is also surjective, and that its composite with the canonical projection $\hat{B}_0 \twoheadrightarrow \widetilde{B}_0$ has kernel $q^{-1}(Q_0) = Q_0^\sharp$. This identifies $\widetilde{B}_0$ with $\hat{B}_0^\sharp/Q_0^\sharp$.

From this new description we easily get the following slight generalization of Lemma 2.5 in the specific case of reflection subgroups of $W = G(de, e, n)$:

Lemma 4.2. *Let W_1, W_2 be two reflection subgroups of $W = G(de,e,n)$, which are conjugate in $W^! = G(de,1,n)$, let $G_i = N_W(W_i)$, and $\hat{B}_i, \widetilde{B}_i, B_i, \widetilde{H}_i$ as in Lemma . Then there exists a group isomorphism $\widetilde{B}_1 \to \widetilde{B}_2$ mapping B_1 to B_2 and an algebra isomorphism $\widetilde{H}_1 \to \widetilde{H}_2$ mapping H_1 to H_2.*

Proof. We set $B_i^\sharp, \hat{B}_i^\sharp$ for $i = 1,2$ the groups $B^\sharp, \hat{B}^\sharp$ defined above for $W_0 = W_i$. We claim that $\hat{B}_1^\sharp$ and $\hat{B}_2^\sharp$ are conjugate in $B^!$.

This can be seen as follows: let $w \in W^!$ such that $wW_1w^{-1} = W_2$. As $W \trianglelefteq W^!$ we have $wG_1w^{-1} = G_2$. Let $b \in (\pi^!)^{-1}(\{w\})$. Note that using the diagram (4.1), we can see each $\hat{B}_i^\sharp$ inside $B^!$. We claim that $b\hat{B}_1^\sharp b^{-1} = \hat{B}_2^\sharp$. Indeed, for $x \in \hat{B}_1^\sharp$, we have that $\pi^!(bxb^{-1}) \in wG_1w^{-1} = G_2$. By (4.1), it implies that there is $b' \in \hat{B}_2^\sharp$ such that $\pi^!(b') = \pi^!(bxb^{-1})$. Since both maps $\pi^\sharp$ and $\pi^!$ have the same kernel we have $\mathrm{Ker}\pi^! = P^! \subseteq \hat{B}_2^\sharp$ and since $b' \in \hat{B}_2^\sharp$, we get that $bxb^{-1} \in \hat{B}_2^\sharp$. Hence $b\hat{B}_1^\sharp b^{-1} \subseteq \hat{B}_2^\sharp$ and conversely we show that $b^{-1}\hat{B}_2^\sharp b \subseteq \hat{B}_1^\sharp$.

The rest of the proof is then the same as in the proof of Lemma 2.5, only replacing Q_i with $Q_i^\sharp = \mathrm{Ker}(\hat{B}_i^\sharp \twoheadrightarrow \widetilde{B}_i)$, and noticing that it is the subgroup of $P^!$ generated by the meridians around the hyperplanes in $\mathcal{A}^!$ which are not reflecting hyperplanes for W_i. □

4.2. **Lifting complements.** From the commutativity of the diagram (4.1) we readily get that $\hat{B}_0^\sharp = (\pi^\sharp)^{-1}(N_0) = \hat{B}_0^! \cap B^\sharp$. Let us now consider the subgroup $Q_0^\sharp$ of $P^!$, which is generated by the meridians around the hyperplanes which are not in $\mathcal{E}$. We have $\widetilde{B}_0 = \hat{B}_0^\sharp / Q_0^\sharp$ and $\widetilde{B}_0^! = \hat{B}_0^! / Q_0^!$. Let us denote $\hat{p}_B$ the restriction of p_B to $\hat{B}_0^!$. Since $Q_0^! \subset \mathrm{Ker}\hat{p}_B$, it induces a morphism $\tilde{p}_B : \widetilde{B}_0^! \to C$ whose kernel is exactly $\widetilde{B}_0^\sharp := \hat{B}_0^\sharp / Q_0^!$, which projects onto $\widetilde{B}_0$ with kernel $Q_0^\sharp / Q_0^!$.

Let $U_0^!$ be a complement of $W_0^!$ inside $N_0^!$. We make the following assumption

$$U_0 := U_0^! \cap W \text{ is complementary to } W_0 \text{ inside } N_0. \tag{4.2}$$

This assumption does *not* hold in general, but one can always choose $U_0^!$ so that it works, as we will see below.

Denote by $\tilde{\pi}^! : \widetilde{B}_0^! \to W^!$, $\tilde{\pi}^\sharp : \widetilde{B}_0^\sharp \to W$, $\tilde{\pi} : \widetilde{B}_0 \to W$ the morphisms induced by $\pi^!$, $\pi^\sharp$ and π, respectively. Note that $\tilde{p}_B = p_W \circ \tilde{\pi}^!$. If we have a morphism $\psi^! : U_0^! \to \widetilde{B}_0^!$ satisfying the following assumption

$$\tilde{\pi}^! \circ \psi^! = \mathrm{Id}_{U_0^!}, \tag{4.3}$$

then we also have $\tilde{p}_B \circ \psi^! = (p_W)_{U_0^!}$. This implies that, under assumptions (4.2) and (4.3), the map $\psi^!$ restricts to a morphism $\psi : U_0 \to \widetilde{B}_0^\sharp \subset \widetilde{B}_0^!$. Indeed, for $g \in U_0^!$, we have

$$\psi^!(g) \in \widetilde{B}_0^\sharp \iff \tilde{p}_B(\psi^!(g)) = 1 \iff p_W(g) = 1,$$

and the last condition is always fulfilled if $g \in U_0 = U_0^! \cap W$. Finally, $\tilde{\pi}^! \circ \psi^! = \mathrm{Id}_{U_0^!}$ immediately implies $\tilde{\pi}^\sharp \circ \psi = \mathrm{Id}_{U_0}$, so that ψ indeed provides a convenient lift into $\widetilde{B}_0^\sharp$.

The complement $U_0^!$ that we chose is itself a complex reflection group, with irreducible components belonging to the general series. Its braid group $\Gamma^!$ is an Artin group with irreducible components of type B, with a projection map $\gamma^! : \Gamma^! \twoheadrightarrow U_0^!$ (see Remark 3.3). Let

$\Gamma = \{g \in \Gamma^! \, ; \, \gamma^!(g) \in W\}$ and γ the restriction of $\gamma^!$ to Γ. Under assumption (4.2), γ is a map $\Gamma \twoheadrightarrow U_0$ and $\mathrm{Ker}(\Gamma^! \twoheadrightarrow U_0^!) = \mathrm{Ker}(\Gamma \twoheadrightarrow U_0)$.

While it is not possible, in general, to obtain a lifting morphism $U_0^! \to \widetilde{B}_0^!$ as above, we were able in the cases satisfying this assumption to construct morphisms $\psi^! : \Gamma^! \to \widetilde{B}_0^!$ such that $\tilde{\pi}^! \circ \psi^! = \gamma^!$.

The restriction of $\psi^!$ to Γ then provides a morphism $\psi : \Gamma \to \widetilde{B}_0^\sharp$ such that $\tilde{\pi}^\sharp \circ \psi = \gamma$.

4.3. **Case where $W_0 = W_0^!$.** In some cases we can directly relate the algebras $\widetilde{H}_0^!$ and $\widetilde{H}_0$ for arbitrary parameters. Recall that both $N_0^!$ and N_0 act on $\mathcal{E}$. In order to get comparable parameters on both sides we need the following assumption

(4.4) The orbits of $\mathcal{E}$ under N_0 and $N_0^!$ are the same.

This condition will be satisfied in the cases in which we are interested. Under this condition, one has a natural morphism from $\widetilde{H}_0$ to some specialization of $\widetilde{H}_0^!$. Indeed, recall that $\widetilde{H}_0$ and $\widetilde{H}_0^!$ are both defined using parameters $u_{s,i}, s \in \mathcal{R}_0^*, 0 \le i < o(s)$, but with the additional condition that $u_{s,i} = u_{wsw^{-1},i}$ for all $w \in N_0 = N_W(W_0)$ in the case of $\widetilde{H}_0$, for all $w \in N_{W^!}(W_0)$ in the case of $\widetilde{H}_0^!$. Since the conjugation action of the normalizers on $\mathcal{R}_0^*$ is the same as their action on $\mathcal{E}$, assumption (4.4) says that the conditions on the parameters are the same and thus the algebras are defined over the same ring.

Then, the composition of $K\widetilde{B}_0^\sharp \to K\widetilde{B}_0^!$ and $K\widetilde{B}_0^! \to \widetilde{H}_0^!$ factorizes through $K\widetilde{B}_0^\sharp \to \widetilde{H}_0$, as the Hecke relations are obviously mapped to 0 inside $\widetilde{H}_0$, and the possibly additional meridians around $\mathcal{A}^! \setminus \mathcal{A}$ are mapped to 1. From this we get an algebra morphism $\widetilde{H}_0 \to \widetilde{H}_0^!$.

The following lemma will be useful in the special case where $W_0 = W_0^!$.

Lemma 4.3. *When $W_0 = W_0^!$, the algebra morphism $\widetilde{H}_0 \to \widetilde{H}_0^!$ is injective.*

Proof. We first notice that the projection maps $\widetilde{B}_0^! \to N_0^!$ and $\widetilde{B}_0^\sharp \to N_0$ induce isomorphisms $\widetilde{B}_0^!/B_0 \simeq N_0^!/W_0^! = N_0^!/W_0$ and a clearly surjective morphism $\widetilde{B}_0^\sharp/B_0 \to N_0/W_0$. This morphism is actually an isomorphism, as its kernel coincides with the kernel of the natural morphism $\widetilde{B}_0^\sharp/B_0 \to \widetilde{B}_0^!/B_0$, which is injective as it is induced by the inclusion map $\hat{B}_0^\sharp \subset \hat{B}_0^!$.

We choose sets of representatives $E^\sharp \subset \widetilde{B}_0^\sharp$ and $E^! \subset \widetilde{B}_0^!$ such that $E^\sharp \subset E^!$. Then we have morphisms of KB_0-modules

$$\bigoplus_{b \in E^\sharp} (KB_0)b \simeq K\widetilde{B}_0^\sharp \to K\widetilde{B}_0^! \simeq \bigoplus_{b \in E^!} (KB_0)b$$

where $K\widetilde{B}_0^\sharp \to K\widetilde{B}_0^!$ is the inclusion map, and the composition is the identity on $E^\sharp \subset E^!$. Dividing out by the defining ideals of $\widetilde{H}_0$ and $\widetilde{H}_0^!$ then yields a sequence of morphisms

$$\bigoplus_{b \in E^\sharp} H_0 b \to \widetilde{H}_0 \to \widetilde{H}_0^! \to \bigoplus_{b \in E^!} H_0 b$$

whose composite is injective. Now, the morphisms $\bigoplus_{b \in E^\sharp} H_0 b \to \widetilde{H}_0$ and $\bigoplus_{b \in E^!} H_0 b \to \widetilde{H}_0^!$ are isomorphisms (see [20] section 2.3.1), whence the natural morphism $\widetilde{H}_0 \to \widetilde{H}_0^!$ is also injective.

□

Notice that in this case, the natural morphism $\widetilde{B}_0^\sharp \to \widetilde{B}_0$ is actually an isomorphism.

4.4. **Lifting complements for standard parabolic subgroups.** Since the reflecting hyperplanes for W are reflecting hyperplanes for $W^!$, every parabolic subgroup W_0 of W is a subgroup of a uniquely defined parabolic subgroup $W_0^!$ of $W^!$ of the same rank. Now, by [22, Theorem 3.9], every parabolic subgroup $W_0^!$ of $W^!$ is a conjugate inside $W^!$ of a standard one. Hence by Lemma 4.2, we can assume that $W_0^!$ itself is standard. In this case W_0 is also standard.

Recall from Section 3 that such a standard parabolic subgroup is determined by a pair (n_0, λ), where $0 \leq n_0 \leq n$, and λ is a partition of $n - n_0$. We will keep the notation introduced in Section 3, except that all the groups related to $W^!$ will have the symbol ! as exponent. The pair (n_0, λ) will be referred to as the *type* of W_0. If W_0 has type (n_0, λ), then

$$W_0 = G(de, e, n_0) \times \prod_{k=1}^{n} G(1,1,k)^{b_k}, \quad W_0^! = G(de, 1, n_0) \times \prod_{k=1}^{n} G(1,1,k)^{b_k}.$$

4.4.1. *Case where* $n_0 = 0$. In this case, we have $W_0 = W_0^!$, and $U_0 = W \cap U_0^!$ is a complement to W_0 inside N_0. The complement $U_0^!$ is a complex reflection group of type $\prod_{k, b_k \neq 0} G(de, 1, b_k)$ with braid group an Artin group $\Gamma^!$ of type $\prod_{k, b_k \neq 0} B_{b_k}$ (with the convention that $B_1 = A_1$ among Coxeter types). The explicit description of U_0 is less straightforward than in the other cases (see [23, Theorem 3.12]), however assumption (4.2) is satisfied and we constructed in Section 3 a morphism $\psi^! : \Gamma^! \to \widetilde{B}_0^!$ satisfying assumption (4.3). Therefore we can apply the results of Section 4.2 and by restriction we get a morphism $\psi : \Gamma \to \widetilde{B}_0^\sharp$ such that $\tilde{\pi}^\sharp \circ \psi = \gamma$. Since assumption (4.4) is also satisfied, the natural compositions towards the Hecke algebras fit into the following commutative diagram (see the paragraph before Lemma 4.3):

$$\begin{array}{ccccc} \Gamma & \longrightarrow & K\widetilde{B}_0^\sharp & \longrightarrow & \widetilde{H}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma^! & \longrightarrow & K\widetilde{B}_0^! & \longrightarrow & \widetilde{H}_0^! \end{array}$$

Now by twisting $\psi^!$, we obtained in Section 3 (see the end of Subsection 3.3.2) a group morphism $(\psi')^! : \overline{N_0^!} \to (\widetilde{H}_0^!)^\times$ satisfying the assumptions of Lemma 2.3. We claim that the restriction ψ' of this map to U_0 provides a group morphism $U_0 \cong \overline{N_0} \to \widetilde{H}_0^\times$ also satisfying the assumptions of Lemma 2.3.

Given $g \in \overline{N_0^!}$, an element $b \in \widetilde{B}_0^!$ as in the statement of Lemma 2.3 is (by the construction made in Section 3) an element of the form $\psi^!(x)$ for some preimage x of g under $\gamma^! : \Gamma^! \to U_0^!$. In particular, if $g \in \overline{N_0}$, then x lies in Γ and $b = \psi^!(x) = \psi(x) \in \widetilde{B}_0^\sharp$. By the above commutative diagram and the fact that the map $\widetilde{H}_0 \to \widetilde{H}_0^!$ is injective, this implies that $bH_0 \subseteq \widetilde{H}_0^!$ lies in fact inside $\widetilde{H}_0^\times$, and thanks to the isomorphism $\widetilde{B}_0^\sharp \to \widetilde{B}_0$ the element b can be seen inside $\widetilde{B}_0$. This shows that the restriction of ψ' to $\overline{N_0}$ satisfies the assumptions of Lemma 2.3.

4.4.2. *Case where* $n_0 \neq 0$. In this case, a complement U_0 to W_0 is obtained as the direct product of the subgroups N_k, where N_k has the same generators as $N_k^!$ except for the first one, which is replaced by $s_0^{-k} s_k^{(0)} \in G(r,r,n) \subseteq W$, $r = de$ (see the proof of Theorem 3.12 of [23]). Note that each N_k is isomorphic to $N_k^!$ – and to the complex reflection group $G(r,1,b_k)$ – so that U_0 is isomorphic to $U_0^!$, and we can also consider it as a quotient of $\Gamma^!$ by the order relations corresponding to the generators of N_k.

These generators all have order 2 except the first one which has order r. In Section 3, we described a lifting $\psi^! : \Gamma^! \to \widetilde{B}_0^!$ such that $\widetilde{\pi}^! \circ \psi^! = \gamma^!$. The images under $\psi^!$ of the standard generators of $\Gamma^!$ were denoted $\sigma_k^{(i)}$.

The computations done in Section 3.3.2 prove that, by replacing each $\sigma_k^{(0)}$ by $\sigma_0^{-k} c_k \sigma_k^{(0)}$ where $c_k = T_{0,k}(\sigma_0^e) T_k(\Delta_{J_{k,1}}) \in K\widetilde{B}_0^!$ for some polynomials $T_{0,k}, T_k \in K[X]$, we get another group morphism $\psi' : \Gamma^! \to K\widetilde{B}_0^!$.

Moreover, notice that the image of each c_k under the projection map $K\widetilde{B}_0^! \to KN_0^!$ lies inside KN_0. It follows that the generators of $\Gamma^!$ have their image under ψ' inside $K\widetilde{B}_0^\sharp$ and that ψ' is actually a morphism $\Gamma^! \to K\widetilde{B}_0^\sharp$.

It remains to prove that we can choose the polynomials $T_{0,k}$ and T_k (for all k) so that the composition of the above morphism with $K\widetilde{B}_0^\sharp \to \widetilde{H}_0$ factorizes through some $\psi : U_0 \to \widetilde{H}_0$, making the following diagram commute

$$\begin{array}{ccc} \Gamma^! & \longrightarrow & K\widetilde{B}_0^\sharp \\ \big\downarrow & & \big\downarrow \\ U_0 & \cdots\cdots\!\!> & \widetilde{H}_0 \end{array}$$

Indeed, the other conditions for applying Lemma 2.3 are obviously satisfied by such a $U_0 \to \widetilde{H}_0$, as each c_k belongs to H_0 and we have a factorization $K\widetilde{B}_0^\sharp \to K\widetilde{B}_0 \to \widetilde{H}_0$.

To this end, as the braid relations are still satisfied by the modified generators, as well as the order relations except possibly for the $\sigma_0^{-k} c_k \sigma_k^{(0)}$, the only missing condition is the order relation $1 = (\sigma_0^{-k} c_k \sigma_k^{(0)})^{de}$. In other words, we need to have the equality

$$1 = (\sigma_0^{-k} c_k \sigma_k^{(0)})^{de} = \sigma_0^{-kde} c_k^{de} (\sigma_k^{(0)})^{de}$$

inside $\widetilde{H}_0$. By Lemma 3.2, this is equal to $\sigma_0^{-kde} c_k^{de} \Delta_{J_k^1}^2$. Hence we need to have $c_k^{de} = \sigma_0^{kde} \Delta_{J_k^1}^{-2}$ inside $H_0 \subseteq \widetilde{H}_0$. Therefore, it is sufficient to have polynomials $T_{0,k}, T_k \in K[X]$ such that $T_{0,k}(\sigma_0^e)^{de} = \sigma_0^{kde} = (\sigma_0^e)^{kd}$ and $T_k(\Delta_{J_{k,1}})^{de} = \Delta_{J_k^1}^{-2}$ inside $\widetilde{H}_0$, and this provides the conditions of the statement.

Remark 4.4. *In the particular case where* $n_0 \neq 0$ *and* $\lambda = 1^{n-n_0}$, *we have that* $\widetilde{B}_0^!$ *is the semidirect product of* $B_0^!$ *with* $U_0^!$, *as shown in* [13, Proposition 5.1] *(the product is even direct in that case). In this case, the only* k *for which* $b_k \neq 0$ *is* 1, *and in this case* $\Delta_{J_k^1} = 1$ *as it is the Garside element in a braid group on one strand (note that Lemma 3.2 still holds in this case, and provides the order relation on* $\sigma_1^{(0)}$ *which allows the splitting of the sequence (2.1)).*

Hence in this case, we only need to find one polynomial $T_{0,1}$ such that $T_{0,1}(\sigma_0^e)^{de} = (\sigma_0^e)^d$, as the constant polynomial $T_1 = 1$ satisfies $T_1(\Delta_{J_1^1})^{de} = \Delta_{J_1^1}^{-2}$.

5. Parabolic subgroups of maximal rank in exceptional types

We assume here that the parabolic subgroup W_0 of the exceptional (irreducible) reflection group W has rank $\mathrm{rk}\, W - 1$. We prove the following.

Theorem 5.1. *Let W be an irreducible complex reflection group of exceptional type, and W_0 a parabolic subgroup of maximal rank. Let z_{B_0} be the canonical positive central element of B_0. Assume that the pair (W, W_0) is not of type (G_{33}, D_4) or $(G_{25}, (\mathbb{Z}/3\mathbb{Z})^2)$. If there exists $T \in K[X]$ such that $T(z_{B_0})^{|Z(W)|} = z_{B_0}^{-|Z(W_0)|}$ inside H_0, then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$. In the two exceptional cases, the same conclusion holds with the condition replaced by $T(z_{B_0})^3 = z_{B_0}^{-1}$.*

In the case of rank 2, all maximal parabolic subgroups have rank 1, and z_{B_0} is the braided reflection associated to the unique distinguished reflection inside W_0. By Lemma 2.8 an immediate consequence of the theorem is the following.

Corollary 5.2. *Assume that W is an irreducible exceptional complex reflection group of rank 2, and W_0 a parabolic subgroup of rank 1 and order m. Then H_0 has parameters $u_0, \ldots, u_{m-1} \in K^\times$. Assume that $i \neq j \Rightarrow u_i - u_j \in K^\times$ and that each u_i admits a r-th root inside K, for $r = |Z(W)|$. Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$.*

According to [23] Theorem 5.5, we have $N_0 = W_0 \times Z(W)$ in almost all cases. In these cases, we have $z_B, z_{B_0} \in \hat{B}_0$ and $z_B^{|Z(W)|} z_{B_0}^{-|Z(W_0)|} \in Q_0$. Therefore, it is sufficient to find $T \in K[X]$ such that $T(z_{B_0})^{|Z(W)|} = z_{B_0}^{-|Z(W_0)|}$ inside H_0, so that $z_B T(z_{B_0})$ has order $|Z(W)|$.

We now consider the exceptions. There are two exceptions in rank 2, for $W \in \{G_{13}, G_{15}\}$, and the ones in higher rank are for $W = G_{35} = E_6$, which is already known by [13] since W is a Coxeter group in this case, and $W \in \{G_{25}, G_{33}\}$. We deal with these cases now.

5.1. $W = G_{13}$.

When W has type G_{13}, a presentation of W (see [9]) is given by generators s, t, u and relations $tust = ustu$, $stust = ustus$, $s^2 = t^2 = u^2 = 1$. The only case when $N_0 \neq Z(W).W_0$ is when W_0 is a conjugate of $\langle s \rangle$. In this case $Z(W).W_0$ has index 2 inside N_0, and $\overline{N_0}$ is cyclic. Moreover $Z(W)$ is cyclic of order 4 generated by $z = (stu)^3$. A presentation of B is given by generators σ, τ, υ and relations $\tau\upsilon\sigma\tau = \upsilon\sigma\tau\upsilon$, $\sigma\tau\upsilon\sigma\tau = \upsilon\sigma\tau\upsilon\sigma$ and a generator of the center is $z_B = (\sigma\tau\upsilon)^3$. The group $B = B_{13}$ is isomorphic to the Artin group of type $I_2(6) = G_2$ with presentation $\langle a, b \mid ababab = bababa \rangle$, an isomorphism being given by $b \mapsto \upsilon$, $a \mapsto (\upsilon\sigma\tau\upsilon)^{-1}$ with inverse $\upsilon \mapsto b$, $\tau \mapsto a^{-1}ba$, $\sigma \mapsto \Delta^{-1}a^2$ with $\Delta = ababab$. Since Δ is central, we have $a \in \hat{B}_0$. One checks that $\pi(a)$ has order 6 inside N_0 and generates $\overline{N_0}$. Moreover, $\Delta = z_B^{-1}$. In order to lift $\pi(a)$, we look for polynomials $T \in K[X]$ such that $(aT(\sigma))^8 = 1$ inside H_0. We have

$$(aT(\sigma))^8 = (a^2)^4 T(\sigma)^8 = \sigma^4 \Delta^{-4} T(\sigma)^8 = \sigma^4 z_B^4 T(\sigma)^8 = \sigma^4 \sigma^2 T(\sigma)^8 = \sigma^6 T(\sigma)^8$$

since $z_B^4 \sigma^{-2} \in Q_0$, and we need to find a polynomial T satisfying $T(\sigma)^8 = \sigma^{-6}$ inside H_0. For this it is enough to get one such that $T(\sigma)^4 = \sigma^{-3}$. Since $r = |Z(W)| = 4$ this follows from Lemma 2.8.

5.2. $W = G_{15}$. When W has type G_{15}, a presentation of W is given by generators s_1, s_2, s_3 and relations $s_1s_2s_3 = s_3s_1s_2$, $s_2s_3s_1s_2s_1 = s_3s_1s_2s_1s_2$, $s_1^2 = s_2^3 = s_3^2 = 1$. The only case when $N_0 \neq Z(W).W_0$ is when W_0 is a conjugate of $\langle s_3 \rangle$. In this case $Z(W).W_0$ has index 2 inside N_0, and $\overline{N_0}$ is cyclic. The braid group has a presentation with generators braided reflections $\sigma_1, \sigma_2, \sigma_3$ and relations $\sigma_1\sigma_2\sigma_3 = \sigma_3\sigma_1\sigma_2$, $\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 = \sigma_3\sigma_1\sigma_2\sigma_1\sigma_2$ and π maps σ_i to s_i. It embeds inside the Artin group $A = \langle \sigma, \tau \mid \sigma\tau\sigma\tau = \tau\sigma\tau\sigma \rangle$ of type B_2 under $\sigma_3 \mapsto \tau^4$, $\sigma_1 \mapsto \sigma$, $\sigma_2 \mapsto \tau\sigma\tau^{-1}$, and can be identified in this way with the kernel of $A \twoheadrightarrow \mathbb{Z}/4\mathbb{Z}$ given by $\tau \mapsto 1$, $\sigma \mapsto 0$. Under this identification, The positive generator of $Z(B)$ is $z_B = \sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 = \sigma_3\sigma_1\sigma_2\sigma_1\sigma_2$ and maps to $(\sigma\tau\sigma\tau)^2$. We notice that $\alpha = \sigma_1\sigma_2\sigma_3 = \sigma_3\sigma_1\sigma_2$ maps to $\tau^2(\tau\sigma\tau\sigma)$, which centralizes $\tau^4 = \sigma_3$. The order of $\pi(\alpha)$ is 24 inside N_0 and in the quotient group $\overline{N_0}$ as well. Therefore $N_0 = W_0 \rtimes \langle \pi(\alpha) \rangle$. Notice that $\alpha^2 = z_B\tau^4 = z_B\sigma_3$.

We then look for polynomials $T \in K[X]$ such that $(\alpha T(\sigma_3))^{24} = 1$ inside H_0. We have

$$(\alpha T(\sigma_3))^{24} = \alpha^{24} T(\sigma_3)^{24} = z_B^{12}\sigma_3^{12} T(\sigma_3)^{24} = \sigma_3^{14} T(\sigma_3)^{24}$$

since $|Z(W)| = 12$ hence $z_B^{12}\sigma_3^{-2} \in Q_0$ and we need a polynomial T satisfying $T(\sigma)^{24} = \sigma_3^{-14}$. For this it is enough to have $T \in K[X]$ with $T(\sigma)^{12} = \sigma_3^{-7}$, and this is again a consequence of Lemma 2.8 under our assumptions since $r = 12$.

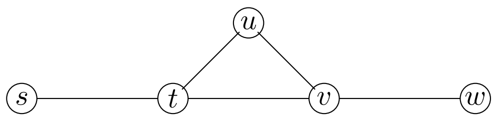
5.3. $W = G_{25}$, $W_0 = (\mathbb{Z}/3\mathbb{Z})^2$. We have that B_{25} is isomorphic to the Artin group of type A_3. We denote its standard Artin generators by $\sigma_1, \sigma_2, \sigma_3$. The Hecke algebra relation is $(\sigma_i - u_0)(\sigma_i - u_1)(\sigma_i - u_2) = 0$. We have $B_0 = \langle \sigma_1, \sigma_3 \rangle$.

We have that $\overline{N_0}$ is cyclic of order 6, generated by the image of $(\sigma_1\sigma_2\sigma_3)^2$. Now, $(\sigma_1\sigma_2\sigma_3)^4 = z_B$, and $z_{B_0} = \sigma_1\sigma_3$ is centralized by $(\sigma_1\sigma_2\sigma_3)^2$. Moreover, conjugation by $(\sigma_1\sigma_2\sigma_3)^2$ exchanges σ_1 with σ_3 (and in particular does *not* centralize B_0). This has for consequence that, although W_0 is not irreducible, we have only 3 parameters and the Hecke algebra relation is $(\sigma_i - u_0)(\sigma_i - u_1)(\sigma_i - u_2) = 0$ for $i \in \{1, 3\}$. Since $|Z(W)| = 3$, we have that z_B^3 and $z_{B_0}^3$ are the full loops for W and W_0, respectively, so that $z_B^3 = z_{B_0}^3$ inside $\widetilde{B}_0$. For $T \in K[X]$ we have

$$\left((\sigma_1\sigma_2\sigma_3)^2 T(z_{B_0})\right)^6 = z_{B_0}^3 T(z_{B_0})^6.$$

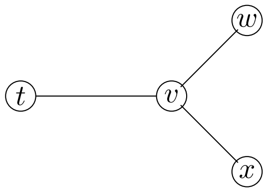
It is therefore enough to find $T \in K[X]$ with $T(z_{B_0})^2 = z_{B_0}^{-1}$ inside H_0. Now, one checks that z_{B_0} is annihilated inside H_0 by the polynomial $\prod_{0 \le i,j \le 2}(X - u_iu_j)$. By Lemma 2.8 such a T exists as soon as this polynomial is square-free and K contains a square root of each of the u_iu_j, $0 \le i, j \le 2$, and this is the case as soon as K contains the $\sqrt{u_i}$.

5.4. $W = G_{33}$, $W_0 = D_4$. A presentation of B_{33} is with generators s, t, u, v, w, Artin relations symbolized by the diagram



and the additional relations $(vtu)^2 = (uvt)^2 = (tuv)^2$ obtained in [4, 3] and implemented in CHEVIE. A presentation of B_{34} is deduced from it by adding another generator commuting with all the other ones except w, and satisfying an Artin relation of length 3 with it.

The center of B_{33} is generated by $z_B = (stuvw)^5$. We let $x = stut^{-1}s^{-1}$. Then, t, v, w, x satisfy the Artin relations of type D_4 – as in easily checked using CHEVIE – and provide generators for the braid group of a parabolic subgroup W_0 of type D_4.



The group $\overline{N_0}$ is cyclic of order 6, and contains the center of W, which has order 2. Let $c_1 = t(stvwuvtu)^{-1}$. One checks that $\bullet \mapsto \bullet^{c_1}$ permutes the generators of the D_4 diagram clockwise. Moreover, its image inside $\overline{N_0}$ has order 6. Now, explicit computations show that $z_B^2 c_1^6 = b^{12}$ where $b = txvw$. Then, considering the generator z_{B_0} of the Artin group B_0 of type D_4, since W_0 has Coxeter number 6 and center of order 2 we have $b^6 = z_{B_0}^2$, whence $c_1^6 = z_{B_0}^4 z_B^{-2} = (z_{B_0}/z_B)^4 z_B^2$. Since both $Z(W)$ and $Z(W_0)$ have order 2, $z_{B_0}^2$ and z_B^2 are the full loops in their respective hyperplane complements, and therefore $(z_{B_0}/z_B)^2$ is mapped to 1 in $\widetilde{H}_0$, and z_B^2 is mapped to $z_{B_0}^2$. We thus need to find $T \in K[X]$ such that $(c_1 T(z_{B_0}))^6 = z_{B_0}^2 T(z_{B_0})^6 = 1$ inside H_0, so it is enough to have $T(z_{B_0})^3 = z_{B_0}^{-1}$.

Using CHEVIE we check that, inside H_0, z_{B_0} is annihilated by the polynomial

$$(X - u_0^8 u_1^4)(X - u_0^9 u_1^3)(X - u_0^{12})(X - u_0^6 u_1^6)(X + u_0^6 u_1^6)(X - u_0^4 u_1^8)(X - u_0^3 u_1^9)(X - u_1^{12})$$

where u_0, u_1 are the eigenvalues of the Artin generators. Notice that the condition that this polynomial is square-free implies that $u_0^6 u_1^6 - (-u_0^6 u_1^6) = 2u_0^6 u_1^6 \in K^\times$, hence the characteristic of K cannot be 2.

6. On the remaining exceptional types

Assume that W is of exceptional type, and W_0 is not maximal. In particular W has rank at least 3. Since the case where W is a real reflection group is known by Theorem 2.4, it only remains 9 exceptional types to consider.

The problem of determining possible liftings using a systematic computer search fails in general for a couple of reasons, one of them being the following one. The only known ways so far to solve the word problem for the complex braid groups are, either to embed them into some Artin group of finite Coxeter type when this is possible, or to use methods from Garside theory, which involve the monoids (or categories) introduced by Bessis in [3]. One of the problems with these monoids is that there is no known method yet to write an element of the pure braid group as a product of natural generators in 1-1 correspondence with the (distinguished) reflections – even worse, it seems that no such collection of generators has ever been determined, and the minimal number of generators of the pure braid group is greater than the number of atoms of the monoid. Therefore, one cannot hope to mimic the methods of Digne and Gomi in [12] for the real case. Finally, the more direct approach given by applying a generic Reidemeister-Schreier method from the morphism $B \twoheadrightarrow W$ also fails most of the time, because the groups W are quite big.

In this section we are nevertheless able to deal with all the rank 3 groups except G_{27}, and with G_{32} (rank 4). This leaves five open cases (G_{27},G_{29},G_{31},G_{33},G_{34}). Notice that, when W_0 has rank 1 (which is the only case to consider when W has rank 3) then the centralizer of W_0 is equal to its normalizer (see [23, Lemma 5.1]).

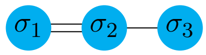
6.1. $W = G_{25}$. The braid group B of W is the Artin group of type A_3, with generators $\sigma_1, \sigma_2, \sigma_3$ and the (ordinary) braid relations between them. Then W is its quotient by the relations $\sigma_i^3 = 1$, and its Hecke algebra is defined by the relations $(\sigma_i - u_0)(\sigma_i - u_1)(\sigma_i - u_2) = 0$. There is a single class of (distinguished) reflections.

We let $W_0 = \langle s_1 \rangle$. A complement of W_0 inside N_0 is given by $U_0 = \langle s_3, \zeta_1 \rangle \simeq G(3,1,2)$ with $\zeta_1 = (s_1 s_2)^3$. The defining relations are $s_3^3 = \zeta_1^2 = 1$ and $s_3\zeta_1 s_3 \zeta_1 = \zeta_1 s_3 \zeta_1 s_3$.

We have $\sigma_3^3 \in Q_0$. We set $z_1 = (\sigma_1\sigma_2)^3$. We check that z_1 and σ_3 satisfy a braid relation of type B_2. Then, z_1^2 is the full loop inside $\langle \sigma_1, \sigma_2 \rangle$, hence $z_1^2 \equiv \sigma_1^3 \mod Q_0$. If there exists $T \in K[X]$ such that $T(\sigma_1)^2 = \sigma_1^{-3}$, which is the case by Lemma 2.8 as soon as the $i \neq j \Rightarrow u_i - u_j \in K^\times$ and K contains $\sqrt{u_i}$, since σ_1 commutes with z_1 we get a lift $z_1 T(\sigma_1)$ of order 2 which still satisfies the B_2 relation with σ_3. Since the image of σ_3 has order 3 this provides a convenient lift $U_0 \simeq \overline{N_0}$.

6.2. $W = G_{26}$.

6.2.1. $W = G_{26}$, $W_0 = \mathbb{Z}/3\mathbb{Z}$. The braid group of W is the Artin group of type B_3, with generators indexed as follows



and the Hecke relations are $(\sigma_1 - v_0)(\sigma_1 - v_1) = 0$, $(\sigma_i - u_0)(\sigma_i - u_1)(\sigma_i - u_2) = 0$ for $i \in \{2,3\}$. Then W is the quotient of B by the relations $\sigma_1^2 = \sigma_2^3 = \sigma_3^3 = 1$, and s_i denotes the image of σ_i inside W. We first look at $W_0 = \langle s_3 \rangle \simeq \mathbb{Z}/3\mathbb{Z}$. Then $\overline{N_0}$ has order 36, and a complement of W_0 inside N_0 is obtained via $U_0 = \langle z_1, z_W, s_1 \rangle$, where $z_1 = (s_2 s_3)^3$, $z_W = (s_1 s_2 s_3)^3$. The orders of s_1, z_1, z_W are $2, 2, 6$ and the subgroup generated by s_1, z_1 is a dihedral group of order 12, which contains $z_W^3 = (s_1 z_1)^3$. It is easily checked that U_0 has for presentation

$$\langle s_1, z_1, z_W \mid s_1^2 = z_1^2 = 1, (s_1 z_1)^3 = (z_1 s_1)^3 = z_W^3, s_1 z_W = z_W s_1, z_1 z_W = z_W z_1 \rangle$$

If we manage to find $\tilde{s}_1 \in s_1 H_0, \tilde{z}_1 \in z_1 H_0, \tilde{z}_W \in z_W H_0 \subset \widetilde{H}_0$ with the same orders and satisfying these relations, we are done.

We choose $\tilde{s}_1 = \sigma_1$, $\tilde{z}_W = z_B T_1(\sigma_3)$, $\tilde{z}_1 = (\sigma_2\sigma_3)^3 T_2(\sigma_3)$ for some $T_1, T_2 \in K[X]$ and with $z_B = (\sigma_1\sigma_2\sigma_3)^3 \in Z(B)$. Then the commutation relations of $\tilde{z}_W$ are satisfied, and $\sigma_1^2 \in Q_0$ hence $\tilde{s}_1$ has order 2 inside $\widetilde{H}_0$. Moreover, z_B^6 is the central full loop hence is equal to σ_3^3 modulo Q_0, and $((\sigma_2\sigma_3)^3)^2$ is the central full loop for the parabolic subgroup $\langle s_2, s_3 \rangle$ of type G_4, hence is also equal to σ_3^3 modulo Q_0. It follows that, if T_1, T_2 can be chosen so that $T_1(\sigma_3)^6 = \sigma_3^{-3}$ and $T_2(\sigma_3)^2 = \sigma_3^{-3}$ then these lifts have the convenient orders. For T_1 it is enough to have $T_1' \in K[X]$ such that $T_1'(\sigma_3)^2 = \sigma_3^{-1}$. By Lemma 2.8 a sufficient condition for these polynomials to exist is that $\prod_i (X - u_i)$ is square-free and K contains a square root of each u_i.

Therefore the only thing remaining to be checked is that $(\tilde{s}_1\tilde{z}_1)^3(\tilde{z}_1\tilde{s}_1)^{-3}$ is $1 \in H_0 \subset \widetilde{H}_0$. Since $T_1(\sigma_3)$ and $T_2(\sigma_3)$ commute with the other terms involved, it is enough to check that $x = (\sigma_1(\sigma_2\sigma_3)^3)^3((\sigma_2\sigma_3)^3\sigma_1)^{-3} \in \hat{B}_0$ actually belongs to Q_0.

For this we do computations in GAP4, using a Reidemeister-Schreier type method to get a generating set for $P = \mathrm{Ker}(B \twoheadrightarrow W)$ and express x (as a lengthy expression) in terms of these generators. The generators obtained by this method are 21 elements which turn out to be conjugates to powers of the generators. These are the following ones.

$\sigma_1^{-2}, {}^{\sigma_2}(\sigma_1^{-2}), \sigma_2^3, (\sigma_1^{-2})^{\sigma_2}, \sigma_3^3, {}^{\sigma_1}(\sigma_2^3), {}^{\sigma_2}(\sigma_3^3), (\sigma_3^3)^{\sigma_2}, {}^{\sigma_3\sigma_2}(\sigma_1^{-2}), (\sigma_1^{-2})^{\sigma_2\sigma_3^{-1}}, (\sigma_1^{-2})^{\sigma_2\sigma_3}, {}^{\sigma_1\sigma_2}(\sigma_3^3),$
$(\sigma_3^3)^{\sigma_2\sigma_1^{-1}}, {}^{\sigma_2\sigma_1\sigma_2}(\sigma_3^3), {}^{\sigma_2^{-1}\sigma_1\sigma_2}(\sigma_3^3), {}^{\sigma_1^2\sigma_3^{-1}\sigma_2}(\sigma_1^{-2}), {}^{\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2}(\sigma_1^{-2}), {}^{\sigma_1^{-1}\sigma_2^2(\sigma_2\sigma_1)^2\sigma_2^{-1}}(\sigma_3^3),$
$(\sigma_1^{-2})^{\sigma_2^{-1}\sigma_3(\sigma_2^{-1}\sigma_1^{-1})^2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^2(\sigma_2\sigma_1)^2\sigma_2^{-1}\sigma_3^3\sigma_2(\sigma_1^{-1}\sigma_2^{-1})^2\sigma_2^{-2}\sigma_1}, (\sigma_3^3)^{\sigma_2\sigma_1^{-1}\sigma_3^3\sigma_2}, (\sigma_3^3)^{\sigma_2\sigma_1^{-1}(\sigma_2\sigma_3^2)^2\sigma_3\sigma_2^{-1}\sigma_3^{-3}}$

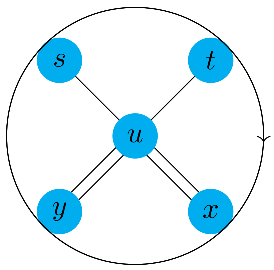
Being conjugates of powers of the generators, they belong to $Q_0 = \mathrm{Ker}(P \twoheadrightarrow P_0 \simeq \mathbb{Z})$ exactly when the corresponding hyperplane is not $\mathrm{Ker}(s_3 - 1)$. This is the case for all of them except for σ_3^3, which is mapped to 1 under $P \twoheadrightarrow P_0 \simeq \mathbb{Z}$. Using GAP4 we get that the image of x under this map is 0, and this defines a morphism $U_0 \to \widetilde{H}_0^\times$ satisfying the assumption of Lemma 2.3.

6.2.2. *$W = G_{26}$, $W_0 = \mathbb{Z}/2\mathbb{Z}$.* We use the notations of section 6.2.1 and let $W_0 = \langle s_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$. In this case $\overline{N_0}$ has order 72, and a complement U_0 of W_0 inside N_0 is easily checked to be generated by s_3 and $z_3 = s_1 s_2 s_1 s_2$. These elements both have order 3, and satisfy the relations $(s_3 z_3)^2 = (z_3 s_3)^2$. These relations are known to be defining relations of the complex reflection group G_5, which has order 72, therefore they indeed provide a presentation of U_0.

It is thus sufficient to find elements $\tilde{s}_3 \in s_3 H_0, \tilde{z}_3 \in z_3 H_0$ satisfying these relations. We set $\tilde{s}_3 = \sigma_3$ and $\tilde{z}_3 = (\sigma_1\sigma_2)^2 T(\sigma_1)$ for $T \in K[X]$ such that $\tilde{z}_3$ has order 3. This is possible when $(\sigma_1\sigma_2)^6 T(\sigma_1)^3$ is 1 inside H_0. Now, since $(\sigma_1\sigma_2)^2$ is the positive generator of the center of the parabolic braid group associated to the parabolic subgroup $\langle s_1, s_2 \rangle$ of type $G(3,1,2)$, whose center has order 3, we know that $(\sigma_1\sigma_2)^6$ is the corresponding full central loop. Therefore, its image inside P_0 is the full loop associated to the parabolic subgroup W_0, that is σ_1^2. It follows that, inside $\widetilde{H}_0$, we have $(\sigma_1\sigma_2)^6 T(\sigma_1)^3 = \sigma_1^2 T(\sigma_1)^3$, and we want T to satisfy $T(\sigma_1)^3 = \sigma_1^{-2}$ inside H_0. By Lemma 2.8 this is possible as soon as $(X - v_0)(X - v_1)$ is square-free and $K^\times$ contains 3-roots of the parameters v_i. Then, it is sufficient to check that $x = (\sigma_3(\sigma_1\sigma_2)^2)^2((\sigma_1\sigma_2)^2\sigma_3)^{-2} \in Q_0$ to get a convenient morphism $U_0 \to \widetilde{H}_0^\times$, and this is checked by the same computational method as in section 6.2.1.

6.3. $W = G_{24}$.

6.3.1. *A new presentation for G_{24}.* The presentation given in CHEVIE, originating from [4], is by generators s, t, u, and relations $stst = tsts$, $sus = usu$, $tut = utu$ and $stustustu = tstustust$. We propose an alternative presentation such that the centralizer of a reflection is easily described. We introduce additional generators $x = s^t = t^{-1}st$, $y = {}^s t = sts^{-1}$. Using CHEVIE it is easily checked that the relations symbolized by the following diagram hold.



In this 'steering wheel' diagram, all edges represent Artin relations, that is $sus = usu$, $tut = utu$, $xuxu = uxux$, $uyuy = yuyu$, and the oriented circle has the same meaning as for the Corran-Picantin presentations of the groups $G(e,e,n)$ (see [11]), namely it symbolizes the relation $st = tx = xy = ys$, originating from the dual braid monoid of dihedral type $I_2(4) = B_2$.

In order to prove that this indeed provides a presentation of B, it is sufficient to check that the relation $stustustu = tstustust$ is a consequence of these relations. This is done as follows :

$$\begin{array}{rclclcl} stustustu & = & stu({}^s t)\underline{sus}tu & = & stuyus\underline{utu} & = & stuyustut \\ & = & stuyu({}^s t)sut & = & st\underline{uyuy}sut & = & styuy\underline{usu}t \\ & = & \underline{st}yuysust & = & t\underline{xy}uy\underline{s}ust & = & tstustust \end{array}$$

A collection of 21 generators for $P = \mathrm{Ker}(B \twoheadrightarrow W)$ is obtained as follows

$$s^{-2}, t^{-2}, u^{-2}, x^{-2}, y^{-2}, {}^s(u^{-2}), {}^t(u^{-2}), {}^u(x^{-2}), {}^u(y^{-2}), {}^x(u^{-2}), {}^y(u^{-2}), {}^{st}(u^{-2}), {}^{su}(x^{-2}),$$
$${}^{su}(y^{-2}), {}^{sx}(u^{-2}), {}^{sy}(u^{-2}), {}^{tu}(x^{-2}), {}^{tu}(y^{-2}), {}^{xu}(y^{-2}), {}^{yu}(x^{-2}), {}^{usx}(u^{-2})$$

6.3.2. *The centralizer of a reflection.* Let $W_0 = \langle \pi(u) \rangle$. Then $\overline{N_0}$ has order 8, and it is easily checked that W_0 admits a complement U_0 inside N_0 generated by $\pi(xux)$ and $\pi(yuy)$. Actually, both elements have order 2 and satisfy a Coxeter relation of length 4. In order to lift this complement it is thus enough to find elements $\tilde{a}, \tilde{b} \in B$ mapping to $\pi(xux), \pi(yuy)$, such that $\tilde{a}^2, \tilde{b}^2, (\tilde{a}\tilde{b})^2(\tilde{b}\tilde{a})^{-2} \in Q_0$.

By simply setting $\tilde{a} = xux$, $\widetilde{B} = yuy$, it is checked computationally that, when written in terms of the generators of P given above, these elements map to 0 under the morphism $P \twoheadrightarrow P_0 \simeq \mathbb{Z}$ where all the generators map to 0 but u^{-2}. In the case of the order relation, it is even possible to check this 'by hand', as

$$u^{-2}.ux^2u^{-1}.u^2.x^2.x^{-1}u^2x = u^{-1}x^2uxu^2x = u^{-1}x(xuxu)ux = (xuxu)u^{-1}xux = (xux)^2$$

and similarly with x replaced with y. This provides a group-theoretic splitting in this case.

6.4. $W = G_{32}$. In this case, B is the Artin group of type A_4, with generators $\sigma_1, \dots, \sigma_4$.

6.4.1. *$W = G_{32}$, $W_0 = \mathbb{Z}/3\mathbb{Z}$.* Let $W_0 = \langle s_1 \rangle$. We have $|\overline{N_0}| = 1296 = \|G_{26}\|$. We set $z_1 = (\sigma_1\sigma_2)^3$. Then, the images of z_1, σ_3, σ_4 inside W centralize W_0, have order $2, 3, 3$, and generate together a subgroup U_0 of order 1296 which does not intersect W_0. Therefore it is a complement to W_0 inside N_0.

Moreover, inside B one checks that z_1, σ_3, σ_4 satisfy the relations of the braid group of G_{26}, which also has order 1296. Since the order of the generators for G_{26} are also $2, 3, 3$ this proves in particular that U_0 is isomorphic to G_{26}. Finally, we have $\sigma_3^3 \in Q_0$, $\sigma_4^3 \in Q_0$, and $z_1^2 \equiv \sigma_1^3$ mod Q_0. Letting $x = z_1 T(\sigma_1)$, $y = \sigma_3$, $z = \sigma_4$ with $T(\sigma_1)^2 = \sigma_1^{-3}$, we get $x^2 = y^3 = z^3 = 1$ and that x, y, z satisfy inside $\widetilde{H}_0$ the braid relations of the braid group of type G_{26} as soon as $\prod_i (X - u_i)$ is square-free and K contains the $\sqrt{u_i}$, by Lemma 2.8.

6.4.2. *$W = G_{32}$, $W_0 = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.* We first consider the case where $W_0 = \langle s_1, s_3 \rangle$. Then $\overline{N_0}$ has order 72 and permutes non-trivially the two distinguished reflections in W_0. Therefore we still have only 3 parameters $u_0, u_1, u_2 \in K^\times$ for the Hecke algebra of W_0. We set $a = (s_1s_2s_3)^2$, $c = (s_3s_4)^3$. Then $\langle a, c \rangle$ does not intersect W_0, normalizes it, and has order 72, like $G(6,1,2)$. Moreover, a and c have order 6 and 2, respectively. We let $\tilde{a} = (\sigma_1\sigma_2\sigma_3)^2$ and $\tilde{c} = (\sigma_3\sigma_4)^3$. These are preimages of a, c, and one checks that $(\tilde{a}\tilde{c})^2 = (\tilde{c}\tilde{a})^2$ inside the braid group B. In particular, we have that $\overline{N_0} \cong G(6,1,2)$. Now, $\tilde{a}^6$ is the full central twist of the parabolic subgroup $\langle s_1, s_2, s_3\rangle$, and thus its image inside $\widetilde{H}_0$ is equal to the image of $z_1 = (\sigma_1\sigma_3)^3$. Similarly, the image of $\tilde{c}^2$ is equal to the image of $z_2 = \sigma_3^3$.

We check that $z_1, z_2, z_2^{\tilde{a}}$,and $\tilde{c}$ commute to each other, that $\tilde{a}^2$ commutes with z_1, z_2, and that $\tilde{a}$ commutes with z_1. Letting $x = \tilde{a}T_x(z_1)$, $y = \tilde{c}T_y(z_2)$ for some $T_x, T_y \in K[X]$, we have

$$\begin{aligned} xyxy &= \tilde{a}T_x(z_1)\tilde{c}T_y(z_2)\tilde{a}T_x(z_1)\tilde{c}T_y(z_2) \\ &= \tilde{a}\tilde{c}\tilde{a}\tilde{c}T_x(z_1)(T_y(z_2)^{\tilde{a}})T_x(z_1)T_y(z_2) \\ &= \tilde{c}\tilde{a}\tilde{c}\tilde{a}(T_y(z_2)^{\tilde{a}^2})T_x(z_1)(T_y(z_2)^{\tilde{a}})T_x(z_1) \\ &= \tilde{c}\tilde{a}(T_y(z_2)^{\tilde{a}})T_x(z_1)\tilde{c}\tilde{a}(T_y(z_2)^{\tilde{a}})T_x(z_1) \\ &= \tilde{c}T_y(z_2)\tilde{a}T_x(z_1)\tilde{c}T_y(z_2)\tilde{a}T_x(z_1) \\ &= yxyx \end{aligned}$$

Moreover, we have $x^6 = \tilde{a}^6T_x(z_1)^6 = z_1T_x(z_1)^6$, while $y^2 = \tilde{c}^2T_y(z_2)^2 = z_2T_y(z_2)^2$, so we need to have $T_x(z_1)^6 = z_1^{-1}$, $T_y(z_2)^2 = z_2^{-1}$, and z_1 and z_2 are obviously annihilated by $\prod_{0\le i,j\le 2}(X - u_i^3u_j^3)$ and $\prod_i(X - u_i^3)$. In order to apply Lemma 2.8 we thus need that these polynomials are square-free and that $\sqrt{u_i} \in K, 0 \le i \le 2$.

6.4.3. *$W = G_{32}$, $W_0 = G_4$.* We now consider the case where $W_0 = \langle s_1, s_2\rangle \simeq G_4$. Then $\overline{N_0}$ still has order 72, but is not isomorphic to $G(6, 1, 2)$. Note that $G(6, 1, 2)$ and G_5 are two complex reflection groups of the same rank and the same order which are not isomorphic as abstract groups.

We set $a = (s_1s_2s_3)^4$, $c = s_4$. They both centralize W_0, have order 3, satisfy $(ac)^2 = (ca)^2$, and $\langle a, c\rangle$ has order 72 and has trivial intersection with W_0. Therefore it is a convenient complement, isomorphic to the complex reflection group G_5. Let $\tilde{a} = (\sigma_1\sigma_2\sigma_3)^4$, $\tilde{c} = \sigma_4$. One checks that $(\tilde{a}\tilde{c})^2 = (\tilde{c}\tilde{a})^2$ inside B, and moreover $\tilde{c}^3 \in Q_0$, $\tilde{a}^3 \equiv z_{B_0}^2 \mod Q_0$ where $z_{B_0} = (\sigma_1\sigma_2)^3$. If there exists $T \in K[X]$ with $z_{B_0}^2T(z_{B_0})^3 = 1$, letting $x = \tilde{a}T(z_{B_0})$, and $y = \tilde{c}$ provides a morphism $\overline{N_0} \to \widetilde{H}_0^\times$ to which we can apply Lemma 2.3.

Now, the eigenvalues of z_{B_0} can be computed on the irreducible representations of the Hecke algebra of G_4. One gets that z_{B_0} acts on the irreducible representations of the generic Hecke algebra of type G_4 with the values $\{u_i^3, -u_i^3u_j^3, (u_0u_1u_2)^2; 0 \le i, j \le 2, i \ne j\}$, and therefore it is annihilated inside H_0 by the polynomial

$$\left(\prod_{0\le i\le 2}(X - u_i^3)\right)\left(\prod_{\substack{0\le i,j\le 2\\ i\ne j}}(X + u_i^3u_j^3)\right)\left(X - (u_0u_1u_2)^2\right) \tag{6.1}$$

Therefore, Lemma 2.6 can be applied when this polynomial is square-free and $(u_0u_1u_2)^2$ admits a 3-root, which is in particular the case when $u_0u_1u_2$ admits one such root.

6.5. Conditions for G_{24}, G_{25}, G_{26}, G_{32}.

Proposition 6.1. *Let $W = G_{24}$, and $u_0, u_1 \in K^\times$ the defining parameters. Let W_0 be a parabolic subgroup of W. Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ in the following cases:*

1. *If W_0 has rank 1; in this case, there is a group-theoretic splitting and the complement is a Coxeter group of type B_2.*

(2) *If W_0 has rank 2 and type A_2, when the polynomial $(X - u_0^6)(X - u_1^6)(X + u_0^3 u_1^3)$ is square-free and $-u_0^3 u_1^3$ admits a square root in K. In this case the complement is $Z(W) \simeq \mathbb{Z}/2\mathbb{Z}$.*
(3) *If W_0 has rank 2 and type B_2; in this case, there is a group-theoretic splitting and the complement is $Z(W) \simeq \mathbb{Z}/2\mathbb{Z}$.*

Proof. In the case where W_0 has rank 1, this has been proved in Section 6.3. When W_0 has rank 2 it is maximal and the complement is $Z(W) \simeq \mathbb{Z}/2\mathbb{Z}$. Therefore, when W_0 has type B_2, since $Z(W_0) \simeq \mathbb{Z}/2\mathbb{Z}$, we get a group-theoretic splitting via $\langle z_B z_{B_0}^{-1} \rangle$. And, when W_0 has type A_2, since $Z(W_0) = 1$, we get an isomorphism $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ by Lemma 2.8 under the condition of the statement, by Table 1 (see the proof of Lemma 2.9 and the comment after it on how to determine the polynomial out of the table). □

Proposition 6.2. *Let $W = G_{25}$, and $u_0, u_1, u_2 \in K^\times$ the defining parameters. Let W_0 be a parabolic subgroup of W. Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ in the following cases:*

(1) *If W_0 has rank 1, when the polynomial $\prod_i (X - u_i)$ is square-free and $\sqrt{u_i} \in K$. In this case the complement is isomorphic to a complex reflection group of type $G(3,1,2)$.*
(2) *If W_0 has rank 2 and type G_4, when the polynomial (6.1) is square-free and $\sqrt[3]{u_0 u_1 u_2} \in K$. In this case the complement is $Z(W) \simeq \mathbb{Z}/3\mathbb{Z}$.*
(3) *If W_0 has rank 2 and type $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, when the polynomial $\prod_{0 \le i,j \le 2} (X - u_i u_j)$ is square-free and $\sqrt{u_i} \in K$. In this case the complement is a cyclic group of order 6.*

Proof. In the case where W_0 has rank 1, this has been proved in Section 6.1. When W_0 has rank 2 and type $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, this has been proved in Section 5.3. We now assume that W_0 is maximal of type G_4. Then $|Z(W)| = 3$ and $|Z(W_0)| = 2$ and the polynomial (6.1) annihilates z_{B_0} as in Section 6.4.3, and we get similarly the additional condition $\sqrt[3]{u_0 u_1 u_2} \in K$. This concludes the proof. □

Proposition 6.3. *Let $W = G_{26}$ and $v_0, v_1, u_0, u_1, u_2 \in K^\times$ the defining parameters. The Hecke relations are $(\sigma_1 - v_0)(\sigma_1 - v_1) = 0$, $(\sigma_i - u_0)(\sigma_i - u_1)(\sigma_i - u_2) = 0$ for $i = 2, 3$. Let W_0 be a parabolic subgroup of W. Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ in the following cases:*

(1) *If W_0 has rank 1 and type $\mathbb{Z}/2\mathbb{Z}$, when $v_1 - v_0 \in K^\times$ and $\sqrt[3]{v_i} \in K$. In this case the complement is isomorphic to a complex reflection group of type $G(3,1,2)$.*
(2) *If W_0 has rank 1 and type $\mathbb{Z}/3\mathbb{Z}$, when $\prod_i (X - u_i)$ is square-free and $\sqrt{u_i} \in K$. In this case the complement is isomorphic to a complex reflection group of type $G(6,2,2)$.*
(3) *If W_0 has rank 2 and type $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$; in this case, we have a group-theoretic splitting, and the complement is $Z(W)$ which is cyclic of order 6.*
(4) *If W_0 has rank 2 and type G_4, when the polynomial (6.1) is square-free and $\sqrt[3]{u_0 u_1 u_2} \in K$. In this case the complement is $Z(W)$ which is cyclic of order 6.*
(5) *If W_0 has rank 2 and type $G(3,1,2)$, when the polynomial*

$$\left(\prod_{(i,j) \in \{0,1\} \times \{0,1,2\}} (X - v_i^2 u_j^2) \right) \left(\prod_{i \neq j} (X + v_0 v_1 u_i u_j) \right)$$

is square-free and $\sqrt{-v_0 v_1 u_i u_j} \in K$ for $i \neq j$. In this case the complement is $Z(W)$ which is cyclic of order 6.

Proof. The proof of (1) is given in Section 6.2.2. The proof of (2) is given in Section 6.2.1, and the identification of the complement with the group $G(6,2,2)$ is done using GAP4 algorithms for identifying the isomorphism type of small groups. We now consider the case of maximal parabolic subgroups. We have $|ZW| = 6$, $|ZG(3,1,2)| = 3$, $|ZG_4| = 2$. When $W_0 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, since $|ZW| = |ZW_0|$, we get a group-theoretic splitting. In case W_0 has type G_4, we have $z_{B_0} = (\sigma_2\sigma_3)^3$, and we know by Section 6.4.3 that it is annihilated by polynomial (6.1), and for applying Lemma 2.8 we need to be able to take 3-roots of its roots, that is, we need to have $\sqrt[3]{u_0u_1u_2} \in K$.

When W_0 has type $G(3,1,2)$, we have $z_{B_0} = \sigma_1\sigma_2\sigma_1\sigma_2$. By computing its value on the irreducible representations of the generic Hecke algebra of $G(3,1,2)$, we get that it is annihilated by the polynomial of the statement. In order to apply Lemma 2.8 we thus need this polynomial to be square-free, and also to have square roots of its roots, that is, we need to have $\sqrt{-v_0v_1u_iu_j} \in K$. □

Proposition 6.4. *Let $W = G_{32}$, and $u_0, u_1, u_2 \in K^\times$ the defining parameters. Let W_0 be a parabolic subgroup of W. Then $\widetilde{H}_0 \simeq \overline{N_0} \ltimes H_0$ in the following cases:*

1. *If W_0 has rank 1, when $\prod_i (X - u_i)$ is square-free and $\sqrt{u_i} \in K$. In this case the complement is isomorphic to a complex reflection group of type G_{26}.*
2. *If W_0 has rank 2 and type G_4, when the polynomial (6.1) is square-free and $\sqrt[3]{u_0u_1u_2} \in K$. In this case the complement is isomorphic to a complex reflection group of type G_5.*
3. *If W_0 has rank 2 and type $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, when $\prod_{0\le i,j\le 2}(X - u_i^3u_j^3)$ and $\prod_i (X - u_i^3)$ are square-free and $\sqrt{u_i} \in K$, $0 \le i \le 2$. In this case the complement is a complex reflection group of type $G(6,1,2)$.*
4. *If W_0 has rank 3 and type $G_4 \times \mathbb{Z}/3\mathbb{Z}$, then there is a group-theoretic splitting and the complement is $Z(W)$ which is cyclic of order 6.*
5. *If W_0 has rank 3 and type G_{25}, when the polynomial*

$$\left(\prod_{\{i,j,k\}=\{0,1,2\}} (X - u_i^2u_j^4u_k^6)(X - u_i^3u_j^3u_k^6)(X - u_i^4u_j^4u_k^4)\right)$$
$$\times \left(\prod_{i\neq j}(X - u_i^4u_j^8)(X - u_i^6u_j^6)\right)\left(\prod_i (X - u_i^{12})\right)\left(X^3 - (u_0u_1u_2)^{12}\right)$$

   is split and square-free (which implies that $\mu_3(\mathbb{C}) \subset K$), and we have $\sqrt{u_i} \in K$, $0 \le i \le 2$. In this case the complement is $Z(W)$ which is cyclic of order 6.

Proof. The cases where W_0 has rank 1 or 2 have been dealt with in Section 6.4; we can thus assume that W_0 is a maximal parabolic subgroup of rank 3. In this case, we need to find a polynomial P such that $P(z_{B_0})^{|Z(W)|} = z_{B_0}^{|Z(W_0)|}$. We first consider the case where $W_0 = G_4 \times \mathbb{Z}/3\mathbb{Z}$.

Then $|ZW_0| = 6 = |ZW|$, and we get a group-theoretic splitting $Z(W_0) \to \widetilde{B}_0$ via $\zeta_6 \mapsto z_B z_{B_0}^{-1}$ with $z_{B_0} = (\sigma_1\sigma_2)^3\sigma_4$.

We then consider the case where W_0 has type G_{25}. Then $|ZW_0| = 3$, $|ZW| = 6$, and from the computation of the values of z_{B_0} on the irreducible representations of the generic Hecke algebra of type G_{25}, we get that z_{B_0} is annihilated by the polynomial of the statement. Therefore the condition of Lemma 2.8 is that this polynomial is split and square-free in K, and K must contain the square roots of all its roots. This translates into the conditions of the statement.

□

References

[1] S. Ariki, *Representation theory of a Hecke algebra of* $G(r,p,n)$, J. Algebra **177** (1995), 164-185.

[2] S. Ariki, K. Koike, *A Hecke algebra of* $(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math. **106** (1994), 216-243.

[3] D. Bessis, *Finite complex reflection arrangement are* $K(\pi,1)$, Ann. of Math. (2) **181** (2015), 809-904.

[4] D. Bessis, J. Michel, *Explicit presentations for exceptional braid groups*, Experiment. Math. **13** (2004), 257–266.

[5] C. Bonnafé, M. Dyer, *Semidirect product decomposition of Coxeter groups*, Comm. Algebra **38** (2010), no. 4, 1549-1574.

[6] K. Bremke, G. Malle, *Reduced words and a length function for* $G(e,1,n)$, Indag. Math. (N.S.) **8** (1997), no. 4, 453-469.

[7] M. Broué, G. Malle, *Zyklotomische Heckealgebren*, in *Représentations unipotentes génériques et blocs des groupes réductifs finis*, Astérisque **212** (1993), 119-189.

[8] B. Brink, R. B. Howlett, *Normalizers of parabolic subgroups in Coxeter groups*, Invent. Math. **136** (1999), 323-351.

[9] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127-190.

[10] E. Chavli, *The BMR freeness conjecture for the tetrahedral and octahedral family*, Comm. Algebra **46** (2018), 386-464.

[11] R. Corran, M. Picantin, *A new Garside structure for braid groups of type* (e,e,r), J. London Math. Soc. **84** 689-711 (2011).

[12] F. Digne, Y. Gomi, *Presentation of pure braid groups*, J. Knot Theory Ramifications **10** (2001), 609–623.

[13] T. Gobet, A. Henderson, I. Marin, *Braid groups of normalizers of reflection subgroups*, preprint, 2020.

[14] M. Geck, G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori-Hecke algebras*, London Math. Soc. Monographes new series 21, Clarendon Press, 2000.

[15] R.B. Howlett, *Normalizers of Parabolic Subgroups of Reflection Groups*, J. of the London Math. Soc. **21** (1980), 62-80.

[16] I. Marin, *The cubic Hecke algebra on at most 5 strands*, J. Pure Appl. Algebra **216** (2012), 2754-2782.

[17] I. Marin, *The freeness conjecture for Hecke algebras of complex reflection groups, and the case of the Hessian group* G_{26}, J. Pure Appl. Algebra **218** (2014), 704-720.

[18] I. Marin, *Proof of the BMR conjecture for* G_{20} *and* G_{21}, J. Symbolic Computations **92** (2019), 1-14.

[19] I. Marin, *Artin groups and Yokonuma–Hecke algebras*, Int. Math. Res. Not. IMRN **2018** (2018), 4022–4062.

[20] I. Marin, *Lattice extensions of Hecke algebras*, J. Algebra **503**, 104-120 (2018).

[21] I. Marin, G. Pfeiffer, *The BMR freeness conjecture for the 2-reflection groups*, Math. of Computation **86** (2017), 2005-2023.

[22] D.E. Taylor, *Reflection subgroups of finite complex reflection groups*, J. Algebra **366** (2012) 218-234.

[23] K. Muraleedaran, D.E. Taylor, *Normalisers of parabolic subgroups in finite unitary reflection groups*, J. Algebra **504** (2018), 479-505.

[24] S. Tsuchioka, *BMR freeness for icosahedral family*, *BMR Freeness for Icosahedral Family*, Exp. Math. **29** (2020), 234-245.

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