Galois actions on complex braid groups

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Dedicated to François Digne and Jean Michel on the occasion of their sixtieth birthday.

Abstract.

We establish the faithfulness of a geometric action of the absolute Galois group of the rationals that can be defined on the discriminantal variety associated to a finite complex reflection group, and review some possible connections with the profinite Grothendieck-Teichmüller group.

§1. Introduction

More than a decade ago, Broué, Malle and Rouquier suggested in [2] that lots of the well-known properties of the braid groups associated to real reflection groups (so-called Artin groups, or Artin-Tits groups, or Artin-Brieskorn groups), which are consequences either of the simplicial structure of the corresponding hyperplane arrangement or of Coxeter theory, could actually be generalized to all complex reflection groups, for which no such comprehensive theory apply. Although a general reason is still missing for most of these properties, and although most proofs rely on a case-by-case approach using the Shephard-Todd classification, this intuition has been confirmed by large in the past decade. It should be noticed that the group-theoretic study of these general braid groups is still at the beginning ; for instance, the determination of nice *presentations* for all these groups has been completed only recently.

At much about the same time, Matsumoto investigated in [20] the geometric Galois actions on the discriminantal variety of Weyl groups and compared it with the ones on mapping class groups. The reason why this article restricted itself to Weyl groups was probably twofold :

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the most visible connections with mapping class groups involved Artin groups of type ADE, and the Weyl groups are the reflection groups which are naturally 'defined over $\mathbb{Q}$'.

In a joint work with J. Michel (see [14]) we proved that the discriminantal varieties arising from complex reflection groups are actually defined over $\mathbb{Q}$. In the first part of this note we review this result, define the corresponding Galois action, and prove its faithfulness. We also survey the state-of-the-art for the basic group-theoretic questions which are relevant to the geometric-algebraic setting. The second part of the paper is more speculative. It investigates these 'complex braid groups', and already usual Artin groups, from a 'Grothendieck-Teichmüller perspective' : can we expect that the Galois actions on these groups factor through $\widehat{GT}$?

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§2. Complex reflection groups

Let $\mathbf{k}$ be field of characteristic 0 and V a finite-dimensional $\mathbf{k}$-vector space. A pseudo-reflection in V is an element s of $\mathrm{GL}(V)$ whose set of fixed points $\mathrm{Ker}(s-1)$ is an hyperplane. A (finite) reflection group W over $\mathbf{k}$ is a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo-reflections. We refer to [13] for basic notions in this area, and recall that these groups have been classified by Shephard and Todd. When $\mathbf{k}$ is algebraically closed, such groups can naturally be decomposed as a product of *irreducible* ones (which act irreducibly on their associated vector space) which belong either to an infinite family $G(de,e,n)$ depending on three integer parameters d,e,n, or to a finite set of 34 exceptions denoted G_4, $\dots$, G_{37}. We let $\mathcal{R}$ denote the set of pseudo-reflections of W, $\mu_n(\mathbf{k})$ the group of n-th roots of 1 in $\mathbf{k}^\times$, and $\mu_n = \mu_n(\mathbb{C})$. Of course we can assume $\mathbf{k} \subset \mathbb{C}$.

When $|\mu_{de}(\mathbf{k})| = de$, that is $\mathbf{k} \supset \mathbb{Q}(\mu_{de})$, the groups $G(de,e,n) < \mathrm{GL}_n(\mathbf{k})$ are the groups of $n \times n$ *monomial* matrices in $\mathbf{k}$ (one nonzero entry per row and per column) whose nonzero entries belong to $\mu_{de}(\mathbf{k})$ and have product in $\mu_d(\mathbf{k})$.

When $\mathbf{k} = \mathbb{C}$, we let $\mathcal{A} = \{\mathrm{Ker}(s-1) \mid s \in \mathcal{R}\}$ denote the hyperplane arrangement associated to $\mathcal{R}$, and $X = V \setminus \bigcup \mathcal{A}$ the corresponding hyperplane complement. By a theorem of Steinberg, the action of W on X is free and defines a Galois (étale) covering $X \to X/W$. Let $\alpha_H, H \in \mathcal{A}$ denote a collection of linear forms such that $H = \mathrm{Ker}\alpha_H$, and e_H the order of the (cyclic) subgroup of W fixing H. By a theorem of Chevalley-Shephard-Todd, $V//W$ is isomorphic over $\mathbb{C}$ to the affine space : identifying V with $\mathbb{C}^n$, there exists $y_1, \dots, y_n \in S(V^*)$ such that $S(V^*)^W = \mathbb{C}[y_1, \dots, y_n]$. Letting $\Delta = \prod_{H \in \mathcal{A}} \alpha_H^{e_H} \in S(V^*)^W$, X/W is then identified to the complement of $\Delta = 0$ in $\mathbb{C}^n$. The groups $B = \pi_1(X/W)$ and $P = \pi_1(X)$ are called the braid group and pure braid group associated to W. As in the case of the usual braid group (corresponding to $W = \mathfrak{S}_n \subset \mathrm{GL}_n(\mathbb{C})$) we have a short sequence $1 \to P \to B \to W \to 1$. When W is a Coxeter group, that is a reflection group for $\mathbf{k} = \mathbb{R}$, the groups B were basically introduced by Tits and extensively studied since then.

2.1. Arithmetic actions

There is a natural number field K associated to a complex reflection group $W < \mathrm{GL}_N(\mathbb{C})$, called the *field of definition* of W : it is the field generated by the traces of the elements of W. It is a classical fact (see e.g. [13], theorem 1.39) that a suitable conjugation in $\mathrm{GL}_N(\mathbb{C})$ maps W into $\mathrm{GL}_N(K)$. Since W is finite, $K \subset \mathbb{Q}(\mu_\infty)$ is an abelian extension of $\mathbb{Q}$. As a consequence, X and X/W can be defined over K. One has $K \subset \mathbb{R}$ if and only if W is a Coxeter group, and $K = \mathbb{Q}$ if and only if W is the Weyl group of some root system.

We proved in [14] the following.

Theorem 1. *(see* [14]*)The varieties X/W, X, and the étale covering map $X \to X/W$ can be defined over $\mathbb{Q}$.*

More precisely, we proved that we can attach to each (complex) reflection group a suitable extension K'' of K (with $K = K''$ for most of the groups), which is still an abelian extension of $\mathbb{Q}$, and which fulfills the following properties :

- Up to global conjugation by some element of $\mathrm{GL}_n(\mathbb{C})$ we have and can assume $W < \mathrm{GL}_n(K'')$ with $\mathrm{Gal}(K''|\mathbb{Q}).W = W$. This defines a natural action by automorphisms of $\mathrm{Gal}(K''|\mathbb{Q})$ on W, and we can assume V has a natural K''-form V_0, namely $V = V_0 \otimes_{K''} \mathbb{C}$ with $W < \mathrm{GL}(V_0)$ – in particular, the α_H can be chosen in V_0^*.
- This action induces injective morphisms $\mathrm{Gal}(K''|\mathbb{Q}) \hookrightarrow \mathrm{Aut}(W)$ and $\mathrm{Gal}(K|\mathbb{Q}) \hookrightarrow \mathrm{Out}(W)$.
- The action of $\mathrm{Gal}(K''|\mathbb{Q})$ normalizes the action of W on $S(V_0^*)$, hence $S(V_0^*)$, $S(V_0^*)^W$ have natural $\mathbb{Q}$-forms $S(V_0^*) = S(V_0^*)^{\mathrm{Gal}(K''|\mathbb{Q})} \otimes K''$, $S(V_0^*)^W = S(V_0^*)^{W,\mathrm{Gal}(K''|\mathbb{Q})} \otimes K''$ defining V, $V//W = \mathrm{Spec} S(V_0^*)^{W,\mathrm{Gal}(K''|\mathbb{Q})}$ over $\mathbb{Q}$, as well as X/W.

As a consequence we get natural compatible maps $G(\mathbb{Q}) \to \mathrm{Out}^*(\widehat{\pi}_1(X/W))$ and $G(\mathbb{Q}) \to \mathrm{Out}^*(\widehat{\pi}_1(X))$, where we denote, for an algebraic variety Y, $\mathrm{Aut}^*\widehat{\pi}_1 Y$ the group of inertia-preserving automorphisms of $\widehat{\pi}_1 Y$, and $\mathrm{Out}^*\widehat{\pi}_1 Y$ its image in $\mathrm{Out}\,\widehat{\pi}_1 Y$ (and we use the notation $G(\mathbb{Q}) = \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ for the absolute Galois group of the rationals). Since the étale covering map $X \to X/W$ is defined over $\mathbb{Q}$, one can check on the fiber above a rational base point that the action of $G(K'') = \mathrm{Gal}(\overline{\mathbb{Q}}|K'')$ is trivial on the covering $X \to X/W$. This induces an action by automorphisms of $\mathrm{Gal}(K''|\mathbb{Q})$ on W which is easily seen to be identical to the one defined above, and thus provides a geometrical origin to the 'Galois automorphisms' of [14]. The situation is depicted in the following commutative diagrams.

$$\begin{array}{ccc} G(\mathbb{Q}) & \longrightarrow & \mathrm{Aut}^*(\widehat{B}) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(K''|\mathbb{Q}) & \longrightarrow & \mathrm{Aut}(W) \end{array} \qquad \begin{array}{ccc} G(\mathbb{Q}) & \longrightarrow & \mathrm{Out}^*(\widehat{B}) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(K|\mathbb{Q}) & \longrightarrow & \mathrm{Out}(W) \end{array}$$

It turns out that these algebraic varieties, beside being rational, are 'anabelian' enough in the sense that the action of the Galois group on their fundamental group is faithful. Informally, this 'lifts' at the level of the braid group the embedding $\mathrm{Gal}(K|\mathbb{Q}) \hookrightarrow \mathrm{Out}(W)$ defined in [14].

Theorem 2. *If W is not abelian, then the maps $G(\mathbb{Q}) \to \mathrm{Out}(\widehat{\pi}_1(X))$ and $G(\mathbb{Q}) \to \mathrm{Out}(\widehat{\pi}_1(X/W))$ are injective.*

Proof. We can choose a K-form $V = V_K \otimes_K \mathbb{C}$ with $W < \mathrm{GL}(V_K)$.

Notice that, since $K \subset \mathbb{Q}(\mu_\infty)$ then K is an abelian (Galois) extension of $\mathbb{Q}$, and is stable under the complex conjugation $z \mapsto \bar{z}$. We can thus endow V_K with a unitary (positive definite hermitian) form which is left invariant under the finite group W, and extend it to V. For $H \subset V$, we denote $H^\perp \subset V$ the orthogonal of H with respect to this unitary form.

Since W is not abelian and generated by $\mathcal{R}$, there exists $s_1, s_2 \in \mathcal{R}$ with $s_1 s_2 \neq s_2 s_1$. Let $H_i = \mathrm{Ker}(s_i - 1)$ and $H = H_1 \cap H_2$. One can assume that s_i generates the (cyclic) group of W which fixes H_i and more precisely, letting d_i denote the order of that group, that the non-trivial eigenvalue of s_i is $\exp(2\mathrm{i}\pi/d_i)$. Such a pseudo-reflection is classically called 'distinguished', and clearly such pseudo-reflections are in 1-1 correspondence (under $s \mapsto \mathrm{Ker}(s-1)$) with the collection of reflecting hyperplanes. Choosing $v_1 \in H^\perp \cap H_2$, $v_2 \in H^\perp \cap H_1$, and $v_3, \dots, v_n$ a basis for H, in such a way that $v_1, \dots, v_n \in V_K$ (this is possible because the hyperplanes are defined over K) one gets a basis $v_1, \dots, v_n$ of V_K and V, hence an identification of V_K and V with K^n and $\mathbb{C}^n$ respectively, and of H_i with the hyperplane defined by $z_i = 0$ if $\underline{z} = (z_1, \dots, z_n)$ denotes a generic point of K^n. Let W_0 be the subgroup of W generated by s_1, s_2. It is a reflection subgroup of W that fixes H. Since W_0 is not abelian, it contains a pseudo-reflection with respect to another hyperplane. Indeed, $s_1 s_2 s_1^{-1} \in W_0$ is a distinguished reflection with reflecting hyperplane $s_1(H_2)$, and $s_1(H_2) \in \{H_1, H_2\}$ would imply $s_1 s_2 s_1^{-1} = s_1$ or $s_1 s_2 s_1^{-1} = s_2$, meaning $s_2 = s_1$ or $s_1 s_2 = s_2 s_1$, which have been ruled out.

The equation of this new hyperplane has the form $z_1 = \alpha z_2$ with $\alpha \neq 0$ and $\alpha \in K \subset \overline{\mathbb{Q}}$. We consider the morphism $V \setminus \{\underline{z} \mid (z_1, z_2) = (0,0)\} \to \mathbb{P}^1$ defined by $\underline{z} \mapsto [z_1 : z_2]$. It maps $\bigcup \mathcal{A}$ to a finite set S of points, including $0 = [0 : 1], \infty = [1 : 0], \alpha = [\alpha : 1]$, hence induces an algebraic morphism $f : X \to \mathbb{P}^1 \setminus S$ defined over K. Moreover, the induced map $f_* : \widehat{\pi}_1 X \to \widehat{\pi}_1 \mathbb{P}^1 \setminus S$ is equivariant under $G(K)$ and *surjective*.

The easiest way to see this is to prove it topologically on the usual π_1's and then use the right-exactness of the profinite completion functor. We do this now. Let $\mathcal{A}' = \{H' \in \mathcal{A} \mid H' \not\supset H\}$, and choose $\underline{\zeta}_0 = (0, 0, \zeta_3, \dots, \zeta_n) \in H \setminus \bigcup \mathcal{A}'$. Since $H \setminus \bigcup \mathcal{A}'$ is open in H, for $\varepsilon > 0$ small enough we have $\underline{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_n) \in X$ whenever $0 < |\zeta_1| < \varepsilon$, $0 < |\zeta_2| < \varepsilon$, and $[z_1 : z_2] \notin S$. We choose such an ε. Now $\mathbb{P}^1 \setminus S$ can be identified with $\mathbb{C} \setminus S_0$ under $[z_1 : z_2] \mapsto z_1/z_2$, where S_0 is some finite set. Let $M = 1 + \max\{|z|; z \in S_0\}$ and choose $z_0 \in \mathbb{C} \setminus S_0$ with $|z_0| \leq M$ for base-point. A loop γ in $\mathbb{C} \setminus S_0$ based at z_0 can be homotoped to a loop satisfying $|\gamma(t)| \leq M + 1$ for all t, and such a loop has a lift of the form $t \mapsto (\varepsilon \frac{z(t)}{M}, \frac{\varepsilon}{M}, \zeta_3, \dots, \zeta_n)$ in $\pi_1(X(\mathbb{C}), \underline{\zeta}_0(\varepsilon))$ with $\underline{\zeta}_0(\varepsilon) = (\varepsilon \frac{z_0}{M}, \frac{\varepsilon}{M}, \zeta_3, \dots, \zeta_n)$, which proves the surjectivity.

Another way is to check that f is faithfully flat and quasicompact, hence universally submersive, and that it has geometrically connected fibers ; the conclusion then follows from [11] (exp. IX cor. 3.4). Indeed, f is affine, more precisely $f = \mathrm{Spec}\varphi$ with

$$\varphi : A = K[x, y, x^{-1}, y^{-1}, (y - \alpha_i x)^{-1}] \hookrightarrow K[z_1, \dots, z_n, \alpha_{L_1}^{-1}, \dots, \alpha_{L_m}^{-1}]$$

defined by $x \mapsto z_1, y \mapsto z_2$, with $\alpha_{H_1}, \dots, \alpha_{H_m}$ some linear forms defining (over K) the hyperplanes. Identifying A with its image under φ, we have $K[z_1, \dots, z_n, \alpha_{H_1}^{-1}, \dots, \alpha_{H_m}^{-1}] = A[z_3, \dots, z_n, L_1^{-1}, \dots, L_r^{-1}] = B$ with each $L_i \in A[z_3, \dots, z_n]$ of degree 1. Now B is faithfully flat over A as the localization of a polynomial algebra $C = A[z_3, \dots, z_n]$ over A at an element $P = L_1 \dots L_r$ with $P \neq 0$ inside $C/\mathfrak{P}C$ for each prime ideal $\mathfrak{P}$ of A. In particular f is surjective ; since it is affine it is quasi-compact if and only if $f^{-1}(\mathrm{Spec}A) = \mathrm{Spec}B$ is quasi-compact, and this holds because B is noetherian. Finally, since f is the composition of an affine open immersion and of the projection $\mathbb{A}^n \to \mathbb{A}_2$, its fibers are open subsets of an affine space, and are thus connected.

Since $|S| \geq 3$, $\mathbb{P}^1 \setminus S$ is an hyperbolic curve over K, so by Belyi's theorem and [19], the natural map $G(K) \to \mathrm{Out}\widehat{\pi}_1\mathbb{P}^1 \setminus S$ is injective, and so is $G(K) \to \mathrm{Out}\widehat{\pi}_1 X = \mathrm{Out}\widehat{P}$ by the surjectivity of $f_* : \widehat{\pi}_1 X \to \widehat{\pi}_1\mathbb{P}^1 \setminus S$. Now X is the complement of an hypersurface in $\mathbb{A}^n$ defined by a reduced equation of the form $\delta = 0$ for some $\delta \in \mathbb{Q}[x_1, \dots, x_n]$ the product of convenient α_H, $H \in \mathcal{A}$. The function δ induces a morphism $g : X \to \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$ defined over $\mathbb{Q}$. Let $\psi : G(\mathbb{Q}) \to \mathrm{Out}\widehat{\pi}_1 X = \mathrm{Out}\widehat{P}$. We prove that the induced morphism $\widehat{\pi}_1 X \to \widehat{\pi}_1\mathbb{A}^1 \setminus \{0\}$ is surjective. It is enough to prove that $f_* : \pi_1(X(\mathbb{C}), \underline{\zeta}) \to \pi_1(\mathbb{C} \setminus \{0\}, f(\underline{\zeta}))$ is surjective. For this, we can assume $V = \mathbb{C}^n$, $\delta = u_0 u_1 \dots u_m = u_0\hat{\delta}$ with $u_i \in V^*$ and $i \neq j \Rightarrow \mathrm{Ker}u_i \neq \mathrm{Ker}u_j$, and that $u_0 : \underline{z} = (z_1, \dots, z_n) \mapsto z_1$ is the first coordinate. Since $\mathrm{Ker}u_i \neq \mathrm{Ker}u_0$ for all $i \neq 0$, there exists an element $\underline{\zeta}_0 \in \mathrm{Ker}u_0 \setminus \bigcup_{i \neq 0} \mathrm{Ker}u_i$. We let $\underline{\zeta}_\varepsilon = (\varepsilon, \zeta_2, \dots, \zeta_n)$ and $\gamma(t) = (\varepsilon e^{2\mathrm{i}\pi t}, \zeta_2, \dots, \zeta_n)$. We have $\delta(\gamma(t)) = \varepsilon e^{2\mathrm{i}\pi t}\hat{\delta}(\gamma(t))$. For $\varepsilon > 0$ small enough, $\hat{\delta}(\gamma)$ is close to $\hat{\delta}(\underline{\zeta}) \neq 0$, hence $t \mapsto \delta(\gamma(t))$ describes a positive turn around 0, whose class generates $\pi_1(\mathbb{C} \setminus \{0\}, \delta(\underline{\zeta}_\varepsilon))$, and is the image of $[\gamma] \in \pi_1(X(\mathbb{C}), \underline{\zeta}_\varepsilon)$.

Since the kernel of the cyclotomic character $G(\mathbb{Q}) \to \mathrm{Out}\widehat{\pi}_1\mathbb{P}^1 \setminus \{0, \infty\}$ is $\mathbb{Q}(\mu_\infty)$, we get that every element in $\mathrm{Ker}\psi$ fixes $\mathbb{Q}(\mu_\infty)$, hence belongs to $G(K)$. Since we know the action of $G(K)$ is faithful this proves $G(\mathbb{Q}) \hookrightarrow \mathrm{Out}\widehat{\pi}_1 X = \mathrm{Out}\widehat{P}$.

We now consider the map $\varphi : G(\mathbb{Q}) \to \mathrm{Out}\widehat{B} = \mathrm{Out}\widehat{\pi}_1 X/W$, and a lift $\tilde{\varphi} : G(\mathbb{Q}) \to \mathrm{Aut}\widehat{B}$ given by the choice of a rational base point. We can assume that this base point lies in the image of $X(\mathbb{Q}) \to (X/W)(\mathbb{Q})$. An element σ in $\mathrm{Ker}\varphi$ should induce on $\widehat{P}$ the conjugation $\mathrm{Ad}(g)$ by an element g of $\widehat{B}$. Let $\bar{g}$ its image in $\widehat{B}/\widehat{P} = W$. If $\sigma \in G(K)$, $\mathrm{Ad}(\bar{g})$ is the identity automorphism, hence $\bar{g} \in Z(W)$. Now the morphism $Z(B) \to Z(W)$ is known to be onto (see [2]) hence we can assume that $\sigma \mapsto \mathrm{Ad}(g)$ with $\bar{g} = 1$, hence $g \in \widehat{P}$; this implies $\sigma = 1$ as we showed before. But using again $\delta : X/W \to \mathbb{P}^1 \setminus \{0, \infty\}$ we have $\sigma \in G(\mathbb{Q}(\mu_\infty))$. Since $K \subset \mathbb{Q}(\mu_\infty)$ this concludes the proof.

Q.E.D.

2.2. A remark on complex conjugation

A consequence of theorem 1 is that the complex conjugation, being a continuous automorphism of $\mathbb{C}$, induces an automorphism σ of $B = \pi_1(X/W, z)$, for z a *real* point in X/W. Such an automorphism setwise stabilizes P and is well-defined up to inner automorphism. In case W is a Coxeter group, one can take for base point an arbitrary point inside the Weyl chamber. Then B has a classical Tits-Brieskorn presentation, with generators $s_1, \dots, s_n$ and Coxeter-like relations. Geometrically, it is clear that σ maps $s_i \mapsto s_i^{-1}$. It coincides with the well-known 'mirror image' automorphism in the theory of Artin-Tits groups.

In the complex setting however, the question of finding Artin-like presentations for B has been the subject of intense studies in the past decade, starting from the partly conjectural presentations of [2]. The complex conjugation is easier to see on some presentations than others (recall that its image in $\mathrm{Aut}(W)$ has been studied in [14]). We do an example here.

For instance, let us consider the group W of rank 2 and type G_{12}. Then B has presentation $< s, t, u \mid stus = tust = ustu >$, and one can check on this presentation that it admits an automorphism of order 2 given by $s \mapsto t^{-1}, t \mapsto s^{-1}, u \mapsto u^{-1}$. The identification of this automorphism with the complex conjugation asks for getting back to the source of this presentation, which lies in [1], and to redo it in an algebraic and Galois-equivariant way. Redoing the computations from the $\mathrm{Gal}(K''|\mathbb{Q})$-invariant models provided in [14], one finds $\mathbb{C}[x_1, x_2]^W = \mathbb{C}[\alpha, \beta]$ with $\alpha = (x_1^2 - x_2^2)(x_1^4 + 12x_1^2x_2^2 + 4x_2^4)$ and $\beta = (x_1^2 - 4x_1x_2 - 2x_2^2)(x_1^2 + 4x_1x_2 - 2x_2^2)(3x_1^4 + 4x_1^2x_2^2 + 12x_2^4)$, and the 'discriminantal equation' defining X/W inside $\mathrm{Spec}\,\mathbb{C}[\alpha, \beta]$ is $0 = \beta^3 - 27\alpha^4$. Up to a rational change in coordinates one gets that the identification of X/W with $\{(z_1, z_2) \mid z_2^3 \neq z_1^4\}$ given by [1] is indeed over $\mathbb{Q}$ and Galois-equivariant. One can then consider the rational morphism $f : X/W \to \mathbb{A}^1$ which maps $(z_1, z_2) \mapsto z_1$, which has for typical fiber $\mathbb{A}^1 \setminus \mu_3$. Then Bannai shows that B is generated by the image of the π_1 of this fiber, and the chosen generators s, t, u are simple loops around the points of $\mu_3(\mathbb{C})$. Taking the base point on the real line of absolute value > 1, we gets that the automorphism described above is indeed the image of the complex conjugation. The issue is that, Bannai does in general *not* choose the base point on the real line. For G_{12} this has little consequences (taking the base point on the real line provides the formula above), but for the other exceptional groups there is a need in getting presentations which are 'compatible' with the real structure.

Problem 1. *Find presentations of the exceptional braid groups with 'nice' conjugation automorphism / Find explicit formulas for the conjugation in terms of the known presentations of these groups.*

For the general series of the $G(e,e,n)$ a nice presentation for B has been obtained by R. Corran and M. Picantin in [5], with generators $t_i, i \in \mathbb{Z}/e\mathbb{Z}$, $s_3, s_4, \dots, s_n$ and relations $t_{i+1}t_i = t_{j+1}t_j$, $s_3 t_i s_3 = t_i s_3 t_i$ and ordinary braid relations between the $s_3, \dots, s_n$.

It is deduced by algebraic methods from one given in [2]. We gave in [3] a topological description of this presentation, with base point in the real part of X/W. Since all the morphisms used there are algebraic over $\mathbb{Q}$, it is easily checked that this presentation behaves nicely with respect to the complex conjugation, and we get the explicit formulas $s_k \mapsto s_k^{-1}$, $t_i \mapsto t_{-i}^{-1}$.

The following question was asked to me by F. Digne. It is well-known that non-isomorphic W may provide the 'same' braid group B. This happens for instance when the spaces X/W are analytically isomorphic, in which case both complex conjugations provide the same element in $\mathrm{Out}(B)$. However, there are some other coincidences where geometry does not seem to help decide.

Question 1. *Is it possible to get two reflection groups W_1, W_2 with isomorphic braid group B, such that the complex conjugation induces two distinct elements in* $\mathrm{Out}(B)$ *?*

This question should be viewed as a test about the comprehension of the coincidence phenomena (see question 3 below).

A good test for a positive answer to this question seemed to be the exceptional group called G_{13} in the Shephard-Todd classification. It has rank 2, and its braid group B is isomorphic to the Artin group of type $I_2(6)$, in a seemingly non-geometric way. Moreover, $\mathrm{Out}(B)$ is rather large (it is precisely isomorphic to $(\mathbb{Z} \rtimes (\mathbb{Z}/2)) \times (\mathbb{Z}/2)$, see [6]). For G_{13}, $Y_1 = X/W = \{(x,y) \mid y(x^3 - y^2) \neq 0\}$, whereas for the dihedral group $I_2(6)$, $Y_2 = X/W = \{(x,y) \mid x^6 \neq y^2\}$. By Zariski-Van Kampen method Bannai found the presentations $< g_1, g_2, g_3 \mid g_1 g_2 g_3 g_1 = g_3 g_1 g_2 g_3, g_3 g_1 g_2 g_3 g_2 = g_2 g_3 g_1 g_2 g_3 >$ coming from Y_1, whereas the presentation arising from Y_2 is the usual $< a, b \mid ababab = bababa >$. An explicit automorphism is given by $a \mapsto g_3 g_1 g_2 g_3$, $b \mapsto g_3^{-1}$.

Although Y_1 and Y_2 are homotopically equivalent (as they are $K(\pi,1)$ with the same π_1) they do not seem to be geometrically related. Surprisingly enough, an explicit computation shows that the complex conjugation provides the same element in $\mathrm{Out}(\hat{B})$. Indeed, from a suitable choice of a base point one gets $g_1 \mapsto g_1^{-1}$, $g_2 \mapsto g_1 g_2^{-1} g_1^{-1}$ and

$g_3 \mapsto g_1g_2g_3g_2^{-1}g_1^{-1}$, which, under the above isomorphism, translates into $a \mapsto (bab)a^{-1}(bab)^{-1}$, $b \mapsto (ba)b^{-1}(ba)^{-1} = (bab)b^{-1}(bab)^{-1}$, which has clearly same image in $\mathrm{Out}(B)$ as the complex conjugation for $I_2(6)$.

2.3. Group-theoretic properties of complex braid groups

We consider now the possible goodness of these groups, in the sense of Serre ([26] §2.6). We recall from there that a group G is called good if, for every finite G-module M and positive integer q, the natural maps $H^q(\widehat{G}, M) \to H^q(G, M)$ are isomorphisms. Goodness is inherited by finite-index subgroups, and a crucial property is that, if G is an extension of Q by $H \lhd G$, then G is good as soon as H, Q are good, when H is finitely generated and its cohomology groups with coefficients in finite H-modules are finite. This last assumption is clearly verified by groups of type FP_m for some m, that is a group H such that the trivial $\mathbb{Z}H$-module $\mathbb{Z}$ admits a resolution of finite length by free modules of finite rank. This obviously holds for the fundamental group of an affine complex algebraic variety of dimension m, as it is homotopically equivalent to a finite CW-complex of (real) dimension m.

A well-known consequence is that iterated extension of free groups are good. Recall that the profinite completion functor is right exact, and that every short exact sequence $1 \to H \to G \to K \to 1$ yields a short exact sequence $1 \to \widehat{H} \to \widehat{G} \to \widehat{K} \to 1$ when H is finitely generated and K is good.

Proposition 1. *For all but possibly finitely many irreducible reflection groups W, the groups P and B are good in the sense of Serre. The possible exceptions are the ones labelled $G_{23}, G_{24}, G_{27}, G_{28}, G_{29}$, G_{30}, G_{31}, $G_{33}, G_{34}, G_{35}, G_{36}, G_{37}$ in the Shephard-Todd classification.*

Proof. Let W be such an irreducible reflection group. Since P has finite index in B the properties for P or B to be good are equivalent. We first prove that P is good when $W = G(de, e, n)$. In this case, we prove that P is actually an iterated extension of free groups of finite rank.

When $d \neq 1$ this is the case because the corresponding hyperplane complement $X_n = \{\underline{z} \in \mathbb{C}^n | z_i \neq 0, z_j/z_i \notin \mu_{de}(\mathbb{C}) \text{ for } j \neq i\}$ is fiber-type in the sense of Falk-Randell (or equivalently: supersolvable, see e.g. [24]).

We briefly recall the definition. The assertion that X_n is fibertype means that there exists a fibration $X_n \to Y$ for some $Y \subset \mathbb{C}^{n-1}$ which is the restriction of a linear map, and such that Y is an hyperplane complement which is itself fibertype — this inductive definition makes sense because the dimension decreases by one, and because in addition every hyperplane arrangement is said to be fibertype in dimension 1.

A classical consequence of this property and of the long exact sequence of a fibration is that (1) X_n is a $K(\pi, 1)$ and (2) there is a short exact sequence $1 \to \pi_1(F) \to \pi_1(X_n) \to \pi_1(Y) \to 1$, with the fiber F being the complement in $\mathbb{C}$ of a finite number of points. From this it easily follows that $\pi_1(X_n)$ is an iterated extension of free groups of finite rank. A classical fact is that central hyperplane complements of rank 2 (i.e. complement of a finite collection of linear hyperplanes in $\mathbb{C}^2$) are always fibertype (a convenient projection $\mathbb{C}^2 \to \mathbb{C} \setminus \{0\}$ is given by any of the linear forms defining an hyperplane of the arrangement).

In our case, when $d \neq 1$, the fibration is given by $X_n \to X_{n-1}$, $\underline{z} \mapsto (z_1, \dots, z_{n-1})$. In case $d = 1$, we cannot use the fibertype machinery. However, the hyperplane complement $X_n = \{\underline{z} \in \mathbb{C}^n |, z_j \notin \mu_e(\mathbb{C}) z_i \text{ for } j \neq i\}$ admits a fibration $\underline{z} \mapsto (z_1^e - z_n^e, \dots, z_{n-1}^e - z_n^e)$ over the space $Y = \{\underline{z} \in \mathbb{C}^n \ | z_i \neq 0, z_i \neq z_j \text{ for } j \neq i\}$, which is fiber-type (consider the natural map $\underline{z} \mapsto (z_1, \dots, z_{n-1})$ on Y). It follows that $\pi_1(Y)$ is good and that Y is a $K(\pi, 1)$, hence from the homotopy sequence of the fibration it follows that P is an extension of $\pi_1(Y)$ by the π_1 of the fiber, which is clearly a smooth algebraic curve. The group $\pi_1(Y)$ is finitely generated of type FP_m for some m (e.g. because Y is a $K(\pi, 1)$ with finite cellular decomposition), hence its cohomology groups with values in any finite module are finite. Since π_1's of curves are good, it follows (see [26]) that P is good (alternatively, an explicit decomposition as an iterated extension of free groups can be found in [2] proposition 3.37).

We now recall that, among the exceptional reflection groups, half of them, namely the ones labelled G_4 to G_{22} by Shephard and Todd [27], have rank 2. As noticed before, this implies that the associated hyperplane complement is necessarily fibertype, hence that P is good.

Among the remaining groups, which are labelled G_{24} to G_{37} in the Shephard-Todd classification, three of them have the same braid group than one in the infinite series. Specifically we get the same braid group B for the pairs $(G_{25}, G(1, 1, 4))$, $(G_{32}, G(1, 1, 5))$, $(G_{26}, G(2, 2, 3))$ (see e.g. the tables of [2]). By the above arguments, this proves that the corresponding braid groups B are good. There remains the list of 12 possible exceptions mentionned in the statement. Q.E.D.

In general, the following conjecture however remains open.

Conjecture 1. *B is good in the sense of Serre.*

We recall from e.g. [24] that the Shephard groups are the symmetry groups of regular complex polytopes. Their braid group is always isomorphic to the braid group of another complex reflection group which belongs to the infinite series, and which is also a Coxeter group. Among the exceptional groups, and in rank at least 3, these are the groups labelled G_{25}, G_{26} and G_{32} in the Shephard-Todd classification. One can prove the following.

Proposition 2. *Let W be an irreducible finite reflection group. The groups B and P are residually finite when $W = G(de, e, n)$ for $d \neq 1$, when W is a Coxeter or a Shephard group, and when W has rank 2.*

Proof. Again, the statements for P and B are equivalent. In case $W = G(de, e, n)$ for $d \neq 1$ or W has rank 2, then P is residually finite because it is residually torsion-free nilpotent ; indeed, this is the case for the π_1 of an arbitrary fibertype hyperplane complement (see [FR]). In case W is a Coxeter group (and thus also when W is a Shephard group) this is a consequence of the linearity of B, proved by F. Digne and A. Cohen - D. Wales (see [8, 4]), since finitely generated linear groups are residually finite. Q.E.D.

In the remaining cases, the groups P are conjectured to be residually torsion-free nilpotent, and the groups B to be linear – these both statements being potentially connected, as shown in [16] and [17], and both of them implying the residual finiteness of B. The following conjecture is thus highly plausible.

Conjecture 2. *B is residually finite.*

Another group-theoretic question concerns the centre of these group and of their profinite completion. It was conjectured in [2] that, when W is irreducible, $Z(P)$ and $Z(B)$ were infinite cyclic, generated by natural elements denoted π and β, and that they are related to each other through an exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$. This conjecture has been recently proved (see [9]). In addition, we proved in [9] that, whenever U is a finite index subgroup of B, one has $Z(U) \subset Z(B)$. This raises the following natural question :

Question 2. *Is $Z(\hat{B})$ (respectively $Z(\hat{P})$) freely generated by β (resp. π) ? Do we have a short exact sequence $1 \to Z(\hat{P}) \to Z(\hat{B}) \to Z(W) \to 1$?*

Finally, from the point of view of capturing the image of $G(\mathbb{Q})$ inside $\mathrm{Out}\hat{B}$, one would like to know the following.

Question 3. *Which reflection groups W actually provide distinct (non-isomorphic) braid groups B, and more importantly distinct profinite groups $\widehat{B}$?*

A partial answer has been obtained in [3], for the case of reflection groups having one conjugacy class of reflections : the map $W \mapsto B$ is injective on irreducible 2-reflection groups with one class of reflections. Actually the methods used there are either cohomological or depend only on $\widehat{B}$, so it also proves the injectivity of $W \mapsto \hat{B}$ under the goodness assumption (conjecture 1 above).

§3. Grothendieck-Teichmüller actions

A few years ago, P. Lochak asked the following question, in the setting of real reflection groups. It now seems natural to extend this question to complex reflection groups.

Question 4. *For W a complex reflection group, is there an action of $\widehat{GT}$ on $\widehat{B}$ which extends an action of $G(\mathbb{Q})$? Or of $\mathbb{I}\Gamma$, $\mathbb{I}\Gamma'$,… ?*

Here $\widehat{GT}$ denotes the usual Grothendieck-Teichmüller, whose definition we recall below, and $\mathbb{I}\Gamma$, $\mathbb{I}\Gamma'$ denote subgroups of $\widehat{GT}$ (whose equality with $\widehat{GT}$ is an open question) which act on the full tower of mapping class groups in all genus. These groups have been defined in [22].

In this section we shall explore this question (without answering it in any way) in the general setting of complex reflection groups, and we shall try to distinguish a few maybe more accessible subquestions.

3.1. The Grothendieck-Teichmüller group

We denote Br_n the usual braid group on n strands, with generators $s_1, \dots, s_{n-1}$, and denote $\widehat{GT}$ the (profinite version of) the Grothendieck-Teichmüller group, introduced by Drinfeld in [10]. Recall that the elements of $\widehat{GT}$ are couples $(\lambda, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{F}_2$, where F_2 denotes the free group of rank 2, and that there exists an embedding $G(\mathbb{Q}) \hookrightarrow \widehat{GT}$, $\sigma \mapsto (\chi(\sigma), f_\sigma)$ with $\chi : G(\mathbb{Q}) \to \widehat{\mathbb{Z}}^{\times}$ the cyclotomic character.

There is a natural (Drinfeld) action of $\widehat{GT}$ on $\widehat{Br_n}$ defined in [10], that associates to $(\lambda, f) \in \widehat{GT}$ the automorphism $s_1 \mapsto s_1^{\lambda}$, $s_i \mapsto f(s_i^2, y_i) s_i^{\lambda} f(y_i, s_i^2)$.

3.2. A remark on the derived subgroup

One of the properties of the couples $(\lambda, f) \in \widehat{GT}$ is that $f \in \widehat{F}_2$ actually belongs to the derived subgroup $(\widehat{F}_2, \widehat{F}_2)$. Here, we let (G, G) denote the (algebraic) derived subgroup of the group G, that is the subgroup algebraically generated by the commutators $(a, b) = aba^{-1}b^{-1}$ for $a, b \in G$. Note that, for a finitely generated group G, there are a priori two natural notions for the 'derived subgroup' of $\widehat{G}$. Letting

$p : G \twoheadrightarrow G^{ab}$ denote the abelianization morphism, and $\widehat{p} : \widehat{G} \twoheadrightarrow \widehat{G^{ab}}$ the induced morphism, $\mathrm{Ker}\widehat{p}$ equals the closure $\overline{(G,G)}$ of (the image of) (G,G) in $\widehat{G}$. Of course $\widehat{G^{ab}}$ is abelian, hence $(\widehat{G},\widehat{G}) \subset \mathrm{Ker}\widehat{p} = \overline{(G,G)}$, and clearly $(G,G) \subset (\widehat{G},\widehat{G})$. When G is finitely generated, by the recent result of [23] theorem 1.4, $(\widehat{G},\widehat{G})$ is closed hence $(\widehat{G},\widehat{G}) = \overline{(G,G)} = \mathrm{Ker}\widehat{p}$. Moreover, G^{ab} being a finitely generated abelian group is good, hence the sequence $1 \to \widehat{(G,G)} \to \widehat{G} \to \widehat{G^{ab}} \to 1$ is exact. In our case $G = F_2$ we thus get $\widehat{(F_2,F_2)} \simeq \overline{(F_2,F_2)} = (\widehat{F}_2,\widehat{F}_2)$

3.3. Artin group of type B_n and the groups $G(de,e,n)$, $d > 1$

For the complex reflection groups $G(de,e,n)$ with $d \neq 1$, the corresponding braid group is a finite index subgroup of the Artin group $Art(B_n)$ of type B_n (see [2] proposition 3.8). Recall that this group is generated by elements $t, s_2, s_3, \dots, s_n$ with usual braid relations between the s_i's, and $ts_2ts_2 = s_2ts_2t$, $ts_i = s_it$ for $i \geq 3$. This group can be embedded in Br_{n+1} through $t \mapsto s_1^2$, $s_i \mapsto s_i$. We denote the image subgroup by Br_{n+1}^1. It has finite index $n+1$ in Br_{n+1}, being the inverse image in Br_{n+1} of the permutations fixing 1. This implies that Br_{n+1}^1 is good, and also that the closure of Br_{n+1}^1 in $\widehat{Br_{n+1}}$ can be identified with $\widehat{Br_{n+1}^1}$. From the formulas defining the $\widehat{GT}$ action on Br_{n+1} it is then clear that $\widehat{Br_{n+1}^1}$ is stabilized. This action of $\widehat{GT}$ is compatible with the morphism $G(\mathbb{Q}) \hookrightarrow \widehat{GT}$ and with the action of $G(\mathbb{Q})$ on $\widehat{\pi}_1(X/W)$, for W the Coxeter group of type B_n, with respect to a natural tangential base-point (see [20]). Related results on $Art(B_n) \simeq Br_{n+1}^1$ can be found in [18].

When $W = G(de,e,n)$ with $d > 1$, the space X depends only on de and n, while X/W depends only on e and n. The corresponding braid group B is then the kernel of $Art(B_n) \twoheadrightarrow \mathbb{Z}/e$ defined by $t \mapsto 1$, $s_i \mapsto 0$, and as before $\widehat{B}$ can be identified with the kernel of the induced morphism $\widehat{Art(B_n)} \twoheadrightarrow \mathbb{Z}/e$. Clearly the action of $\widehat{GT}$ on $\widehat{Art(B_n)}$ stabilizes this kernel, hence an action of $\widehat{GT}$ on $\widehat{B}$.

3.4. Artin group of type D_n and the groups $G(e,e,n)$

3.4.1. *On the Artin group of type D_n* The Artin group of type D_n is generated by elements $s_1, s_1', s_2, \dots, s_{n-1}$ with both $s_1, s_2, \dots, s_{n-1}$ and $s_1', s_2, \dots, s_{n-1}$ satisfying the relations of Br_n, plus $s_1s_1' = s_1's_1$. When $2 \leq r \leq n$ we identify $Art(D_r)$ with the subgroup of $Art(D_n)$ generated by $s_1, s_1', s_2, \dots, s_{r-1}$.

We let $w_r = s_1s_1's_2 \dots s_r$ when $r \geq 2$. The Garside element $\Delta_r \in Art(D_{r+1})$ is w_r^{r-1}. It is central in $Art(D_{r+1})$ when r is even, and in all cases Δ_r^2 is central in $Art(D_r)$. When r is odd, conjugation by Δ_r in $Art(D_{r+1})$ induces the 'diagram' automorphism ϕ which exchanges s_1 with s'_1 and fixes the s_i for $i > 1$. We let $\eta_r = s_{r-1} \dots s_2 s_1 s'_1 s_2 \dots s_{r-1}$.

The proof of the following technical lemma is easy by induction, and left to the reader. The items (1) and (2) only make easier the proof of (3), which is the item we need for the sequel.

Lemma 1. (1) $\forall r \geq 3 \;\; w_{r+1} s_{r-1} = s_r w_{r+1}$
(2) $\forall r > i > 2 \;\; w_{r+1} s_{r-1} \dots s_i = s_r \dots s_{i+1} w_{r+1}$
(3) $\forall r \geq 2 \;\; \Delta_r \eta_{r+1} = \Delta_{r+1}$.

The following lemma is then an easy consequence of item (3) above and of the properties of Δ_r recalled earlier.

Lemma 2. (1) *If $x \in Art(D_r)$ centralizes s_1, s'_1, then so does (η_r^m, x) for all $m \in \mathbb{Z}$, whenever $r \geq 3$.*
(2) *If $f \in (\widehat{F}_2, \widehat{F}_2)$ and $r \geq 3$, then $f(\eta_r, s_r^2)$ centralizes s_1, s'_1 in $\widehat{Art(D_r)}$.*

Proof. (1) By lemma 1 (3) we have $\eta_r = \Delta_{r-1}^{-1}\Delta_r$, hence $\eta_r y \eta_r^{-1} = \phi(y)$ for all y. Note that $\phi^2 = \mathrm{Id}$ and thus $\phi(y) = \eta_r^{-1} y \eta_r$.

It follows that, for $y \in \{s_1, s'_1\}$ and $m \in \mathbb{Z}$,

$$\begin{array}{rclcl} (\eta_r^m, x) y & = & \eta_r^m x \eta_r^{-m} x^{-1} y & = & \eta_r^m x \eta_r^{-m} y x^{-1} \\ & = & \eta_r^m x \eta_r^{-m} y \eta_r^m \eta_r^{-m} x^{-1} & = & \eta_r^m x \phi^m(y) \eta_r^{-m} x^{-1} \\ & = & \eta_r^m \phi^m(y) x \eta_r^{-m} x^{-1} & = & \eta_r^m \phi^m(y) \eta_r^{-m} (\eta_r^m, x) \\ & = & \phi^{2m}(y)(\eta_r^m, x) & = & y(\eta_r^m, x) \end{array}$$

(2) We denote C the centralizer in $Art(D_r)$ of $< s_1, s'_1 >$. We let H_k denote the subgroup of (F_2, F_2) generated by the commutators of length at most k in powers of s_r^2 and η_r, and $\varphi : (F_2, F_2) \to Art(D_r)$ defined by $f \mapsto f(\eta_r, s_r^2)$. By induction on $k \geq 2$, we show that $\varphi(H_k) \subset C$. The case $k = 2$ is a consequence of (1), as $(\eta_r^u, s_r^v) \in C$ for all $u, v \in \mathbb{Z}$ by (1). Let now c be (the image under φ of) a commutator of length $k+1$. Either $c = (u, v)$ with $u, v \in H_k \subset C$, and then $c \in C$, or $c = (s_r^{2m}, v)$ for some $m \in \mathbb{Z} \setminus \{0\}$ with $v \in H_k \subset C$ and then $c \in C$ because $s_r^2 \in C$, or $c = (\eta_r^m, v)$ for some $m \in \mathbb{Z} \setminus \{0\}$ with $v \in H_k \subset C$, and then $c \in C$ by (1). This proves that $\varphi((F_2, F_2)) \subset C$, that is $\varphi(f) s_1 = s_1 \varphi(f)$, $\varphi(f) s'_1 = s'_1 \varphi(f)$ for all $f \in (F_2, F_2)$, hence $f(\eta_r, s_r^2)$ centralizes s_1, s'_1 in $\widehat{Art(D_r)}$ for all f in the closure of (F_2, F_2) in $\widehat{F}_2$. Q.E.D.

3.4.2. *Lifting the Galois action for D_n to $\widehat{GT}$.* In [20], Matsumoto showed that, for $W = D_n = G(2, 2, n)$, the action of $G(\mathbb{Q})$ on X with respect to a natural tangential basepoint can be expressed with the formulas $s_1 \mapsto s_1^\lambda, s'_1 \mapsto (s'_1)^\lambda$, $s_i \mapsto f(s_i^2, \eta_i) s_i^\lambda f(\eta_i, s_i^2)$ for $i \geq 2$, with (λ, f)

the image in $\widehat{GT}$ of the element of $G(\mathbb{Q})$. The question is thus to check whether the morphism F from the free group on $s_1, s_1', s_2, \dots, s_{n-1}$ to $\widehat{Art(D_n)}$ defined by these equations, for an arbitrary $(\lambda, f) \in \widehat{GT}$, factors through the natural morphisms $F \to Art(D_n)$ and $Art(D_n) \to \widehat{Art(D_n)}$. Using the previous results we find that the *commutation relations* are satisfied.

Proposition 3. *$F(s_i)F(s_j) = F(s_j)F(s_i)$ when $|j-i| \geq 2$ and $F(s_1')F(s_j) = F(s_j)F(s_1')$ when $j \geq 3$*

Proof. When $i, j \geq 2$ for the first equation it follows from the fact that η_i commutes with s_j in these cases. We can thus assume $i = 1$ for the first equation. It is then a consequence of the fact that $f \in (\widehat{F_2}, \widehat{F_2})$ by lemma 2 (2). Q.E.D.

The braid relation $F(s_1)F(s_2)F(s_1) = F(s_2)F(s_1)F(s_2)$ does not, at least readily, follow from the defining equation of $\widehat{GT}$. Indeed, the restriction to $< s_1, s_2 > \simeq Art(A_2)$ from $Art(D_n)$ of the action of $G(\mathbb{Q})$ is $s_1 \mapsto s_1^{\lambda}$, $s_2 \mapsto f(s_2^2, s_1 s_1') s_2^{\lambda} f(s_1 s_1', s_2^2)$, whereas the natural (Drinfeld) action of $\widehat{GT}$ on Br_3 is $s_1 \mapsto s_1^{\lambda}$, $s_2 \mapsto f(s_2^2, s_1^2) s_2^{\lambda} f(s_1^2, s_2^2)$. However, identifying $Art(D_3) =< s_1, s_2, s_1' >$ with $Art(A_3) = Br_4$ we notice that the two actions almost coincide when (λ, f) belong to the subgroup $\mathbb{I}\Gamma' \subset \widehat{GT}$ defined in [22]. Indeed, recall from [22] lemma 4.2 that, when $(\lambda, f) \in \mathbb{I}\Gamma'$, its natural action on the image of $\widehat{Br_4}$ modulo center [1] inside the profinite mapping class group $\widehat{\Gamma}_0^{[5]}$ composed with $\mathrm{Ad} f(x_{45}, x_{34})$ satisfies $s_2 \mapsto f(s_2^2, s_1 s_3) s_2^{\lambda} f(s_1 s_3, s_2^2)$. Here $x_{34} = s_3^2$ and $x_{45} = (s_1 s_2)^3 = s_2 s_1^2 s_2 . s_1^2$. By the formulas (4.1) of [22] we also have $s_1 \mapsto s_1^{\lambda}$. Under the isomorphism $Art(D_3) \simeq Art(A_3) \simeq Br_4$ this means $F(s_1)F(s_2)F(s_1) = F(s_2)F(s_1)F(s_2)\omega_4^{\mu}$ for some $\mu \in \widehat{\mathbb{Z}}$. Taking the image under $\widehat{Br_4} \twoheadrightarrow \widehat{\mathbb{Z}}$ one gets $3\lambda = 3\lambda + 12\mu$ in $\widehat{\mathbb{Z}}$ hence $\mu = 0$. In particular, the formulas of Matsumoto for the Galois action on $\widehat{Art(D_3)}$ actually define an action of $\mathbb{I}\Gamma'$.

Since the condition denoted (III)' in [22] is the starting point for extending the action of $\widehat{GT}$ to mapping class groups of higher genus, extending the action of $G(\mathbb{Q})$ on $Art(D_n)$ to $\widehat{GT}$ seems related to the open question '$\widehat{GT} = \mathbb{I}\Gamma = \mathbb{I}\Gamma'$?' of [22, 12]. Explanations for this

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[1] This embedding is defined in §3 of [22] p. 518. The ambiguity here is only apparent, as the short exact sequence $1 \to Z(Br_4) \to Br_4 \to Br_4/Z(Br_4) \to 1$ survives the profinite completion : $Br_n/Z(Br_n)$ is good, because of the short exact sequences $1 \to P_n/Z(P_n) \to Br_n/Z(Br_n) \to \mathfrak{S}_n \to 1$ and $1 \to F_{n-1} \to P_n/Z(P_n) \to P_{n-1}/Z(P_{n-1}) \to 1$

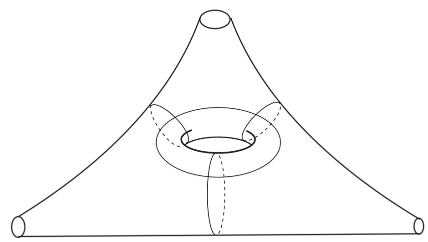

Fig. 1. D_4 and $\Gamma_{1,3}$

connection can be found in the relationship between the mapping class group $\Gamma_{1,3}$ and $Art(D_4)$ (using the Dehn twists pictured in figure 1) as well as in the known embeddings of $Art(D_n)$ into mapping class groups of higher genus.

Other connections between mapping class groups and Artin groups of type ADE can be found in [20] and [21]. Since nice presentations for the braid groups of type $G(e,e,n)$ are now at disposal, which generalize nicely the case of type D_n, a natural question is the following.

Subquestion 4.1. *What is the analogous picture for the more general series of groups $G(e,e,n)$? Is there a related connection with mapping class groups ?*

We hope to come back to this question in a forthcoming work. A first step towards answering it would be to tackle seriously the case of the groups $W = G(e,e,2)$. In this case B is generated by 2 generators s,t with $stst\cdots = tsts\dots$ (m factors on both sides), and the pure braid group P has for derived subgroup a free group on m generators. A group capturing the Galois action will typically map $s \mapsto s^{\lambda}$, $t \mapsto g^{-1}t^{\lambda}g$ for some $g \in \widehat{P}'$. One can then define, following Drinfeld recipe for $\widehat{GT}$, a group $\widehat{GD_m}$ made of elements (λ, g) in $\widehat{\mathbb{Z}}^{\times} \times \widehat{P}'$ such that there exists a morphism $\tilde{g} \in \mathrm{Hom}(B, \widehat{B})$ satisfying

(1) $\tilde{g}$ induces an automorphism of $\widehat{B}$
(2) $\tilde{g}$ maps $s \mapsto s^{\lambda}$, $t \mapsto g^{-1}t^{\lambda}g$
(3) $\tilde{g}$ maps $\Delta \mapsto \Delta^{\lambda}g$ if m odd, $\Delta \mapsto \Delta^{\lambda}$ is m even
(4) $\tilde{g}$ maps $\Delta^2 \mapsto \Delta^{2\lambda}$.

where $\Delta = stst\cdots = tsts\dots$ (m factors). In this group, composition is given by $(\lambda_1, g_1) \star (\lambda_2, g_2) = (\lambda_1\lambda_2, g_1\tilde{g}_1(g_2))$, and the conditions (2) – (4) can be translated in algebraic conditions on the couple (λ, g), as in the analogous unipotent setting described in [15]. The task is then to combine this setup with the one of the usual braid group, in order to get a description for the types $G(e,e,n)$, in such a way that it fits with the special case of the $G(2,2,n)$. Note that a connection between these braid groups and mapping class groups has not been found yet.

3.4.3. *The exceptional and Shephard groups.* Of course, the remaining exceptional groups deserve equal interest, but they are tricky to solve, as in essence they reflect very special geometric situations. The case of Artin groups of type E_n, $n \in \{6,7,8\}$ were described in [20] : a description of the Galois action is given, as well as a connection with mapping class groups of higher genus. It is shown in [20] that the Galois action is compatible, in case of W of type E_7, with the embedding of B inside the mapping class group in genus 3. Even in that case, the question of whether this action factorizes through $\widehat{GT}$ seems to be open – which is not surprising at all, as the Artin group of type E_7 admits a parabolic subgroup of type D_4.

However, there are some exceptional groups for which one *knows* that there exists an action of $\widehat{GT}$. These are groups for which the discriminantal variety X/W is isomorphic to the one for W a symmetric group, or even W a Coxeter group of type B_n.

These groups are the groups of symmetries of regular complex polytopes, and are known as Shephard groups. They cover all the cases where X/W also arises from a Coxeter group.

More precisely, we have $W_1, W_2 < \mathrm{GL}(V)$ two different reflection groups, one of the two (say W_1) being a Coxeter group of type A_n or B_n, such that $X_1/W_1 \simeq X_2/W_2$ (no such phenomenon appear for other Coxeter groups). When W_1 is a symmetric group, of course the profinite completion of $B_2 \simeq B_1$ inherits an action of $\widehat{GT}$. This happens exactly for W_2 of type $G_4, G_{25}, G_{32}, G_8, G_{16}$; these groups are the finite quotients of the usual braid group Br_n by the relations $s_i^k = 1$ for some k, as shown by Coxeter in [7]. (The case where W_1 is a Coxeter group of type B_n corresponds to W_2 being of type $G_5, G_{10}, G_{18}, G_{26}$). We refer again to [2] for tables on these exceptional reflection groups and their braid groups, and to [24] §6.6 for the isomorphism $X_1/W_1 \simeq X_2/W_2$ in case of Shephard groups (see in particular cor. 6.126).

Problem 2. *Give a geometric interpretation of the $\widehat{GT}$-action on X/W, for $W = G_4, G_{25}, G_{32}, G_8, G_{16}$. Same question for $G_5, G_{10}, G_{18}, G_{26}$.*

Example. Let us consider the case of $W = G_4 < \mathrm{GL}_2(\mathbb{C})$ generated by $\overline{s_1} = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}$ $\overline{s_2} = \frac{1}{\sqrt{-3}} \begin{pmatrix} -1 & j \\ 2 & j \end{pmatrix}$ with $j = \exp\frac{2i\pi}{3}$ and $\sqrt{-3} = j - j^2$. Its braid group is generated by lifts s_1, s_2 with relations $s_1 s_2 s_1 = s_2 s_1 s_2$, that is $B = Br_3$. We detail in this example a natural Belyi covering map for X/W. The complex conjugation acts by $s_1 \mapsto s_1^{-1}, s_2 \mapsto s_1 s_2^{-1} s_1^{-1}$, and one finds by explicit computation that $\mathbb{C}[z_1, z_2]^W = \mathbb{C}[g_1, g_2]$ with $g_1 = z_1^4 - z_1 z_2^3$, $g_2 = z_1^6 + (5/2) z_1^3 z_2^3 - (1/8) z_2^6$, and an equation for $Y = X(G_4)/G_4$ inside $\mathrm{Spec}\mathbb{C}[g_1, g_2]$ is $g_1^3 \neq g_2^2$. This has to be compared with the situation with $\mathfrak{S}_3 < \mathrm{GL}_2(\mathbb{C})$, generated by $\hat{s}_1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $\hat{s}_2 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ with $\mathbb{C}[z_1, z_2]^{\mathfrak{S}_3} = \mathbb{C}[f_1, f_2]$, $f_1 = z_1^2 - z_1 z_2 + z_2^2$, $f_2 = z_1^3 - (3/2) z_1^2 z_2 - (3/2) z_1 z_2^2 + z_2^3$, $Y = X(\mathfrak{S}_3)/\mathfrak{S}_3$ having equation $f_1^3 \neq f_2^2$ in $\mathrm{Spec}\mathbb{C}[f_1, f_2]$; for the $\mathfrak{S}_3$-case the space $X = X(\mathfrak{S}_3)$ is isomorphic to $C \times P$ under $(z_1, z_2) \mapsto (z_1, z_2/z_1)$ with $C = \mathbb{A}^1 \setminus \{0\}$ and $P = \mathbb{A}^1 \setminus \{0, \infty\} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The étale covering $X(G_4) \to Y$ can then be pulled back to an étale covering map $E = X(\mathfrak{S}_3) \times_Y X(G_4) \to X(\mathfrak{S}_3)$ and then restricted to an étale covering map $E_0 \to P = \{1\} \times P$ with fiber G_4.

(For the algebraically inclined reader, we notice that all computations can be done explicitely at the level of coordinate rings, as all varieties used are affine. For instance, $E = \mathrm{Spec} S^{-1} A$ for $A = \mathbb{C}[u, v] \otimes \mathbb{C}[z_1, z_2]/g_i(u, v) = f_i(z_1, z_2)$ and $S = (g_1^3 - g_2^2)$, and $S_0 = \mathrm{Spec} A_0$ with $A_0 = (S^{-1}A) \otimes_{\mathbb{C}[z_1]} \mathbb{C}$ under $z_1 \mapsto 1$. Given the explicit values of g_1, g_2, f_1, f_2 chosen above, this means $A_0 = \mathbb{C}[z^{-1}, (z-1)^{-1}, z, u, v]/I$, where I is generated by $u^4 - uv^3 - (1 - z + z^2)$ and $u^6 + (5/2) u^3 v^3 - (1/8) v^6 - (1 - (3/2) z - (3/2) z^2 + z^3)$.)

The situation can be summarized as the following sequence of morphisms between étale G_4-coverings

$$\begin{tikzcd}
E_0 \arrow[r, hook] \arrow[d, "G_4"'] & E \arrow[r, equal] \arrow[d, "G_4"'] & E \arrow[r, "\mathfrak{S}_3"] \arrow[d, "G_4"'] \arrow[dr, phantom, "\square"] & X(G_4) \arrow[d, "G_4"'] \\
\{1\} \times P \arrow[r, hook] & C \times P \arrow[r, equal] & X(\mathfrak{S}_3) \arrow[r, "\mathfrak{S}_3"'] & Y
\end{tikzcd}$$

In particular, the action of $\pi_1(P)$ on the fiber of $E_0 \to P$ can be computed on the fiber of $X(G_4) \to Y$. For this, and since the morphism $Br_3 \to G_4$ is known, one only needs to compute the image of $\pi_1(P) \to \pi_1(X/W) = Br_3$. We take $\frac{1}{2}$ for base-point in P, and denote x, y, z the usual generators of $\pi_1(P, \frac{1}{2})$ (see figure 2). Figure 3 shows the image of x and y in $\pi_1(X(\mathfrak{S}_3), (1, \frac{1}{2}))$ as well as homotopies (in green rays) with the

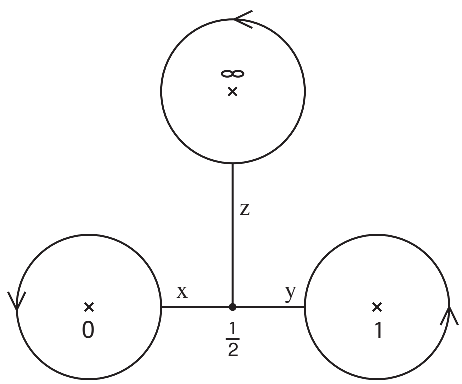

Fig. 2. Generators of the π_1 of $\mathbb{C} \setminus \{0,1\} = \mathbb{P}^1 \setminus \{0, 1\infty\}$

standard loops around the walls of the Weyl chamber containing $(1, \frac{1}{2})$. The two points singled out in the pictures are the basepoint and its image under the relevant reflection. We denoted $\hat{s}_3 = \hat{s_2}\hat{s}_1\hat{s_2} = \hat{s_1}\hat{s}_2\hat{s_1}$. The left-hand side of figure 3 lives in $X(\mathfrak{S}_3) \cap \mathbb{R}^2 \oplus \mathrm{i}\mathbb{R}(0,1)$, while the right-hand side lives in $X(\mathfrak{S}_3) \cap \mathbb{R}^2 \oplus \mathrm{i}\mathbb{R}(1,-1)$. It follows that, up to Br_3-conjugation, x and y are mapped to the squares of the standard generators of Br_3. In particular, the usual generators $x, y, z = (xy)^{-1}$ act (again, up to global conjugacy) by $\bar{s}_1^2, \bar{s}_2^2, (\bar{s}_1^2\bar{s}_2^2)^{-1}$.

Extending this étale covering of P to a ramified covering map $\overline{E_0} \to \mathbb{P}^1$ we get the ramification at $0, 1, \infty$ by considering the orders of these elements : $\bar{s}_1^2$ and $\bar{s}_2^2$ have order 3, whereas $(\bar{s}_1^2\bar{s}_2^2)^{-1}$ has order 6. This means that the fiber over 0 and 1 has 8 elements with ramification index 3, whereas the fiber over infinity has 4 elements with ramification index 6. Riemann-Hurwitz formula shows that $\overline{E_0}$ has genus 4.

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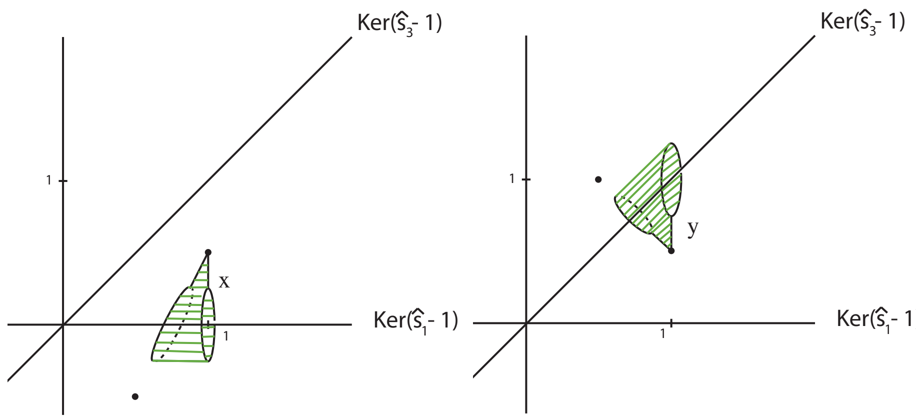

Fig. 3. Images of x, y in $X(\mathfrak{S}_3)$ and homotopies with the standard loops.

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