RANDOMIZED SIMPLICIAL SETS

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ABSTRACT. We construct new geometric realizations of simplicial and pre-simplicial sets where the standard n-simplex, viewed as the space of probability measures on $n+1$ elements, is replaced by the space of $(n+1)$-valued random variables, with the topology of probability convergence. We prove that the map which associates to a random variable its probability law is an homotopy equivalence from these new geometric realizations to the classical ones. Finally, we prove that this realization provides a new Quillen equivalence between simplicial sets and topological spaces.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. **Context.** We continue the exploration of Simplicial Random Variables, as initiated in [16]. The observation at the starting point of [16] was that the usual geometric realization $|\mathcal{K}|$ of a simplicial complex

Date: October 2, 2022.

$\mathcal{K}$ is given by the collection of all the probability measures on a vertex set S whose support provides a face of the simplicial complex. Of course the usual topology of this geometric realization is the weak topology, which is not that natural in the realm of measure theory, but Dowker's theorem tells us that the choice of topology is not that relevant in terms of homotopy theory, and that one can choose a metric topology instead. Then, from the viewpoint of probability theory, a natural object living above this geometric realization is the (metric) space $L(\mathcal{K}) \subset L^1(\Omega, S)$ of random variables with values in the vertex set S whose essential image is a face of $\mathcal{K}$. Here (Ω, λ) is an atomless (complete) probability space, the chosen metric on $L^1(\Omega, S)$ is $d(f, g) = \int d(f(t), g(t)) \mathrm{d}\lambda(t)$, where S is endowed with the discrete metric $d(x, y) = 1 - \delta_{x,y}$. When S is finite, the underlying topology of $L^1(\Omega, S)$ corresponds to the concept of convergence in probability of a sequence of random variables. We proved in [16] that this space has the same homotopy type as the ordinary realization, and that the natural 'probability law' map $L(\mathcal{K}) \to |\mathcal{K}|$ is a Serre fibration and a homotopy equivalence. Therefore these spaces of random variables provide alternative constructions for the geometric realization of the simplicial complex $\mathcal{K}$.

These alternative constructions have the following merit. Given a vertex set S, and $\mathcal{K}$ a simplicial complex with vertices inside S, then a free action of group G on S does not in general induce a free action on $|\mathcal{K}|$, but it *does* provide a free action on $L(\mathcal{K})$.

As a consequence, if G is an arbitrary group, then one can consider with new eyes the obvious candidate for a universal simplicial complex being acting upon by G, the full collection $\mathcal{K}_G = \mathcal{P}^*_{\mathrm{f}}(G)$ of non-empty finite subsets of G. The induced action of G on $|\mathcal{K}_G|$ is not free, thus preventing the construction of a classifying space for G as $|\mathcal{K}_G|/G$. But the induced action of G on $L(\mathcal{K}_G)$ *is* free, providing an easy construction of a classifying space for G, as $L(\mathcal{K}_G)/G = L^1(\Omega, G)/G$. The properties of this construction have been studied separately in detail in [15].

1.2. **Constructions and results.** Here we consider the other standard concept of simpliciality, namely simplicial sets. Replacing in each case the n-simplex Δ_n, again considered as a space of probability measures, by the space of ∇_n of random variables with values in $\{0, \dots, n\}$ yields new realizations of these simplicial sets as spaces of simplicial random variables.

In order to be more precise, we first recall the basic concepts in the realm of simplicial sets. We let **Top** be a convenient category of topological spaces containing the metrizable ones, for instance the category of weakly Hausdorff and compactly generated topological spaces, and **Set** the category of sets. In our conventions, compact (and paracompact) spaces are Hausdorff. We denote $\mathbf{\Delta}$ the category with objects the $[n] = \{0, \dots, n\}, n \in \mathbf{Z}_{\geq 0}$ and (weakly) increasing maps as morphisms. A simplicial set is a contravariant functor $F \in \mathbf{Fun}(\mathbf{\Delta}^{op}, \mathbf{Set})$, or equivalently a graded set $F = \bigsqcup_{n\geq 0} F_n$ with $F_n = F([n])$, equiped with an action on the right of the category $\mathbf{\Delta}$. The elements of F_n are called the n-simplices of F.

The usual geometric realization has been defined by Milnor [22] as follows. Let $\Delta_n = \{(x_0, \dots, x_n) \in [0, 1]^n; \sum_i x_i = 1\}$ endowed with the product topology of $[0, 1]^n$. It defines a (covariant) functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ via $\Delta([n]) = \Delta_n$ and, for $\sigma \in \mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$,

$$\Delta(\sigma) : (x_0, \dots, x_n) \mapsto \left(\sum_{j \in \sigma^{-1}(i)} x_j \right)_{i=0,\dots,m}.$$

From this functor, the geometric realization $|F|$ is classically defined as the quotient space of $E = \bigsqcup_n \left(F_n \times \Delta_n \right)$ with $F_n = F([n])$ considered as a discrete topological space, by the equivalence relation $\sim$ generated by the relations $(\alpha\sigma, a) \sim (\alpha, \Delta(\sigma)(a))$, for $\sigma \in \mathbf{\Delta}$. It is a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$ admitting for right adjoint the singular functor $X \mapsto \mathrm{Sing}(X)$ with $\mathrm{Sing}(X)_n$ equal to the set of maps $\Delta_n \to X$. It can be seen as the colimit of the functor $\Delta \circ D_F : C_F \to \mathbf{\Delta}$, where C_F is the category of simplices of F (see [8] §4.2) and $D_F : C_F \to \mathbf{\Delta}$ the forgetful functor.

Now set $\nabla_n = L^1(\Omega, [n])$ considered as a (paracompact) topological space, with topology defined by convergence in probability, or equivalently as the underlying topology of the L^1 metric. We introduce the probability-law map $p_n : \nabla_n \to \Delta_n$, mapping $f \in \nabla_n$ to $(\lambda(f^{-1}(i)))_{i=0,\dots,n} \in \Delta_n$. The first statement suggesting that these concepts of probability theory are well adapted to the topological simplicial context is the following one.

Proposition 1.1. *The map $n \mapsto \nabla_n$ extends to a functor $\mathbf{\Delta} \to \mathbf{Top}$, and the $(p_n)_{n\geq 0}$ define a natural transformation $\nabla \rightsquigarrow \mathbf{\Delta}$.*

Proof. For $\sigma : [n] \to [m]$, the map $\nabla(\sigma)$ maps $f \in \nabla_n = L^1(\Omega, [n])$ to $\sigma\circ f \in L^1(\Omega, [m]) = \nabla_m$. Moreover, for $f, g \in \nabla_n$, we have

$$d\left(\nabla(\sigma)(f), \nabla(\sigma)(g)\right) = \int d\left(\sigma\circ f(t), \sigma\circ g(t)\right) \leqslant \int d\left(f(t), g(t)\right) = d(f, g)$$

which proves that $\nabla(\sigma)$ is 1-Lipschitz and in particular continuous. The property $\nabla(\sigma\circ\tau) = \nabla(\sigma)\circ\nabla(\tau)$ is clear, hence ∇ defines a functor $\mathbf{\Delta} \to \mathbf{Top}$.

In order to prove that $n \mapsto p_n$ is a natural transformation, we need to compare the elements $p_m\circ\nabla(\sigma)(f)$ and $\Delta(\sigma)\circ p_n(f)$ of Δ_m for $\sigma \in \mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$ and $f \in \nabla_n$. For $i \in [m]$, we have

$$(p_m\circ\nabla(\sigma)(f))_i = \lambda((\sigma\circ f)^{-1}(i)) = \lambda(f^{-1}(\sigma^{-1}(i))) = \sum_{j\in\sigma^{-1}(i)} \lambda(f^{-1}(j))$$

while

$$(\Delta(\sigma)\circ p_n(f))_i = \Delta(\sigma)((\lambda(f^{-1}(j)))_{j=0,\dots,n})_i = \sum_{j\}\in\sigma^{-1}(i)} \lambda(f^{-1}(j))$$

and this proves the claim. □

From this, the realization $L(F)$ of F as a random variable space functorially associates to F the quotient of $\coprod_n \left(F_n \times \nabla_n\right)$ by the equivalence relation $\sim$ generated by the relations $(\alpha\sigma, a) \sim (\alpha, \nabla(\sigma)(a))$, for $\sigma \in \mathbf{\Delta}$. Although ∇_n is *not* locally compact, the topology of $L(F)$ has the same degree of tameness as $|F|$: it is paracompact, compactly generated and perfectly normal (see Proposition 3.7). As in the classical case, $L(F)$ can be seen as the colimit of the functor $\nabla\circ D_F : C_F \to \mathbf{Top}$. By the general machinery (see e.g. [18], [13]) this functor $L : \mathbf{sSet} \to \mathbf{Top}$ obviously admits a right adjoint $X \mapsto \mathrm{Sing}_{RV}(X)$ with $\mathrm{Sing}_{RV}(X)_n$ equal to the set of maps $\nabla_n \to X$.

Moreover, the natural transformation $p : \nabla \rightsquigarrow \Delta$ immediately provides a map $p_F : L(F) \to |F|$ and commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Top}}(|F|, X) & \longleftrightarrow & \mathrm{Hom}_{\mathbf{sSet}}(F, \mathrm{Sing} X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{Top}}(L(F), X) & \longleftrightarrow & \mathrm{Hom}_{\mathbf{sSet}}(F, \mathrm{Sing}_{RV} X) \end{array}$$

Our first main result is the following one.

Theorem 1.2. *(see Theorem 3.1 and Section 5.1) For F a simplicial set, the probability-law map p_F : $L(F) \to |F|$ is an homotopy equivalence. In particular, $L(F)$ has the homotopy type of a CW-complex. For X a topological space, $\mathrm{Sing}_{RV} X$ is a Kan complex.*

Let $\mathbf{M}$ denote the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Hom}_{\mathbf{M}}([n], [m])$ is the set of injective applications inside $\mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$. The elements of $\mathrm{Hom}_{\mathbf{M}}([n], [m])$ are called face maps.

The elements of $\mathbf{psSet} = \mathbf{Fun}(\mathbf{M}^{op}, \mathbf{Set})$ are called pre-simplicial sets (other common terminologies: semi-simplicial sets, Δ-sets). In particular, to each pre-simplicial set F can be associated a geometric realization $\|F\|$ defined as the quotient of $\coprod_n \left(F_n \times \Delta_n\right)$ by the equivalence relation $\sim$ generated by the relations $(\alpha\sigma, a) \sim (\alpha, \Delta(\sigma)(a))$, for $\sigma \in \mathbf{M}$.

As before, we can construct the quotient $\mathbb{L}(F)$ of $\coprod_n \left(F_n \times \nabla_n\right)$ by the equivalence relation generated by $(\alpha\sigma, a) \sim (\alpha, \nabla(\sigma)(a))$, for $\sigma \in \mathbf{M}$. Again, the natural transformation $p : \nabla \rightsquigarrow \Delta$ immediately provides a probability-law map $p_F : \mathbb{L}(F) \to \|F\|$. Our second main result is the following one.

Theorem 1.3. *(see Theorem 4.1) For F a pre-simplicial set, the probability-law map $\mathbb{L}(F) \to \|F\|$ is an homotopy equivalence. In particular, $\mathbb{L}(F)$ has the homotopy type of a CW-complex.*

In this case, we are able to construct an *explicit* homotopy, which we will use in exploring the homotopic properties of the construction on simplicial sets (for instance in the proof of Theorem 1.7 below).

Finally, recall that if $\mathcal{K}$ is a simplicial complex over a *totally ordered* set S of vertices, then one can associate to it a simplicial set $S\mathcal{K}$ and a pre-simplicial set $M\mathcal{K}$ (see Section 5.2). The next result says that the constructions of this paper are compatible with the constructions of [16] in this case.

Theorem 1.4. *(see Section 5.2) Let $\mathcal{K}$ be an ordered simplicial complex. Then $L(S\mathcal{K})$ and $L(M\mathcal{K})$ are homeomorphic, and $L(S\mathcal{K})$, $L(\mathcal{K})$ and $|\mathcal{K}|$ are homotopically equivalent.*

Comparing the above results with the results of [16] for simplicial complexes, this suggests the following conjecture.

Conjecture 1.5. *The maps $L(F) \to |F|$ and $\mathbb{L}(F) \to \|F\|$ Hurewicz fibrations.*

We then prove that this new adjunction L : **sSet** $\leftrightarrows$ **Top** : Sing_{RV} is well-behaved with respect to homotopy. Our first result in this direction compares the homotopy types of $\mathrm{Sing}X$ and $\mathrm{Sing}_{RV}X$.

Theorem 1.6. *(see Theorem 6.4) For X a topological space, the natural map $|\mathrm{Sing}X| \to |\mathrm{Sing}_{RV}X|$ is an homotopy equivalence. In particular, $|\mathrm{Sing}_{RV}X|$ and X have the same weak homotopy type.*

Thus the natural map $\mathrm{Sing}X \to \mathrm{Sing}_{RV}X$ is a weak homotopy equivalence inside **sSet**. If $\mathrm{Sing}X$ is viewed as the ∞-groupoid $\pi_{\leq\infty}X$ of X, then $\mathrm{Sing}_{RV}X$ provides another construction for it. More concretely, this Theorem implies in particular that the induced morphism $\mathbf{Z}\mathrm{Sing}X \to \mathbf{Z}\mathrm{Sing}_{RV}X$ of simplicial abelian groups is also a weak equivalence (see [11] Proposition III 2.16), so that the obvious 'randomized singular chain complex' of X, constructed in the same way as the classical singular chain complex by replacing the collection of all maps $\Delta_n \to X$ with the collection of all maps $\nabla_n \to X$, has for homology the classical singular homology of X.

Then, we endow **Top** with M. Cole's more flexible version of the standard (Quillen) model category structure. The homotopy equivalences for this structure are the usual weak homotopy equivalences. The classical functors $|\bullet|$ and Sing together provide a Quillen equivalence between this structure and the standard model structure on **sSet**. We get the following 'randomization' of this classical result.

Theorem 1.7. *(see Theorem 7.1) The functors L : **sSet** $\to$ **Top** and Sing_{RV} : **Top** $\to$ **sSet** provide a Quillen equivalence between **sSet** and **Top**. In particular they induce an equivalence of categories between the corresponding homotopy categories.*

Acknowledgements. I thank S. Douteau and D. Chataur for useful discussions. I thank especially D. Chataur for his help in the proof of Theorem 1.6.

2. Preliminaries on measure algebras

In the paper, Ω is a atomless (complete) probability space. It admits a *measure algebra* $\mathfrak{M}$, defined (see [7]), as the collection of all measurable sets modulo neglectable ones, with the operations of intersection $\cap$ and symmetric difference Δ, together with the measure map $\lambda : \mathfrak{M} \to [0,1]$. It is naturally endowed with a metric $d(X,Y) = \lambda(X\Delta Y)$, so that as a metric space it is naturally isomorphic to ∇_2. The atomless condition implies, thanks to Sierpinsky's theorem ([24]), that there exists an *exhaustion map* $t \mapsto \Omega_t$, which is a continuous map $[0,1] \to \mathfrak{M}$ such that $t_1 \leq t_2 \Rightarrow \Omega_{t_1} \subset \Omega_{t_2}$ and $\lambda(\Omega_t) = t = t\lambda(\Omega)$. We fix this exhaustion map once and for all. When Ω is a *standard* probability space, one can identify Ω with $[0,1]$ endowed with the Lebesgue measure and set $\Omega_t = [0,t]$.

The following useful technical results were proven in [16], under the unnecessary assumption that Ω is standard. More generally, all the results of [16] are true without this assumption, with essentially the same proofs (with the exhaustion map replacing the choices of intervals). The suspicious reader may however impose this additional assumption that Ω is standard on the current paper as well. Hopefully the detailed proofs for the statements of [16] in this more general setting will appear in [17].

The first useful map constructed in [16] is the following one. It is a continuous map $\mathbf{g} : \mathfrak{M}\times[0,1] \to \mathfrak{M}$ such that

(1) for every $A \in \mathfrak{M}$, $u \in [0,1]$, $\mathbf{g}(A,0) = A$, $\lambda(\mathbf{g}(A,u)) = \lambda(A)(1-u)$
(2) for every $A \in \mathfrak{M}$, $0 \leq u \leq v \leq 1$, $\mathbf{g}(A,u) \supset \mathbf{g}(A,v)$
(3) setting $\breve{\mathbf{g}}(A,u) = \mathbf{g}(A,1-u)$, so that $\lambda(\breve{\mathbf{g}}(A,u)) = u\lambda(A)$, we have $\breve{\mathbf{g}}(A,uv) = \breve{\mathbf{g}}(\breve{\mathbf{g}}(A,u),v)$ for every $A \in \mathfrak{M}$ and $u,v \in [0,1]$.
(4) for every $u,v \in [0,1]$, $\mathbf{g}(\Omega_v,u) = \Omega_v \setminus \Omega_{uv}$
(5) for all $E,F \in \mathfrak{M}$ and $u,v \in [0,1]$,

$$\lambda\left(\mathbf{g}(E,u)\Delta\mathbf{g}(F,v)\right) \leq 4\lambda(E\Delta F) + |v-u|\max(\lambda(E),\lambda(F)) \leq 4\lambda(E\Delta F) + |v-u|$$

This map is constructed in [16], Lemma 6. The additional statements we provide here are proven in the course of the proof of the Lemma given there.

From this map, we can immediately build two other useful ones.

(1) Setting $\mathbf{h}(A,u) = {}^c\mathbf{g}({}^cA,u)$, one gets a continuous companion map $\mathbf{h} : \mathfrak{M} \times [0,1] \to \mathfrak{M}$ such that $\mathbf{h}(A,0) = A$, $\mathbf{h}(A,1) = \Omega$, $\lambda(\mathbf{h}(A,u)) = u + (1-u)\lambda(A)$ and $\mathbf{h}(A,u) \subset \mathbf{h}(A,v)$ for all A and $u \leq v$. Moreover it satisfies $\mathbf{h}(\Omega_t,u) = \Omega_{u+(1-u)t}$ and
$$\lambda\left(\mathbf{h}(E,u)\Delta\mathbf{h}(F,v)\right) \leq 4\lambda(E\Delta F) + |v-u|$$
for all $E,F \in \mathfrak{M}$ and $u,v \in [0,1]$.

(2) The map $(t,A) \mapsto tA = \breve{\mathbf{g}}(A,t)$ provides a topological 'retracting' action of the monoid $[0,1]$ (for the multiplication law) on $\mathfrak{M}$, that is $t_1(t_2A) = (t_1t_2)A$, with the property that $\lambda(tA) = t\lambda(A)$.

A third, more elaborate map is constructed in [16] from $\mathbf{g}$. For X a topological space, let us denote $\mathrm{P}(X)$ the *path space* made of continuous maps $[0,1] \to X$ endowed with the compact-open topology. Then, there is a continuous map
$$\Phi : \mathrm{P}([0,1]) \times \mathrm{P}(\mathfrak{M}) \times \mathfrak{M} \to \mathrm{P}(\mathfrak{M})$$
mapping $(q, E_\bullet, A)$ to $B_\bullet$, so that

- if $A \subset E_0$ and $q(0)\lambda(E_0) = \lambda(A)$, then $B_0 = A$
- for all $u \in [0,1]$, $B_u \subset E_u$ and $\lambda(B_u) = q(u)\lambda(E_u)$
- if q and $E_\bullet$ are constant maps, then so is $B_\bullet$

Informally this says that, when $E_\bullet \in \mathrm{P}(\mathfrak{M})$ is a path inside $\mathfrak{M}$ with $A \subset E_0$, then we can find another path $B_\bullet \in \mathrm{P}(\mathfrak{M})$ such that $B_u \subset E_u$ for every u, and the ratio $\lambda(B_\bullet)/\lambda(E_\bullet)$ follows any previously specified variation starting at $\lambda(A)/\lambda(E_0)$ – and, moreover, that this can be done continuously.

The map is constructed as follows. We extend by constants the map $\breve{\mathbf{g}} : \mathfrak{M}\times[0,1] \to \mathfrak{M}$ so that to define a continuous map $\mathfrak{M} \times \mathbf{R} \to \mathfrak{M}$. We have $\breve{\mathbf{g}}(A,t) = \breve{\mathbf{g}}(A,1) = A$ for every $t \geq 1$, and $\breve{\mathbf{g}}(A,t) = \breve{\mathbf{g}}(A,0) = \emptyset$ for every $t \leq 0$. Then, setting $a(u) = q(u)\lambda(E_u)$, the image of $(q, E_\bullet, A)$ is defined by the formula
$$u \mapsto \breve{\mathbf{g}}\left(A \cap E_u, \frac{a(u)}{\lambda(A \cap E_u)}\right) \cup \breve{\mathbf{g}}\left(E_u \setminus A, \frac{a(u) - \alpha(u)}{\lambda(E_u \setminus A)}\right)$$
A detailed elementary proof that this map is indeed continuous can be found in [16] (see Proposition 5 there).

Finally, we notice that the topological space ∇_n depends only on the measure algebra $\mathfrak{M}$, as it can be defined as
$$\nabla_n = \{\underline{A} = (A_k)_{k=0..n} \in \mathfrak{M}^{n+1} \mid i \neq j \Rightarrow A_i \cap A_j = 0 \ \& \ \sum_{k=0}^{n} \lambda(A_k) = 1\}$$
the correspondance with the description of $f \in \nabla_n$ as a map $f : \Omega \to [n]$ being given by $A_k = f^{-1}(k)$. Therefore, our construction $L(F)$ depends only on $\mathfrak{M}$, and not on the probability space Ω itself. As a consequence, from Maharam's theorem (see e.g. [7] ch. 33), we could assume w.l.o.g. that Ω is a countable union of (renormalized) probability spaces of the form $\{0,1\}^\alpha$ with the α infinite cardinals. We shall not need this fact, though, in the course of our proofs.

Another remark is that all the constructions made here make sense if $\mathfrak{M}$ is replaced by *any* subalgebra of $\mathfrak{M}$ containing the subsets $\Omega_t, t \in [0,1]$: typically, from the construction in [16], we get immediately that the set $\mathbf{g}(A,u)$ belongs to the subalgebra generated by the Ω_t and A. As an example of such a subalgebra, in the case where $\mathfrak{M} = \mathfrak{M}([0,1])$ is the measure algebra of the unit interval and the exhaustion map is the map $t \mapsto [0,t]$, the algebra $\mathfrak{M}$ could be replaced by any subalgebra containing the unions of any finite number of open (or closed) intervals of $[0,1]$. Therefore, all the results of the present paper remain valid if, in the above definition of ∇_n, the algebra $\mathfrak{M}$ is replaced by any of these subalgebras.

3. Simplicial Random Variables

The purpose of this section is to prove the following theorem.

Theorem 3.1. *The probability-law map $L(F) \to |F|$ is a homotopy equivalence.*

In order to prove it, we first need to prove that $L(F)$ has similar structural properties as $|F|$, so we need to browse the proofs describing the structure of $|F|$ as they appear in standard textbooks and prove that they can be adapted to $L(F)$ (without, in particular, using the theory of CW-complexes). We use [8] for this purpose throughout.

We fix some $F \in \mathbf{sSet}$, and let $F_n^\# \subset F_n$ the collection of non degenerate simplices, that is the ones not inside $F_{n-1}.\sigma$ for some $\sigma \in \mathbf{\Delta}$. Recall that $|F|$ is a quotient of the subspace $\bigsqcup F_n^\# \times \Delta_n$ ([8] cor. 4.3.2) and that it is a CW-complex ([8], Theorem 4.3.5), the n-cells being given by the $\{x\} \times \Delta_n \simeq \Delta_n \simeq B^n$ for $x \in F_n^\#$ with attaching maps $c_x : \Delta_n \to |F|$ mapping $u \in \Delta_n$ to the class of (x, u).

For technical purposes, we need to introduce standard subcategories of $\mathbf{\Delta}$. We let $\mathbf{M}$ (resp. $\mathbf{E}$) denote the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Hom}_{\mathbf{M}}([n], [m])$ (resp. $\mathrm{Hom}_{\mathbf{E}}([n], [m])$) is the set of injective (resp. surjective) applications inside $\mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$. The elements of $\mathrm{Hom}_{\mathbf{M}}([n], [m])$ are called the face maps and the elements of $\mathrm{Hom}_{\mathbf{E}}([n], [m])$ are called the degeneracy maps.

We first notice that the composition of the functor $\triangledown : \mathbf{\Delta} \to \mathbf{Top}$ with the forgetful functor $\mathbf{Top} \to \mathbf{Set}$ provides a cosimplicial set, which is immediately checked to have the Eilenberg-Zilber property. Recall from e.g. [8] Proposition 4.2.6 that this property means $\forall x \in \triangledown_0\ \triangledown(\delta_0)(x) \neq \triangledown(\delta_1(x))$ with $\{\delta_0, \delta_1\} = \mathrm{Hom}_{\mathbf{\Delta}}([0], [1])$, and has for consequence that every element of $L(F)$ admits a unique representative of the form (α, a) with $\alpha \in F_n^\#$ and a an interior point of $\triangledown_n$ (that is, a point not inside the image of $\triangledown(\sigma)$ for $\sigma : [m] \to [n]$, $m < n$). This representative is called the minimal representative.

3.1. **The boundary and interior of $\triangledown_n$.** Recall from [16] that to every simplicial complex $\mathcal{K}$ one associates the metric space $L(\mathcal{K})$, defined as a subspace of $L^1(\Omega, S)$ for $S = \bigcup \mathcal{K}$ the union of all the elements of $\mathcal{K}$, that is its vertex set. This subspace is made of the (up to neglectability, measurable) maps $f : \Omega \to \bigcup \mathcal{K}$ such that $f(\Omega) \in \mathcal{K}$, where

$$f(\Omega) = \{s \in S \mid \lambda(f^{-1}(s)) > 0\}$$

is what is called the *essential image* of f. In this context, $\Delta_n = |\mathcal{P}_{\mathrm{f}}^*([n])|$ and $\triangledown_n = L(\mathcal{P}_{\mathrm{f}}^*([n]))$, where $\mathcal{P}_{\mathrm{f}}^*(S)$ denotes the collection of all nonempty finite subsets of the set S.

For any simplicial complex $\mathcal{K}$, let us consider the set $\mathcal{K}_{\max}$ of its maximal elements. Then it is easily checked that $\partial\mathcal{K} = \mathcal{K} \setminus \mathcal{K}_{\max}$ is a simplicial complex as well. We set $\partial\triangledown_n = L(\partial\mathcal{P}_{\mathrm{f}}^*([n])) = L(\mathcal{P}_{\mathrm{f}}^*([n]) \setminus \{[n]\})$ and

$$\triangledown_n^\circ = \triangledown_n \setminus \partial\triangledown_n = \{f \in L^1(\Omega, [n]);\ f(\Omega) \subsetneq [n]\}.$$

It is easily checked that $\triangledown_n^\circ$ is equal to the interior of $\triangledown_n$ in the sense of the cosimplicial set $\triangledown$ as defined above.

We can now notice the following properties of $\triangledown$, whose easy proofs are left to the reader. Part (3) can be proved either directly or, applying $p : \triangledown \twoheadrightarrow \Delta$, immediately deduced from the classical case.

Lemma 3.2.

(1) *For every $S \in \mathbf{E}$, the map $\triangledown(S)$ maps interior points to interior points.*
(2) *For every $D \in \mathbf{M}$, the map $\triangledown(D)$ is a closed map.*
(3) *If $\sigma \in \mathrm{Hom}_{\mathbf{\Delta}}([m], [n])$ and $a \in \triangledown_m^\circ$, then $\triangledown(\sigma).a$ determines σ.*

Together with the Eilenberg-Zilber property, part (1) of the lemma has the following easy consequence. Let $(\beta, b) \in \bigsqcup F_n \times \triangledown_n$ having (α, a) for minimal representative. We have $b = \triangledown(D).a'$ for some $D \in \mathbf{M}$ and some interior point a' by the Eilenberg-Zilber property, hence $(\beta, b) \sim (\beta.D, a')$; now, $\beta.D = \alpha'.S$ for some $S \in \mathbf{E}$ and $\alpha' \in F^\#$, so that $(\beta.D, a') = (\alpha'.S, a') \sim (\alpha', \triangledown(S).a')$. But since a' is an interior point so is $\triangledown(S).a'$ and $(\alpha', \triangledown(S).a')$ is the unique minimal representative, which proves $\alpha' = \alpha$, $a = \triangledown(S).a'$. In particular we have $D \in \mathbf{M}$ and $S \in \mathbf{E}$ such that $\beta.D = \alpha.S$.

By [16] we know that the probability-law map provides a Hurewicz fibration $\partial\triangledown_n = L(\partial\mathcal{P}_{\mathrm{f}}^*([n])) \to |\partial\mathcal{P}_{\mathrm{f}}^*([n])| = \partial\Delta_n$ with homotopically trivial fiber (consider the preimage of a vertex of $\partial\Delta_n$), which is an homotopy equivalence. Since Δ_n is homeomorphic to a n-sphere, we get that $\partial\triangledown_n$ has the (strong) homotopy type of a n-sphere. Moreover, $\partial\triangledown_n$ is equal to the preimage of $\partial\Delta_n$ under the probability-law map.

3.2. **The cofibration $\partial\triangledown_n \to \triangledown_n$.** The purpose of this section is to prove the following.

Proposition 3.3. *The inclusion map $\partial\triangledown_n \to \triangledown_n$ is a closed cofibration.*

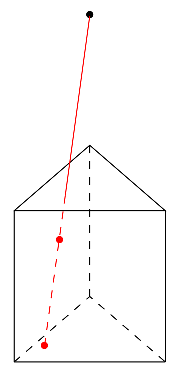


FIGURE 1. The cofibration $\partial\Delta_n \hookrightarrow \Delta_n$

As $\partial\nabla_n$ is a closed subset of ∇_n, in order to prove the proposition we need (see e.g. [8], Proposition A.4.1 p.250) to construct a retract $\nabla_n \times I \to (\nabla_n \times \{0\}) \cup ((\partial\nabla_n) \times I)$ of the reverse natural inclusion, with $I = [0,1]$.

We first follow the classical geometric receipe ([8], p. 7-8) for proving that the inclusion of the n-sphere inside the $(n+1)$-ball is a closed fibration, except that we do it on the n-simplex (see Figure 1). For this we construct a retract $\Delta_n \times I \to (\Delta_n \times \{0\}) \cup ((\partial\nabla_n) \times I)$ of the natural inclusion in the other direction. The elements of $\Delta_n \times I$ are the $(\underline{u}; a) \in \Delta_n \times I$ for $\underline{u} = (u_0, \ldots, u_n)$ with $u_i \geq 0$ and $\sum u_i = 1$. The line from $(\underline{v}; 2)$ to $(\underline{u}; a)$ with $\underline{v} = (v_i)_{i=0..n}$ and $v_i = \frac{1}{n+1}$ crosses $(\Delta_n \times \{0\}) \cup ((\partial\Delta_n) \times I)$ at exactly one point. The corresponding (continuous) map from $\Delta_n \times I$ is explicitely given by the following formulas

$$(\underline{u}; a) \mapsto \begin{cases} (\frac{1}{2-a}(2u_i - \frac{a}{n+1})_{i=0,\ldots,n}; 0) & \text{if } a \leq 2(n+1)m(\underline{u}) \\ ((\frac{u_i - m(\underline{u})}{1-(n+1)m(\underline{u})})_{i=0,\ldots,n}; \frac{a-2(n+1)m(\underline{u})}{1-(n+1)m(\underline{u})}) & \text{if } a \geq 2(n+1)m(\underline{u}) \end{cases}$$

where $m(\underline{u}) = \min(u_0, u_1, \ldots, u_n)$.

We now want to lift the map $\Delta_n \times I \to (\Delta_n \times \{0\}) \cup ((\partial\Delta_n) \times I)$ to a map $\nabla_n \times I \to (\nabla_n \times \{0\}) \cup ((\partial\nabla_n) \times I)$. For this we use the following result from [16] (Proposition 4.4 and Remark 4.5).

Proposition 3.4. *Let X be a topological space. Then the probability-law map $p_n : \nabla_n \to \Delta_n$ has the homotopy lifting property w.r.t. X, that is, for any (continuous) maps $H : X \times [0,1] \to \Delta_n$, $h : X \to \nabla_n$ such that $p_n \circ h = H(\bullet, 0)$, there exists a map $\tilde{H} : X \times [0,1] \to \nabla_n$ such that $p_n \circ \tilde{H} = H$ and $\tilde{H}(\bullet, 0) = h$. Moreover, for any $x \in X$ such that $H(x, \bullet)$ is constant, then so is $\tilde{H}(x, \bullet)$.*

We then start from the map $f : \Delta_n \times [0,1] \to (\Delta_n \times \{0\}) \cup (\partial\Delta_n \times I) \subset \Delta_n \times I$ constructed above and we consider the projection map $p_1 : \Delta_n \times I \to \Delta_n$ as well as the composed map $p_1 \circ f = f^1 : \Delta_n \times I \to \Delta_n$. Let us consider the probability-law map $p_n : \nabla_n \to \Delta_n$ and set $H = f^1 \circ (p_n \times \mathrm{Id}) : \nabla_n \times I \to \Delta_n$. We have $H(x,t) = p_1(f(p_n(x), t))$, and $H(x, 0) = p_1(f(p_n(x), 0)) = p_n(x) = p_n(h(x))$ for $h = \mathrm{Id}_{\nabla_n}$. Applying Proposition 3.4 with $X = \nabla_n$, we get $\tilde{H} : \nabla_n \times [0,1] \to \nabla_n$ such that $p_n \circ \tilde{H} = H$ and $\tilde{H}(\bullet, 0) = h = \mathrm{Id}_{\nabla_n}$. Moreover, for any $x \in \partial\nabla_n$, since $f^1(p_n(x)) = p_n(x)$ we get that $H(x, \bullet) = f^1(p_n(x), \bullet)$ is constant, since $f^1(y,t) = y$ for all $y \in \partial\Delta_n$. This yields $\tilde{H}(x,t) = \tilde{H}(x,0) = h(x) = x$ for all $x \in \partial\nabla_n$, $t \in I$. Let us now consider $\varphi = \pi_2 \circ f : \Delta_n \times [0,1] \to I$ where π_2 is the second projection and set $\Psi(x,t) = (\tilde{H}(x,t), \varphi(x,t))$. This defines a continuous map $\Psi : \nabla_n \times I \to \nabla_n \times I$ such that $p_n \times \mathrm{Id} \circ \Psi$ coincides with f. As a consequence it takes values inside

$$(\nabla_n \times \{0\}) \cup ((\partial\nabla_n) \times I) = (p_n \times \mathrm{Id})^{-1} \left((\Delta_n \times \{0\}) \cup ((\partial\Delta_n) \times I)\right)$$

and it makes the following diagram commute, where the vertical maps are restrictions of $p_n \times \mathrm{Id}$.

$$\begin{array}{ccc} \nabla_n \times I & \longrightarrow & (\nabla_n \times \{0\}) \cup ((\partial\nabla_n) \times I) \\ \downarrow & & \downarrow \\ \Delta_n \times I & \longrightarrow & (\Delta_n \times \{0\}) \cup ((\partial\Delta_n) \times I) \end{array}$$

It remains to prove that Ψ is the identity both on $\triangledown_n \times \{0\}$, which is clear because $\Psi(x,0) = (\tilde{H}(x,0), \varphi(x,0)) = (h(x),0) = (x,0)$, and on $(\partial\triangledown_n)\times I$, which holds true because $\varphi(x,t) = t$ and $\tilde{H}(x,t) = \tilde{H}(x,0) = x$ whenever $x \in \partial\triangledown_n$. This concludes the proof of Proposition 3.3.

3.3. **Preliminaries on attachments.** In the remaining part of this section we adopt the point of view of a simplicial set G as a graded set $\bigsqcup_n G_n$ endowed with a right action of the category $\mathbf{\Delta}$. A simplicial subset of G is a simplicial set $D = \bigsqcup_n D_n$ with $D_n \subset G_n$ such that the inclusion $D \subset F$ is a simplicial map.

We briefly recall the definition of a simplicial attachment (see [8] p. 144). Let A and G be two simplicial sets, and D a simplicial subset of G. That is, D is a simplicial set $\bigsqcup_n D_n$ with $D_n \subset G_n$ and the inclusion maps $D_n \to G_n$ commute with the face and degeneracy maps. Moreover, let $f : D \to A$ be a simplicial map. In order to avoid confusions, we temporarily denote $x \star \rho$ the action of $\rho \in \mathbf{\Delta}$ on $x \in G_n$. From this the simplicial attachement F is such that $F_n = A_n \sqcup (G_n \setminus D_n)$ and, for any $\rho \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$ and $x \in F_n$, we define $\rho.x$ from the action of $\mathbf{\Delta}$ on A if $x \in A_n$, from the action $\star$ on G if $x \in G_n \setminus D_n$ and $x.\rho \notin D_m$, and finally as $f_m(x \star \rho)$ if $x.\rho \in G_n \setminus D_n$ and $x \star \rho \in D_m$. Checking that this construction is well-defined is straightforward.

Now, we recall from e.g. [8] Corollary 4.2.4 that the n-skeleton F^n of a simplicial set F is obtained from its $(n-1)$-skeleton by attaching the simplicial set $\bigsqcup_{x\in F_n^\#} \Delta_x$, where Δ_x is a copy of the simplicial set Δ_n, via the simpleCial map $\bigsqcup_{x\in F_n^\#} \varphi_x$ with $\varphi_x : \partial\Delta_x \to F^{n-1}$ given by $\varphi_x(\alpha) = x\alpha$ where the simplicial set $\partial\Delta_n$ is by definition the $(n-1)$-skeleton of Δ_n.

3.4. **Properties of the functor L : sSet → Top.** Let F, G two simplicial sets, and $f : F \to G$ a simplicial map. This means that f is a collection of maps $f_n : F_n \to G_n$ commuting with the right action of the category $\mathbf{\Delta}$, in the sense that, for every $\sigma \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$ and $\alpha \in F_n$, we have $f_m(\alpha.\sigma) = f_n(\alpha).\sigma$. It induces continuous maps $\hat{f}_n = f_n \times \mathrm{Id}_{\triangledown_n} : F_n \times \triangledown_n \to G_n \times \triangledown_n$. For $\sigma \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$ and $(\alpha, a) \in F_n \times \triangledown_n$ we have

$$\hat{f}_m(\alpha.\sigma, a) = (f_m(\alpha.\sigma), a) = (f_n(\alpha).\sigma, a) \sim (f_n(\alpha), \triangledown(\sigma)(a)) = \hat{f}_n(\alpha, \triangledown(\sigma)(a))$$

hence $\bigsqcup_n \hat{f}_n$ induces a continuous map $L(f) : L(F) \to L(G)$, and clearly $L(f \circ g) = L(f) \circ L(g)$, $L(\mathrm{Id}_F) = \mathrm{Id}_{L(F)}$. Therefore L defines a functor L : **sSet** → **Top**.

The composite of $\triangledown$ with the forgetful functor V : **Top** → **Set** is a cosimplicial set, and clearly $V \circ L(F) = F \otimes V \circ \triangledown$ with the notations of e.g. [8]. Moreover, it is immediately checked that $V \circ \triangledown$ has the Eilenberg-Zilber property, and therefore $V \circ L$ preserves and reflects monomorphisms ([8], corollary 4.2.9). In particular, if D is a simplicial subset of G, then the induced map $L(D) \to L(G)$ is injective.

As in the classical case, we have the following property.

Lemma 3.5. *If G is a simplicial subset of the simplicial set F, then the natural map $L(G) \to L(F)$ embeds $L(G)$ as a closed subset of $L(F)$.*

Proof. Let $\bar{y} \in L(G)$ and $\bar{x}$ its image in $L(F)$. There exists unique minimal representatives of $\bar{x}$ and $\bar{y}$ inside $\bigsqcup_n F_n \times \triangledown_n$ and $\bigsqcup_n G_n \times \triangledown_n$, respectively. Since $G^\# \subset F^\#$ they are the same, and this implies that $\bar{x}$ determines $\bar{y}$, whence $L(G) \subset L(F)$.

Let now C be a closed subset of $L(G)$, $q : \bigsqcup F_n \times \triangledown_n \to L(F)$ the natural projection map, and $C_\alpha = q^{-1}(C) \cap \{\alpha\} \times \triangledown_n$ for each $\alpha \in F_n$. We need to prove that each C_α is closed. This is clear when $\alpha \in G$, so we assume otherwise, and consider $(\alpha, y) \in C_\alpha$. Then $y = \triangledown(D).y_0$ for some interior point y_0 and $D \in \mathbf{M}$, and $\alpha.D = \beta.S$ for some $S \in \mathbf{E}$ and $\beta \in F^\#$. Then

$$(\alpha, y) = (\alpha, \triangledown(D).y_0) \sim (\alpha.D, y_0) = (\beta.S, y_0) \sim (\beta, \triangledown(S).y_0)$$

and β is non-degenerate, $\triangledown(S).y_0$ is interior (Lemma 3.2 (1)) , hence $(\beta, \triangledown(S).y_0)$ is the minimal representative in the class, which implies $\beta \in G$. Then $\alpha.D = \beta.S \in G$ and $y = \Delta(D).y_0$ with $(\alpha.D, y_0) \in C_{\alpha.D}$. This implies

$$C_\alpha = \bigcup_{\substack{D\in\mathbf{M} \\ \alpha.D\in G}} \triangledown(D)(C_{\alpha.D}).$$

Now, each $C_{\alpha.D}$ is closed, each $\triangledown(D)$ is a closed map (Lemma 3.2 (2)) and the collection of all $\Delta \in \mathbf{M}$ that can be applied to α is finite, whence C_α is closed. ☐

We show that, when F is a simplicial set, then $L(F)$ can be constructed as a limit of successive attachments. Notice that, because of Proposition 3.3, the natural maps $F_n^\# \times \partial\nabla_n \to F_n^\# \times \nabla_n$ implied in the attachment are closed cofibrations.

Proposition 3.6. *Let F be a simplicial set, then $(L(F^{(n)}))_{n\geq 0}$ is a filtration of $L(F)$ which determines the topology of F. Moreover, for every n,*

$$L(F^{(n)}) = L(F^{(n-1)}) \cup_{L(\varphi)} \left(\coprod_{x\in F_n^\#} \nabla_x \right).$$

where $\varphi : \coprod_{x\in F_n^\#} \partial\Delta_x \to F^{(n-1)}$ is the simplicial attaching map, and the natural map $\coprod_n F_n^\# \times \nabla_n \to L(F)$ is a quotient map.

Proof. This statement is adapted from the classical analogous statement for the geometric realization functor, so we need to check that the classical proof uses only properties of the cosimplicial *space* $[n] \mapsto \Delta_n$ (that we still denote $\mathbf{\Delta}$) that are also satisfied by the cosimplicial space $[n] \mapsto \nabla_n$ (that we still denote L). For this we follow the steps described in [8], §4.3. First of all, since the cosimplicial set L also has the Eilenberg-Zilber property, then any element of the tensor product $F \otimes L = L(F)$ can be represented by a unique so-called minimal pair ([8], Proposition 4.2.7). At the set-theoretical level this implies (see [8], Proposition 4.3.3) that every element of $L(F)$ has a unique representative of the form (α, a) with α a non-degenerate simplex of F of some dimension n, and $a \in \nabla_n \setminus \partial\nabla_n$, and also that, for f a simplicial map, f is injective iff $L(f)$ is injective. Moreover, the topology of $L(F)$ is the final topology w.r.t. the family of maps $\overline{c}_x : \nabla_n \to L(F)$, induced by $a \mapsto (x, a) \in F_n \times \nabla_n \to L(F)$ (compare with [8] p. 153). As a consequence (see [8], Proposition 4.3.1) we get that, if some subset $E \subset \coprod_n F_n$ generates the simplicial set F, then $L(F)$ is a quotient space of the subspace $\coprod_n (E \cap F_n) \times \nabla_n$ of $\coprod_n F_n \times \nabla_n$; in particular, $L(F)$ is a quotient space of $\coprod_n F_n^\# \times \nabla_n$.

From this and Lemma 3.5 we can adapt the proof of [8], Theorem 4.3.5. First of all, we get from the previous point that the $L(F^{(n)})$ are closed subspaces of $L(F)$ and form a filtration of it. The fact that the topology of the space $L(F)$ is determined by the family $(L(F^{(n)}))_{n\in\mathbf{N}}$ has the same proof as for $|F|$: if a map $f : L(F) \to Z$ is such that all its restrictions to $L(F^{(n)})$ are continuous, then so are the composites $f \circ \overline{c}_x$ since any $\overline{c}_x$ factorizes through $L(F^{(n)}) \to L(F)$ for some n ; since $L(F)$ has the final topology with respect to the $\overline{c}_x$, this fact follows. Therefore, we can fix n, and consider $F^{(n)}$ as a simplicial attachment. Then the proof of Theorem 4.3.5 of [8] can be applied verbatim to our case, and $L(F^{(n)})$ can be described as a topological attachment as in the statement. □

From this we get the following result.

Proposition 3.7. *For F a simplicial set, $L(F)$ is paracompact and perfectly normal. It is also compactly generated.*

Proof. Recall that a space X is called perfectly normal if every closed subset is the vanishing locus of some map $X \to \mathbf{R}_+$. Since discrete spaces and the metrizable spaces $F_n^\# \times \nabla_n$ are perfectly normal, by induction on n we get from Proposition 3.6 that all the $L(F^{(n)})$ are perfectly normal (see [8] Proposition A.4.8 (iv)). Moreover, the embeddings $L(F^{(n)}) \hookrightarrow L(F^{(n+1)})$ are closed cofibrations (see [8] Proposition A.4.8 (ii)) and we checked in the proof of Proposition 3.6 that the topology of $L(F)$ is the topology of the union of the $L(F^{(n)})$. From this it follows ([8] Proposition A.5.1 (iv)) that $L(F)$ is perfectly normal. In particular it is Hausdorff. In order to prove that it is paracompact, it is by the same arguments enough to check that each of the $L(F^{(n)})$ is paracompact ([8] Proposition A.5.1 (v)). By Michael's theorem ([20], pages 791-792; [21]) this follows by induction on n from Proposition 3.6 and the fact that each of the $F_n^\# \times \nabla_n$ is metrizable hence paracompact.

We now prove that $L(F)$ is compactly generated. The fact that each $L(F^{(n)})$ is compactly generated follows from the proposition by induction on n (see e.g. [19] ch. 5.2), as $F_n^\# \times \nabla_n$ is (metrizable hence) first countable hence compactly generated. Since the filtration $L(F^{(n)})$ determines the topology of $L(F)$ this implies that $L(F)$ is compactly generated.

□

Remark 3.8. *Recall that every subset of a space which is both paracompact and perfectly normal is also paracompact and perfectly normal (*[14] *Appendice I Theorem 6), so this property of CW-complexes is also shared by $L(F)$.*

3.5. **Proof of Theorem 3.1.** We follow the scheme of the proof of the comparison theorem of [8] Theorem 4.3.20, and adapt it to our case. Let F be a simplicial set. The natural map $p_F^{(0)} : L(F^{(0)}) \to |F^{(0)}|$ is the identity map on a disjoint union of points, therefore it is a homotopy equivalence. Let us assume that we know that the probability-law maps $p_F^{(k)} : L(F^{(k)}) \to |F^{(k)}|$ for $k \leq n-1$ are homotopy equivalences and commute with the natural inclusion maps $|F^{(k-1)}| \subset |F^{(k)}|$ and $L(F^{(k-1)}) \subset L(F^{(k)})$. By Proposition 3.6 we have a commutative diagram

$$\begin{array}{ccccc} \bigsqcup_{x \in F_n^\#} \nabla_n & \longleftarrow & \bigsqcup_{x \in F_n^\#} \partial \nabla_n & \longrightarrow & L(F^{(n-1)}) \\ \downarrow & & \downarrow & & \downarrow \\ \bigsqcup_{x \in F_n^\#} \Delta_n & \longleftarrow & \bigsqcup_{x \in F_n^\#} \partial \Delta_n & \longrightarrow & |F^{(n-1)}| \end{array}$$

where the vertical maps are homotopy equivalences and the horizontal maps going left are closed cofibrations by Proposition 3.3. By the gluing theorem ([8] Theorem A.4.12 and [1] 7.5.7) this implies that $p_F^{(n)} : L(F^{(n)}) \to |F^{(n)}|$ is a homotopy equivalence. By induction this proves that $p_F^{(n)}$ is a homotopy equivalence for every n. This provides a commutative ladder of homotopy equivalences $(p_F^{(n)})_{n \geq 0}$, thus the induced map $p_F : L(F) \to |F|$ between the union spaces is a homotopy equivalence ([8], Proposition A.5.11). This concludes the proof of the theorem.

4. Pre-simplicial random variables

We use [8] and [5] for reference, and recall the notations of the introduction. Recall from Section 3 that $\mathbf{M}$ and $\mathbf{E}$ denotes the subcategories of $\mathbf{\Delta}$ made of the face and degeneracy maps, respectively. The objects of $\mathbf{psSet} = \mathbf{Fun}(\mathbf{M}^{op}, \mathbf{Set})$ are called pre-simplicial sets (or semi-simplicial sets, or Δ-sets).

The categories $\mathbf{M}$ and $\mathbf{E}$ are generated by the elementary face and degeneracy maps, respectively. These are defined as

$$\begin{array}{ccccccc} D_i^c & : & [n-1] & \to & [n] & & \\ & & k & \mapsto & k & \text{if } k < i \\ & & & & k+1 & \text{if } k \geq i \end{array} \qquad \begin{array}{ccccccc} S_i^c & : & [n+1] & \to & [n] & & \\ & & k & \mapsto & k & \text{if } k \leq i \\ & & & & k-1 & \text{if } k > i \end{array}$$

In particular, the geometric realization $\|F\|$ defined in the introduction can also be defined as the quotient of $\bigsqcup_n \left(F_n \times \Delta_n\right)$ by the equivalence relation generated by the $(D_i \alpha, a) \approx (\alpha, D^i a)$ where $D_i = F(D_i^c) : F_n \to F_{n-1}$ and $D^i = \Delta(D_i^c)$. Similarly, $\mathbb{L}(F)$ is the quotient of $\bigsqcup_n \left(F_n \times \nabla_n\right)$ by the equivalence relation generated by the $(D_i \alpha, a) \approx (\alpha, D^i_{RV} a)$ where $D^i_{RV} = \nabla(D_i^c)$.

In this section we will prove the following

Theorem 4.1. *For every pre-simplicial set F, the probability-law map $\mathbb{L}(F) \to \|F\|$ is an homotopy equivalence.*

Recall from [16] (Proposition 4.1 and its proof) that the probability-law map $p_n : \nabla_n \to \Delta_n$ has a (continuous) section $\sigma_n : \Delta_n \to \nabla_n$ characterized by

$$\sigma_n(\alpha)(x) = a \quad \text{if} \quad x \in \Omega_{\sum_{u \leq a} \alpha(u)} \setminus \Omega_{\sum_{u < a} \alpha(u)}$$

for $\alpha : [n] \to [0,1]$ with $\sum_{k=0}^n \alpha(k) = 1$. The main theorem will readily follow from the following proposition, which provides an explicit homotopy equivalence. This will be also used in Section 7.

Proposition 4.2. *The composition $\sigma_n \circ p_n : \nabla_n \to \nabla_n$ is homotopic to the identity map, by an homotopy $H_n : [0,1] \times \nabla_n \to \nabla_n$ which commutes with the face maps D^i_{RV}, that is*

$$\forall n \, \forall f \in \nabla_n \, \forall i \in [0, n+1] \, \forall u \in [0,1] \; H_{n+1}(u, D^i_{RV} f) = D^i_{RV} H_n(u, f)$$

and such that

- $p_n(H_n(u,f)) = p_n(f)$ *for all* $f \in \nabla_n, u \in [0,1]$.

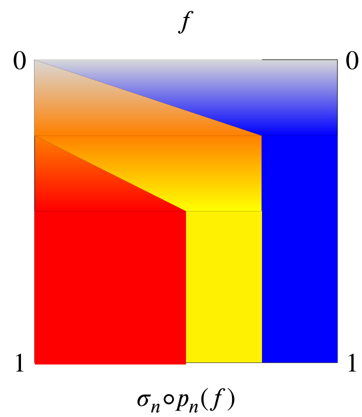


FIGURE 2. Homotopy between $\sigma_n \circ p_n$ and Id_{∇_n}

- $H_n(u, \sigma_n(\alpha)) = \sigma_n(\alpha)$ *for all* $\alpha \in \Delta_n, u \in [0,1]$.

The proof of the Theorem then goes as follows. The maps σ_n, p_n and H_n for $n \geq 0$ together define maps $\tilde{\sigma}_F : A \to B$, $\tilde{p}_F : B \to A$ and $\tilde{H}_F : I \times B \to B$ with $A = \bigsqcup_n F_n \times \Delta_n$ and $B = \bigsqcup_n F_n \times \nabla_n$. Because of the compatibility with the face maps, they induce maps $\sigma_F : \|F\| \to \mathbb{L}(F)$, $p_F : \mathbb{L}(F) \to \|F\|$ and $H_F : I \times \mathbb{L}(F) \to |F|$. It is readily checked that $\tilde{p}_F \circ \tilde{\sigma}_F = \mathrm{Id}$, hence $p_F \circ \sigma_F = \mathrm{Id}$. By the proposition, H_F is an homotopy between the identity and $\sigma_F \circ p_F$, and this proves the claim.

In the remaining part of this section, we prove Proposition 4.2.

4.1. **Homotopy equivalence : definition in the case** $n = 1$**.**
We use the retracting action $(t, A) \mapsto tA$ of the topological monoid $[0,1]$ on $\mathfrak{M} = \nabla_2$ defined in Section 2.

We first consider the case $n = 1$ and set, for $u \in [0,1]$ and $f \in \nabla_1$,

$$\begin{aligned} \hat{H}_1(u,f) &= 0 \quad \text{over } \tfrac{\alpha_0}{u+(1-u)\alpha_0}\mathbf{h}(f^{-1}(\{0\}), u) \\ &= 1 \quad \text{over its complement.} \end{aligned}$$

and set $H_1(u,f) = \hat{H}_1(\frac{u}{\alpha_1}, f)$ for $u \leq \alpha_1$, $H_1(u,f) = \sigma_1 \circ p_1(f) = \hat{H}_1(1,f)$ for $u > \alpha_1$, where $\alpha_k = \lambda(f^{-1}(\{k\}))$. Then $H_1 : [0,1] \times \nabla_1 \to \nabla_1$ provides an homotopy between $f \mapsto H_1(0,f) = \hat{H}_1(0,f) = f$ and $f \mapsto H_1(1,f) = \hat{H}_1(1,f) = \sigma_1 \circ p_1(f)$. Moreover, $p_1(H_1(u,f)) = p_1(f)$ since

$$\lambda\left(\frac{\alpha_0}{u+(1-u)\alpha_0}\mathbf{h}(f^{-1}(\{0\}), u)\right) = \frac{\alpha_0}{u+(1-u)\alpha_0}\left(u + (1-u)\lambda(f^{-1}(\{0\}))\right) = \alpha_0$$

Finally, since $\mathbf{h}(\Omega_a, u) = \Omega_{u+(1-u)a}$ we have $\hat{H}_1(u, \sigma_1(\alpha)) = \sigma_1(\alpha)$ hence $H_1(u, \sigma_1(\alpha)) = \sigma_1(\alpha)$ for all $\alpha \in \Delta_1$ and $u \leq \alpha_1$.

4.2. **Homotopy equivalence : definition in the case of higher** n**.** We then construct an homotopy $H_n : [0,1] \times \nabla_n \to \nabla_n$ between Id_{∇_n} and $\sigma_n \circ p_n$ by induction on n. We use the map Φ of Section 2.

Let $f \in \nabla_n$ with $\underline{\alpha} = p_n(f) \in \Delta_n$. We denote $\underline{X} = (X_0 \subset X_1 \subset \dots)$ defined as $X_k = f^{-1}(\{0, \dots, k\})$ and $\underline{x} = (x_0 \leq x_1 \leq)$ defined as $x_k = \alpha_0 + \dots + \alpha_k$. We set

$$E_u^n = \frac{x_{n-1}}{u + (1-u)x_{n-1}}\mathbf{h}(X_{n-1}, u), \quad \Omega_u^n = \Omega \setminus E_u^n$$

and, by descending induction, for $2 \leq k$ and $u \in [0,1]$,

$$E_u^{k-1} = \Phi\left(\frac{x_{k-2}}{x_{k-1}}, E_\bullet^k, X_{k-2}\right)(u), \quad \Omega_u^{k-1} = E_u^k \setminus E_u^{k-1}$$

and $\Omega_\bullet^0 = E_\bullet^1$.

From the defining properties of the maps Φ and $\mathbf{h}$ we immediately get

- For all u, $\lambda(E^n_u) = x_{n-1}$ and $\lambda(\Omega^n_u) = \alpha_n$
- By induction on k, for all $k \geq 1$, $\lambda(E^k_\bullet) = x_{k-1}$
- hence , for all $k \geq 0$, $\lambda(\Omega^k_\bullet) = \alpha_k$, with $\Omega^k_0 = f^{-1}(\{k\})$.
- For all $r < k$, $\Omega^r_u \subset E^{r+1}_u \subset \cdots \subset E^k_u$ hence $\Omega^r_u \cap \Omega^k_u = \emptyset$ and, when u is fixed, the Ω^k_u form a partition of Ω, as $\sum_k \lambda(\Omega^k_u) = 1$.
- $\Omega^n_1 =]x_{n-1}, 1] =]1-\alpha_n, 1]$

hence we can define $\check{H}_n(u,f)(t) = k$ if $t \in \Omega^k_u$, and $H_n(u,f) = \check{H}_n(\frac{u}{\alpha_n}, f)$ for $0 \leq u \leq \alpha_n$, and, for $\alpha_n \leq u \leq 1$,

$$\begin{array}{rcll} H_n(u,f)(t) & = & H_{n-1}(\frac{u-\alpha_n}{1-\alpha_n}, \check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n}) & \text{for } 0 \leq t \leq 1-\alpha_n \\ & = & n & \text{for } 1-\alpha_n \leq t \leq 1 \end{array}$$

where $\check{H}_n(1,f)_{[0,1-\alpha_n]} \in \nabla_{n-1}$ is defined by

$$u \mapsto \check{H}_n(1,f)\left((1-\alpha_n)u\right).$$

In the case $n = 2$, this homotopy is depicted in Figure 2.

We check that $H_n(0,f) = \check{H}_n(0,f) = f$ since $\Omega^k_0 = f^{-1}(\{k\})$, and, using the induction assumption, that, for $t \leq 1-\alpha_n$, we have

$$H_n(1,f)(t) = H_{n-1}(1, \check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n}) = \sigma_{n-1} \circ p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n})$$

and, since $\lambda(\Omega^k_\bullet) = \alpha_k$, we know that, for $k < n$, $p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})_k = \frac{\alpha_k}{1-\alpha_n}$ hence

$$\sigma_{n-1} \circ p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n}) = \sigma_n \circ p_n(f)(t).$$

Moreover, if $t \geq 1-\alpha_n$, then $H_n(1,f)(t) = n = \sigma_n \circ p_n(f)(t)$, whence $H_n(1,f) = \sigma_n \circ p_n(f)$.

4.3. **Proof of Proposition 4.2 : generalities.** We check by induction on n that the maps H_n satisfy the properties of Proposition 4.2, and first of all that it indeed provides an homotopy from Id_{∇_n} to $\sigma_n \circ p_n$.

For $f \in \nabla_n$, we have $H_n(0,f) = \check{H}_n(0,f) = f$, since $\Omega^k_0 = f^{-1}(\{k\})$. Moreover, using the induction assumption, we have that, for $t \leq 1-\alpha_n$,

$$H_n(1,f)(t) = H_{n-1}(1, \check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n}) = \sigma_{n-1} \circ p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n})$$

and, since $\lambda(\Omega^k_\bullet) = \alpha_k$, we know that, for $k < n$, $p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})_k = \frac{\alpha_k}{1-\alpha_n}$ hence

$$\sigma_{n-1} \circ p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})(\frac{t}{1-\alpha_n}) = \sigma_n \circ p_n(f)(t).$$

Finally, if $t \geq 1-\alpha_n$, then $H_n(1,f)(t) = n = \sigma_n \circ p_n(f)(t)$, whence $H_n(1,f) = \sigma_n \circ p_n(f)$ and H_n is indeed an homotopy from Id_{∇_n} to $\sigma_n \circ p_n$.

We now prove that $p_n(H_n(u,f)) = p_n(f)$ for all $u \in [0,1]$. For $u \leq \alpha_n$ this is clear as $p_n(H_n(u,f)) = p_n(\check{H}_n(\frac{u}{\alpha_n}, f)) = p_n(f)$ since $\lambda(\Omega^k_\bullet) = \alpha_k$ for all $k \geq 1$. For $u \geq \alpha_n$ we have $p_n(H_n(u,f))_n = \alpha_n = p_n(f)_n$ and, for $k < n$,

$$p_n(H_n(u,f))_k = (1-\alpha_n)p_{n-1}(H_{n-1}(\frac{u-\alpha_n}{1-\alpha_n}, \check{H}_n(1,f)_{[0,1-\alpha_n]}))_k = (1-\alpha_n)p_{n-1}(\check{H}_n(1,f)_{[0,1-\alpha_n]})_k = \alpha_k$$

hence $p_n(H_n(u,f)) = p_n(f)$ and this proves the claim.

Let us now consider the case $f = \sigma_n(\alpha)$, and prove $H_n(u,f) = f$ for every $u \in [0,1]$. From the property $\mathbf{h}(\Omega_a, u) = \Omega_{u+(1-u)a}$ we get $E^n_\bullet = \Omega_{x_{n-1}}$. From the explicit construction of Φ in Section 2 we get that, for $0 \leq a \leq b \leq 1$,

$$\Phi\left(\frac{a}{b},\Omega_b,\Omega_a\right)(u)=\check{\mathbf{g}}\left(\Omega_a\cap\Omega_b,\frac{a}{\lambda(\Omega_a\cap\Omega_b)}\right)=\check{\mathbf{g}}\left(\Omega_a,\frac{a}{a}\right)=\Omega_a$$

It follows by descending induction on k that $E^k_\bullet = \Omega_{x_{k-1}}$ whence $\check{H}_n(u,\sigma_n(\alpha)) = \sigma_n(\alpha)$. Clearly then $H_n(u,\sigma_n(\alpha)) = \sigma_n(\alpha)$ for $u \leq \alpha_n$. Letting $\beta_k = \alpha_k/(1-\alpha_n)$ for $k < n$, we have $\sigma_n(\alpha)_{[0,1-\alpha_n]} = \sigma_{n-1}(\beta)$ hence, for $u \geq \alpha_n$,

$$H_n(u,\sigma_n(\alpha))(t) = H_{n-1}(\frac{u-\alpha_n}{1-\alpha_n},\sigma_{n-1}(\beta))(\frac{t}{1-\alpha_n}) = \sigma_{n-1}(\beta)(\frac{t}{1-\alpha_n}) = \sigma_n(\alpha)(t)$$

for $t \leq 1-\alpha_n$, and $H_n(u,\sigma_n(\alpha))(t) = n = \sigma_n(\alpha)(t)$ for $t > 1-\alpha_n$, and we get $H_n(u,\sigma_n(\alpha)) = \sigma_n(\alpha)$ for all u,α.

It remains to prove that H_n commutes with the face maps.

4.4. **Proof of Proposition 4.2 : face maps.** We consider $D^i_{RV} : \nabla_n \to \nabla_{n+1}$ for $0 \leq i \leq n+1$ and denote $\hat{E}^k_u$, $\hat{\Omega}^k_u$ the sets E^k_u, Ω^k_u for $D^i_{RV}(f)$. We set $B_k = f^{-1}([k])$, $\hat{B} = (D^i_{RV}f)^{-1}([k])$, $\alpha_k = \lambda(f^{-1}(\{k\}))$, $\check{\alpha}_k = \lambda((D^i_{RV}f)^{-1}(\{k\}))$, $A_k = \lambda(B_k)$, $\hat{A}_k = \lambda(\hat{B}_k)$. We have $A_k = \alpha_0+\cdots+\alpha_k$, $\hat{A}_k = \hat{\alpha}_0+\cdots+\hat{\alpha}_k$.

We have $\hat{\alpha}_k = \alpha_k$ for $k \leq i-1$, $\hat{\alpha}_i = 0$, and $\hat{\alpha}_{k+1} = \alpha_k$ for $k \geq i$. Also, $\hat{B}_0 = B_0, \dots, \hat{B}_{i-1} = B_{i-1}$ and $\hat{B}_i = \hat{B}_{i-1} = B_{i-1}$, $\hat{B}_{i+1} = B_i, \dots, \hat{B}_{n+1} = B_n$. As a consequence, $\hat{A}_0 = A_0, \dots, \hat{A}_{i-1} = A_{i-1}$ and $\hat{A}_i = \hat{A}_{i-1} = A_{i-1}$, $\hat{A}_{i+1} = A_i, \dots, \hat{A}_{n+1} = A_n$.

We want to prove that

$$\forall n\,\forall i\in[0,n+1]\,\forall u\in[0,1]\; H_{n+1}(u,D^i_{RV}f) = D^i_{RV}H_n(u,f)$$

We prove this by induction on n.

For a given n we then prove this by descending induction on i. Therefore we start assuming $i = n+1$. Then D^i_{RV} is the inclusion $\nabla_n \subset \nabla_{n+1}$, and we have $\check{\alpha}_{n+1} = 0$, hence $1-\check{\alpha}_{n+1} = 1$ and

$$H_{n+1}(u,D^{n+1}_{RV}f) = \check{H}_n(u,D^{n+1}_{RV}f) = H_n(u,f) = D^{n+1}_{RV}H_n(u,f)$$

by definition.

We then assume that $i < n+1$. Then

$$\hat{E}^{n+1}_u = \frac{\hat{A}_n}{u+(1-u)\hat{A}_n}\mathbf{h}(\hat{B}_n,u) = \frac{A_{n-1}}{u+(1-u)A_{n-1}}\mathbf{h}(B_{n-1},u) = E^n_u$$

hence $\hat{E}^{n+1}_\bullet = E^n_\bullet$ and $\hat{\Omega}^{n+1}_\bullet = \Omega^n_\bullet$. Then, by descending induction on k, we have that, provided $k > i$,

$$\hat{E}^k_\bullet = \Phi\left(\frac{\hat{A}_{k-1}}{\hat{A}_k},\hat{E}^{k+1}_\bullet,\hat{B}_{k-1})\right) = \Phi\left(\frac{A_{k-2}}{A_{k-1}},E^k_\bullet,B_{k-2}\right) = E^{k-1}_\bullet$$

Therefore, $\hat{E}^{k+1}_\bullet = E^k_\bullet(f)$ when $k \geq i$.

Then,

$$\hat{E}^i_\bullet = \Phi\left(\frac{\hat{A}_{i-1}}{\hat{A}_i},\hat{E}^{i+1}_\bullet,\hat{B}_{i-1}\right) = \Phi\left(\frac{A_{i-1}}{A_{i-1}},E^i_\bullet,B_{i-1}\right) = \Phi\left(1,E^i_\bullet,B_{i-1}\right) = E^i_\bullet$$

since one always has $\Phi(1,E_\bullet,A) = E_\bullet$.

Then, for $k < i$, we prove that

$$\hat{E}^k_\bullet = \Phi\left(\frac{\hat{A}_{k-1}}{\hat{A}_k},\hat{E}^{k+1}_\bullet,\hat{B}_{k-1}\right) = \Phi\left(\frac{A_{k-1}}{A_k},E^{k+1}_\bullet,B_{k-1}\right) = E^k_\bullet$$

again by descending induction on k. Summarizing, one gets $\hat{E}^{k+1}_\bullet = E^k_\bullet$ for $k \geq i$, $\hat{E}^k_\bullet = E^k_\bullet$ for $k \leq i$. It follows that $\hat{\Omega}^k_\bullet = E^k_\bullet \setminus E^{k-1}_\bullet = \Omega^{k-1}_\bullet$ for $k > i$, $\hat{\Omega}^i_\bullet = \emptyset$, $\hat{\Omega}^k_\bullet = \Omega^k_\bullet$ for $k < i$. Thus, from the definition we get $\check{H}_{n+1}(u,D^i_{RV}f) = D^{RV}_i\check{H}_n(u,f)$ for all $u \in [0,1]$ hence and

$$H_{n+1}(u,D^i_{RV}f) = \check{H}_{n+1}(\frac{u}{\check{\alpha}_{n+1}},D^i_{RV}f) = \check{H}_{n+1}(\frac{u}{\check{\alpha}_n},D^i_{RV}f) = D^i_{RV}\check{H}_n(\frac{u}{\check{\alpha}_n},f) = D^i_{RV}H_n(u,f)$$

for $0 \le u \le \check{\alpha}_{n+1} = \alpha_n$. Then, for $\check{\alpha}_{n+1} = \alpha_n \le u \le 1$, we have over $[0, 1-\check{\alpha}_{n+1}]$ that

$$\begin{aligned}H_{n+1}(u, D^i_{RV} f)(t) &= H_n\left(\frac{u-\check{\alpha}_{n+1}}{1-\check{\alpha}_{n+1}}, \check{H}_{n+1}(1, (D^i_{RV} f)_{|[0,1-\check{\alpha}_{n+1}]})\right)\left(\frac{t}{1-\check{\alpha}_{n+1}}\right)\\ &= H_n\left(\frac{u-\alpha_n}{1-\alpha_n}, \check{H}_{n+1}(1, D^i_{RV}(f_{|[0,1-\alpha_n]}))\right)\left(\frac{t}{1-\check{\alpha}_{n+1}}\right)\\ &= H_n\left(\frac{u-\alpha_n}{1-\alpha_n}, D^i_{RV}\check{H}_n(1, f_{|[0,1-\alpha_n]})\right)\left(\frac{t}{1-\check{\alpha}_{n+1}}\right)\\ &= D^i_{RV} H_{n-1}\left(\frac{u-\alpha_n}{1-\alpha_n}, \check{H}_n(1, f_{|[0,1-\alpha_n]}))\right)\left(\frac{t}{1-\alpha_n}\right) = D^i_{RV} H_n(u,f)(t)\end{aligned}$$

and for $t \ge 1-\check{\alpha}_{n+1} = 1-\alpha_n$ we have $H_{n+1}(u, D^i_{RV} f)(t) = n+1$ while $D^i_{RV} H_n(u,f)(t) = D^c_i(n) = n+1$ whence $H_{n+1}(u, D^i_{RV} f) = D^i_{RV} H_n(u,f)$ and this proves the claim.

Remark 4.3. *It can be checked by explicit computations that the above construction is* not *compatible with the degeneracy maps, already for $f \in \triangledown_2$ being $f([0,1/3[) = 2, f([1/3,2/3]) = 1$ and $f(]2/3,1]) = 0$, with $\Omega_t = [0,t]$ and $[0,1]$ is endowed with the Lebesgue measure.*

5. Additional properties

5.1. **Kan condition for $\mathrm{Sing}_{RV} X$.** Here we prove that, for X a topological space, the simplicial set $\mathrm{Sing}_{RV} X$ satisfies the Kan condition. For this, consider some $n \ge 1$, $k \in [n]$, $z_i \in (\mathrm{Sing}_{RV} X)_{n-1} = (\mathrm{Sing}_{RV} X)([n-1])$ for $i \in [n]$ with $i \neq k$, such that $z_i.D^c_j = z_j.D^c_{i-1}$ for $0 \le j < i \le n$ and $k \notin \{i,j\}$. In order to prove that $\mathrm{Sing}_{RV} X$ satisfies the Kan condition, we need (see e.g. [8] §4.5) to find $z \in (\mathrm{Sing}_{RV} X)_n$ such that $z_i = z.D^c_i$ for $i \neq k$.

For $\underline{a} = (a_0, \dots, a_n) \in \Delta_n$, denote $m_k(\underline{a}) = \min_{r\neq k} a_r$, and define $q(\underline{a}) \in \Delta_n$ by $q(\underline{a})_i = a_i - m_k(\underline{a})$ for $i \neq k$, $q(\underline{a})_k = a_k + n m_k(\underline{a})$. This defines a continuous map which is actually a retraction from Δ_n onto its k-th horn $\Lambda^k_n = \{(a_0,\dots,a_n) \mid \exists i \neq k \ \ a_i = 0\}$. We want to define $z_n : \triangledown_n \to X$. Let us represent elements of $\triangledown_n$ by $(n+1)$-tuples $\underline{A} = (A_0, \dots, A_n)$ providing a partition of Ω into measurable sets, with A_i the preimage of $i \in [n]$ by the corresponding random variable $\Omega \to [n]$ and introduce

$$V^k_n = p_n^{-1}(\Lambda^k_n) = \{\underline{A} = (A_0,\dots,A_n) \in \triangledown_n \mid \exists i \neq k \lambda(A_i) = 0\}$$

The $z_i \in (\mathrm{Sing}_{RV} X)_{n-1}$ for $i \neq k$ can immediately be glued together into a map $\check{z} : V^k_n \to X$.

Then, consider the map $\triangledown_n \to \triangledown_n$ defined by mapping $\underline{A} = (A_0,\dots,A_n)$ to $\underline{B} = (B_0,\dots,B_n)$ with B_k equal to the complement in Ω of $\bigcup_{r\neq k} B_r$ and, for $i \neq k$, $B_i = A_i$ if $\lambda(A_i) = 0$, and $B_i = \mathbf{g}(A_i, \min_{r\neq k} \lambda(A_r)/\lambda(A_i))$ otherwise, where $\mathbf{g}$ is as in Section 4.1. We have $B_i \subset A_i$ for $i \neq k$, hence the B_i's still form a partition of Ω, and it is easily checked that $\lambda(B_i) = q(\underline{a})_i$ for $\underline{a} = p_n(\underline{A}) \in \Delta_n$. In particular $\underline{B} \in V^k_n$ and this defines a continuous retraction $\triangledown_n \to V^k_n$. Composing it with $\check{z} : V^k_n \to X$ provides $z \in \mathrm{Sing}_{RV} X$ such that $z_i = z.D^c_i$ for $i \neq k$, and this proves the claim.

Although it is not needed for the proof, as in the classical case one can prove that V^k_n is a *deformation* retract of $\triangledown_n$, see Proposition 7.4 below.

5.2. **Ordered simplicial complexes.** Let $\mathcal{K}$ be a simplicial complex over a *totally ordered* set S of vertices. Then a simplicial set $S\mathcal{K}$ can be obtained in a standard way by repeating vertices, namely

$$S\mathcal{K}_n = \{(s_0,\dots,s_n) \in S^{n+1}; s_0 \le s_1 \le \dots \le s_n \ \& \ \{s_0,\dots,s_n\} \in \mathcal{K}\}$$

and, for $f \in \mathrm{Hom}_\Delta([m],[n])$, $S\mathcal{K}(f)$ maps $(s_0,\dots,s_n)$ to $(s_{f(0)},\dots,s_{f(m)})$. A non-degenerate n-simplex is characterized by the property $s_0 < \dots < s_n$.

In [16] we defined a (metric) space of simplicial random variables $L(\mathcal{K})$ associated to the simplicial complex $\mathcal{K}$. Letting $|\mathcal{K}|_w$ and $|\mathcal{K}|_m$ be the geometric realizations of $\mathcal{K}$ equipped with the weak and strong (or metric) topology, respectively, the proof of Theorem 1.4 consists in the following chain of homotopy equivalences

$$L(S\mathcal{K}) \sim |S\mathcal{K}| \sim |\mathcal{K}|_w \sim |\mathcal{K}|_m \sim L(\mathcal{K})$$

where the first one is given by Theorem 3.1, the second one is standard (see e.g. [10] I.2.13), the third one is Dowker's theorem ([3]; see also [23]), and the fourth one is our Theorem 1 of [16].

Let us consider $M\mathcal{K}$ the associated pre-simplicial set, namely the restriction of $S\mathcal{K}$ to the category $\mathbf{M}$. As expected, the topological spaces $L(M\mathcal{K})$ and $L(S\mathcal{K})$ are actually the same.

Proposition 5.1. *The spaces $L(M\mathcal{K})$ and $L(S\mathcal{K})$ are homeomorphic.*

Proof. Let us denote $F = S\mathcal{K}$. Then $|F|$ is a quotient space of $\bigsqcup F_n^\# \times \Delta_n$, equal to the image of $\bigsqcup F_n \times \Delta_n$ modulo the equivalence relations $(\alpha f, a) \sim (\alpha, \Delta(f)(a))$ for $f \in \mathbf{\Delta}$. Now let us assume that $(\alpha f, a) \sim (\alpha, \Delta(f)(a))$ for some $\alpha \in F_n^\#$, $\alpha f \in F_m^\#$ and $f \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$. We have $\alpha = (s_0, \dots, s_n)$ and $\alpha f = (s_{f(0)}, \dots, s_{f(m)})$. But $\alpha f \in F_m^\#$ means $s_{f(0)} < \cdots < s_{f(m)}$ and this implies that f is injective. Therefore $|S\mathcal{K}| = \|M\mathcal{K}\| = (\bigsqcup F_n^\# \times \Delta_n)/\sim$ with $\sim$ generated by $(\alpha f, a) \sim (\alpha, \Delta(f)(a))$ for $f \in \mathbf{M}$.

The same argument shows that $L(M\mathcal{K})$ and $L(S\mathcal{K})$ are homeomorphic.

The (weak) geometric realization of $\mathcal{K}$ is the topological union of the $|\mathcal{K}^{(n)}|$ where $\mathcal{K}^{(n)}$ is the n-skeleton of $\mathcal{K}$, defined as the collection of elements of $\mathcal{K}$ of cardinality at most $n+1$. From the cellular structure of $\mathcal{K}^{(n)}$ and since $\mathbf{M}$ is generated by the face maps we get immediately that the identity map of $\bigsqcup_{k \le n} F_k \times \Delta_k$ induces an homeomorphism between $|\mathcal{K}^{(n)}|$ and $\|(M\mathcal{K})^{(n)}\|$. From this we get a commutative ladder of homeomorphisms which induces an homeomorphism between $|\mathcal{K}|$ and $\|M\mathcal{K}\|$. □

5.3. **Finite products and equalizers.** Let F, G be two simplicial sets, and $F \times G$ their product in the category **sSet**. It is defined (see e.g. [8]) by $(F \times G)_n = F_n \times G_n$, and $(\alpha, \beta).\sigma = (\alpha.\sigma, \beta.\sigma)$ for $\sigma \in \mathbf{\Delta}$. From the product property we have natural maps $|F \times G| \to |F| \times |G|$ and $L(F \times G) \to L(F) \times L(G)$. The former is known to be an homeomorphism ([8] Proposition 4.3.15), and more generally the geometric realization functor preserves finite limits. Here we show that the latter map $L(F \times G) \to L(F) \times L(G)$ is *not* an homeomorphism in general.

Our example is the following one. We consider the simplicial complex $\mathcal{P}_{\mathrm{f}}^*([1])$ on the vertex set $S = [1] = \{0,1\}$ given by the collection of all non-empty subsets of S. The simplicial set F associated to it is the 1-simplex, considered as a simplicial set, and $|F| = \Delta_1 = I$, $L(F) = \nabla_1 = \mathfrak{M}$.

The simplicial set $F \times F$ can be described as $F_{\mathcal{K}}$ for $\mathcal{K}$ the 2-dimensional simplicial complex on the set $(F \times F)_0 = S \times S$ whose maximal simplices are $\{(0,0),(0,1),(1,1)\}$ and $\{(0,0),(1,1),(1,0)\}$. An element of $L(F \times F)$ can thus be described as an element of $L^1(\Omega, S \times S)$, or equivalently by a partition of Ω (up to neglectability) into 4 parts $\underline{A} = (A_{00}, A_{01}, A_{10}, A_{11})$. The condition that it belongs to $L(F \times F)$ reads $A_{10} = \emptyset$ or $A_{01} = \emptyset$.

We now consider its image under the projection map $L(F \times F) \to \nabla_1 \times \nabla_1 = \mathfrak{M} \times \mathfrak{M}$, where elements of ∇_1 are identified with (classes of measurable) subsets of Ω. It is easily checked that $\underline{A}$ is mapped to $(A_{10} \cup A_{11}, A_{01} \cup A_{11})$, which is equal either to $(A_{11}, A_{01} \cup A_{11})$ if $A_{10} = \emptyset$, or to $(A_{10} \cup A_{11}, A_{11})$ if $A_{01} = \emptyset$. From this one gets that the image of $L(F \times F)$ inside $L(F) \times L(F)$ is the collection of pairs $(U, V) \in \mathfrak{M} \times \mathfrak{M}$ such that either $U \subset V$ or $V \subset U$. Therefore the map is not surjective, and this proves that $L : \mathbf{sSet} \to \mathbf{Top}$ does *not* preserve finite products.

A positive property however is that L preserves equalizers.

Proposition 5.2. *Let F, G be two simplicial sets, $\varphi, \psi : F \to G$ two simplicial maps, and $H \subset F$ their equalizer in* **sSet**. *Then $L(H)$ is the equalizer of the maps $L(\varphi), L(\psi) : L(F) \to L(G)$.*

Proof. Clearly $L(H)$ is included inside this equalizer. Conversely, let us consider one of its elements, and a representative (α, a) of it with a an interior point. By definition $(f(\alpha), a)$ and $(g(\alpha), a)$ are equivalent inside $L(F)$. Let $\beta, \beta' \in F^\#$ and $S, S' \in \mathbf{E}$ such that $f(\alpha) = \beta.S$, and $g(\alpha) = \beta'.S'$. We have $(f\alpha), a) = (\beta.S, a) \sim (\beta, \nabla(S).a)$ and similarly $(g(\alpha), a) \sim (\beta', \nabla(S').a)$. Now by definition $(f(\alpha), a)$ and $(g(\alpha), a)$ are equivalent inside $L(F)$, so their minimal representatives are the same, that is $\beta = \beta'$ and $\nabla(S).a = \nabla(S').a$. Since a is interior one gets $S = S'$ by Lemma 3.2 (3) and this proves $f(\alpha) = \beta.S = \beta'.S' = g(\alpha)$, which proves the claim. □

Actually L also *reflects* equalizers, as is immediate by application of the natural transformation $p : L \rightsquigarrow |\bullet|$ and the similar result in the classical case (see e.g. [8] Proposition 4.3.13).

5.4. **Standard variation.** There is a well-known standard variation on the definition of a n-simplex, which is better behaved for some purposes, as $\llcorner\!\!\!\diagdown_n = \{(x_1, \dots, x_n) \in [0,1]; x_1 \le \cdots \le x_n\}$. More precisely, it defines a functor $◺ : \mathbf{\Delta} \to \mathbf{Top}$, mapping $\sigma \in \mathbf{\Delta}$ to $◺(\sigma)$ defined by

$$◺(\sigma)(x_1, \dots, x_n) = \left(x_{1+\sup \sigma^{-1}(\{0,\dots,i-1\})}\right)_{i=1,\dots,m}$$

with the conventions $\sup \emptyset = -\infty$ and $x_{-\infty} = 0$.

There is a (bicontinuous) bijection between $◺_n$ and $\Delta_n = \{(a_0, \dots, a_n) \in [0,1]^n; a_0 + \cdots + a_n = 1\}$, given by $x_0 = a_0$, $x_1 = a_0 + a_1$, …, $x_n = a_0 + \cdots + a_{n-1}$ and conversely $a_0 = x_0$, $a_i = x_i - x_{i-1}$ for $0 < i < n$, $a_n = 1 - x_n$. It is immediately checked that this bijection defines an isomorphism between the functors Δ and $◺$.

There is a similar variation for simplicial random variables. There is a cosimplicial space $◸ : \mathbf{\Delta} \to \mathbf{Top}$ corresponding to

$$◸_n = \{\underline{A} = (A_1, \dots, A_n); A_1 \subset A_2 \subset \cdots \subset A_n \subset \Omega\}$$

where the subsets of Ω are always understood up to subsets of measure 0, and the topology is for instance given by the metric $d(\underline{A}^1, \underline{A}^2) = \max_i \lambda(A_i^1 \Delta A_i^2)$ where $U \Delta V = (U \setminus V) \cup (V \setminus U)$. For $\sigma \in \mathbf{\Delta}$,

$$◸(\sigma)(A_1, \dots, A_n) = \left(A_{1+\sup \sigma^{-1}(\{0,\dots,i-1\})}\right)_{i=1,\dots,m}$$

This functor is isomorphic to $\triangledown$. To see this, recall that $\triangledown_n = L^1(\Omega, [n])$ can be identified with $\{(B_0, \dots, B_n); \Omega = B_0 \sqcup \cdots \sqcup B_n\}$ (where set-theoretic equalities are always understood up to neglectable subsets) through $B_i = f^{-1}(\{i\})$ for $i \in [n]$ and $f \in \triangledown_n$. Then, the correspondence is given via $A_k = B_0 \cup \cdots \cup B_{k-1}$, and it is easily checked to be simplicial. Finally, the probability-law maps $p'_n : ◸_n \to ◺_n$, $(A_1, \dots, A_n) \mapsto (\lambda(A_1), \dots, \lambda(A_n))\}$ are continuous, fit together, and define a natural transformation $◸ \rightsquigarrow ◺$. It is easily checked that all this fits into a commutative diagram of functors as follows.

$$\begin{array}{ccc} \triangledown & \longleftrightarrow & ◸ \\ \downarrow & & \downarrow \\ \Delta & \longleftrightarrow & ◺ \end{array}$$

6. Homotopy invariance

In this section we investigate the relation between the functors Sing and Sing_{RV}. Recall that $p_n : \triangledown_n \to \Delta_n$ denotes the probability-law map. For Z a topological space, $\mathrm{Sing}_{RV} Z$ is defined as the simplicial set with n-vertices the (continuous) maps $\triangledown_n \to Z$, and $\sigma \in \mathbf{\Delta}$ acts on $f \in (\mathrm{Sing}_{RV} Z)_n$ as $f.\sigma = f \circ \triangledown(\sigma)$. If (Z, z_0) is a pointed space, $\mathrm{Sing}_{RV} Z$ is a pointed simplicial set, with distinguished vertex the map $\triangledown_0 \to Z$ mapping (the single element of) $\triangledown_0$ to z_0, and Sing_{RV} clearly defines a functor $\mathbf{Top}_* \to \mathbf{sSet}_*$, right adjoint to $L : \mathbf{sSet}_* \to \mathbf{Top}_*$. Also recall that the monomorphisms of $\mathbf{sSet}$ are the injective simplicial maps between the underlying graded sets, so we call them injective morphisms.

Proposition 6.1. *For every (pointed) topological space Z, the maps $f \mapsto f \circ p_n$, $(\mathrm{Sing} Z)_n \to (\mathrm{Sing}_{RV} Z)_n$ induce an injective morphism of (pointed) simplicial sets $R_Z : \mathrm{Sing} Z \to \mathrm{Sing}_{RV} Z$. The collection $(R_Z)_Z$ defines a natural transformation* Sing $\rightsquigarrow$ Sing_{RV}.

Proof. Let $f \in (\mathrm{Sing} Z)_n$ and $\sigma \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]) = \mathrm{Hom}_{\mathbf{\Delta}^{op}}([n],[m])$. We want to prove $R_Z(f.\sigma) = R_Z(f).\sigma$. We have $R_Z(f.\sigma) = R_Z(f \circ \Delta(\sigma)) = f \circ \Delta(\sigma) \circ p_m$ and $R_Z(f).\sigma = R_Z(f) \circ \triangledown(\sigma) = f \circ p_n \circ \triangledown(\sigma)$. In the proof of Proposition 1.1 we have got that $p_n \circ \triangledown(\sigma) = \Delta(\sigma) \circ p_m$ hence $R_Z(f.\sigma) = R_Z(f).\sigma$ and we have a morphism of simplicial sets. Its injectivity is an immediate consequence of the surjectivity of the maps $p_n : \triangledown_n \twoheadrightarrow \Delta_n$. Finally, if Z is pointed, it clearly maps base point to base point.

Let $\varphi : Z \to T$ a continuous map. We now want to prove that $R_T \circ \mathrm{Sing}(\varphi) = \mathrm{Sing}_{RV}(\varphi) \circ R_Z$. Let $f \in (\mathrm{Sing} Z)_n$. We have $\mathrm{Sing}(\varphi)(f) = \varphi \circ f$ and $R_T(\mathrm{Sing}(\varphi)(f)) = (\varphi \circ f) \circ p_n$ and $\mathrm{Sing}_{RV}(\varphi)(R_Z(f)) = \mathrm{Sing}_{RV}(\varphi)(f \circ p_n) = \varphi \circ (f \circ p_n)$ and this proves the claim. □

For $\mathcal{K}$ and $\mathcal{L}$ two ordered simplicial complexes, we have $F_{\mathcal{K}} \times F_{\mathcal{L}} = F_{\mathcal{K} \times \mathcal{L}}$, where $\mathcal{K} \times \mathcal{L}$ is the simplicial complex with vertices (x, y) for $x \in \bigcup \mathcal{K}$ and $y \in \bigcup \mathcal{L}$ and simplices the $\{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$ such that $x_0 \leq x_1 \leq \cdots \leq x_n$, $y_0 \leq y_1 \leq \cdots \leq y_n$. The total ordering chosen on the vertices is for instance the lexicographic ordering – or any other total ordering refining the diagonal partial ordering $(a, b) \leq (a', b')$ iff $a \leq b$ and $a' \leq b'$.

In particular, let $\mathcal{I} = \mathcal{P}_f^*([1])$ the simplicial complex whose geometric realization is Δ_1. Let S be the (ordered) vertex set of $\mathcal{K}$. For $x \in S$ we set $x^- = (x, 0)$, $x^+ = (x, 1)$. The simplices of $\mathcal{K} \times \mathcal{I}$ are the

$$\{x_0^-, x_1^-, \dots, x_{r-1}^-, x_r^+, x_{r+1}^+, \dots, x_m^+\}$$

such that $x_0 \leq \cdots \leq x_m$ and $\{x_0, \dots, x_m\} \in \mathcal{K}$.

We have natural maps $L(\mathcal{K}) \times \{0, 1\} \to L(\mathcal{K} \times \mathcal{I})$ and $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ providing a factorization of the first projection map $L(\mathcal{K}) \times \{0, 1\} \to L(\mathcal{K})$. In the forthcoming sections we are going to prove the following property.

Proposition 6.2. *Let $\mathcal{K}$ be a finite simplicial complex. Then $L(\mathcal{K} \times \mathcal{I})$ is a cylinder object for $L(\mathcal{K})$ inside Strøm's (closed) model category structure.*

Recall from [25] and e.g. [11] that this means that

(1) The natural projection map $\pi : L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ is an homotopy equivalence.
(2) The natural inclusion map $L(\mathcal{K}) \times \{0, 1\} \to L(\mathcal{K} \times \mathcal{I})$ is a closed cofibration.

Thus, this is a *good cylinder* in the terminology of [4].

Assume that S is well-ordered, denote x_0 the minimal vertex of $\mathcal{K}$. We considered $\mathcal{K}$ (or $F_{\mathcal{K}}$) as based at x_0. Consider $\{0\}$ as a base vertex for $\mathcal{I}$. The coproduct $F_{\mathcal{K}} \vee F_{\mathcal{K}}$ of $F_{\mathcal{K}}$ with itself inside $\mathbf{sSet}_*$ has the form $F_{\mathcal{K} \vee \mathcal{K}}$ with $\mathcal{K} \vee \mathcal{K}$ a simplicial complex on the vertex set obtained by dividing $S \times \{0, 1\}$ by the relation $x_0^+ = x_0^-$. On this vertex set, $\mathcal{K} \vee \mathcal{K}$ has for simplices the $\{x_{i_0}^-, x_{i_1}^-, \dots, x_{i_r}^-\}$ and the $\{x_{i_0}^+, x_{i_1}^+, \dots, x_{i_r}^+\}$ for $x_{i_0} \leq \cdots \leq x_{i_r}$ and $\{x_{i_1}, \dots, x_{i_r}\} \in \mathcal{K}$.

Recall that the smash product $X \wedge Y$ is in $\mathbf{Top}_*$ the quotient of $X \times Y$ by the image of $X \vee Y$. It is also the push-out

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X \wedge Y \end{array}$$

and this definition also applies in $\mathbf{sSet}_*$. Denote $\mathcal{I}^+$ the simplicial complex $\mathcal{I}$ together with an isolated vertex, taken as basepoint (for instance $\mathcal{I}^+ = \{\{0\}, \{1\}, \{0, 1\}, \{-1\}\}$).

We have a similar statement for pointed sets (recall from [25] that a map between pointed spaces is a cofibration, fibration, or homotopy equivalence if and only if the underlying map between non-pointed spaces is one).

Proposition 6.3. *Let $\mathcal{K}$ be a finite simplicial complex with minimal vertex x_0. Then $L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+})$ is a good cylinder object for $L(\mathcal{K})$ inside Strøm's (closed) model category structure on $\mathbf{Top}_*$.*

From these propositions, which are together equivalent to the the propositions 6.7 and 6.8 proven below, we deduce the following Theorem. Recall (see e.g. [11] p. 73) that homotopy equivalence is a well-defined concept between objects of a (closed) model category which are both fibrant and cofibrant. The strategy used in its proof has been suggested to me by D. Chataur.

Theorem 6.4. *Let Z be a topological space. Then the natural morphism $\mathrm{Sing}(Z) \to \mathrm{Sing}_{RV}(Z)$ is an homotopy equivalence. In particular the induced map $|\mathrm{Sing}(Z)| \to |\mathrm{Sing}_{RV}(Z)|$ is an homotopy equivalence.*

Proof. Since $\mathrm{Sing}(Z)$ and $\mathrm{Sing}_{RV}(Z)$ are both Kan complexes (see Section 5.1), they are both fibrant and cofibrant in the standard model structure on $\mathbf{sSet}$. It is then it is enough (see e.g. [11] Theorem 1.10) to prove that the induced maps $\pi_0(\mathrm{Sing}(Z)) \to \pi_0(\mathrm{Sing}_{RV}(Z))$ and $\pi_n(\mathrm{Sing}(Z), z_0) \to \pi_n(\mathrm{Sing}_{RV}(Z), z_0)$ are bijections for every $n \geq 1$ and vertex $z_0 \in \mathrm{Sing}(Z)_0 = \mathrm{Sing}_{RV}(Z)_0 = Z$. This is what we are going to prove.

We first claim that, for $X = F_{\mathcal{K}}$ the simplicial set associated to a finite simplicial complex $\mathcal{K}$, then the natural map from $\mathrm{Hom}(X, \mathrm{Sing}(Z)) \simeq \mathrm{Hom}(|X|, Z)$ to $\mathrm{Hom}(X, \mathrm{Sing}_{RV}(Z)) \simeq \mathrm{Hom}(L(X), Z)$ becomes a bijection up to a homotopy, where up to homotopy means the genuine homotopy relation on $\mathrm{Hom}(|X|, Z)$ inside the topology category, and on $\mathrm{Hom}(L(X), Z)$ means the image under L of the homotopy relation of $\mathbf{sSet}$. In other terms, this latter equivalence relation corresponds to the cylinder $L(\mathcal{K} \times \mathcal{I})$.

We prove this claim now. Since by Proposition 6.2 $L(\mathcal{K}\times\mathcal{I})$ is a good cylinder for Strøm's stucture, it induces (see e.g. [11] ch. II.1 Corollary 1.9) the same homotopy equivalence as the standard cylinder $L(\mathcal{K})\times[0,1]$, which is the genuine homotopy relation. Therefore the map we want to prove it is a bijection is nothing else than the natural map $[|\mathcal{K}|, Z] \to [L(\mathcal{K}), Z]$ inside the naive homotopy category. Since this map is induced by the probability-law map $L(\mathcal{K}) \to |\mathcal{K}|$ which is an homotopy equivalence, it is a bijection, and this proves the claim.

An immediate consequence of this claim, taking for $\mathcal{K}$ a point, is that the induced map $\pi_0(\mathrm{Sing}(Z)) \to \pi_0(\mathrm{Sing}_{RV}(Z))$ is a bijection.

Now assume that the vertex set of $\mathcal{K}$ is linearly ordered with a minimal element x_0, and that some basepoint $z_0 \in Z$ is chosen. We then make the similar claim that, for $X = F_{\mathcal{K}}$ and $\mathcal{K}$ finite, the natural map from $\mathrm{Hom}_{\mathbf{sSet}_*}(X, \mathrm{Sing}(Z)) \simeq \mathrm{Hom}_{\mathbf{Top}_*}(|X|, Z)$ to $\mathrm{Hom}_{\mathbf{sSet}_*}(X, \mathrm{Sing}_{RV}(Z)) \simeq \mathrm{Hom}_{\mathbf{Top}_*}(L(X), Z)$ becomes a bijection up to a homotopy, where up to homotopy means the genuine pointed homotopy relation on $\mathrm{Hom}_{\mathbf{Top}_*}(|X|, Z)$ inside the pointed topology category, and on $\mathrm{Hom}_{\mathbf{Top}_*}(L(X), Z)$ means the image under L of the homotopy relation of $\mathbf{sSet}_*$. In other terms, this equivalence relation corresponds to the cylinder $L(F_{\mathcal{K}} \wedge F_{\mathcal{I}})$. This claim results from Proposition 6.3 and the same argument as before, because two pointed spaces are homotopically equivalent if and only if they are freely equivalent (see [25] and the references there).

We apply this result to $\mathcal{K} = \partial\mathcal{P}^*_{\mathrm{f}}([n])$ and $X = F_{\mathcal{K}}$ equal to the boundary of the n-simplex, for $n \geq 1$, with base point 0. Recall that $\pi_n(F, 0) = \mathrm{Hom}_{\mathbf{HosSet}_*}(X, F)$ whenever F is a Kan complex (see e.g. [9] §9). Then the natural map

$$\pi_n(\mathrm{Sing}(Z), z_0) \simeq \mathrm{Hom}_{\mathbf{HosSet}_*}(X, \mathrm{Sing}(Z)) \to \mathrm{Hom}_{\mathbf{HosSet}_*}(X, \mathrm{Sing}_{RV}(Z)) \simeq \pi_n(\mathrm{Sing}_{RV}(Z), z_0)$$

is identified with $\mathrm{Hom}_{\mathbf{HoTop}_*}(|X|, Z) \to \mathrm{Hom}_{\mathbf{HoTop}_*}(L(X), Z)$. Again because the probability-law map induces an homotopy equivalence, this map is an isomorphism and we get that the natural map

$$\pi_n(\mathrm{Sing}(Z), z_0) \to \pi_n(\mathrm{Sing}_{RV}(Z), z_0)$$

is an isomorphism for all $n \geq 1$. This proves the theorem.

□

6.1. **Preliminary tools.** Notice that the topology on $L(\mathcal{K})$ is induced by the metric topology on $L^1(\Omega, S)$ when $\mathcal{K}$ is finite, and $\mathcal{K} \subset \mathcal{P}^*_{\mathrm{f}}(S)$. We shall need the following lemma only when S is finite, however the extra cost for the full statement is minimal, so we state it in full generality.

Lemma 6.5. *Let X be a space, S a set. For $s \in S$ we define $p_s : L^1(\Omega, S) \to \triangledown_1$ given by $f \mapsto f^{-1}(s)$. Then $F : X \to L^1(\Omega, S)$ is continuous iff $\forall s \in S$ $p_s \circ F : X \to \triangledown_1$ is continuous.*

Proof. Recall that the topology on $L^1(\Omega, S)$ is given by the metric $d(f, g) = \int_\Omega d(f(t), g(t))\mathrm{d}t$, and the one on $\triangledown_1 = \mathfrak{M}$ by the metric $d(U, V) = \lambda(U\Delta V)$. Since $\lambda(f^{-1}(s)\Delta g^{-1}(s)) \leq \int_\Omega d(f(t), g(t))\mathrm{d}t$ we get that each of the $p_s, s \in S$ is 1-Lipschitz hence continuous. This implies that, if $F : X \to L^1(\Omega, S)$ is continuous, then so are the $p_s \circ F$, $s \in S$.

Conversely, let us choose $x_0 \in X$ and set $f_0 = F(x_0)$. We want to prove that F is continuous at x_0. Let $\varepsilon > 0$. We want to prove that there exists an open neighborhood U of x_0 such that $x \in U \Rightarrow d(F(x), f_0) \leq \varepsilon$.

Since $\sum_{s\in S} \lambda(f^{-1}(s)) = 1$, there exists $S_0 \subset S$ finite such that $\lambda(f^{-1}(S_0)) > 1 - \varepsilon/2$. We set $m = \#S_0$. By continuity of $p_s \circ F$, for every $s \in S_0$ there exists an open neighborhood U_s of x_0 such that $x \in U_s$ implies that $f_x = F(x)$ satisfies $\lambda(f_x^{-1}(s)\Delta f_0^{-1}(s)) \leq \varepsilon/2m$. Setting $U := \bigcap_{s\in S_0} U_s$ we get an open neighborhood U of x_0 such that, for every $x \in U$,

$$d(f_0, f_x) = \sum_{s\in S} \lambda(f_0^{-1}(s) \setminus f_x^{-1}(s)) \leqslant \frac{\varepsilon}{2} + \sum_{s\in S_0} \lambda(f_0^{-1}(s) \setminus f_x^{-1}(s)) \leqslant \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon$$

and this proves the claim. □

Recall that $\triangledown_n = \{\underline{X} = (X_1, \dots, X_n) \mid X_1 \subset X_2 \subset \cdots \subset X_n \subset \Omega\}$. For $\underline{X} \in \triangledown_n$, we set by convention $X_0 = \emptyset$ and $X_{n+1} = \Omega$. Also recall from Section 2 the map $\mathbf{g} : \triangledown_1 \times I \to \triangledown_1$ such that $\mathbf{g}(A, 0) = A$, $\lambda(\mathbf{g}(A, u)) = \lambda(A)(1 - u)$ and $\mathbf{g}(A, u) \supset \mathbf{g}(A, v)$ whenever $u \leq v$. It moreover satisfies that $\lambda(\mathbf{g}(E, u)\Delta\mathbf{g}(E, v)) \leqslant 4\lambda(E\Delta F) + |v - u|$.

Lemma 6.6. *There exists a continuous* interpolation map $J : \nabla_n \times I \to \nabla_1$ *such that, for* $\underline{X} = (X_1, \dots, X_n) \in \nabla_n$, $c \in I$, *and* k *such that* $\lambda(X_k) \le c < \lambda(X_{k+1})$, *we have* $X_k \subset J(\underline{X}, c) \subset X_{k+1}$ *and* $\lambda(J(\underline{X}, c)) = c$.

Proof. We define such a map J by letting $J(\underline{X}, c) = Y$ with $Y = X_k \cup \mathbf{g}(X_{k+1} \setminus X_k, a)$ with k as in the statement, and with $a \in I$ such that $\lambda(Y) = \lambda(X_k) + (\lambda(X_{k+1}) - \lambda(X_k))(1-a) = \lambda(X_{k+1}) - a\lambda(X_{k+1} \setminus X_k) = c$, that is $a = (\lambda(X_{k+1}) - c)/\lambda(X_{k+1} \setminus X_k)$.

It remains to prove that J is a continuous map. Let $\underline{X}^0 \in \nabla_n$ and $c_0 \in I$. Let k_0 be such that $\lambda(X^0_{k_0}) \le c_0 < \lambda(X^0_{k_0+1})$. We separate two cases.

First assume $\lambda(X^0_{k_0}) < c_0$. Then, for $(\underline{X}, c)$ close enough to $(\underline{X}_0, x_0)$ we can assume that $\lambda(X_{k_0}) < c < \lambda(X_{k_0+1})$, and the continuity of J at $(\underline{X}^0, c_0)$ is an easy consequence of the continuity of $\mathbf{g}$ and of the elementary set-theoretic operations.

We now assume $\lambda(X^0_{k_0}) = c_0$. Then we have some r_0 with $1 \le r_0 \le k_0$ such that $\lambda(X^0_{r_0-1}) < \lambda(X^0_{r_0}) = \lambda(X^0_{r_0+1}) = \cdots = \lambda(X^0_{k_0}) = c_0$. Of course this implies $X^0_{r_0} = X^0_{r_0+1} = \cdots = X^0_{k_0}$. Then, for $(\underline{X}, c)$ close enough to $(\underline{X}_0, x_0)$ we can assume that $\lambda(X_{r_0-1}) < c < \lambda(X_{k_0+1})$, and we have $\lambda(X_k) \le \lambda(Y) < \lambda(X_{k+1})$ for some $k \in [r_0 - 1, k_0]$.

Therefore $Y \subset X^+_{k_0}$ for some $X^+_{k_0} \supset X_{k_0}$ with either $X^+_{k_0} = X_{k_0}$ or $\lambda(X^+_{k_0}) - \lambda(X_{k_0}) = c - \lambda(X_{k_0})$. But then

$$\lambda(X^+_{k_0}) - \lambda(X_{k_0}) \le |c - \lambda(X_{k_0})| \le |c - c_0| + |c_0 - \lambda(X_{k_0})| \le |c - c_0| + \lambda(X^0_{k_0} \Delta X_{k_0})$$

Similarly, $Y \supset X^-_{r_0}$ with $X^-_{r_0} \subset X_{r_0}$ satisfying

$$\lambda(X_{r_0}) - \lambda(X^-_{r_0}) \leqslant |\lambda(X_{r_0}) - c| \leqslant |c - c_0| + \lambda(X^0_{r_0} \Delta X_{r_0})$$

Assuming $|c - c_0| < \varepsilon/6$ and $\lambda(X_k \Delta X^0_k) < \varepsilon/6$ for every k, and setting $J(\underline{X}, c) = Y$, $J(\underline{X}^0, c_0) = Y_0$ we get from $X^-_{r_0} \subset Y \subset X^+_{k_0}$ that

$$\begin{aligned}
\lambda(Y_0 \Delta Y) &\le \lambda(Y_0 \Delta X^-_{r_0}) + \lambda(Y_0 \Delta X^+_{k_0}) \\
&\le \lambda(X^0_{r_0} \Delta X^-_{r_0}) + \lambda(X^0_{k_0} \Delta X^+_{k_0}) \\
&\le (\lambda(X^0_{r_0} \Delta X_{r_0}) + \lambda(X_{r_0} \Delta X^-_{r_0})) + (\lambda(X^0_{k_0} \Delta X_{k_0}) + \lambda(X_{k_0} \Delta X^+_{k_0})) \\
&\le 2|c - c_0| + 2\lambda(X^0_{r_0} \Delta X_{r_0}) + 2\lambda(X^0_{k_0} \Delta X_{k_0}) \\
&\le \varepsilon
\end{aligned}$$

and this proves the continuity of J at any given $(\underline{X}^0, c_0)$

□

6.2. **Proof of item (1) : homotopy equivalence.** We first prove that $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ is an homotopy equivalence, and the analogous statement for pointed complexes.

Proposition 6.7. *Let* $\mathcal{K}$ *be a simplicial complex over the vertex set* $[n]$. *The natural projection map* $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ *is an homotopy equivalence, and* $L(\mathcal{K}) \times \{0\}$ *is a deformation retract of* $L(\mathcal{K} \times \mathcal{I})$. *If* $\mathcal{K}$ *is considered with* 0 *as basepoint, the natural projection map* $L(F_{\mathcal{K}} \wedge F_{\mathcal{I}}) \to L(F_{\mathcal{K}}) = L(\mathcal{K})$ *is an homotopy equivalence.*

Proof. Let us denote $\pi : L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ the projection map. It is induced by the map $[n] \times [1] \to [n]$, $i^\pm \mapsto i$. This map admits an inverse on the right $i \mapsto i^-$, which induces a map $L^1(\Omega, [n]) \to L^1(\Omega, [n] \times [1])$ mapping $L(\mathcal{K})$ into $L(\mathcal{K} \times \mathcal{I})$. We claim that this map $\sigma : L(\mathcal{K}) \to L(\mathcal{K} \times \mathcal{I})$ is an homotopy inverse for π. Since $\pi \circ \sigma = \mathrm{Id}_{L(\mathcal{K})}$ this amounts to proving that $\sigma \circ \pi \sim \mathrm{Id}_{L(\mathcal{K} \times \mathcal{I})}$, and this will prove at the same time that $L(\mathcal{K}) \times \{0\}$ is a deformation retract of $L(\mathcal{K} \times \mathcal{I})$.

We now justify the claim by constructing an explicit homotopy $H : L(\mathcal{K} \times \mathcal{I}) \times I \to L(\mathcal{K} \times \mathcal{I})$. Let $\underline{A} = (A^\pm_j)_{j \in [n]}$ and $t \in I$. Write $t = \frac{1}{n+1}(i + u)$ for some $i \in [n]$ and $u \in [0, 1]$, and set $H(\underline{A}, t) = \underline{B}$ with $B^\pm_j = A^\pm_j$ for $j > i$, $B^-_j = A^+_j \cup A^-_j$ and $B^+_j = \emptyset$ for $j < i$ and finally $B^-_i = A^-_i \cup \mathbf{g}(A^+_i, 1 - u)$, $B^+_i = A^+_i \setminus B^-_i$.

It is well-defined, as for $t = i/(n+1) = \frac{1}{n+1}(i+0) = \frac{1}{n+1}((i-1)+1)$ we have $B^-_i = A^-_i \cup \mathbf{g}(A^+_i, 1-0) = A^-_i$ and $B^-_{i-1} = A^-_{i-1} \cup A^+_{i-1} = A^-_{i-1} \cup \mathbf{g}(A^+_{i-1}, 0)$. More concisely, we have for every j and t the formulas

$B_j^- = A_j^- \cup \mathbf{g}(A_j^+, \eta(j+1-(n+1)t))$, $B_j^+ = A_j^+ \setminus B_j^-$ where $\eta : \mathbf{R} \to \mathbf{R}$ is defined by $\eta(x) = 0$ for $x \leq 0$, $\eta(x) = 1$ for $x \geq 1$ and $\eta(x) = x$ otherwise.

Clearly $H(\underline{A}, 0) = \underline{A}$, $H(\underline{A}, 1) = \sigma(\pi(\underline{A}))$ and we have $H(\underline{A}, t) \in L(\mathcal{K} \times \mathcal{I})$ for all $\underline{A}, t$. Therefore there only remains to prove that H is continuous. By Lemma 6.5 this is then a consequence of the continuity of $\mathbf{g}$ and η, and of the elementary set-theoretic operations.

We now consider the pointed case, with $0 \in [n]$ chosen for basepoint of $\mathcal{K}$. By construction the smash product $F_{\mathcal{K}} \wedge F_{\mathcal{I}}$ is a push-out hence a colimit. Since L is a left adjoint it commutes with colimits, and from this we get that $L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+})$ is the push-out of

$$\begin{array}{ccc} L(\mathcal{K} \vee \mathcal{I}^+) & \longrightarrow & L(\mathcal{K} \times \mathcal{I}^+) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bullet \end{array}$$

that is the quotient of $L(\mathcal{K} \times \mathcal{I}^+)$ by the subspace $L(\mathcal{K} \vee \mathcal{I}^+)$. Now notice that the projection map $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ factorizes through the maps $L(\mathcal{K} \times \mathcal{I}) \to L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+})$ and $L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+}) \to L(\mathcal{K})$. We want to prove that the latter is an homotopy equivalence. Since we just proved that the composite $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ is an homotopy equivalence, it is equivalent to prove that the former map $L(\mathcal{K} \times \mathcal{I}) \to L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+})$ is an homotopy equivalence. Since $\mathcal{K} \times \mathcal{I}^+ = (\mathcal{K} \times \mathcal{I}) \cup (\mathcal{K} \times \{*\})$ and $\mathcal{K} \times \{*\} \subset \mathcal{K} \vee \mathcal{I}^+$ we get that this map is the same as the quotient map $\pi_0 : L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K} \times \mathcal{I})/L_0$ with $L_0 = L^1(\Omega, \{0^-, 0^+\})$.

We consider the map $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K} \times \mathcal{I})$ mapping $\underline{A}^{\pm}$ to $\underline{B}^{\pm}$ with $B_i^{\pm} = A_i^{\pm}$ for $i > 0$,

$$B_0^- = A_0^- \cup \mathbf{g}\left(A_0^+, \sum_{\substack{i \neq 0 \\ \varepsilon = \pm}} \lambda(A_i^{\pm}) \right) = A_0^- \cup \mathbf{g}\left(A_0^+, 1 - \lambda(A_0^- \cup A_0^+) \right)$$

and $B_0^+ = A_0^+ \setminus B_0^-$. If $\underline{A}^{\pm} \in L_0$, that is if $i > 0 \Rightarrow A_i^{\pm} = \emptyset$, then $B_0^- = A_0^- \cup A_0^+ = \Omega$, $B_0^+ = \emptyset$. It follows that the map factorizes through $L(\mathcal{K} \times \mathcal{I})/L_0$ and induces a map $\sigma_0 : L(\mathcal{K} \times \mathcal{I})/L_0 \to L(\mathcal{K} \times \mathcal{I})$.

Consider the map $H : L(\mathcal{K} \times \mathcal{I}) \times I \to L(\mathcal{K} \times \mathcal{I})$ mapping $(\underline{A}^{\pm}, t)$ to $\underline{B}^{\pm}$ with $B_i^{\pm} = A_i^{\pm}$ for $i > 0$ and

$$B_0^- = A_0^- \cup \mathbf{g}\left(A_0^+, t(1 - \lambda(A_0^- \cup A_0^+)) + (1-t) \right)$$

It provides an homotopy between $\sigma_0 \circ \pi_0$ and the identity map. Now notice that the image of $L_0 \times I$ is equal to L_0, so that it induces a map $L(\mathcal{K} \times \mathcal{I})/L_0 \times I \to L(\mathcal{K} \times \mathcal{I})/L_0$. This map defines an homotopy between $\pi_0 \circ \sigma_0$ and the identity map. This proves that π_0 is an homotopy equivalence.

□

6.3. **Proof of item (2) : cofibration property.** We assume that $\mathcal{K}$ is a subcomplex of $\mathcal{P}_f^*([n])$, so that $L(\mathcal{K} \times \mathcal{I})$ (resp. $L(\mathcal{K} \wedge \mathcal{I})$) is a closed subset of $L(\mathcal{P}_f^*([n]) \times \mathcal{I}) \subset L^1(\Omega, [n] \times [1])$ (resp. $L(\mathcal{P}_f^*([n]) \wedge \mathcal{I}) \subset L^1(\Omega, [n] \times [1])$). Precisely, it is made of the $f \in L(\mathcal{P}_f^*([n]) \times \mathcal{I})$ (resp. $f \in L(\mathcal{P}_f^*([n]) \wedge \mathcal{I})$) such that $\pi(f) \in L(\mathcal{K})$, for $\pi : L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K})$ (resp. $\pi : L(\mathcal{K} \wedge \mathcal{I}) \to L(\mathcal{K})$) the natural projection map, that is such that $\pi(f)(\Omega) \in \mathcal{K}$. We now prove the following.

Proposition 6.8. *Let $\mathcal{K}$ be a simplicial complex with vertex set $[n]$. Then the inclusion map $L(\mathcal{K}) \times \{0,1\} \to L(\mathcal{K} \times \mathcal{I})$ is a closed cofibration. For $\mathcal{K}$ considered as a pointed simplicial complex, the inclusion map $L(\mathcal{K} \vee \mathcal{K}) \to L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+})$ is also a closed cofibration.*

Proof. The space $L(\mathcal{K}) \times \{0,1\}$ is equal to $L(\mathcal{K} \times \{0,1\})$, where we denote $\mathcal{K} \times \{0,1\} = \mathcal{K} \sqcup \mathcal{K}$ the disjoint union of two copies of $\mathcal{K}$, with vertex sets $\{x^+, x \in [n]\}$ and $\{x^-, x \in [n]\}$. Since $L(\mathcal{K} \times \{0,1\})$ is a closed subset of $L(\mathcal{K} \times \mathcal{I})$, it is enough to construct a retract of the cylinder $L(\mathcal{K} \times \mathcal{I}) \times I$ onto $(L(\mathcal{K}) \times \{0,1\} \times I) \cup L(\mathcal{K} \times \mathcal{I}) \times \{0\}$. We do this now.

We endow the space $\nabla_n = \{\underline{X} = (X_1, \dots, X_n); \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \Omega\}$ with the metric $d(\underline{X}^1, \underline{X}^2) = \max_i \lambda(X_i^1 \Delta X_i^2)$, and $L(\mathcal{P}_f^*([n]) \times \mathcal{I})$ with the L^1 metric.

Let $\mathcal{F}_r = \mathcal{P}_f^*(\{0^-, 1^-, \dots, r^-, r^+, (r+1)^+, \dots, n^+\}$ for $0 \leq r \leq n$. These are the maximal simplices of $\mathcal{P}_f^*([n]) \times \mathcal{I}$, and $L(\mathcal{P}_f^*([n]) \times \mathcal{I}) = \bigcup_r L(\mathcal{F}_r)$.

We define a map $D : L(\mathcal{P}_f^*([n]) \times \mathcal{I}) \to \nabla_{n+1}$ as follows. To $f \in L(\mathcal{F}_r)$ we associate $\underline{X} \in \nabla_n$ such that $X_i = f^{-1}([0^-, (i-1)^-])$ if $i \leq r+1$, and $X_i = f^{-1}([0^-, (i-2)^+])$ if $i > r+1$. We check that these

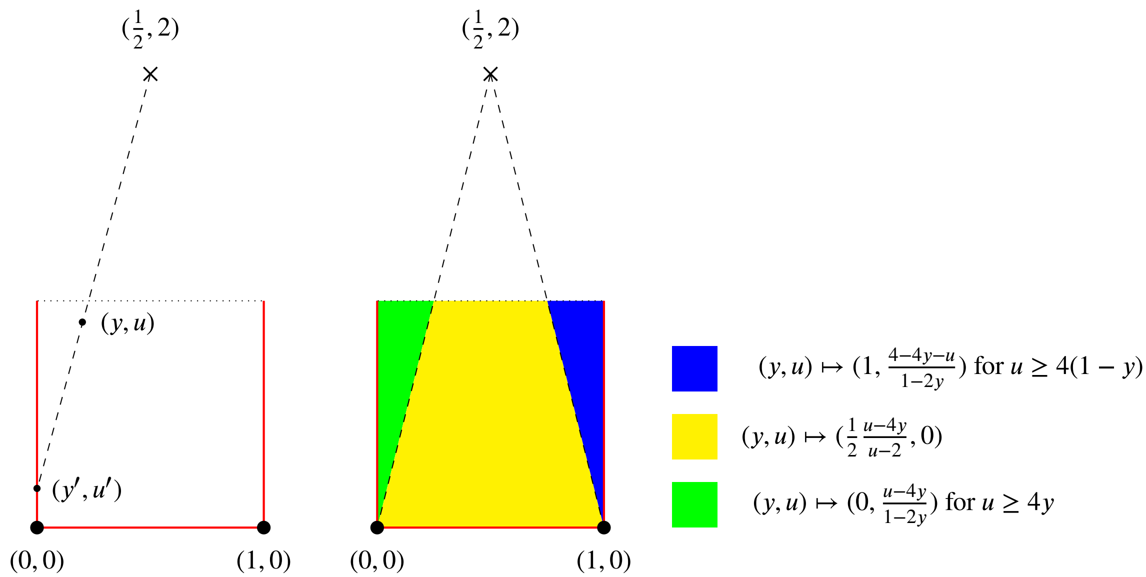


FIGURE 3. The geometric retract $q : [0,1]^2 \to \{(y,u); u = 0 \text{ or } y \in \{0,1\}\}$

definitions agree on $L(\mathcal{F}_r) \cap L(\mathcal{F}_s)$ for $r < s$ as follows. If $i \leq r+1$ or $i > s+1$ the two possible definitions of X_i clearly agree, so we can assume $r+1 < i \leq s+1$. According to the definition on $L(\mathcal{F}_r)$ we have $X_i = f^{-1}([0^-, (i-2)^+])$, whereas according to the definition on $L(\mathcal{F}_s)$ we have $X_i = f^{-1}([0^-, (i-1)^-])$. But since $f \in L(\mathcal{P}^*_{\mathrm{f}}([n]) \times \mathcal{I})$ we have $\# f(\Omega) \cap \{(i-2)^+, (i-1)^-\} \leq 1$ hence $f^{-1}([0^-, (i-2)^+]) = f^{-1}([0^-, (i-1)^-])$ and this proves that these maps can be glued together. Finally it is easily checked that the map D is 2-Lipschitz on each $L(\mathcal{F}_r)$, as we have, for $\underline{X} = D(f)$ and $\underline{X}^0 = D(f_0)$,

$$\lambda(X_i \Delta X_i^0) = \lambda(X_i \setminus X_i^0) + \lambda(X_i^0 \setminus X_i) = \int_{X_i} d(f(t), f_0(t))\mathrm{d}t + \int_{X_i^0} d(f(t), f_0(t))\mathrm{d}t \leqslant 2d(f, f_0)$$

and this proves that it is continuous on its whole domain. Moreover it is easily checked that every restriction $D_{|L(\mathcal{F}_r)} : L(\mathcal{F}_r) \to \nabla_{n+1}$ is a bijection.

Let $\tau_+ : [n] \times [1] \to [n]$ defined by $i^\pm \mapsto i^+$ and similarly $\tau_- : i^\pm \mapsto i^-$. We then define two maps $G_\pm : L(\mathcal{P}^*_{\mathrm{f}}([n]) \times \mathcal{I}) \times I \to L(\mathcal{P}^*_{\mathrm{f}}([n]) \times \mathcal{I})$ as follows. Let $(f, c) \in L(\mathcal{P}^*_{\mathrm{f}}([n]) \times \mathcal{I}) \times I$. We use the map J of Lemma 6.6.

For $x \in J(D(f), u)$ we set $G_+(f, u)(x) = f(x)$, and for $x \notin J(D(f), u)$, we set $G_+(f, u)(x) = \tau_+(f(x))$. For $x \notin J(D(f), u)$ we set $G_-(f, u)(x) = f(x)$, and for $x \in J(D(f), u)$, we set $G_-(f, u)(x) = \tau_-(f(x))$.

Let $(f_0, c_0) \in L(\mathcal{P}^*_{\mathrm{f}}([n]) \times \mathcal{I}) \times I$. We prove that $G_\pm$ is continuous at (f_0, c_0). Let us consider $(f, c) \in L(\mathcal{P}^*_{\mathrm{f}}([n]) \times \mathcal{I}) \times I$ and set $J_0 = J(D(f_0), c_0)$, $J = J(D(f), c)$. Let $\varepsilon > 0$, and assume $d(f, f_0) < \varepsilon/3$. By continuity of D and J we know that, for (f, c) close enough to (f_0, c_0), we have $\lambda(J \Delta J_0) \leq \varepsilon/3$. Then, by definition

$$d(G_+(f, c), G_+(f_0, c_0)) \leqslant \lambda(J \Delta J_0) + \int_{J \cap J_0} d(f(t), f_0(t))\mathrm{d}t + \int_{\Omega \setminus (J \cup J_0)} d(\tau_+(f(t)), \tau_+(f_0(t)))\mathrm{d}t$$

Since $d(\tau_+(x), \tau_+(y)) \leq d(x, y)$ for every x, y, this implies $d(G_+(f, c), G_+(f_0, c_0)) \leq \varepsilon/3 + 2d(f, f_0) \leq \varepsilon$ and this proves the continuity of G_+ at (f_0, c_0). The proof of continuity for G_- is similar and left to the reader.

We extend the maps $\tau_\pm$ to $L^1(\Omega, [n] \times [1])$ by setting $\tau_\pm(f) = x \mapsto \tau_\pm(f(x))$. It is clear from the definitions that

- $G_+(f, 1) = f$ and $G_+(f, 0) = \tau_+(f)$.
- $G_-(f, 1) = \tau_-(f)$ and $G_-(f, 0) = f$.

We now construct a continuous map $L(\mathcal{P}_{\mathrm{f}}^*([n]) \times \mathcal{I}) \times I \to L(\mathcal{P}_{\mathrm{f}}^*([n]) \times \mathcal{I}) \times I$ as follows. To $(f,u) \in L(\mathcal{P}_{\mathrm{f}}^*([n]) \times \mathcal{I}) \times I$ we associate $(p(f),u) = (\underline{x}, y, u) \in (\llcorner\!\!\diagdown_n \times \llcorner\!\!\diagdown_1) \times \Delta_1$, and then $q(y,u) = (y',u')$, where q is the projection map of Figure 3. Notice that, if $y = 1/2$, then $y' = 1/2$ and there exists a component X_k of $D(f)$ of measure $1/2$, which implies $J(D(f), 1/2) = X_k$ and finally $G_{\pm}(f, 1/2) = f$. Therefore, setting $H(f,u) = (G_+(f,y'),u')$ for $y \leq 1/2$ and $H(f,u) = (G_-(f,y'),u')$ for $y > 1/2$, we get a well-defined continuous map with the properties that :

- either $u' = 0$, in which case $H(f,u) \in L(\mathcal{P}_{\mathrm{f}}^*([n]) \times \mathcal{I}) \times \{0\}$,
- or $u' > 0$ in which case either $y' = 0$ and $H(f,u) = (G_+(f,0),u') = (\tau_+(f),u') \in L(\mathcal{P}_{\mathrm{f}}^*([n] \times \{1\})$, or $y' = 1$ and $H(f,u) = (G_-(f,1),u') = (\tau_-(f),u') \in L(\mathcal{P}_{\mathrm{f}}^*([n] \times \{0\})$.
- if $u = 0$, then $u' = 0$ and $y' = y$, in which case $J(D(f),y) = X_k$ for some component X_k of measure $y' = y$, and this yields $H(f,u) = f$.
- if $y = 0$, then $f = \tau_+(f)$, $y' = y$ and $H(f,u) = \tau_+(f) = f$
- if $y = 1$, then $f = \tau_-(f)$, $y' = y$ and $H(f,u) = \tau_-(f) = f$

This proves that H provides a suitable retract, for $\mathcal{K} = \mathcal{P}_{\mathrm{f}}^*([n])$. In the general case, starting from the map $H : L(\mathcal{P}_{\mathrm{f}}^*([n]) \times \mathcal{I}) \times I \to L(\mathcal{P}_{\mathrm{f}}^*([n]) \times \mathcal{I}) \times I$ we constructed, we consider its restriction $H_{\mathcal{K}}$ to $L(\mathcal{K} \times \mathcal{I}) \times I$. Starting from $(f,u) \in L(\mathcal{K} \times \mathcal{I}) \times I$, that is with $\pi(f)(\Omega) \in \mathcal{K}$, we need to check that $\pi(G_{\pm}(f,y'))(\Omega) \in \mathcal{K}$ so that $H_{\mathcal{K}}$ provides a suitable retract. Since by construction $\pi \circ G_{\pm}(f,y') = f$, this is obvious.

We now consider the pointed case. From the decomposition $\mathcal{K} \times \mathcal{I}^+ = (\mathcal{K} \times \mathcal{I}) \cup (\mathcal{K} \times \{*\})$ observed in the previous section, we can again identify $L(F_{\mathcal{K}} \wedge F_{\mathcal{I}^+})$ with the quotient space $L(\mathcal{K} \times \mathcal{I})/L_0$ where $L_0 = L^1(\Omega, \{0^-, 0^+\}) = L(\mathcal{P}_{\mathrm{f}}^*(\{0^-, 0^+\}))$.

Then $L(\mathcal{K} \vee \mathcal{K})$ can be identified with a closed subset of $L(\mathcal{K} \times \mathcal{I})/L_0$ via $0^{\pm} \mapsto 0^-$, $i^{\pm} \mapsto i^{\pm}$ for $i > 0$, and it is precisely the image of $L(\mathcal{K} \times \{0,1\})$ under the composite of the natural maps $L(\mathcal{K} \times \{0,1\}) \to L(\mathcal{K} \times \mathcal{I})$ and $L(\mathcal{K} \times \mathcal{I}) \to L(\mathcal{K} \times \mathcal{I})/L_0$. Now notice that, for every f, u, x we have $G_{\pm}(f,u)(x) \in \{f(x), \tau_+(f(x)), \tau_-(f(x))\}$ and this proves that $G_{\pm}(L_0 \times I) \subset L_0$. This implies that $H_{\mathcal{K}}$ induces a map $\bar{H}_{\mathcal{K}} : (L(\mathcal{K} \times \mathcal{I})/L_0) \times I \to (L(\mathcal{K} \times \mathcal{I})/L_0) \times I$. It is easily checked that this map is a retract onto $(L(\mathcal{K} \vee \mathcal{K}) \times I) \cup (L(\mathcal{K} \times \mathcal{I})/L_0) \times \{0\}$, and this proves the claim in the pointed case.

□

7. Quillen equivalences between **sSet** and **Top**

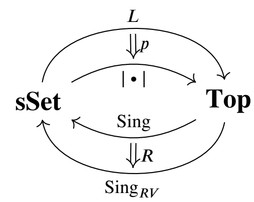
In this section we want to prove that the functors L and $|\bullet|$ from **sSet** to **Top** induce the same functor **HosSet** → **HoTop**, where **HosSet** and **HoTop** denotes the homotopy categories obtained from **sSet** and **Top** by formally inverting the weak homotopy equivalences.

In order to this, we use the classical (closed) model category structure on **sSet**, and endow **Top** with a model category structure more flexible than Quillen's classical one, which nevertheless defines the same homotopy category. Such a model category has been introduced by M. Cole in [2]. Actually, Cole presents his construction as an intermediate between Quillen's and Strøm's model category structures on **Top**. Our point of view here is more that is provides a 'flexibilization' of Quillen's model category structure on **Top**, relevant to the same homotopy category. Notice for instance that the subcategory of cofibrant objects is Milnor's category $\mathcal{W}$ of spaces which are *homotopically equivalent* to CW-complexes, instead of the category of genuine CW-complexes.

We recall the main characteristics of this model category structure below. With **sSet** and **Top** endowed with these structures of model categories, the main theorem of this section is the following one.

Theorem 7.1. *The functors* L : **sSet** → **Top** *and* Sing_{RV} : **Top** → **sSet** *together provide a Quillen equivalence. In particular they provide an equivalence of categories between their homotopy categories.*

We shall make clear in Section 7.1 that the classical functors $|.|$ and Sing also provide a Quillen equivalence for the *same* model category structures. The situation is thus summarized by the following diagram, where $p : L \to |.|$ and $R : \mathrm{Sing} \to \mathrm{Sing}_{RV}$ are the obvious natural transformations induced by the probability-law maps, so that the probability-law map somewhat provides a *natural transformation between the two Quillen equivalences*.



7.1. **Cole's model category structure on Top.** From [2] Theorem 2.1 one gets that one can mix Quillen's classical model category structure on **Top**, which has for fibrations and weak equivalences the Serre fibrations and weak homotopy equivalences, respectively, with Strøm's structure, which has for fibrations and weak equivalences the Hurewicz fibrations and (strong) homotopy equivalences, respectively. The resulting model category, which we call Cole's model category, has for fibrations the Hurewicz fibrations and for weak equivalences the weak homotopy equivalences. In particular, it has the same homotopy category as Quillen's original one, and moreover a Quillen adjunction (resp. equivalence) between **sSet** and **Top** for Quillen's original model category structure is a Quillen adjunction (resp. equivalence) between **sSet** and **Top** for Cole's model category structure : indeed, every (trivial) cofibration for Quillen's model category structure is in particular a (trivial) cofibration for Cole's model category – where we use the terminology that a (co)fibration is said to be a trivial one if it is in addition a weak homotopy equivalence.

More precisely, recall that the cofibrations of Quillen's classical model category structure on **Top** are (retracts of) CW-attachments (also called relative CW complexes). We call them Quillen-cofibrations. The cofibrations in Cole's model category structure are the closed cofibrations (in the classical sense) f which can be written as $\xi \circ f'$ where f' is a Quillen-cofibration and ξ is an homotopy equivalence (see [2] Proposition 3.6 and Example 3.8). We call them Cole cofibrations.

The first application of this is for the classical adjunction $|\bullet|$: **sSet** $\leftrightarrow$ **Top** : Sing, which provides a Quillen equivalence for Quillen's model category structure on **Top**, and therefore *also* for Cole's model category structure.

7.2. **Proof of Theorem 7.1.** We shall prove the following strengthening of Proposition 3.3.

Proposition 7.2. *For every $n \geq 0$ the inclusion map $\partial\nabla_n \to \nabla_n$ is a Cole cofibration.*

Then recall that the r-th hook of the n-dimensional simplex has for geometric realization

$$\Lambda_n^r = \{(a_0, \dots, a_n) \in \Delta_n \mid \exists i \neq r \ \ a_i = 0\} \subset \partial\Delta_n \subset \Delta_n.$$

Its image under the functor L is

$$V_n^r = \{(A_0, \dots, A_n) \in \nabla_n \mid \exists i \neq r \ \ A_i = \emptyset\} \subset \partial\nabla_n \subset \nabla_n.$$

In this subsection we are going to prove the following. Recall that, in any given model category, a (co)fibration is said to be a trivial one if it is in addition a weak homotopy equivalence.

Proposition 7.3. *For $n > 0$ the inclusion $V_n^r \subset \nabla_n$ is a trivial cofibration for Cole's model category structure on* **Top**.

From these two propositions and our previous results we finally can prove Theorem 7.1.

Proof. Recall (see e.g. [12] Theorem 3.6.5) that the category of simplicial sets is a finitely generated closed model category with generating cofibrations the inclusions of simplicial complexes $\partial\mathcal{P}_{\mathrm{f}}^*([n]) \subset \mathcal{P}_{\mathrm{f}}^*([n])$ for $n \geq 0$ and generating trivial cofibrations the inclusion of simplicial complexes $\Lambda_n^r \subset \mathcal{P}_{\mathrm{f}}^*([n])$ for $n > 0$ et $r \in [n]$. By the above propositions we know that their image under L are cofibrations and trivial cofibrations, respectively. Therefore ([12] Lemma 2.1.20) L : **sSet** $\to$ **Top** and Sing_{RV} : **Top** $\to$ **sSet** together provide a Quillen adjunction. In order to prove that they provide a Quillen equivalence, we need to prove that L maps cofibrations to cofibrations and trivial cofibrations to trivial cofibrations for Cole's model category structure hence provides a Quillen adjunction. Since every object of **sSet** is cofibrant and every object of **Top** is fibrant for Cole's model category (the constant maps are Hurewicz fibrations), it remains to prove ([12] Corollary 1.3.16) that L reflects weak equivalences and that $L(\mathrm{Sing}_{RV} X) \to X$ is always a weak equivalence.

Let $f : F \to G$ be such a weak equivalence in **sSet**. By definition this is equivalent to asking for the induced map $|f| : |F| \to |G|$ to be one. Applying the natural transformation $p : L \rightsquigarrow | \bullet |$, we get the commutative square

$$\begin{array}{ccc} |F| & \xrightarrow{|f|} & |G| \\ \uparrow{\scriptstyle p_F} & & \uparrow{\scriptstyle p_G} \\ L(F) & \xrightarrow{L(f)} & L(G) \end{array}$$

and we know by Theorem 3.1 that p_F and p_G are homotopy equivalences. Therefore $L(f)$ is a weak equivalence iff $|f|$ is a weak equivalence.

We finally need to prove that $L(Sing_{RV} X) \to X$ is a weak equivalence for every $X \in$ **Top**. We consider the following diagram, where the horizontal maps are induced from the weak equivalence $\mathrm{Sing}X \to \mathrm{Sing}_{RV} X$ of Theorem 6.4, the vertical maps by the natural transformation p, and the diagonal ones are the obvious co-unit maps.

$$\begin{array}{ccccc} L(\mathrm{Sing}X) & \longrightarrow & L(\mathrm{Sing}_{RV} X) & & \\ \downarrow & & \downarrow & \searrow & \\ |\mathrm{Sing}X| & \longrightarrow & |\mathrm{Sing}_{RV} X| & & \\ & \searrow & & & X \end{array}$$

Since $|\mathrm{Sing}X| \to X$ is a classical weak equivalence and since $L(\mathrm{Sing}X) \to |\mathrm{Sing}X|$ is also one by Theorem 3.1, we get that $L(\mathrm{Sing}X) \to X$ is one. Therefore the composite of the maps $L(\mathrm{Sing}X) \to L(\mathrm{Sing}_{RV} X)$ and $L(\mathrm{Sing}_{RV} X) \to X$ is a weak-equivalence. Since $\mathrm{Sing}X \to \mathrm{Sing}_{RV} X$ is a weak equivalence by Theorem 6.4 so is $L(\mathrm{Sing}X) \to L(\mathrm{Sing}_{RV} X)$ as we just showed, and this has for consequence that $L(\mathrm{Sing}_{RV} X) \to X$ is a weak-equivalence, which concludes the proof.

□

7.3. **Proof of Proposition 7.2.** Since $\partial \triangledown_0 = \emptyset$ the statement is trivial for $n = 0$ and we can assume $n \geq 1$. We already know from Proposition 3.3 that the inclusion map $\partial \triangledown_n \to \triangledown_n$ is a cofibration. We prove here that it can be written as the composition of a cellular attachement with an homotopy equivalence.

We consider the section $\sigma_n : \Delta_n \to \triangledown_n$ of the probability-law map $p_n : \triangledown_n \to \Delta_n$ used in Section 4. We recall from there that $\sigma_n \circ p_n$ is homotopic to the identity map, by an homotopy H_n which commutes to the face maps for various n's (see Proposition 4.2). Clearly, σ_n, p_n, H_n preserve the boundaries of $\triangledown_n$ and Δ_n. In particular we get an attaching map $\partial \sigma_n : \partial \Delta_n \to \partial \triangledown_n$, enabling us to build a relative cellular complex $\partial \triangledown_n \subset (\partial \triangledown_n) \cup_{\partial \sigma_n} \Delta_n$. Moreover, our map $\partial \triangledown_n \to \triangledown_n$ is the composition of this Quillen cofibration with the map $f' : (\partial \triangledown_n) \cup_{\partial \sigma_n} \Delta_n \to \triangledown_n$ obtained by gluing together the inclusion map $\partial \triangledown_n \to \triangledown_n$ with the map $\sigma_n : \Delta_n \to \triangledown_n$. It remains to prove that f' is an homotopy equivalence.

Consider the map $p_n \circ f' : (\partial \triangledown_n) \cup_{\partial \sigma_n} \Delta_n \to \Delta_n$. It is equal to the identity map on Δ_n, and to p_n on $\partial \triangledown_n$. It admis a section given by σ_n on $\partial \Delta_n$ and to the identity on Δ_n. Moreover, since the homotopy H_n preserves the boundary, we get that $p_n \circ f'$ is an homotopy equivalence. Since p_n is also an homotopy equivalence, from the 2-out-of-3 property of homotopy equivalences we get that f' is an homotopy equivalence, which concludes the proof.

7.4. **Proof of Proposition 7.3.** We prove it in several steps.

Proposition 7.4. *For every $n \geq 1$ the inclusion $V_n^r \subset \triangledown_n$ is a strong deformation retract. In particular, it is an homotopy equivalence.*

Proof. We define $H : \triangledown_n \times I \to \triangledown_n$ by $H(\underline{A}, t) = \underline{B}$ with

$$B_i = \mathbf{g}\left(A_i, t \frac{\min_{j \neq i, r} \lambda(A_j)}{\lambda(A_i)} \right)$$

for $i \neq r$ with the convention $0/0 = 1$, and $B_r = \Omega \setminus \bigcup_{i\neq r} B_i$. From the continuity of $\mathbf{g}$ and of the elementary set-theoretic operations one gets immediately that H is continuous. Since $B_i \subset A_i$ for $i \neq r$, we have $H(V_n^r \times I) \subset V_n^r$. Moreover, for $t = 0$ we have $B_i = A_i$ for all i that is $H(\underline{A}, 0) = \underline{A}$ for every $\underline{A} \in \nabla_n$.

Consider $\underline{A} \in V_n^r$. By definition there exists $i_0 \neq r$ with $A_{i_0} = \emptyset$. From $A_{i_0} = \emptyset$ one gets $B_{i_0} = \emptyset$, and also $\min_{j\neq i,r} \lambda(A_j) = 0$ hence $B_j = \mathbf{g}(A_j, 0) = A_j$ for every $j \neq r$. It follows that $B_r = A_r$ hence $H(\underline{A}, t) = \underline{A}$ for every $\underline{A} \in V_n^r$ and $t \in I$.

Now assume $t = 1$. Notice that there exists $i \neq r$ such that $\lambda(A_i) = \min_{j\neq i,r} \lambda(A_j)$. Then, for $t = 1$, $B_i = \mathbf{g}(A_i, 1) = \emptyset$, and this proves $H(\underline{A}, 1) \in V_n^r$.

Therefore H indeed provides a strong deformation retract.

□

Proposition 7.5. *For $n \geq 1$ the inclusion $V_n^r \subset \nabla_n$ is a closed cofibration.*

Proof. We already know that V_n^r is a closed subset of ∇_n. We need to exhibit a retract of $\nabla_n \times I$ onto $(\nabla_n \times \{0\}) \cup (V_n^r \times I)$. We define $H : \nabla_n \times I \to (\nabla_n \times \{0\}) \cup (V_n^r \times I)$ as follows. First of all, define a continuous map $m : \nabla_n \to [0, 1]$ by $m(\underline{A}) = \min_{i\neq r} \lambda(A_i)$, and then define $H(\underline{A}, u) = ((H(\underline{A}, u)_i)_{i\in[n]}, \max\left(u - m(\underline{A}), 0\right))$ by setting

$$H(\underline{A}, u)_i = \mathbf{g}\left(A_i, \frac{m(\underline{A}) + \min(u - m(\underline{A}), 0)}{\lambda(A_i)}\right)$$

for $i \neq r$, $H(\underline{A}, u)_r = \Omega \setminus \bigcup_{i\neq r} H(\underline{A}, u)_i$. This formula is made so that $H(\underline{A}, u)_i$ for $i \neq r$ has measure $\lambda(A_i) - m(\underline{A})$ for $u \geq m(\underline{A})$ and $\lambda(A_i) - u$ for $u \leq m(\underline{A})$. In particular, $H(\underline{A}, u) \in V_n^r \times I$ for $u \geq m(\underline{A})$ and $H(\underline{A}, u) \in \nabla_n \times \{0\}$ for $u \leq m(\underline{A})$. Moreover, for $\underline{A} \in V_n^r$ we have $m(\underline{A}) = 0$ hence $H(\underline{A}, u)_i = \mathbf{g}(A_i, 0) = A_i$ for every $i \neq r$, whence $H(\underline{A}, u) = (\underline{A}, u)$. Similarly, for $u = 0$, we have $\max\left(u - m(\underline{A}), 0\right) = 0$ and $H(\underline{A}, u)_i = \mathbf{g}(A_i, 0) = A_i$ for every $i \neq r$, $H(\underline{A}, 0) = (\underline{A}, 0)$. Since H is continuous as a composite of continuous maps, this provides a suitable retract. □

In order to prove that these inclusions $V_n^r \to \nabla_n$ are Cole cofibrations, it remains to prove that they can be obtained as the composition of a Quillen cofibration with an homotopy equivalence. We consider the restriction $\sigma_{n-1}^{(r)} : \Delta_{n-1} \to \nabla_n$ of $\sigma_n : \Delta_n \to \nabla_n$ to the r-th face of its boundary, and then its restriction $\partial\sigma_{n-1}^{(r)}$ to the boundary of the r-th face. This provides an attachment map $\partial\sigma_{n-1}^{(r)} : \partial\Delta_{n-1} \to V_n^r$, mapping $\underline{a} \in \partial\Delta_{n-1}$ to $\sigma_{n-1}^{(r)}(\underline{a}) = \sigma_n(\Delta(D_r^c)(\underline{a}))$.

Let $Y = V_n^r \cup_{\partial\sigma_{n-1}^{(r)}} \Delta_{n-1}$. Since V_n^r and the r-th face ∇_{n-1} of $\partial\nabla_n$ are closed inside $\partial\nabla_n$, the set-theoretic decomposition $\partial\nabla_n = V_n^r \cup \nabla_{n-1}$ is a topological union. Therefore there is a map $f' : Y \to \partial\nabla_n$ obtained by mapping an element of V_n^r to itself, and $\underline{a} \in \Delta_{n-1}$ to $\sigma_n(\Delta(D_r^c)(\underline{a}))$.

We then consider a map $\partial'\sigma_n : \partial\Delta_n \to Y$ given by mapping the r-th face identically to the copy of Δ_{n-1} just added, and by applying σ_n to the other faces. In formulae, $\underline{a} = (a_0, \dots, a_n) \in \partial\Delta_n$ is mapped to $(a_0, \dots, a_{r-1}, a_{r+1}, \dots, a_n) \in \Delta_{n-1}$ if $a_r = 0$, and to $\sigma_n(\underline{A}) \in V_n^r$ otherwise. We let $Z = Y \cup_{\partial'\sigma_n} \Delta_n$. Again because $\partial\nabla_n$ is closed inside ∇_n, we can write ∇_n as a topological union $(V_n^r \cup \nabla_{n-1}) \cup \nabla_n$ and there is a map $f'' : Z \hookrightarrow \nabla_n$ obtained by gluing $f' : Y \to \nabla_n$ with $\sigma_n : \Delta_n \to \nabla_n$. Clearly, the restriction of f'' to V_n^r is the inclusion map $V_n^r \to \nabla_n$.

The inclusion map $V_n^r \hookrightarrow Z$ being a relative CW-complex, proving that the inclusion $V_n^r \to \nabla_n$ is a Cole cofibration amounts to proving that f'' is an homotopy equivalence.

We post-compose it with $p_n : \nabla_n \to \Delta_n$ and get a map

$$Z = (V_n^r \cup \Delta_{n-1}) \cup \Delta_n \xrightarrow{p'} (\Lambda_n^r \cup \Delta_{n-1}) \cup \Delta_n = \Delta_n$$

equal to p_n on V_n^r and to the identity map on the rest This map p' admits a section σ' defined by σ_n on Λ_n^r and to the identity map on the rest. In order to prove that $\sigma' \circ p'$ is homotopic to the identity map, we build $H_Z : I \times Z \to Z$ via the decomposition

$$I \times Z = (I \times V_n^r) \cup (I \times \Delta_{n-1}) \cup (I \times \Delta_n)$$

gluing the map H_n of Proposition 4.2 on $I \times V_n^r$ with the identity map on the two other parts. This is possible because $\forall \underline{a} \in \Delta_n\ H_n(t, \sigma_n(\underline{a})) = \sigma_n(\underline{a})$ by Proposition 4.2. This provides an homotopy proving that $p' = p_n \circ f''$ is an homotopy equivalence. Since p_n is an homotopic equivalence so is f'' by the 2-out-of-3 principle and this proves the claim.

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