INFINITESIMAL HECKE ALGEBRAS III

IVAN MARIN

Institut de Mathématiques de Jussieu
Université Paris 7
175 rue du Chevaleret
F-75013 Paris

Abstract. We define Lie subalgebras of the group algebra of a finite *pseudo-reflection* group that are involved in the definition of the Cherednik KZ-systems, and determine their structure. We provide applications for computing the Zariski closure of the image of generalized (pure) braid group B inside the representations of the corresponding Hecke algebras. We also get unitarizability results for the representations of B originating from Hecke algebras for suitable parameters, and relate our Lie algebras with the topological closure of B in these compact cases.

1. INTRODUCTION

Let V be a finite-dimensional complex vector space, and $W < \mathrm{GL}(V)$ a finite subgroup generated by pseudo-reflections, namely invertible endormorphisms of V which fix some hyperplane. Such a group is called a (pseudo-)reflection group. We denote by $\mathcal{R}$ the set of (pseudo-)reflections of W.

Among them are the 2-reflection groups, namely when $\forall s \in \mathcal{R} \ s^2 = 1$. Examples of 2-reflection groups include the finite Coxeter groups. For a 2-reflection group, we introduced in [Ma03] the Lie subalgebra of the group algebra $\mathbb{C}W$ generated by $\mathcal{R}$, and call it the infinitesimal Hecke algebra. This is a reductive Lie algebra that appears naturally in the monodromy constructions of braid groups representations, also known as KZ systems. This Lie algebra was decomposed in [Ma07a] for the case of the symmetric group, and in [Ma09a] for the general case of 2-reflection groups. We proved in [Ma09a] that it can be identified with the Lie algebra of an interesting algebraic group, namely the Zariski closure of the image of the (generalized) braid group inside the generic Hecke algebra associated to W. It also admits a compact form that is relevant to the topological closure of the same group, under some conditions on the parameters involved.

The present paper is a continuation of [Ma09a]. Here we consider the general case of (pseudo-)reflection groups.

There are several natural generalizations of the infinitesimal Hecke algebras. The first one is the Lie subalgebra $\mathcal{H}$ of $\mathbb{C}W$ generated by the set

$\mathcal{R}$ of all reflections. A second one is the Lie subalgebra $\mathcal{H}_s$ generated by a subset $\mathcal{S} \subset \mathcal{R}$ which admit the following properties : it is invariant under W-conjugation, and $s \mapsto \mathrm{Ker}\,(s-1)$ induces a bijection from $\mathcal{S}$ to the reflecting hyperplanes.

For a conjugacy class $c \subset \mathcal{S}$, we let e_c denote the order of the elements of c, and subdivide $\mathcal{S} = \mathcal{S}_+ \sqcup \mathcal{S}_0$ with $e_c = 2$ for $s \in \mathcal{S}_0$ and $e_c > 2$ for $s \in \mathcal{S}_+$. We let $\mathcal{C}_+$ denote the set of conjugacy classes in $\mathcal{S}_+$, and $\mathbb{A}(W) = \prod_{c \in \mathcal{C}_+} \Bbbk^{e_C - 1}$ for a fixed field $\Bbbk$ of characteristic 0. A typical element of $\mathbb{A}(W)$ is denoted $\underline{\boldsymbol{\lambda}} = (\underline{\lambda}^c)_{c \in \mathcal{C}_+}$, with $\underline{\lambda}^c = (\lambda_1^c, \ldots, \lambda_{e_c-1}^c)$. We thus get a family of generalizations, that contains $\mathcal{H}_s$, by letting $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ denote the Lie subalgebra of $\Bbbk W$ generated by $\mathcal{S}_0$ and the $\lambda_1^c s + \cdots + \lambda_{e_c-1}^c s^{e_c-1}$, for $s \in c \subset \mathcal{S}_+$.

From now on we assume that W is not a 2-reflection group, this case having been dealt with in [Ma09a], and also that it is irreducible.

We introduce subsets $\overline{\mathrm{QRef}}, \mathcal{E}, \mathcal{F}$ of the set $\mathrm{Irr}(W)$ of irreducible representations of W, as well as an equivalence relation $\approx$ in $\mathrm{Irr}(W)$. The central result is a structure theorem for $\mathcal{H}$, where we denote V_ρ the underlying vector space of $\rho \in \mathrm{Irr}(W)$.

Theorem 1. $\mathcal{H}$ *is a reductive Lie algebra whose center has dimension the cardinality $|\mathcal{R}/W|$ of conjugacy classes of reflections. Its semisimple part is*

$$\mathcal{H}' = \left(\bigoplus_{\rho \in \overline{\mathrm{QRef}}} \mathfrak{sl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{E}/\approx} \mathfrak{sl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{F}/\approx} \mathfrak{osp}(V_\rho) \right)$$

We then prove that there exists a dense open subset of $\mathbb{A}(W)$ over which $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ has the same semisimple part as $\mathcal{H}$.

Theorem 2. *The Lie algebras $\mathcal{H}$ and $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ are reductive for all values of $\underline{\boldsymbol{\lambda}}$. There exists an hyperplane complement $\mathbb{A}(W)^\times \subset \mathbb{A}(W)$ such that, for all $\underline{\boldsymbol{\lambda}} \in \mathbb{A}(W)^\times$, $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ has a center of dimension $|\mathcal{S}/W|$, and semisimple part $\mathcal{H}(\underline{\boldsymbol{\lambda}})' = \mathcal{H}'$*

We then establish the connection between $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ and the Zariski closure of the (pure) braid groups inside the Hecke algebra associated to W, and finally investigate several interesting cases for $\underline{\boldsymbol{\lambda}} \notin \mathbb{A}(W)^\times$:

(1) for the Lie algebra $\mathcal{H}_s$, whose parameter $\underline{\boldsymbol{\lambda}}$ does not always belong to $\mathbb{A}(W)^\times$;
(2) for the exceptional groups G_4, G_{25}, which are connected to the usual braid groups, we investigate the structure of $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ for arbitrary $\underline{\boldsymbol{\lambda}} \in \mathbb{A}(W)$;
(3) when $\underline{\boldsymbol{\lambda}}$ satisfies $\lambda_i^c = \lambda_j^c$, which corresponds to the 'spetsial' Hecke algebra of [BMM], we investigate $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ for the groups $G(d, 1, r)$.

It has been noted in [Ma09a] that the infinitesimal Hecke algebras for 2-reflection groups are actually generated by any set of reflections that generate the groups, e.g. a set of simple reflections for W a finite Coxeter groups. Other generalizations to consider are then, for $\mathcal{U}$ a subset of $\mathcal{S}$ that generates the group and $\tilde{\mathcal{U}} = \{s^k \mid s^k \neq 1, s \in \mathcal{U}\} \subset \mathcal{R}$, the Lie algebras $\mathcal{H}(\mathcal{U})$ and $\mathcal{H}(\tilde{\mathcal{U}})$ generated by these subsets. Both Lie algebras are reductive.

We have $\mathcal{H}(\mathcal{U}) \subset \mathcal{H}_s$, and in general $\mathcal{H}'_s \neq \mathcal{H}'$, so $\mathcal{H}(\tilde{\mathcal{U}})$ is more likely to be considered as a natural generalization than $\mathcal{H}(\mathcal{U})$. Our last result on this topic is the following

Theorem 3. *If $\mathcal{U} \subset \mathcal{S}$ generates W, then $\mathcal{H}(\mathcal{U})$ and $\mathcal{H}(\tilde{\mathcal{U}})$ are reductive, and $\mathcal{H}(\tilde{\mathcal{U}})' = \mathcal{H}'$.*

In order to get this result, we study the following situation. For a finite group G, there are natural Lie-theoretic endomorphisms of $\text{End}(\mathbb{C}G)$ associated to $g \in G$, namely $\text{ad}(g) : x \mapsto gx - xg$ and $\text{Ad}(g) : x \mapsto gxg^{-1}$. The technical result proved in the final section is the following.

Theorem 4. *Let $k \geq 1$. There exists a rational polynomial $P \in \mathbb{Q}[X_1, \ldots, X_{k-1}]$ such that, for every finite group G and $g \in G$ of order k, then $\text{Ad}(g) = P(\text{ad}(g), \text{ad}(g^2), \ldots, \text{ad}(g^{k-1}))$. Moreover P can be taken in $\mathbb{Q}[X_1]$ if k is odd or equal to 2.*

The applications of these structure results on the Zariski closures of braid groups in their corresponding Hecke algebras, are discussed in §5 (see theorem 5.2). In §6 we prove that, for convenient values of the parameters and, if W is irreducible, with the possible exception of a finite number of types, the representation of the Hecke algebra are unitarizable as representations of the corresponding braid group B (theorem 6.1 and its corollary), and we relate the Lie algebra of the topological closure of B in these compact cases with the Lie algebras introduced here.

2. Infinitesimal Hecke algebras

Let W be a (pseudo-)reflection group, $\mathcal{R}$ its set of (pseudo-)reflections and $\mathcal{C}$ the set of conjugacy classes of reflection hyperplanes. We consider $\mathcal{S} \subset \mathcal{R}$ such that $\mathcal{S}$ is stabilized by conjugation under W and generates W as a group. We will show below that a consequence of this assumption and of Stanley's theorem (see [St], theorem 3.1) is that

$$\forall s \in \mathcal{R} \; \exists s_0 \in \mathcal{S} \; \exists r \geq 1 \; s = s_0^r.$$

Let $\Bbbk$ be a field of characteristic 0. We introduce the Lie subalgebra $\mathcal{H}(\mathcal{S})$ of $\Bbbk W$ generated by $\mathcal{S}$ and, for c a conjugacy class of W, we let $T_c = \sum_{w \in c} w \in \Bbbk W$.

We recall from [Ma09a] the following.

Proposition 2.1. *(see [Ma09a] prop. 2.2) The Lie algebra $\mathcal{H}(\mathcal{S})$ is reductive, and a basis of its center is given by the T_c for $c \subset \mathcal{S}$. Every irreducible representation of W induces an irreducible representation of $\mathcal{H}(\mathcal{S})$. The derived Lie algebra $\mathcal{H}(\mathcal{S})'$ is generated by the $s' = s - (T_c)/(\#c)$ for $s \in c$ a conjugacy class included in $\mathcal{S}$.*

We say that a reflection is *primitive* if it generates the fixer of its reflecting hyperplane.

We assume that $\Bbbk$ is a field such that every ordinary representation of W is realizable over $\Bbbk$ (this means that $\Bbbk$ contains the so-called field of definition

of W, by [Bd, Bes]). Without loss of generality, we can assume that $\Bbbk$ is a number field, with $\Bbbk \subset \mathbb{C}$. Since $\mathcal{H}(\mathcal{S}) \subset \Bbbk W$, every such representation ρ of W induces a representation $\rho_{\mathcal{H}(\mathcal{S})}$ of $\mathcal{H}(\mathcal{S})$. Similarly, its restriction to $\mathcal{H}(\mathcal{S})'$ is denoted $\rho_{\mathcal{H}(\mathcal{S})'}$.

We let $\mathrm{Irr}(\rho)$ denote the set of irreducible representations of W, and define for $\rho \in \mathrm{Irr}(W)$ the set

$$\mathrm{X}_{\mathcal{S}}(\rho) = \{\eta \in \mathrm{Hom}(W, \Bbbk^{\times}) \mid \forall s \in \mathcal{S} \ \ \eta(s) \neq 1 \Rightarrow \rho(s) \in \Bbbk^{\times}\}.$$

Proposition 2.2.   (1) *For any $s \in \mathcal{R}$ there exists $s_0 \in \mathcal{S}$ and $r \geq 1$ such that $s = s_0^r$.*
(2) *For any $\rho \in \mathrm{Irr}(W)$, $\mathrm{X}(\rho) = \mathrm{X}_{\mathcal{S}}(\rho)$ does not depend on $\mathcal{S}$.*

Proof. For C of conjugacy class of hyperplanes in W we denote e_C the order of the fixer of an hyperplane in C. By Stanley's theorem there exists an isomorphism $\mathrm{Hom}(W, \mathbb{C}^{\times}) \simeq \prod_C \mathbb{Z}/e_C\mathbb{Z}$ where a primitive pseudo-reflection around H in C is mapped to a generator of $\mathbb{Z}/e_C\mathbb{Z}$. It follows that every pseudo-reflection $s \in \mathcal{R}$ around such an H is mapped to some $x \in \mathbb{Z}/e_C\mathbb{Z}$. Since $\mathcal{S}$ generates W, there exists $s_0 \in \mathcal{S}$ and $r \geq 1$ with s_0^r having the same image as s in $\mathbb{Z}/e_C\mathbb{Z}$. Since $\mathcal{S}$ is stable by conjugation we can assume that s_0 fixes the same hyperplane as s hence $s_0^r = s$. This proves (1). We let $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{R}$, $\rho \in \mathrm{Irr}(W)$ and $\eta \in \mathrm{X}_{\mathcal{S}_1}(\rho)$. Let $s \in \mathcal{S}_2$ such that $\rho(s) \notin \Bbbk^{\times}$. By (1) we have $r \geq 1$ and $s_0 \in \mathcal{S}_1$ such that $s = s_0^r$, so $\rho(s) \notin \Bbbk^{\times}$ implies $\rho(s_0) \notin \Bbbk^{\times}$, hence $\eta(s_0) = 1$ and $\eta(s) = 1$. It follows that $\mathrm{X}_{\mathcal{S}_1}(\rho) \subset \mathrm{X}_{\mathcal{S}_2}(\rho)$ and $\mathrm{X}_{\mathcal{S}_1}(\rho) = \mathrm{X}_{\mathcal{S}_2}(\rho)$ whence (2). $\qquad\square$

For $\mathcal{U} \subset \mathcal{R}$ not necessarily stable under conjugation, we may consider the Lie subalgebra $\mathcal{H}(\mathcal{U})$ of $\Bbbk W$ generated by $\mathcal{U}$. Letting $\tilde{\mathcal{U}} = \{s^k \in \mathcal{R} \mid s \in \mathcal{U}, k \geq 1\}$, we have the following, which shows that the stability under W plays a role mainly for the center.

Proposition 2.3. *If $\mathcal{U}$ generates W, then $\mathcal{H}(\mathcal{U})$ and $\mathcal{H}(\tilde{\mathcal{U}})$ are reductive, and $\mathcal{H}(\tilde{\mathcal{U}})' = \mathcal{H}'$.*

Proof. We decompose $\Bbbk W = Z(\Bbbk W) \oplus (\Bbbk W)'$ as a Lie algebra. Then $Z(\Bbbk W)$ and $(\Bbbk W)'$ are ideals of $\Bbbk W$ as an associative algebra, and there exists a W-equivariant idempotent $p = \Bbbk W \twoheadrightarrow Z(\Bbbk W)$. Since $\mathcal{U}$ and $\tilde{\mathcal{U}}$ generate $\mathcal{H}(\mathcal{U})$ and $\mathcal{H}(\tilde{\mathcal{U}})$, respectively, both Lie algebras are reductive (see [Ma09a] prop. 2.2). Now $\mathcal{R} = \{wsw^{-1} \mid s \in \tilde{\mathcal{U}}, w \in W\}$ by Stanley theorem, hence $\mathcal{H}$ is generated by the $\mathcal{H}(w\tilde{\mathcal{U}}w^{-1})$ for $w \in W$. Since $\tilde{\mathcal{U}}$ generates W, $Z(\mathcal{H}(\tilde{\mathcal{U}})) \subset Z(\Bbbk W)$, hence $p(Z(\mathcal{H}(\tilde{\mathcal{U}}))) = \{0\}$. From the reductiveness of $\mathcal{H}(\tilde{\mathcal{U}})$ we get $\mathcal{H}(\tilde{\mathcal{U}}) = Z(\mathcal{H}(\tilde{\mathcal{U}})) \oplus \mathcal{H}(\tilde{\mathcal{U}})'$, hence $p(\mathcal{H}(\tilde{\mathcal{U}})) = p(\mathcal{H}(\tilde{\mathcal{U}})') = \mathcal{H}(\tilde{\mathcal{U}})'$, as $\mathcal{H}(\tilde{\mathcal{U}})' \subset (\Bbbk W)'$. More generally,

$$p(\mathcal{H}(w\tilde{\mathcal{U}}w^{-1})) = \mathcal{H}(w\tilde{\mathcal{U}}w^{-1}) = w\mathcal{H}(\tilde{\mathcal{U}})'w^{-1} = \mathrm{Ad}(w)(\mathcal{H}(\tilde{\mathcal{U}})').$$

Let now $s \in \mathcal{U}$. The endomorphism $\mathrm{ad}(s^k) \in \mathrm{End}(\Bbbk W)$ stabilizes $\mathcal{H}(\tilde{\mathcal{U}})$ hence $\mathcal{H}(\tilde{\mathcal{U}})'$, as $\mathcal{H}(\tilde{\mathcal{U}}) = Z(\mathcal{H}(\tilde{\mathcal{U}})) \oplus \mathcal{H}(\tilde{\mathcal{U}})'$ and $s^k \in \tilde{\mathcal{U}}$. We will prove later (theorem 8.6) that $\mathrm{Ad}(s)$ can be written as a rational polynomial in $\mathrm{ad}(s), \ldots, \mathrm{ad}(s^k), \ldots$, hence $s\mathcal{H}(\tilde{\mathcal{U}})'s^{-1} \subset \mathcal{H}(\tilde{\mathcal{U}})'$. Since $\mathrm{Ad}(s)$ is invertible, it follows that $s\mathcal{H}(\tilde{\mathcal{U}})'s^{-1} = \mathcal{H}(\tilde{\mathcal{U}})$. Since $\mathcal{U}$ generates W we get

$w\mathcal{H}(\tilde{\mathcal{U}})'w^{-1} = \mathcal{H}(\tilde{\mathcal{U}})'$ for all $w \in W$. Then $p(\mathcal{H}(w\tilde{\mathcal{U}}w^{-1})) = \mathcal{H}(\tilde{\mathcal{U}})'$ hence $\mathcal{H}'$ is generated by $\mathcal{H}(\tilde{\mathcal{U}})'$, that is $\mathcal{H}' = \mathcal{H}(\tilde{\mathcal{U}})'$. $\square$

For $\rho \in \mathrm{Irr}(W)$ we let V_ρ denote its underlying vector space. We define the following subsets of $\mathrm{Irr}(W)$.

$\mathrm{Ref} = \{\rho \in \mathrm{Irr}(W) \mid \dim \rho \geq 2 \text{ and } \forall s \in \mathcal{S} \; \rho(s) \neq 1 \Rightarrow \rho(s) \text{ is a reflection }\}$
$\mathrm{QRef} = \{\eta \otimes \rho \mid \rho \in \mathrm{Ref}, \eta \in \mathrm{Hom}(W, \Bbbk^\times)\}$
$\Lambda\mathrm{Ref} = \{\eta \otimes \Lambda^k \rho \mid \rho \in \mathrm{Ref}, \eta \in \mathrm{Hom}(W, \Bbbk^\times), k \geq 0\}$

The statements and proofs of the propositions 2.13 and 2.15 in [Ma09a] admit a natural generalization.

Proposition 2.4.

(1) *If* $\rho \in \mathrm{Ref}(W)$ *and* $\eta \in \mathrm{Hom}(W, \Bbbk^\times)$, *then* $\Lambda^k\left((\rho \otimes \eta)_{\mathcal{H}(\mathcal{S})'}\right) = (\eta \otimes \Lambda^k\rho)_{\mathcal{H}(\mathcal{S})'}$.
(2) *If* $\rho \in \mathrm{QRef}(W)$ *then* $\rho(\mathcal{H}(\mathcal{S})') = \mathfrak{sl}(V_\rho)$.

Proof. Let $\rho \in \mathrm{Ref}(W)$, and $s \in \mathcal{S}$. If $\rho(s) = 1$ then $(\eta \otimes \Lambda^k\rho)(s) = \eta(s)$ and $\Lambda^k(\rho \otimes \eta)(s) = k\eta(s) = (k-1)\eta(s) + (\eta \otimes \Lambda^k\rho)(s)$. Otherwise, there exists a basis $e_1, \ldots, e_n$ of V_ρ such that $s.e_1 = \zeta e_1$ and $s.e_i = e_i$ if $i \neq 1$ for ζ some nontrivial root of 1. Taking for basis of $\Lambda^k V_\rho$ the basis $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $I = \{i_1, \ldots, i_k\} \subset [1..n]$ of cardinality k with $i_1 < \cdots < i_k$ we get that $(\eta \otimes \Lambda^k\rho)(s)$ maps e_I to $\eta(s)e_I$ if $1 \notin I$ and to $\eta(s)\zeta e_I$ if $1 \in I$. Similarly, $\Lambda^k((\rho\otimes\eta)_{\mathcal{H}(\mathcal{S})})$ maps e_I to $k\eta(s)e_I$ if $1 \notin I$ and to $\eta(s)(k-1+\zeta)e_I$ otherwise. It follows that $\Lambda^k((\rho \otimes \eta)_{\mathcal{H}(\mathcal{S})}(s) = \eta(s)(k-1)\mathrm{Id} + (\eta \otimes \Lambda^k\rho)(s)$ for all $s \in \mathcal{S}$, which easily implies $\Lambda^k((\rho \otimes \eta)_{\mathcal{H}(\mathcal{S})'} = (\eta \otimes \Lambda^k\rho)_{\mathcal{H}(\mathcal{S})'}$ and proves (1). It is clearly enough to prove (2) for $\rho \in \mathrm{Ref}$. Since $\rho(W)$ is an irreducible reflection group, we can also assume that ρ is faithful. We proceed by induction on the rank $n = \dim V_\rho$ of W for $n \geq 1$, the case $n = 1$ being trivial, so we assume $n \geq 2$. Let $W_0 \subset W$ be a maximal parabolic subgroup which acts irreducibly on some hyperplane H of V_ρ (see [Ma09a] lemma 2.17). The image $\mathfrak{g}$ of $\mathcal{H}(\mathcal{S})'$ in $\mathfrak{sl}(V_\rho)$ contains $\mathfrak{sl}(H)$, hence a Cartan subalgebra of rank $n-2$. Since $n \geq 2$, there exists $s \in \mathcal{S} \cap W_0$ with $\rho(s) \notin \Bbbk^\times$. Let $\zeta \neq 1$ with $\zeta \in \mathrm{Sp}(s)$. We denote c_0 the conjugacy class of s in W_0 and c its class in W, $T_0 = \sum_{g \in c_0} g$, $T = \sum_{g \in c} g$. Since T is central in $\mathcal{H}(\mathcal{S})$ and ρ is irreducible, $\rho(T)$ is a scalar determinated by its trace $(\#c)(n-1+\zeta)$. Similarly, $\rho(T_0)$ has for trace $(\#c_0)(n-1+\zeta)$, acts on H by $(\#c_0)(n-2+\zeta)/(n-1)$ and on its orthogonal supplement by $\#c_0$. Letting $X = (\#c)T_0 - (\#c_0)T$ we get that $x = \rho(X)$ has zero trace, belongs to $\mathfrak{sl}(V_\rho) \cap \rho(\mathcal{H}(\mathcal{S})) = \rho(\mathcal{H}(\mathcal{S})')$ (since $\mathcal{H}(\mathcal{S})$ is reductive), is semisimple and centralizes the Cartan of $\mathfrak{sl}(H)$. It follows that the semisimple Lie algebra $\mathfrak{g}$ contains a Cartan subalgebra of rank $n-1$. Since $\mathfrak{sl}_n$ contains no proper root system of rank $n-1$ (see [Ma09a], lemma 2.16) it follows that $\mathfrak{g} = \mathfrak{sl}(V_\rho)$, and this proves (2) by induction. $\square$

We now focus on two special cases, fixing $\mathcal{S}$ to be minimal, for instance by letting $\mathcal{S}$ being the set of distinguished pseudo-reflections. We denote $\mathcal{H}_s = \mathcal{H}(\mathcal{S})$, the letter 's' being understood as the 'special' infinitesimal Hecke algebra. The second one is for $\mathcal{S}$ maximal; we let $\mathcal{H} = \mathcal{H}(\mathcal{R})$ and call

it the *ambient* infinitesimal Hecke algebra. We denote $\mu_\infty(\mathbb{C})$ the group of roots of 1.

Lemma 2.5. *Let $\mathcal{X} \subset \mu_\infty(\mathbb{C})$ and $\omega \in \mathbb{C}^\times$ such that $\omega + \mathcal{X} \subset \mu_\infty$. Then $|\mathcal{X}| \leq 2$ and, if $|\mathcal{X}| = 2$ with $\mathcal{X} = \{\alpha, \beta\}$, then $\omega = -\alpha - \beta$.*

Proof. This is a consequence of the fact that, if $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$, then the equation $|\alpha + z|^2 = |\beta + z|^2 = 1$ has at most two solutions ; for $|\alpha| = |\beta| = 1$ these solutions are 0 and $-\alpha - \beta$. Since $\omega \neq 0$, $\alpha, \beta, \gamma \in \mathcal{X}$ would imply $\alpha + \beta = \alpha + \gamma$ hence $\beta = \gamma$ and $|\mathcal{X}| \leq 2$. $\square$

Proposition 2.6.   (1) *If $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ then $\rho^1 \simeq \rho^2 \Leftrightarrow (\rho^1)_{\mathcal{H}} \simeq (\rho^2)_{\mathcal{H}}$.*
   (2) *If $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ with $\dim \rho^i > 1$, then $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$ iff $\rho^2 \simeq \rho^1 \otimes \eta$ for some $\eta \in \mathrm{X}(\rho)$.*

Proof. (1) is a direct consequence of the fact that $\mathcal{H}$ and W are both generated by $\mathcal{R}$. For (2), we assume that $(\rho^1)_{\mathcal{H}'}$ and $(\rho^2)_{\mathcal{H}'}$ are irreducible and identify the underlying vector spaces of ρ^1, ρ^2. Denoting it by V, there exists $P \in \mathrm{GL}(V)$ with $P\rho^2(s')P^{-1} = \rho^1(s')$ for each $s \in \mathcal{R}$, with $s' = s - (1/N)T_s$, N the cardinality of the conjugacy class c of s and $T_s = \sum_{g \in c} g$. Since ρ^1, ρ^2 are irreducibles we have $T_s \in \Bbbk$. We let $\omega_s = (\rho^2(T_s) - \rho^1(T_s))/N$, so that $P\rho^2(s)P^{-1} = \rho^1(s) + \omega_s$. Raising the previous equation to the square we get $P\rho^2(s^2)P^{-1} = \rho^1(s^2) + 2\omega_s\rho^1(s) + \omega_s^2$. We apply this to some $s \in \mathcal{S}$. If $s^2 = 1$, then we have $1 = 1 + 2\omega_s\rho^1(s) + \omega_s^2$ hence $\omega_s = 0$ or $\rho^1(s) = -\omega_s/2 \in \Bbbk^\times$. Otherwise, $s^2 \in \mathcal{R}$ hence $P\rho^2(s^2)P^{-1} = \rho^1(s^2) + \omega_{s^2}$ and $\omega_{s^2} = 2\omega_s\rho^1(s) + \omega_s^2$, so also in this case $\omega_s = 0$ or $\rho^1(s) \in \Bbbk^\times$.

For C the set of conjugacy classes of hyperplanes we define $J \subset C$ by $c \in J$ iff the corresponding $s \in \mathcal{S}$ satisfy $\omega_s \neq 0$. For such an s we define $\eta(s) \in \Bbbk^\times$ by $\rho^2(s) = \rho^1(s)\eta$, and define $\eta(s) = 1$ otherwise. Stanley's theorem extends this formula to a character $\eta \in \mathrm{Hom}(W, \Bbbk^\times)$ such that $\rho^2 \simeq \rho^1 \otimes \eta$ with $\eta \in \mathrm{X}(\rho^1)$. Conversely, if $\rho^2 = \rho^1 \otimes \eta$ with $\eta \in \mathrm{X}(\rho^1)$ and $s \in \mathcal{R}$, then either $\rho^2(s) = \rho^1(s)$, or $\rho^2(s), \rho^1(s) \in \Bbbk^\times$ hence $\rho^1(s)\eta(s) = \rho^1(s) + \omega$ for some $\omega \in \mathbb{C}^\times$, which implies $\rho^2(s') = \rho^1(s')$. This concludes the proof of (2). $\square$

Proposition 2.7. *Let $\rho \in \mathrm{Irr}(W)$. There exists $\rho^1 \in \mathrm{Irr}(W)$ such that $\rho^1_{\mathcal{H}'_s} \simeq (\rho_{\mathcal{H}'_s})^*$ if and only if, for any $s \in \mathcal{S}$, $\rho(s)$ has at most two eigenvalues. In that case, $\rho^1 \simeq \rho^* \otimes \chi$ with $\chi \in \mathrm{Hom}(W, \Bbbk^\times)$ and $\chi(s)$ for $s \in \mathcal{S}$ is equal to the product of the eigenvalues of $\rho(s)$ whenever $\rho(s)$ has two distinct ones ; moreover, one also has $\rho^1_{\mathcal{H}'} \simeq (\rho^*)_{\mathcal{H}'}$.*

Proof. We assume $\rho^1_{\mathcal{H}'_s} \simeq (\rho_{\mathcal{H}'_s})^*$, and identify the underlying vector space with a common $\Bbbk^n$. Let $s \in \mathcal{S}$ and $X = \rho(s)$, $Y = \rho^1(s)$. We have $P \in \mathrm{GL}_n(\Bbbk)$ independent from s with $P\rho^1(s')P^{-1} = -{}^t\rho(s')$ i.e. $PYP^{-1} = -{}^tX + \omega_s$ with $\omega_s = (1/\#c)(\rho^1(T) + \rho(T))$, c being the conjugacy class of s, $T = \sum_{g \in c} g$.

We assume $\omega_s \neq 0$ and let n denote the order of s. From $PYP^{-1} = -{}^tX + \omega_s$ and $\omega_s - \mathrm{Sp}(X) \subset \mu_\infty$ we get from lemma 2.5 that $|\mathrm{Sp}X| \leq 2$ and, if $\mathrm{Sp}(X) = \{\alpha, \beta\}$ with $\alpha \neq \beta$, we have $\omega_s = \alpha + \beta$ hence, since X is semisimple, $\omega_s - X = \beta\alpha X^{-1}$, whence $PYP^{-1} = u_s {}^tX^{-1}$ for some $u_s \in \mu_n$; similarly, if $|\mathrm{Sp}(X)| = 1$, then $PYP^{-1} = u_s {}^tX^{-1}$ for some $u_s \in \mu_n$. Then $s^k \mapsto u_s^k$ defines a character of the subgroup generated by s.

Let now $\chi \in \text{Hom}(W, \Bbbk^\times)$ defined by $\chi(s) = -1$ if $\omega_s = 0$ or $s^2 = 1$, $\chi(s) = u_s$ for $\rho(s) \in \Bbbk^\times$ with u defined as above, and otherwise $\chi(s)$ equal to the product of the eigenvalues of $\rho(s)$. We then have $\rho^1 \simeq \chi \otimes \rho^*$ and conversely, if $\rho^1 = \chi \otimes \rho^*$ with such a χ, then it is easily checked that $(\rho^1)_{\mathcal{H}'} \simeq (\rho^*)_{\mathcal{H}'}$. $\qquad\square$

Note that any two χ as in the above statement are deduced from each other by tensoring by some $\eta \in \text{X}(\rho^1)$.

We define an equivalence relation $\approx$ on $\text{Irr}(W)$ generated by $\rho^1 \approx \rho^2$ for $\rho^2 = \rho^1 \otimes \eta$, $\eta \in \text{X}(\rho^1)$, and $\rho^2 = (\rho^1)^* \otimes \chi$ for a χ as in proposition 2.7. We let $\overline{\text{QRef}}$ a set of representatives in QRef of QRef$/ \approx$, and subdivide $\text{Irr}(W) \backslash \Lambda\text{Ref} = \mathcal{E} \sqcup \mathcal{F}$ with $\mathcal{F}$ the ρ such that $\rho_{\mathcal{H}'} \simeq (\rho'_{\mathcal{H}})^*$. We identify $\mathcal{E}/ \approx$ and $\mathcal{F}/ \approx$ with some subset of representatives. Finally, if $\rho_{\mathcal{H}'} \simeq \rho^*_{\mathcal{H}'}$, there exists a nondegenerate bilinear form over V_ρ preserved by $\rho(\mathcal{H}')$. We say that $\rho \in \text{Irr}(W)$ is of *orthogonal type* if this bilinear form is symmetric, and of *symplectic type* otherwise, namely if this bilinear form is skew-symmetric. In both cases, we denote the corresponding orthogonal or symplectic algebra by $\mathfrak{osp}(V_\rho)$.

We now define the following Lie algebra

$$\mathcal{M} = \left(\bigoplus_{\rho \in \overline{\text{QRef}}} \mathfrak{sl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{E}/\approx} \mathfrak{sl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{F}/\approx} \mathfrak{osp}(V_\rho) \right)$$

and embed $\mathcal{M}$ inside $\Bbbk W$ in the obvious way. For instance, if ρ_0 is a representative of some class in $\mathcal{E}$, then the $\mathfrak{sl}(V_\rho)$ factor in $\mathcal{M}$ is identified with a diagonal factor $\bigoplus_{\rho \approx \rho_0} \mathfrak{sl}(V_\rho)$ using the isomorphisms $\rho_{\mathcal{H}'} \simeq \rho^0_{\mathcal{H}'}$ or $\rho^*_{\mathcal{H}'} \simeq (\rho^0_{\mathcal{H}'})^*$.

By the above propositions, we have $\mathcal{H}' \subset \mathcal{M}$. In the next section we will prove the following theorem.

Theorem 2.8. *If W is irreducible and $W \neq H_4$, then $\mathcal{H}' = \mathcal{M}$.*

The proof of the following lemma is postponed to the next section.

Lemma 2.9. *Let W be irreducible and not a 2-reflection group. Let $\rho \notin \Lambda\text{Ref}$ with $\rho_{\mathcal{H}'} \simeq (\rho_{\mathcal{H}'})^*$. Then $\dim \rho$ is an even integer and, if $\dim \rho \in \{4, 6, 8\}$, then ρ is of symplectic type.*

Let $\mathfrak{g} \subset \mathcal{M}(\rho)$ a semisimple Lie subalgebra of $\mathcal{M}$. For all $\rho \in \text{Irr}(W)$, we denote $\mathcal{M}(\rho)$ and $\mathfrak{g}(\rho)$ the simple Lie ideal of $\mathcal{M}$ and $\mathfrak{g}$, respectively, that corresponds to ρ, i.e. the orthogonal w.r.t. the Killing form of the kernel of ρ for the corresponding Lie algebras. We have clearly $\mathcal{M}(\rho) \simeq \mathfrak{sl}(V_\rho)$ for $\rho \in \mathcal{E}$ and $\mathcal{M}(\rho) \simeq \mathfrak{osp}(V_\rho)$ for $\rho \in \mathcal{F}$. By the above lemma, all these Lie ideals are simple, as the non-simple case $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$ does not arise.

Lemma 2.10. *Let W be irreducible and not a 2-reflection group. If we have*

(1) *For all $\rho \in \text{Irr}(W)$, $\rho(\mathcal{M}) \simeq \rho(\mathfrak{g})$*
(2) *$\rho^1_\mathfrak{g} \simeq \rho^2_\mathfrak{g}$ iff $\rho^1_\mathcal{M} \simeq \rho^2_\mathcal{M}$*
(3) *$\rho^1_\mathfrak{g} \simeq (\rho^2_\mathfrak{g})^*$ iff $\rho^1_\mathcal{M} \simeq (\rho^2_\mathcal{M})^*$*

then $\mathfrak{g} = \mathcal{M}$.

Proof. Let $\mathcal{X} = \overline{\mathrm{QRef}} \cup \mathcal{E}/\approx \cup \mathcal{F}/\approx$, so that $\mathcal{M} = \bigoplus_{\rho \in \mathcal{X}} \mathcal{M}(\rho)$.

Each of the $\rho \in \mathrm{Irr}(W)$ provides a Lie ideal $\mathfrak{g}(\rho)$ of $\mathfrak{g}$, namely the orthogonal of $\mathrm{Ker}\,\rho_{\mathfrak{g}}$ w.r.t. the Killing form of $\mathfrak{g}$, which is isomorphic to $\rho(\mathfrak{g})$, which is isomorphic to $\rho(\mathcal{M})$ by (1) and simple by the above lemma, as the case $\mathfrak{so}_4$ is excluded. Moreover, the type of $\rho(\mathfrak{g})$, for $\mathfrak{g} \in \mathcal{X}$, determines $\dim \rho$ by the above lemma, as the exceptional isomorphisms $\mathfrak{so}_3 \simeq \mathfrak{sl}_2$ and $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$ are excluded by the lemma for $\rho \notin \Lambda\mathrm{Ref}$, and $\Lambda\mathrm{Ref} \cap \mathcal{X} = \overline{\mathrm{QRef}}$.

Let $\rho^1, \rho^2 \in \mathcal{X}$ such that $\mathfrak{g}(\rho^1) = \mathfrak{g}(\rho^2)$. This is possible if and only if ρ^1 and ρ^2 have the same dimension, say N. But the simple Lie ideals involved here have at most 2 irreducible representations of dimension N, as the exceptional case $\mathfrak{so}_8$ is excluded. Moreover, when there are two of them, one is the dual of the other. Thus $\rho^1_{\mathfrak{g}} \simeq \rho^2_{\mathfrak{g}}$ or $\rho^1_{\mathfrak{g}} \simeq (\rho^2_{\mathfrak{g}})^*$. If $\rho^1_{\mathfrak{g}} \simeq \rho^2_{\mathfrak{g}}$ or $\rho^1_{\mathfrak{g}} \simeq (\rho^2_{\mathfrak{g}})^*$, by (2) and (3) we have $\rho^1 = \rho^2$, as $\rho^1, \rho^2 \in \mathcal{X}$. Then $\mathfrak{g} = \bigoplus_{\rho \in \mathcal{X}} \mathfrak{g}(\rho) \simeq \bigoplus_{\rho \in \mathcal{X}} \rho(\mathcal{M}) \simeq \mathcal{M}$, hence $\dim \mathfrak{g} = \dim \mathcal{M}$ and $\mathfrak{g} = \mathcal{M}$. $\square$

Example for G_{25}. We first describe $\mathrm{Irr}(W)$ for W of type G_{25}. Let $j = \exp(2\mathrm{i}\pi/3)$, and choose $s \in \mathcal{S}$. It has order 3, and there is only one conjugacy class of hyperplanes for W, so there are exactly 3 one-dimensional $S_\alpha : W \to \Bbbk^\times$ and $\mathrm{X}(\rho) = \{S_1\}$ for every $\rho \in \mathrm{Irr}(W)$ with $\dim \rho \geq 2$. The 24 irreducible representations are described in the following table.

Dim.	Name	Parameters	$\mathrm{Sp}(s)$
1	S_α	$\alpha \in \mu_3$	(α)
2	$U_{\alpha,\beta}$	$\{\alpha,\beta\} \subset \mu_3, \alpha \neq \beta$	(α,β)
3	$U'_{\alpha,\beta}$	$(\alpha,\beta) \in (\mu_3)^2, \alpha \neq \beta$	(α,α,β)
3	V		$(1,j,j^2)$
6	$V_{\alpha,\beta}$	$(\alpha,\beta) \in (\mu^3)^2, \alpha \neq \beta$	$(\alpha,\alpha,\alpha,\beta,\beta,\gamma)$
8	W_α	$\alpha \in \mu_3$	$(\alpha,\alpha,\alpha,\alpha,\beta,\beta,\gamma,\gamma)$
9	X		$(\alpha,\alpha,\alpha,\beta,\beta,\beta,\gamma,\gamma,\gamma)$
9	X^*		$(\alpha,\alpha,\alpha,\beta,\beta,\beta,\gamma,\gamma,\gamma)$

From these datas, we get that $\Lambda\mathrm{Ref} = \{S_\alpha, U_{\alpha,\beta}, U'_{\alpha,\beta} \mid \alpha,\beta \in \mu_3\}$, and $\mathrm{QRef} = \{U_{\alpha,\beta}, U'_{\alpha,\beta} \mid \alpha,\beta \in \mu_3\}$. Moreover $U'_{\alpha,\beta} \approx U'_{\beta,\alpha}$ is the only nontrivial identification in $\mathrm{QRef}/\approx$. We will prove later on (see proposition 7.1) that the identifications are the same for $\mathcal{H}'_s$ and $\mathcal{H}'$, since every pseudo-reflection has odd order. It follows that the reflection ideal is made out of simple ideals $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ for each pair $\{\alpha,\beta\}$ and is isomorphic to $\mathfrak{sl}_2^3 \times \mathfrak{sl}_3^3$. Outside $\Lambda\mathrm{Ref}$, each N-dimensional representation corresponds to a distinct ideal $\mathfrak{sl}_N$, $N \in \{6,8,9\}$, so

$$\mathcal{H}'_s \simeq \mathcal{H}' \simeq \mathfrak{sl}_2^3 \times \mathfrak{sl}_3^4 \times \mathfrak{sl}_6^6 \times \mathfrak{sl}_8^3 \times \mathfrak{sl}_9^2$$

3. Proof of the structure theorem

We prove theorem 2.8 separately for the infinite series and for the exceptional groups. Since this theorem is known to hold for W a 2-reflection group by [Ma09a], we assume that W is not such a group.

In order to prove the theorem, by lemma 2.10, we only need to prove lemma 2.9 as well as the following one.

Lemma 3.1. *Let W be an irreducible reflection group which is not a 2-reflection group. For any $\rho \notin \Lambda\mathrm{Ref}$, $\rho(\mathcal{H}'_s) = \mathfrak{osp}(V_\rho)$ if $\rho_{\mathcal{H}'} \simeq (\rho_{\mathcal{H}'})^*$ and $\rho(\mathcal{H}'_s) = \mathfrak{sl}(V_\rho)$ otherwise.*

We prove both lemmas in the two cases of the general series and the exceptional cases.

3.1. **The infinite series** $G(de, e, r)$**.** We let $W = G(de, e, r)$ and $W_0 = G(de, de, r)$. Since W_0 is a 2-reflection group, the structure of its infinitesimal Hecke algebre $\mathcal{H}_0 \subset \mathcal{H}_s$ is known by [Ma09a]. Recall that W_0 is an index d subgroup of W, and that $W = W_0 \rtimes < t >$ where $t \in \mathcal{S}$ is the pseudo-reflection $\mathrm{diag}(\zeta, 1, \dots, 1)$ for ζ some primitive d-th root of 1. We let $\alpha : W \to \mu_d$ with kernel W_0, $\alpha(t) = \zeta$. When needed, we will use the standard labelling by de-tuples of partitions of total size r of the irreducible representations of $G(de, 1, r)$, as in e.g. [Ar, ArKo, MM].

We extract from [Ma09a] the following criterium. For $\mathfrak{h}$ a semisimple Lie algebra, we let $\mathrm{rk}\,\mathfrak{h}$ denote its semisimple rank.

Lemma 3.2. *Let $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{sl}_N$ be semisimple Lie algebras with $\mathfrak{g}$ acting irreducibly on $\Bbbk^N$. If $\mathrm{rk}\,\mathfrak{h} > N/2$, or if $\mathrm{rk}\,\mathfrak{h} = N/2$ with $\Bbbk^N$ not selfdual as a $\mathfrak{g}$-module, then $\mathfrak{g} = \mathfrak{sl}_N$*

Proof. If $\mathrm{rk}\,\mathfrak{g} > N/2$ this is [Ma09a] lemma 3.1. If $\mathrm{rk}\,\mathfrak{g} = N/2$, then by [Ma09a] lemma 3.3, since $N < (N/2+1)^2$ for all N, we get that $\mathfrak{g}$ is simple. Then [Ma09a] lemma 3.4 shows that the only possibility is that $\mathfrak{g} = \mathfrak{sl}_N$. □

Let $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$. If $\rho(t) = u \in \Bbbk^\times$, then $u = \zeta^{-k}$ for some k and $\rho \otimes \alpha^k(W) = \rho(W_0)$ so the result follows from [Ma09a], as $(\rho \otimes \alpha^k)(\mathcal{H}'_s) = \rho(\mathcal{H}'_0)$. From now on we assume that $\rho(t) \notin \Bbbk^\times$, and we decompose $\mathrm{Res}_{W_0}\rho = \rho_1 + \dots + \rho_m$ in irreducible components. By Clifford theory, $\rho_i \not\simeq \rho_j$ for $i \neq j$. We denote ϵ the sign character on W_0. Since the ρ_i are deduced from each other through conjugation by t, the condition $\forall i\ \rho_i \notin \Lambda\mathrm{Ref}(W_0)$ is equivalent to $\rho_1 \notin \Lambda\mathrm{Ref}(W_0)$. We note that we can choose an ordering on $\rho_1, \dots, \rho_m$ such that $V_\rho = V_{\rho_1} \oplus \dots \oplus V_{\rho_m}$ with $V_{\rho_{b+k}} = \rho(t^k)V_{\rho_b}$ and $\rho_{k+1} \simeq \rho_k \circ \mathrm{Ad}(t)$. A consequence is that $p = |\mathrm{Sp}\rho(t)| \geq m$. Recall that, if $p \geq 3$, then $\rho_{\mathcal{H}'_s}$ is not selfdual. This is thus the case if $m \geq 3$.

3.1.1. *The case $m = 1$, $\rho_1 \notin \Lambda\mathrm{Ref}$.* If $\rho_1 \not\simeq \rho_1^* \otimes \epsilon$, then by [Ma09a] we already have $\rho(\mathcal{H}'_s) = \mathfrak{sl}(V_\rho)$, so we can assume $\rho_1 \simeq \rho_1^* \otimes \epsilon$, and let $\mathfrak{h} = \rho(\mathcal{H}'_0)$. By [Ma09a] this is a simple Lie algebra $\mathfrak{osp}_0(V_\rho)$ of rank $(\dim \rho)/2$ that preserves some nondegenerate bilinear form over V_ρ and acts irreducibly on V_ρ.

If $(\rho_{\mathcal{H}'})^* \not\simeq \rho_{\mathcal{H}'}$ then by lemma 3.2 we get $\mathfrak{g} = \mathfrak{sl}(V_\rho)$, so we can assume that $(\rho_{\mathcal{H}'})^* \simeq \rho_{\mathcal{H}'}$ (in particular, $\rho(t)$ admits exactly two eigenvalues u, v), hence $\mathfrak{g} \subset \mathfrak{osp}(V_\rho)$. Since $\mathfrak{h}$ acts irreducibly on V_ρ, it can preserve only one such form (up to scalar), so from $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{osp}(V_\rho)$ and $\mathfrak{h} \subset \mathfrak{osp}_0(V_\rho)$ we get $\mathfrak{osp}(V_\rho) = \mathfrak{osp}_0(V_\rho)$ and $\mathfrak{h} = \mathfrak{g} = \mathfrak{osp}(V_\rho)$.

3.1.2. *The case $m \geq 2$, $\rho_1 \notin \Lambda\mathrm{Ref}$, $\rho_{\mathcal{H}'_s} \not\simeq (\rho_{\mathcal{H}'_s})^*$.* We need to show that $\mathfrak{g} = \mathfrak{sl}(V_\rho)$.

First assume $\rho_1^* \otimes \epsilon \simeq \rho_1$. Since $\rho_1 \notin \Lambda\mathrm{Ref}$ we have $\dim \rho_1 \geq 3$ (see [Ma09a] lemma 2.18) hence $\dim \rho \geq 6$. This implies $r \geq 3$, as it is easily

checked that the irreducible representations of $G(de, e, 2)$ have dimension at most 2. It follows that W_0 has a single class of reflections. Since $W_0 \neq H_4$, the ρ_i then correspond to distinct ideals of rank $(\dim \rho_i)/2$ and $\mathfrak{g}$ contains a semisimple Lie algebra of rank $m(N/2m) = N/2$. Then lemma 3.2 implies $\mathfrak{g} = \mathfrak{sl}(V_\rho)$.

Now assume $\rho_1^* \otimes \epsilon \not\simeq \rho_1$. If $\forall i, j \quad \rho_i \not\simeq \rho_j^* \otimes \epsilon$, then the rank of $\mathfrak{h}$ is $m(N/m - 1) = N - m > N/2$ iff $\dim \rho_i = N/m > 2$, which holds true since $\rho_i \notin \Lambda\mathrm{Ref}$.

Otherwise, the map $\rho_i \mapsto \rho_i^* \otimes \epsilon$ induces a permutation of the ρ_i, as $(\rho_1 \circ \mathrm{Ad}(t^i))^* \otimes \epsilon = (\rho_1^* \otimes \epsilon) \circ \mathrm{Ad}(t^i)$, hence $m = 2q$ for some $q \geq 1$, and $\mathfrak{h}$ has rank $h = q(N/m - 1) = N/2 - m/2$. We have $N < (h + 1)^2$ iff $N^2 - 2(m - 4)N + (m - 2)^2 > 0$. This trinomial in N has for reduced discriminant $-2(2m-6) < 0$ for $m \geq 4$. Checking separately the case $m = 2$, we get that $N < (h + 1)^2$. It follows that $\mathfrak{g}$ is simple by [Ma09a] lemma 3.3 (I). Moreover, since $N/m = \dim \rho_i > 2$ we have $h = N/2 - m/2 > N/4$ hence $\mathrm{rk}\, \mathfrak{g} > N/4$. Moreover $N = m \dim \rho_1$ with $\dim \rho_1 \geq 3$ and $m \geq 2$ hence $N \geq 8$. Since V_ρ is not selfdual as a $\mathfrak{g}$-module and $N \geq 8$ is even, by [Ma09a] lemma 3.4 we get that the only possibility for $\mathfrak{g} \neq \mathfrak{sl}_N$ implies $N = 10$ and $\mathfrak{g} \simeq \mathfrak{sl}_5$, or $N = 16$ and $\mathrm{rk}\, \mathfrak{g} = 5$. The latter case is excluded, since it implies $\dim \rho_i = 4$, $m = 4$ and $\mathrm{rk}\, \mathfrak{g} \geq (N - m)/2 = 6$; the former is excluded because then $\dim \rho_i = 5$ and $\mathfrak{g} \simeq \mathfrak{sl}_5 \simeq \rho(\mathcal{H}_0')$ implies $\mathfrak{g} = \rho(\mathcal{H}_0')$, which contradicts the fact that $\rho_\mathfrak{g}$ is irreducible whereas $\rho'_{\mathcal{H}_0}$ is not. It follows that $\mathfrak{g} = \mathfrak{sl}(V_\rho)$ in this case.

3.1.3. *The case $\rho_1 \notin \Lambda\mathrm{Ref}, m \geq 2$ and $\rho_{\mathcal{H}'_s} \simeq (\rho_{\mathcal{H}'_s})^*$.* This implies $m = 2, p = 2$. We have $\mathfrak{g} \subset \mathfrak{osp}(V_\rho)$ and want to show $\mathfrak{g} = \mathfrak{osp}(V_\rho)$.

By Clifford theory we have that $d = \#\alpha(W)$ even, so $d = 2q$, that $\rho \otimes \alpha^q \simeq \rho$, and that $\rho_1 \circ \mathrm{Ad}(t) = \rho_2$. Since $\alpha^q(t) = -1$, we get from $\alpha^q \otimes \rho \simeq \rho$ and $p = 2$ that $\rho(t)$ has for eigenvalues $u, -u$ with the same multiplicities. Letting $u = \zeta^{-k}$ we get that $\rho' = \rho \otimes \alpha^k$ has the same restriction to W_0 than ρ with $\rho'(t)^2 = 1$. Hence ρ' factorizes through the classical morphism $G(de, e, r) \twoheadrightarrow G(de, de/2, r) = W_1$ that preserves $\mathfrak{S}_r$ and maps t to t. Since W_1 is a 2-reflection group,

the conclusion that $\mathfrak{g} = \rho(\mathcal{H}'_s)$ thus follows from [Ma09a], as $\rho_{\mathcal{H}_s}$ is selfdual iff $\rho' \simeq (\rho')^* \otimes \epsilon$ for ϵ the sign character of W_1, and $\rho \notin \Lambda\mathrm{Ref} \Rightarrow \rho' \notin \Lambda\mathrm{Ref}(W_1)$.

3.1.4. *The case $\rho_1 \in \Lambda\mathrm{Ref}(W_0)$.* The list of representations in $\Lambda\mathrm{Ref}(W_0)$ can be found in [Ma09a]. We check on this list that the only possibilities for $\rho(t)$ not to be a scalar (in which case we would have $\rho \in \Lambda\mathrm{Ref}$) are the following two, with $\dim \rho = 6$.

The former one is when $r = 4$, de is even, $m = 2$, ρ_1 has dimension 3 and, up to tensoring by some power of α, ρ is the restriction to W of a representation $(\lambda, \emptyset, \ldots, \lambda, \emptyset, \ldots)$ of $G(de, 1, 4)$, with $\lambda \in \{[2], [1, 1]\}$. But then $\rho(t)^2 = 1$ and $\mathfrak{g}$ coincides with the image of (the semisimple part of) the infinitesimal Hecke algebra of $G(de, de/2, 4)$, so the conclusion follows from [Ma09a].

The latter is when $r = 3$, $m = 3$, $\dim \rho_1 = 2$ and, up to tensoring by some power of α, ρ is the restriction to W of the representation

$([1], \emptyset, \ldots, [1], \emptyset, \ldots, [1], \emptyset, \ldots,)$ of $G(de, 1, 3)$. It is easily checked (e.g. through the character table of $G(3, 3, 3)$) that $\rho_i \simeq \rho_i^* \otimes \epsilon$, hence the ρ_i correspond to three distinct ideals of $\mathfrak{h}$ and $\mathfrak{h}$ has rank 3. Then $\mathrm{rk}\,\mathfrak{h} \geq 6/2$, and $\rho_{\mathcal{H}'_s}$ is not selfdual, as $m > 2$ and by the argument in the previous section. By lemma 3.2 it follows that $\mathfrak{g} = \mathfrak{sl}(V_\rho)$, and this concludes the proof for the $G(de, e, r)$.

3.1.5. *Proof of lemma 2.9.* We let $\rho \notin \Lambda\mathrm{Ref}$ with $\rho_{\mathcal{H}'} \simeq (\rho_{\mathcal{H}'})^*$ and denote ρ_0 the restriction of ρ to W_0. Assume first $\dim \rho = 4$. If ρ_0 is irreducible, then $\rho_0 \simeq \rho_0^* \otimes \epsilon$, where ϵ denotes the sign character of W_0. Then ρ is of orthogonal type if and only if $\epsilon \hookrightarrow S^2 \rho_0$, which is possible only for W_0 of type F_4 by [Ma09a] lemma 7.3. Since $W_0 = G(de, de, r)$ is not of this type this excludes this case. If ρ_0 is not irreducible, then its irreducible components have dimension 2 or 1, and in particular belong to $\Lambda\mathrm{Ref}(W_0)$. We saw above that this situation does not occur.

Now assume $\dim \rho = 8$. As before, the irreducible components of ρ_0 do not belong to $\Lambda\mathrm{Ref}(W_0)$, and this excludes the case where ρ_0 admits 4 irreducible components. The case of ρ_0 irreducible (then $\epsilon \hookrightarrow S^2 \rho_0$) is excluded by [Ma09a] proposition 7.6. If ρ_0 admits 2 irreducible components ρ_1, ρ_2 of dimension 4, then, since $(\rho_0)_{\mathcal{H}'}$ would preserve a nondegenerate symmetric bilinear form, so would either $(\rho_1)_{\mathcal{H}'_0}$ or $(\rho_2)_{\mathcal{H}'_0}$. This is excluded by [Ma09a] lemma 7.3, which concludes the case $\dim \rho = 8$.

Finally, assume that $\dim \rho = 6$. If ρ_0 is irreducible, we are done by [Ma09a] lemma 7.4. Otherwise, it admits either 2 or 3 irreducible components. The case of 3 irreducible components is excluded by $(\rho_{\mathcal{H}'})^* \simeq \rho_{\mathcal{H}'}$, hence ρ_0 is the sum of 2 irreducible 3-dimensional components. Then d is even and, up to tensoring by some multiplicative character, ρ corresponds to a multipartition of the form $([2], \ldots, [2], \ldots)$ that factorize through $W_1 = G(de, de/2, r)$. This situation is then dealt with again by [Ma09a] lemma 7.4.

Now assume $\dim \rho = 2N + 1 > 1$. Then ρ is of symmetric type if $\rho(\mathcal{H}') \simeq \mathfrak{so}_{2N+1}$. Let ρ_1 denote an irreducible component of ρ_0. We have $\dim \rho_1 > 1$, and also $\rho_1(W_0) \not\subset \mathbb{k}^\times$, otherwise $\rho(W)$ is abelian and $\dim \rho = 1$. It follows that W_0 act by the sign character ϵ on this form, hence this form admits involutive skewisometries afforded by the 2-reflections in W_0. This implies that this form is hyperbolic (see [Ma09a], lemma 2.5), hence $2N + 1$ is even, a contradiction that concludes the proof of lemma 2.9 for the general series.

3.2. **Exceptional groups.** Among the 34 exceptional groups in the Shephard-Todd classification, 15 of them are 2-reflection groups, and were dealt with in [Ma09a]. The others mostly have rank 2, plus 3 groups of higher rank, namely G_{25}, G_{26} and G_{32}.

For a given $\rho \in \mathrm{Irr}(W)$, given an explicit matrix model $\rho : W \to \mathrm{GL}(V_\rho)$, the dimension of $\rho(\mathcal{H}'_s)$ can be computed by the following algorithm : start from the $\rho(s)$ for $s \in \mathcal{S}$, extract a basis for the spanned subspace in $\mathfrak{gl}(V_\rho)$, add to this subspace the images of this basis under the $\mathrm{ad}(\rho(s))$ for $s \in \mathcal{S}$, and then iterate the process until the dimension stops increasing ; this gives the dimension d of $\rho(\mathcal{H}_s)$, and $\rho(\mathcal{H}'_s)$ has dimension d or $d - 1$ if one of the $\rho(s)$ for $s \in \mathcal{S}$ has nonzero trace. Of course this dimension is the same for

ρ and for $\rho \otimes \eta$ if $\eta \in \mathrm{Hom}(W, \Bbbk^{\times})$, which reduces sometimes drastically (notably for W of type G_{19}) the number of representations to check.

This algorithm is tractable provided that $\dim \rho$ is reasonably small, and that explicit matrix models are available. These two conditions are satisfied by all these exceptional groups, except for G_{32}.

For $W \neq G_{32}$, the irreducible representations have dimension at most 9, explicit matrix models were computed by various authors, and are now included in the CHEVIE package for the GAP3 software. We check using the above algorithm that $\dim \rho(\mathcal{H}'_s) = \dim(\rho)^2 - 1$, that is $\rho(\mathcal{H}'_s) = \mathfrak{sl}(V_\rho)$, for all $\rho \notin \Lambda\mathrm{Ref}$ (in particular this shows that $\rho_{\mathcal{H}'} \not\simeq (\rho_{\mathcal{H}'})^*$ for all these representations). This proves lemmas 2.9 and 3.1 in these cases, and the structure theorem by lemma 2.10.

The only group W remaining to be dealt with is G_{32}. In that case we use a parabolic subgroup W_0 of type G_{25} and, assuming the theorem proved in type G_{25}, we consider, for each $\rho \in \mathrm{Irr}(W)$, the (reductive) Lie algebra generated by the $\rho(s)$, $s \in \mathcal{S} \cap W_0$, and let $\mathfrak{h}(\rho)$ denote its semisimple part (that is, its intersection with $\mathfrak{sl}(V_\rho)$). By the analysis carried out above of the infinitesimal Hecke algebra of type G_{25}, we know to which simple Lie ideal corresponds each irreducible representation of W_0. It follows that the rank of $\mathfrak{h}$ can be deduced from the knowledge of the induction table from W_0 to W, which is known and included in CHEVIE. Using it, we get that $\mathfrak{h}(\rho)$ has rank greater than $\dim \rho/2$ for all $\rho \notin \Lambda\mathrm{Ref}$, that is $\rho(\mathcal{H}'_s) = \mathfrak{sl}(V_\rho)$ by lemma 3.2. This completes the proofs of lemmas 2.9 and 3.1, and of the structure theorem.

4. GENERIC HECKE ALGEBRAS

We let $\mathcal{C}_+$ denote the set of conjugacy classes c of hyperplanes with $e_c > 2$, and $\mathbb{A}(W) = \prod_{c \in \mathcal{C}_+} \Bbbk^{e_C - 1}$. A typical element of $\mathbb{A}(W)$ is denoted $\underline{\boldsymbol{\lambda}} = (\underline{\lambda}^c)_{c \in \mathcal{C}_+}$, with $\underline{\lambda}^c = (\lambda_1^c, \ldots, \lambda_{e_c-1}^c)$. We denote $\mathcal{S}_+ = \{s \in \mathcal{S} \mid s^2 \neq 1\}$, $\mathcal{S}_0 = \{s \in \mathcal{S} \mid s^2 = 1\}$, and let $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ denote the Lie subalgebra of $\Bbbk W$ generated by $\mathcal{S}_0$ and the $\lambda_1^c s + \cdots + \lambda_{e_c-1}^c s^{e_c-1}$ for $s \in \mathcal{S}_+$ with reflecting hyperplane in $c \in \mathcal{C}_+$.

4.1. **Preliminaries.** Let $\zeta \in \Bbbk$ a primitive n-th root of 1. Note that, if n is the order of some pseudo-reflection of W, then $\Bbbk$ contains $\mu_n(\mathbb{C})$ (e.g. because the defining representation is realizable over $\Bbbk$).

In $\Bbbk^n$ we define $v_i = (1, \zeta^i, (\zeta^i)^2, \ldots, (\zeta^i)^{n-1})$ for $i \in [0, n-1]$. Since the corresponding (Vandermonde) determinant is invertible, these elements form a basis of $\Bbbk^n$. It follows that, for $(i,j) \neq (k,l)$ with $i \neq j$ or $k \neq l$, $H_{i,j,k,l} = \{\underline{\lambda} \in \Bbbk^n \mid \, <\underline{\lambda}|v_i - v_j - v_k + v_l> = 0\}$ is an hyperplane of $\Bbbk^n$.

Proposition 4.1. *Let $P \in A = \Bbbk[X]/(X^n - 1)$ and $\zeta \in \Bbbk$ a primitive n-th root of 1. Then*

(1) *P generates A as a unital algebra if and only if, for all $r, s \in [0, n-1]$, $r \neq s \Rightarrow P(\zeta^r) \neq P(\zeta^s)$.*

(2) *the unital subalgebra of $A \otimes_\Bbbk A = \Bbbk[X, Y]/(X^n - 1, Y^n - 1)$ generated by $P(X) + P(Y)$ contains $X + Y$ if and only if, for all i, j, k, l, $\zeta^i + \zeta^j \neq \zeta^k + \zeta^l \Rightarrow P(\zeta^i) + P(\zeta^j) \neq P(\zeta^k) + P(\zeta^l)$.*

(3) *letting* $P = P_{\underline{\lambda}} = \lambda_0 + \lambda_1 X + \cdots + \lambda_{n-1} X^{n-1}$, *for* $\underline{\lambda} \notin H_{i,j,k,l}$, $P(\zeta^i) - P(\zeta^j) = P(\zeta^k) - P(\zeta^l)$ *implies* $(i,j) = (k,l)$, *or* $i = j$ *and* $k = l$.

Proof. We consider $\pi_i : A \twoheadrightarrow \Bbbk[X]/(X - \zeta^i) = \Bbbk$ that maps X to ζ^i. Then $\pi = \bigoplus_{i=0}^{n-1} \pi_i : A \to \Bbbk^n$ is a ring isomorphism, and $\pi(P)$ generates $\Bbbk^n$ if and only if, by Lagrange interpolation, $\pi_r(P) \neq \pi_s(P) \neq 0$ whenever $r \neq s$. Then (1) follows. We then consider $\pi_{i,j} : A \otimes_{\Bbbk} A \twoheadrightarrow \Bbbk[X,Y]/(X - \zeta^i, Y - \zeta^j) = \Bbbk$ mapping $X \mapsto \zeta^i$, $Y \mapsto \zeta^j$ and $\pi = \bigoplus_{i,j} : A \otimes_{\Bbbk} A \simeq \Bbbk^{n^2}$. Once again by Lagrange interpolation we get (2). Then (3) follows from the identification $P = \lambda_0 + \lambda_1 X + \cdots + \lambda^{n-1} X \mapsto \underline{\lambda}$ of A with $\Bbbk^n$, for which $P(\zeta^i) = <\underline{\lambda}|v_i>$. $\square$

We will also need the following lemma

Lemma 4.2. *Let* $\mathfrak{g} \subset \mathfrak{sl}(V)$ *be a complex semisimple Lie algebra acting irreducibly on* V, *and* $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$ *the defining representation such that* $\rho^{\otimes 2}$ *admits at most 3 irreducible components. Then* $\mathfrak{g}$ *is a simple Lie algebra. Moreover, if* $\rho^{\otimes 2}$ *admits two irreducible components, then* $\mathfrak{g} = \mathfrak{sl}(V)$; *if* $\rho^{\otimes 2}$ *admits three irreducible components and* $\rho^* \simeq \rho$, *then* $\mathfrak{g}$ *preserves some nondegenerate bilinear form, and* $\mathfrak{g} = \mathfrak{osp}(V_\rho)$ *w.r.t. this form.*

Proof. If $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ with the $\mathfrak{g}_i$ nontrivial semisimple Lie algebras, then $V = V_1 \otimes V_2$ with V_i an irreducible representation of $\mathfrak{g}_i$. But then $\rho^{\otimes 2} = V_1^{\otimes 2} + V_1 \otimes V_2 + V_2 \otimes V_1 + V_2^{\otimes 2}$ would admit at least 4 irreducible components. It follows that $\mathfrak{g}$ is simple. The conclusion follows from [Ma01] prop. 1. $\square$

4.2. Generic infinitesimal Hecke algebras.

For $c \in \mathcal{C}_+$ we choose a primitive e_c-root ζ_c of 1, and we denote $v_i^c \in \Bbbk^{e_c-1}$ the vector $\zeta_c^i, (\zeta_c^i)^2, \ldots, (\zeta_c^i)^{e_c-1}$. Identifying $\mathbb{A}(W)$ with $\Bbbk^{\sum_{c \in \mathcal{C}_+}(e_c-1)}$ we denote $< \cdot | \cdot >$ the natural scalar product on $\mathbb{A}(W)$. We introduce in $\mathbb{A}(W)$ the following hyperplane arrangements

$\mathcal{L}_1 = \{\text{Ker} < v_r^c - v_s^c | \cdot > \mid r \neq s, c \in \mathcal{C}_+\}$
$\mathcal{L}_2 = \{\text{Ker} < v_i^c + v_j^c - v_k^c - v_l^c | \cdot > \mid \zeta_c^i + \zeta_c^j \neq \zeta_c^k + \zeta_c^l, c \in \mathcal{C}_+\}$
$\mathcal{L}_3 = \{\text{Ker} < v_i^c - v_j^c - v_k^c + v_l^c | \cdot > \mid (i,j) \neq (k,l) \text{ and } (i \neq j \text{ or } k \neq l), c \in \mathcal{C}_+\}$

Note that these three sets do not depend on the choice of the primitive e_c-roots ζ_c. Also note that $\mathcal{L}_1 \subset \mathcal{L}_2 \cap \mathcal{L}_3$. For $\rho \in \text{Irr}(W)$, we let $\rho_{\mathcal{H}(\underline{\lambda})}, \rho_{\mathcal{H}(\underline{\lambda})'}$ the representations of $\mathcal{H}(\underline{\lambda}), \mathcal{H}(\underline{\lambda})'$ respectively, that are induced by ρ. We let $\mathbb{A}^{\times}(W) = \mathbb{A}(W) \setminus \bigcup(\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) = \setminus \bigcup(\mathcal{L}_2 \cup \mathcal{L}_3)$.

Let c be a conjugacy class in W. We let $p : \Bbbk W \twoheadrightarrow Z(\Bbbk W)$ denote the natural projection $p(g) = (1/|W|) \sum_{h \in g} hgh^{-1}$.

Let c be a conjugacy class of reflections in $\mathcal{S}$. We denote $\mathcal{S}/W$ the set of such classes. For $c \in \mathcal{S}_0$ we denote $T_c(\underline{\lambda}) = T_c = \sum_{s \in c} s \in Z(\Bbbk W)$ and for $c \in \mathcal{S}_+$ we let

$$T_c(\underline{\lambda}) = \sum_{s \in c} \sum_{i=1}^{e_c-1} \lambda_i^c s^i = \sum_{i=1}^{e_c-1} \lambda_i^c \sum_{s \in c} s^i \in Z(\Bbbk W).$$

For $s \in \mathcal{S}_0$ we let $s(\underline{\lambda}) = s$ and for $s \in \mathcal{S}_+$ we let $s(\underline{\lambda}) = \sum_{i=1}^{e_c-1} \lambda_i s^i$. Clearly $T_c(\underline{\lambda}) = p(s(\underline{\lambda}))|c|$ for any $s \in c \subset \mathcal{S}$.

Proposition 4.3. *For every value of $\underline{\boldsymbol{\lambda}}$, the Lie algebra $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ is reductive.*

Proof. We consider the direct sum of all the irreducible representations of W. This provides a faithful representation of $\mathcal{H}(\underline{\boldsymbol{\lambda}})$. In order to prove that $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ is reductive it is sufficient to prove that this representation is semisimple which means that, for every $\rho \in \mathrm{Irr}(W)$, the representation of the envelopping Lie algebra $\mathsf{U}\mathcal{H}(\underline{\boldsymbol{\lambda}})$ induced by ρ is semisimple. Up to extending scalars we can assume $\Bbbk = \mathbb{C}$, and then assume that $\rho(W)$ preserves some unitary form over $\mathrm{End}(V_\rho)$. Then the elements $x = \lambda_1^c s + \cdots + \lambda_{e_c-1}^c s^{e_c-1}$ act by normal operator (i.e. endomorphisms commuting with their adjoints), which means that $\rho(x)$ is a polynomial of its adjoint. In particular, if $U \subset V_\rho$ is stable under $\rho(\mathsf{U}\mathcal{H}(\underline{\boldsymbol{\lambda}}))$, then its orthogonal is stable under each of the elements $\rho(x)$, hence under $\rho(\mathsf{U}\mathcal{H}(\underline{\boldsymbol{\lambda}}))$. As a consequence V_ρ is completely reducible and the conclusion follows. $\square$

Proposition 4.4.   (1) *For each $\rho \in \mathrm{Irr}(W)$, $\rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})}$ and $\rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'}$ are irreducible as soon as $\underline{\boldsymbol{\lambda}} \notin \bigcup \mathcal{L}_1$.*
   (2) *If $\underline{\boldsymbol{\lambda}} \notin \bigcup \mathcal{L}_1$, the center of $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ has dimension $|\mathcal{S}/W|$, is spanned by the $T_c(\underline{\boldsymbol{\lambda}})$ for $c \in \mathcal{S}/W$ and $\mathcal{H}(\underline{\boldsymbol{\lambda}})'$ is generated by the $s(\underline{\boldsymbol{\lambda}}) - T_c(\underline{\boldsymbol{\lambda}})/|c|$.*
   (3) *Let $\rho \in \mathrm{QRef}$ or $\rho \notin \Lambda\mathrm{Ref}$, and $\underline{\boldsymbol{\lambda}} \notin \bigcup(\mathcal{L}_1 \cup \mathcal{L}_2)$. Then $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}})') = \rho(\mathcal{H}')$.*
   (4) *For $\underline{\boldsymbol{\lambda}} \in \mathbb{A}(W)^\times$, $\mathcal{H}(\underline{\boldsymbol{\lambda}})' = \mathcal{H}'$.*

Proof. (1) For each $s \in \mathcal{S}_+$ with reflection hyperplane $H \in c \in \mathcal{C}_+$, $\rho(s)$ is a polynomial in $\sum \lambda_i^c s^i$ by the lemma if $\underline{\boldsymbol{\lambda}} \notin \bigcup \mathcal{L}_1$. This implies that the algebra generated by $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$ and $\rho(\mathcal{H}_s)$ are the same, hence $\rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})}$ is irreducible. and $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$ is reductive. This implies that $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}})') = \rho(\mathcal{H}(\underline{\boldsymbol{\lambda}}))'$ is semisimple and that $\rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'}$ is irreducible. (2) Since, under this conditition, $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ generates $\Bbbk W$ as a unital algebra, we have $Z(\mathcal{H}(\underline{\boldsymbol{\lambda}})) \subset Z(\Bbbk W)$. Since $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ is reductive we have $\mathcal{H}(\underline{\boldsymbol{\lambda}}) = Z(\mathcal{H}(\underline{\boldsymbol{\lambda}})) \oplus \mathcal{H}(\underline{\boldsymbol{\lambda}})'$ hence $p(\mathcal{H}(\underline{\boldsymbol{\lambda}})) = p(Z(\mathcal{H}(\underline{\boldsymbol{\lambda}}))) = Z(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$. Let $E \subset \Bbbk W$ be the subspace spanned by the generators of $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ and, for a conjugacy class c, denote δ_c the linear form on $\Bbbk W$ defined by $\delta_c(g) = 1$ if $g \in c$, $\delta_c(g) = 0$ otherwise. We have $(\Bbbk W)' = \cap_c \mathrm{Ker}\,\delta_c = \mathrm{Ker}\,p$. Since $\mathcal{H}(\underline{\boldsymbol{\lambda}})$ is generated by E we have $\mathcal{H}(\underline{\boldsymbol{\lambda}}) = E + \mathcal{H}(\underline{\boldsymbol{\lambda}})'$ hence $p(\mathcal{H}(\underline{\boldsymbol{\lambda}})) = p(E)$, hence $Z(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$ is spanned by the $T_c(\underline{\boldsymbol{\lambda}})$ for $c \subset \mathcal{S}$. Finally, the $s(\underline{\boldsymbol{\lambda}}) - p(s(\underline{\boldsymbol{\lambda}}))$ generate a Lie algebra containing $\mathcal{H}(\underline{\boldsymbol{\lambda}})'$, as $p(s(\underline{\boldsymbol{\lambda}})) \in Z(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$ and contained in $\mathcal{H}(\underline{\boldsymbol{\lambda}}) \cap \mathrm{Ker}\,p = \mathcal{H}(\underline{\boldsymbol{\lambda}})'$, which concludes the proof of (2). We now prove (3). For such a ρ, we let $\tilde{\rho}$ denote the representation $\rho_{\mathcal{H}'_s} \otimes \rho_{\mathcal{H}'_s}$ extended to the envelopping algebra $\mathsf{U}\mathcal{H}'_s$. First assume that $\rho(\mathcal{H}') = \rho(\mathcal{H}'_s) = \mathfrak{sl}(V_\rho)$. Then $\tilde{\rho}(\mathsf{U}\mathcal{H}'_s) \simeq \mathrm{End}(S^2 V_\rho) \oplus \mathrm{End}(\Lambda^2 V_\rho)$. Since $\underline{\boldsymbol{\lambda}} \notin \bigcup \mathcal{L}_2$ we get by the lemma that, for each $s \in \mathcal{S}_+$ with reflection hyperplane $H \in c \in \mathcal{C}_+$, $\rho(\tilde{s})$ is a polynomial in $\tilde{\rho}(\sum \lambda_i^c s^i)$. Then $\tilde{\rho}(\mathsf{U}\mathcal{H}(\underline{\boldsymbol{\lambda}})') = \tilde{\rho}(\mathsf{U}\mathcal{H}'_s) = \mathrm{End}(S^2 V_\rho) \oplus \mathrm{End}(\Lambda^2 V_\rho)$. In particular, $\rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'} \otimes \rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'}$ admits two irreducible components, hence $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}})') = \mathfrak{sl}(V_\rho) = \rho(\mathcal{H}')$ by lemma 4.2. Now assume that $\rho(\mathcal{H}') = \rho(\mathcal{H}'_s) = \mathfrak{osp}(V_\rho)$. By the same argument, $\rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'} \otimes \rho_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'}$ admits three irreducible components and $\rho_{\mathcal{H}'} \simeq (\rho_{\mathcal{H}'})^* \Rightarrow \rho_{\mathcal{H}'_s} \simeq (\rho_{\mathcal{H}'_s})^*$, hence $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}})') = \mathfrak{osp}'(V_\rho)$ for some nondegenerate bilinear form by lemma 4.2. On the other hand, $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}})') \subset \rho(\mathcal{H}') = \mathfrak{osp}(V_\rho)$, hence $\mathfrak{osp}'(V_\rho) \subset \mathfrak{osp}(V_\rho)$ which implies $\mathfrak{osp}'(V_\rho) = \mathfrak{osp}(V_\rho)$ and proves (3). Let $s \in \mathcal{S}$ and $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ with $\rho^1_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'} \simeq \rho^2_{\mathcal{H}(\underline{\boldsymbol{\lambda}})'}$.

Identifying $V_{\rho^1} = V_{\rho^2} = V$, there exists $Q \in \mathrm{GL}(V)$ and $\omega_s \in \Bbbk$ with $QP_{\underline{\lambda}^c}(\rho^2(s))Q^{-1} = P_{\underline{\lambda}^c}(\rho^1(s)) + \omega_s$ if $s \in \mathcal{S}_+$, and $Q\rho^2(s)Q^{-1} = \rho^1(s) + \omega_s$ otherwise. If $\rho^1(s) \notin \Bbbk^\times$, for $s^2 = 1$ this implies $\omega_s = 0$ as in the proof of prop. 2.6 ; otherwise, $\rho^1(s)$ admits at least two eigenvalues $\zeta_c^i \neq \zeta_c^j$. Then $P(\zeta_c^i) - \omega_s = P(\zeta_c^k)$ and $P(\zeta_c^j) - \omega_s = P(\zeta_c^l)$ for some k, l hence $P(\zeta_c^i) - P(\zeta_c^k) - P(\zeta_c^j) + P(\zeta_c^l) = 0$. If $\omega_s \neq 0$ we have $i \neq k$ and $j \neq l$. Since $i \neq j$ we get a contradiction for $\underline{\lambda} \in \mathbb{A}(W)^\times$, thus $\omega_s = 0$. We thus get $\rho^2 = \rho^1 \otimes \eta$ for some $\eta \in \mathrm{X}(\rho^1)$ and $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$. We now assume $(\rho^1_{\mathcal{H}(\underline{\lambda})'})^* \simeq \rho^2_{\mathcal{H}(\underline{\lambda})'}$. For $s \in \mathcal{S}$ with reflection hyperplane in $c \in \mathcal{C}_+$, the equation $QP_{\underline{\lambda}^c}(\rho^2(s))Q^{-1} = -{}^tP(\rho^1(s)) + \omega_s$ with $\rho^1(s) \notin \Bbbk^\times$ with eigenvalues $\zeta_c^i \neq \zeta_c^j$, implies that there exists $\zeta_c^k \neq \zeta_c^i$, $\zeta_c^l \neq \zeta_c^i$ with $-P(\zeta_c^i) + \omega_s = P(\zeta_c^k)$, $-P(\zeta_c^j) + \omega_s = P(\zeta_c^l)$ whence $P(\zeta_c^i) - P(\zeta_c^j) + P(\zeta_c^k) - P(\zeta_c^l) = 0$. Since $\zeta_c^i \neq \zeta_c^j$ and $\zeta_c^k \neq \zeta_c^l$, we get from $\underline{\lambda} \in \mathbb{A}(W)^\times$ that $(\zeta_c^i, \zeta_c^j) = (\zeta_c^k, \zeta_c^l)$. But then $\omega_s = 2P(\zeta_c^i) = 2P(\zeta_c^j)$ which implies $\zeta_c^i = \zeta_c^j$, a contradiction. We conclude as above that $\rho^2_{\mathcal{H}'} \simeq (\rho^1_{\mathcal{H}})^*$, and lemma 2.10 implies that $\mathcal{H}(\underline{\lambda})' = \mathcal{H}'$, as $\mathcal{H}' = \mathcal{M}$ by theorem 2.8. $\qquad\square$

Note that, if $\underline{\lambda}, \underline{\mu} \in \mathbb{A}(W)$ are such that, for all $c \in \mathcal{S}_+/W$, we have $\mu^c = u_c \lambda^c$ for some $u_c \in \Bbbk^\times$, clearly $\mathcal{H}(\underline{\lambda}) = \mathcal{H}(\underline{\mu})$. Now denote $\alpha_c \in \mathrm{Hom}(W, \Bbbk^\times)$ defined by $\alpha_c(s) = \zeta_c$ for $s \in c \subset \mathcal{S}$, and $\alpha_c(s) = 1$ for $s \in \mathcal{S} \setminus c$. Any tuple $\underline{n} = (n_c)_{c \in \mathcal{S}/W}$ of integers with $0 \leq n_c < e_c - 1$ then defines a character $\alpha_{\underline{n}} = \prod_{c \in \mathcal{S}/W} \alpha_c^{n_c}$ and we can define $\Phi_{\underline{n}} \in \mathrm{Aut}(\Bbbk W)$ by $\Phi_{\underline{n}}(g) = \alpha_{\underline{n}}(g)g$ for $g \in W$. For any $\rho \in \mathrm{Irr}(W)$ we have $\rho \circ \Phi \simeq \alpha_{\underline{n}} \otimes \rho$ and, for any $\underline{\lambda} \in \mathbb{A}(W)$ we have $\Phi_{\underline{n}}(\mathcal{H}(\underline{\lambda})) = \mathcal{H}(\underline{\mu})$ with $\mu_i^c = \zeta_c^{in_c} \lambda_i^c$ for each c, i. The hyperplane arrangements $\mathcal{L}_i$ are clearly invariant under these operations.

5. Applications to Zariski closures

We recall that $W < \mathrm{GL}(V)$ is a finite group generated by a set $\mathcal{R}$ of (pseudo-)reflections that defines an hyperplane complement $\mathcal{A} = \{\mathrm{Ker}\,(s - 1) \mid s \in \mathcal{R}\} = \{\mathrm{Ker}\,(s - 1) \mid s \in \mathcal{S}\}$. The (generalized) pure braid group and braid group associated to them are $P = \pi_1(X)$ and $B = \pi_1(X/W)$, where X is the hyperplane complement $V \setminus \bigcup \mathcal{A}$ and a base point in X (hence in X/W) is chosen once and for all.

We refer to [BMR] to the basic properties of these groups, only updating some terminology and recalling a few basic facts.

The composition of paths $(\alpha, \beta) \mapsto \alpha\beta$ is given by following first β and then α. With this convention, the Galois covering $X \to X/W$ defines a natural morphism $B \twoheadrightarrow W$ with kernel P. We call the generators-of-the-monodromy in B which were associated to elements of $\mathcal{R}$ in [BMR] *braided reflections*. Recall from [BMR] that they generate B.

An important structure associated to the hyperplane complement X is its holonomy Lie algebra $\mathcal{T}$ (see e.g. [Ko]). It is generated by one element t_H for each hyperplane $H \in \mathcal{A}$, and there is a natural action of W by automorphisms of $\mathcal{T}$, given by $w.t_H = t_{w(H)}$. To each $H \in \mathcal{A}$ is associated a (well-defined) logarithmic 1-form $\omega_H = \mathrm{d}\alpha_H/\alpha_H$, for an arbitrary 1-form $\alpha_H \in V^*$ with kernel H.

To any representation $\varphi : \mathcal{T} \to \mathfrak{gl}_N(\mathbb{C})$ is associated a (family of) integrable 1-form(s)

$$\omega = h \sum_{H \in \mathcal{A}} \varphi(t_H) \omega_H$$

over X with values in $\mathfrak{gl}_N(\mathbb{C})$, depending on some complex parameter h. By monodromy this family of 1-forms defines a family of representations of $P = \pi_1(X)$, that can be considered as a representation $P \to \mathrm{GL}_N(A)$ with $A = \mathbb{C}[[h]] \subset K = \mathbb{C}((h))$.

If $\mathbb{C}^N = V_\rho$ for some linear representation ρ of W such that $\varphi : \mathcal{T} \to \mathfrak{gl}(V_\rho)$ is equivariant w.r.t. the natural actions of W, then the representation of P naturally extends to a representation $B \to \mathrm{GL}(V_\rho \otimes A)$. A change of basepoint gives rise to a representation which is isomorphic to the former one.

In [BMR] is introduced a representation of $\mathcal{T}$, depending on a collection $\tau_{c,j}$ of complex numbers for $c \in \mathcal{A}/W = \mathcal{S}/W$, where $\mathcal{S}$ here denotes the set of distinguished reflections, and $0 \leq j \leq e_c - 1$. For $H \in c$ and $s_H \in \mathcal{S}$ with $\mathrm{Ker}\,(s - 1) = H$, it is given by

$$\varphi(t_H) = \sum_{j=0}^{j=e_c-1} \tau_{H,j} \frac{1}{e_c} \sum_{k=0}^{k=e_c-1} \zeta_c^{-jk} \rho(s_H^k) = \sum_{k=0}^{e_c-1} \frac{1}{e_c} \left(\sum_{j=0}^{j=e_c-1} (\zeta_c^{-k})^j \tau_{H,j} \right) \rho(s_H)^k$$

with $\zeta_c = \det(s_H) = \exp(2\mathrm{i}\pi/e_c)$. The representation of B associated to that satisfies

$$\prod_{j=0}^{e_c-1} (\sigma - q_{H,j} \zeta_c^j) = 0$$

for any braided reflection associated to s_H and $q_{H,j} = \exp(-h\tau_{H,j}/e_c)$, and deforms ρ into a representation of the Hecke algebra of W, as defined in [BMR], with these parameters.

We note that the specialization of interest in the Broué-Malle-Michel 'Spetses' program (see [BMM]) is for $q_{H,0} = q$ and $q_{H,j} = 1$ for $j \neq 0$. This specialization corresponds to the choice of parameters $\tau_{H,j} = 0$ for $j \neq 0$.

We let $R : P \to \mathrm{GL}(V_\rho \otimes K)$ denote the representation of P associated to ρ. We refer to [Ma07b] for a proof of the next proposition.

Proposition 5.1. *For any representation ρ of W, the Lie algebra of the Zariski closure of $R(P)$ contains $\varphi(\mathcal{T}) \otimes K$. If $\varphi(\mathcal{T})$ preserves some bilinear form over V_ρ, then $R(P)$ preserves the induced bilinear form over $V_\rho \otimes K$.*

The group P is generated by the σ^{e_c} for σ running among the braided reflections, with c denoting the class of the hyperplane attached to σ. We have

$$\det R(\sigma^{e_c}) = \exp(he_c \mathrm{tr}\varphi(t_H)) = \exp h \sum_{k=0}^{e_c-1} \left(\sum_{j=0}^{j=e_c-1} (\zeta_c^{-k})^j \tau_{H,j} \right) \mathrm{tr}\rho(s_H)^k$$

We let $\lambda_k^c = \sum_{j=0}^{j=e_c-1} (\zeta_c^{-k})^j \tau_{c,j}$. Note that this defines a bijection between the (λ_k^c) and the $(\tau_{c,k})$ by invertibility of the Vandermonde determinant. With this notation, $\varphi(t_H) = (1/e_c) \sum_{k=0}^{e_c-1} \lambda_k^c \rho(s_H)^k$.

Also note that $\lambda_0^c = \sum_{j=0}^{e_c-1} \tau_{c,j}$. Up to tensoring R by the 1-dimensional representation that maps a braided reflexion around $H \in \mathcal{A}$ of class c to $\exp\left(h(\sum_{j=0}^{e_c-1} \tau_{c,j})/e_c\right)$, we can assume $\lambda_0^c = 0$, which we do from now on.

With this convention, we have $\varphi(\mathcal{T}) = \rho(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$, hence $\varphi(\mathcal{T})$ is a reductive Lie algebra. From proposition 5.1 and theorem 2.8 one readily gets the following.

Theorem 5.2. *If $\underline{\boldsymbol{\lambda}} \in \mathbb{A}(W)^\times$ then $\varphi(\mathcal{T})$ is reductive, $\varphi(\mathcal{T})' = \rho(\mathcal{H}')$ and $R(P)$ has connected Zariski closure, with Lie algebra $\rho(\mathcal{H}(\underline{\boldsymbol{\lambda}}))$.*

6. Unitarizability questions

The determination of the Zariski closure is specially useful when the representations involved are unitarizable. The monodromy construction as described above provides a morphism $B \to W \ltimes \exp \widehat{\mathcal{T}}$, where $\mathcal{T}$ is completed with respect to the graduation $\deg t_H = 1$.

Recall from [Ko] that $\mathcal{T}$ can be defined over an arbitrary field $\Bbbk$ of characteristic 0. We state the following conjecture :

Conjecture 1. *For an arbitrary complex (pseudo-)reflection group W and field $\Bbbk$ of characteristic 0, there exists morphisms $\Phi : B \to W \ltimes \exp \mathcal{T}$, with $\mathcal{T}$ defined over $\mathbb{Q}$, such that $\Phi(\sigma)$ is conjugated to $s \exp t_H$ by some element in $\exp \widehat{\mathcal{T}}$ for every braided reflection σ associated to s_H and $H \in \mathcal{A}$.*

Evidences for this conjecture include the fact that it holds for $W = \mathfrak{S}_n$ by using a rational associator as defined by Drinfeld (see [Dr]), for W of type $G(d, 1, n)$ by using the analogous gadgets by Enriquez ([En]; see also the appendix of [Ma06]), and for W of type $G(e, e, 2)$ (see [Ma06]). Another evidence is that all the varieties involved are defined over $\mathbb{Q}$, as proved in [MM].

Recall that it is conjectured in [BMR] that the Hecke algebra of W is always a flat deformation of the group algebra of W. In the sequel, we call this the BMR-conjecture. It is known to hold for all but possibly a finite number of exceptional cases.

Notice that a necessary condition for the representation R defined in a previous section to be unitary is that the $q_{H,j}\zeta_c^j$ have modulus 1. When h is specialized to a purely imaginary number, this happens exactly when the $\tau_{H,j}$ are real numbers, for $0 \le j \le e_c - 1$. Since, for $e_c - 1 \ge k \ge 1$,

$$\lambda_k^c - \overline{\lambda_{e_c-k}^c} = \sum_{j=1}^{e_c-1} (\zeta_c^{-k})^j (\tau_{c,j} - \overline{\tau_{c,j}}),$$

this imposes that λ_k^c is the complex conjugate of $\lambda_{e_c-k}^c$.

Theorem 6.1. *If conjecture 1 and the BMR-conjecture hold for W when $\Bbbk = \mathbb{R}$ and, for all $c \in \mathcal{C}_+$ and $0 \le j \le e_c - 1$, $\tau_{c,j} \in \mathbb{R}$, then R is unitarizable for $h \in i\mathbb{R}$ small enough.*

Proof. Up to tensoring the representation by a 1-dimensional unitary one, we can assume that $\lambda_0^c = 0$ for every $c \in \mathcal{C}^+$. The condition then exactly means that, for all $1 \le k \le e_c - 1$, λ_k^c is the complex conjugate of $\lambda_{e_c-k}^c$.

We introduce the following automorphisms of $K = \mathbb{C}((h))$. Let $\epsilon \in \text{Aut}(K)$ being defined through $f(h) \mapsto f(-h)$, and $\bar{\epsilon}$ be the its composed by the complex conjugation $\sum_{r \geq r_0} a_r h^r \mapsto \sum_{r \geq r_0} \overline{a_r}(-h)^r$. We can assume $\rho(W) \subset U_N$. We let $U_N^{\bar{\epsilon}}(K) = \{x \in \text{GL}_N(K) \mid {}^t\bar{\epsilon}(x) = x^{-1}\}$. Since $\varphi(t_H) = (1/e_c)\sum_{k=0}^{e_c-1} \lambda_k^c \rho(s_H)^k$ and the $\rho(s_H)^k$ are unitary, our assumption implies that $\varphi(t_H)$ is selfadjoint, hence $\exp h\varphi(t_H) \in U_N^{\bar{\epsilon}}(K)$. Let σ be a braided reflection corresponding to s_H, and $\tilde{\varphi} : \widehat{\mathcal{T}} \to \mathfrak{gl}_N(K)$ be defined through $t_H \mapsto h\varphi(t_H)$. Now consider the automorphism $A \in \text{Aut}(\mathfrak{gl}_N(K))$ given by $x \mapsto -{}^tx$. We have $A \circ \tilde{\varphi}(t_H) = -{}^th\varphi(t_H) = -h\overline{\varphi(t_H)} = \bar{\epsilon} \circ \tilde{\varphi}(t_H)$. It follows that $A \circ \tilde{\varphi} = \bar{\epsilon} \circ \tilde{\varphi}$ and in particular $-{}^t\tilde{\varphi}(\psi) = A \circ \tilde{\varphi}(\psi) = \bar{\epsilon} \circ \tilde{\varphi}(\psi) = \bar{\epsilon}(\tilde{\varphi}(\psi))$. Taking exponentials, we get

$$\left({}^t\exp(\tilde{\varphi}(\psi))\right)^{-1} = \exp(-{}^t\tilde{\varphi}(\psi)) = \exp(\bar{\epsilon}(\tilde{\varphi}(\psi))) = \bar{\epsilon}\left(\exp((\tilde{\varphi}(\psi))\right)$$

whence $\exp \tilde{\varphi}(\psi) \in U_N^{\bar{\epsilon}}(K)$. Finally note that $\rho(s_H) \in U_N \subset U_N^{\bar{\epsilon}}(K)$. Now $(\rho, \tilde{\varphi})$ provide a morphism $W \ltimes \exp\widehat{\mathcal{T}} \to \text{GL}_N(K)$. Combining it with Φ we thus get a representation $R' : B \to U_N^{\bar{\epsilon}}(K)$, that factorizes through the Hecke algebra. Under the BMR-conjecture, we have $R \simeq R'$, since both R and R' specialize to ρ through $h = 0$.

Let L be a subfield of K containing $\mathbb{R}$, the entries of the $R'(b)$ for $b \in B$ as well as $i \in \mathbb{C}$ and h. Since B is finitely generated we can take for L a finitely generated extension of $\mathbb{R}(h)$. Up to considering $L + \epsilon(L)$ we can assume $\epsilon(L) = L$. Since $i \in L$ we have $L = L_0 \oplus iL_0$ with $L_0 \in \mathbb{R}((h))$ and $\epsilon(L_0) = L_0$. By [Ma09b] proposition 3.1 we know that L_0 is isomorphic to some finitely generated extension L_0^* of $\mathbb{R}(h)$ inside the field of convergent Laurent series $\mathbb{R}(\{h\})$, through a ϵ-equivariant isomorphism. It extends to $\bar{\epsilon}$-equivariant isomorphism between $L = L_0 + iL_0$ and $L^* = L_0^* + iL_0^*$. Since $R'(B) \subset U_N^{\bar{\epsilon}}(L) \simeq U_N^{\bar{\epsilon}}(L^*)$ we get a new representation $R'' : B \to U_N^{\bar{\epsilon}}(\mathbb{C}(\{h\}))$. The isomorphism $L_0 \to L_0^*$ can be chosen such that the entries of the $R(b)$, for b running inside a finite set of generators for B, are unchanged modulo h. By the same argument as above, under the BMR-conjecture, R'' is isomorphic to R' hence to R over $\mathbb{C}((h))$. Since R'' and R are both defined over $\mathbb{C}(\{h\})$, they are conjugated over this field. Since they coincide modulo h they are both defined over the ring of convergent power series, so their specialisations to small h are isomorphic over $\mathbb{C}$. Now, for $h \in i\mathbb{R}$ small enough, the specialization of R'' is unitary, and this concludes the proof. $\square$

Corollary 6.2. *If W is a complex reflection group of type $G(de, e, r)$, then R is unitarizable under the above conditions on the parameters.*

Proof. The BMR-conjecture is known to hold for W of these types. Since conjecture 1 holds for W of type $G(de, 1, r)$ by [En], we get unitarity for these groups. Now the Hecke algebra of type $G(de, e, r)$ is a subalgebra of the one of type $G(de, 1, r)$, with images of braided reflexions mapped to images of braided reflections (see [RR]); since any irreducible representation of the Hecke algebra of type $G(de, e, r)$ appears in the restriction of one of type $G(de, 1, r)$ (see [RR]), this concludes the proof of the corollary. $\square$

We recall from [Ma09a] that the unitarizability is known for Coxeter groups, and was also proved for the reflection representation by geometric methods in [CHL].

In the unitary cases, and for transcendant values of the parameters, the Lie algebra of the topological closure is then given by a compact form of the Lie algebra of the Zariski closure. Such a compact form is described by the following generalization of [Ma09a] prop. 2.27.

Proposition 6.3. *Assume that, for all $c \in \mathcal{C}_+$ and $1 \leq k \leq e_c - 1$, λ_k^c is the complex conjugate of $\lambda_{e_c-k}^c$. Then the real Lie subalgebra $\mathcal{H}_c(\underline{\lambda})$ of $\mathbb{C}W$ generated by the $\mathrm{i}s, s \in \mathcal{S}_0$ and the $\mathrm{i}(\sum \lambda_k^c s^k)$ for $s \in c \subset \mathcal{S}_+$ is a compact real Lie subalgebra of $\mathcal{H}(\underline{\lambda})$. Moreover, $\mathcal{H}_c(\underline{\lambda})'$ is a compact real form of $\mathcal{H}(\underline{\lambda})'$.*

Proof. We define on $\mathbb{C}W$ a sesquilinear form by $(w_1, w_2) = \delta_{w_1,w_2}$ for $w_1, w_2 \in W$. This form is clearly positive definite. By left action we have $\mathcal{H}(\underline{\lambda}) \subset \mathbb{C}W \subset \mathrm{End}(\mathbb{C}W)$, and the generators of $\mathcal{H}_c(\underline{\lambda})$ are easily checked to satisfy, under our assumption, the equality $x^* = -x$, where x^* denotes the adjoint of $x \in \mathrm{End}(\mathbb{C}W)$ with respect to our form. This proves that $\mathcal{H}_c(\underline{\lambda})$ is a real Lie subalgebra of the compact Lie algebra $\mathfrak{u}(\mathbb{C}W)$, and therefore is a compact Lie algebra. Now $\mathcal{H}_c(\underline{\lambda})' + \mathrm{i}\mathcal{H}_c(\underline{\lambda})$ is a complex Lie subalgebra of $\mathcal{H}(\underline{\lambda})'$, which equals $\mathcal{H}(\underline{\lambda}')$ because every iterated bracket $[x_1, [x_2, \ldots, [x_{r-1}, x_r] \ldots]$ in the defining generators of $\mathcal{H}(\underline{\lambda})$ can be written as $(-\mathrm{i})^r[y_1, [y_2, \ldots, [y_{r-1}, y_r] \ldots]$ where the $y_k = \mathrm{i}x_k$ are the defining generators of $\mathcal{H}_c(\underline{\lambda})$. Since $\mathcal{H}_c(\underline{\lambda})'$ is a compact Lie subalgebra of the semisimple Lie algebra $\mathcal{H}(\underline{\lambda})'$, its real dimension is at most $\dim_{\mathbb{C}} \mathcal{H}(\underline{\lambda})' = (\dim_{\mathbb{R}} \mathcal{H}(\underline{\lambda})')/2$ (otherwise it would define a compact subgroup of the semisimple group $\exp \mathcal{H}(\underline{\lambda})'$, of real dimension larger than its maximal compact subgroups). It follows that $\mathcal{H}_c(\underline{\lambda})' \cap \mathrm{i}\mathcal{H}_c(\underline{\lambda})' = \{0\}$ hence $\mathcal{H}_c(\underline{\lambda})'$ is a compact real form of $\mathcal{H}(\underline{\lambda})'$. $\qquad\square$

7. Special situations

7.1. **The special Hecke algebra.** Let W an irreducible reflection group, and $\rho^1, \rho^2 \in \mathrm{Irr}(W)$. We assume that $\rho_{\mathcal{H}_s'}^1 \simeq \rho_{\mathcal{H}_s'}^2$. We can assume that $\rho^2(s) = \rho^1(s) + \omega_s$ for all $s \in \mathcal{S}$. Note that $\omega_s \in \Bbbk$ only depends on the conjugacy class of $\mathcal{S}$. If $\mathcal{X} = \mathrm{Sp}\rho^1(s)$ we have $\mathcal{X} + \omega_s = \mathrm{Sp}\rho^2(s) \subset \mu_n$, where n is the order of s. By lemma 2.5 we have $\omega_s \neq 0 \Rightarrow |\mathrm{Sp}(\rho^2(s))| \leq 2$. Up to tensoring ρ^1 by some character in $\mathrm{X}(\rho^1)$, we can assume that, for all $s \in \mathcal{S}$, either $\rho^2(s) = \rho^1(s)$, or $|\mathrm{Sp}(\rho^2(s))| = 2$ when $\omega_s \neq 0$. When $\mathrm{Sp}(\rho^2(s)) = \{\alpha, \beta\}$ with $\alpha \neq \beta$, then $\omega_s = \alpha + \beta$ and $\mathrm{Sp}\rho^1(s) = \{-\alpha, -\beta\}$. This is only possible for n an even integer. Also note that, if $n = 2$, then necessarily $\omega_s = 0$. We can thus assume that $n > 2$ and n is even.

Since $\rho^1(s)$ is semisimple, $\rho^1(s^{-1}) = (-\alpha^{-1}\beta^{-1})(\rho^1(s) + \omega_s)$, hence there exists $\chi \in \mathrm{Hom}(W, \Bbbk^\times)$ defined by $\chi(s) = 1$ if $\omega_s = 0$ and $\chi(s) = (-1/\alpha\beta)$ when $\omega_s \neq 0$ and $\mathrm{Sp}\rho^1(s) = \{\alpha, \beta\}$ with $\alpha \neq \beta$, such that $\rho^3 = \rho^2 \otimes \chi$ is an irreducible representation of W that satisfies $\rho^3(s) = \rho^1(s^{-1})$ for all $s \in \mathcal{S}$ with $\omega_s \neq 0$ and $\rho^3(s) = \rho^1(s)$ when $\omega_s = 0$.

Using the Shephard-Todd classification we can check on the representations of the exceptional groups that these situations do not occur for W an exceptional group. The first remark is that the orders of the pseudo-reflections for such a group are at most 5, so the only cases to consider are when W admits a pseudo-reflection of order 4. There are four such groups,

namely G_8, G_9, G_{10} and G_{11}. In these four cases, there are six pseudo-reflections of order 4, and only one class of such pseudo-reflections in $\mathcal{S}$. We then restrict to the $\rho^1 \in \mathrm{Irr}(W)$ such that the formulas $s \mapsto \rho^1(s^{-1})$ if $s \in \mathcal{S}$ has order 4 and $s \mapsto \rho^1(s)$ otherwise define a representation of W. All such representations have dimensions 1 or 2. Then the representation ρ^2 is given by $\rho^2(s) = -(\det \rho^1(s))\rho^1(s^{-1})$ when $s \in \mathcal{S}$ has order 4 and $\rho^2(s) = \rho^1(s)$ otherwise, and it is readily checked that ρ^2 has the same character as ρ^1 in all cases. This proves the following.

Proposition 7.1. *Let W be an exceptional irreducible reflection group. For $\rho^1, \rho^2 \in \mathrm{Irr}(W)$, $\rho^1_{\mathcal{H}'_s} \simeq \rho^2_{\mathcal{H}'_s}$ if and only if $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$, that is $\rho^2 = \rho^1 \otimes \eta$ for some $\eta \in \mathrm{X}(\rho^1)$.*

We now consider $W = G(de, e, r)$, for which we can assume with $d > 2$. The elements in $\mathcal{S}$ of order more than 2 have order d and form a single conjugacy class $c \subset \mathcal{S}$. We can thus assume d even, and let $\zeta = \zeta_c$. The elements of c are then the $t_i = \mathrm{diag}(1, \ldots, 1, \zeta, 1, \ldots, 1)$ for $1 \le i \le r$. Is is easily checked (see e.g. [MM]) that the formulas $t_i \mapsto t_i^{-1}$ define (uniquely) an automorphism $\mathfrak{c}$ of W that fixes $\mathfrak{S}_r \subset W$.

It follows that $\rho^2_{\mathcal{H}'_s} \simeq \rho^1_{\mathcal{H}'_s}$ if and only if either $\rho^2_{\mathcal{H}'} \simeq \rho^1_{\mathcal{H}'}$ or $\rho^1(t_1)$ has two eigenvalues $\alpha \ne \beta$ and, up to tensoring by some character in $\mathrm{X}(\rho^1)$, $\rho^2 = \chi \otimes \rho^1 \circ \mathfrak{c}$, where $\chi \in \mathrm{Hom}(W, \Bbbk^\times)$ is defined by $\chi(t_i) = -\alpha\beta$, $\chi(s) = 1$ for $s \notin c$.

Proposition 7.2. *Let $W = G(de, e, r)$ with $d > 2$ and $\rho^1, \rho^2 \in \mathrm{Irr}(W)$. If d is odd or $\mathrm{Sp}(\rho^1(t_1)) \ne 2$ then $\rho^1_{\mathcal{H}'_s} \simeq \rho^2_{\mathcal{H}'_s}$ iff $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$.*

The following is then a consequence of lemma 2.10.

Corollary 7.3. *For $G(de, e, r)$ with d an odd integer, or W an exceptional group, $\mathcal{H}'_s = \mathcal{H}'$.*

We consider the special case $e = 1$, and ρ^1, ρ^2 with $\rho^1_{\mathcal{H}'} \not\simeq \rho^2_{\mathcal{H}'}$ but $\rho^1_{\mathcal{H}'_s} \simeq \rho^2_{\mathcal{H}'_s}$. By the arguments above $d > 2$ is even, and $\rho^1(t)$ admits two eigenvalues. This means that ρ^1 is labelled by a multipartition $(a_0, \ldots, a_{d-1})$ with two non-empty parts a_i, a_j with $i \ne j$, and $\mathrm{Sp}\rho^1(t_1) = \{\zeta^{a_i}, \zeta^{a_j}\}$ with $\zeta = \exp(2i\pi/d)$. Then $\rho^3 = \rho^1 \circ \mathfrak{c}$ corresponds to the multipartition $(b_0, b_1, \ldots, b_{d-1}) = (a_0, a_{d-1}, a_{d-2}, \ldots, a_2, a_1)$, and tensoring by χ with $\chi(t_1) = -a_i a_j = \zeta^k$ with $\zeta^k = -\zeta^{i+j} = \zeta^{d/2+i+j}$, e.g. $k = d/2 + i + j$, leads to $\rho^2 = \rho^3 \otimes \chi$ labelled by $(b_{0-k}, b_{1-k}, \ldots, b_{d-2-k}, b_{d-1-k})$. For instance, if $d = 4, r = 3$ and ρ^1 is labelled by $([2], [1], \emptyset, \emptyset)$, we have $\mathrm{X}(\rho^1) = \{\mathbb{1}\}$ and the only possibility is for ρ^2 labelled by $(\emptyset, \emptyset, [1], [2])$. More generally, if ρ^1 is the defining representation of W, labelled by $([r-1], [1], \emptyset)$, then $\mathrm{X}(\rho^1) = \{\mathbb{1}\}$, but $\rho^1_{\mathcal{H}'_s} \simeq \rho^2_{\mathcal{H}'_s}$ with $\rho^1 \not\simeq \rho^2$ as the formulas above guarantee, for $d > 2$ and $r \ge 3$, that the first part of (the multipartition labelling) ρ^2 will not be $[r-1]$.

For $e > 1$, this example also provides ρ^1, ρ^2 with $\rho^1_{\mathcal{H}'} \not\simeq \rho^2_{\mathcal{H}'}$ but $\rho^1_{\mathcal{H}'_s} \simeq \rho^2_{\mathcal{H}'_s}$. As a consequence, we get that the above corollary is sharp.

Proposition 7.4. *If $d > 2$ is even and $r \ge 3$, then $\mathcal{H}'_s \not\simeq \mathcal{H}'$.*

7.2. **Cubic hecke algebras.** We assume that, for all $s \in \mathcal{S}$, $s^3 = 1$, and that there exists a single class c of reflecting hyperplanes for W. For irreducible reflection group of rank at least 2, this happens exactly for the exceptional types G_4, G_{25}, G_{32}. In that case we have two parameters λ_1, λ_2, $\mathbb{A}(W) = \{(\lambda_1, \lambda_2) \in \Bbbk^2\}$ and $\mathcal{H}(\lambda_1, \lambda_2)$ is generated by the $\lambda_1 s + \lambda_2 s^2$ for $s \in \mathcal{S}$. Leting $j = \zeta_c = \exp(2\mathrm{i}\pi/3)$, we have $v_0 = (1,1)$, $v_1 = (j, j^2)$, $v_2 = (j^2, j)$. It is easily checked that we have here $\mathcal{L}_3 \subset \mathcal{L}_2$ hence $\mathbb{A}(W)^\times = \mathbb{A}(W) \setminus \bigcup \mathcal{L}_2$. We have

$$
\begin{aligned}
\mathcal{L}_1 &= \{\mathrm{Ker} <v_0 - v_1|\cdot>, \mathrm{Ker} <v_0 - v_2|\cdot>, \mathrm{Ker} <v_1 - v_2|\cdot>\} \\
\mathcal{L}_2 &= \mathcal{L}_1 \sqcup \{\mathrm{Ker} <v_0 + v_1 - 2v_2|\cdot>, \mathrm{Ker} <v_0 + v_2 - 2v_1|\cdot>, \\
&\quad \mathrm{Ker} <v_1 + v_2 - 2v_0|\cdot>\}
\end{aligned}
$$

It is clear that $\mathcal{H}(0,0) = \{0\}$, and we know $\mathcal{H}(0,1) \simeq \mathcal{H}(1,0) = \mathcal{H}_s$, which has already been studied (moreover $(1,0)$ and $(0,1)$ clearly belong to $\mathbb{A}(W)^\times$), so we can assume $\lambda_1 \lambda_2 \neq 0$. Then $\mathcal{H}(\lambda_1, \lambda_2) = \lambda_1 \mathcal{H}(1, \lambda_2/\lambda_1)$, so we can assume $\lambda_1 = 1$ and $\lambda_2 = a \in \Bbbk^\times$ and let $\mathcal{H}(a) = \mathcal{H}(1, a)$. A straightforward computation shows $(1, a) \in \bigcup \mathcal{L}_1$ iff $a \in \mu_3$, and $(1, a) \in \bigcup \mathcal{L}_2$ iff $a \in \mu_3 \cup (-\mu_3)$. In particular $(1, a) \in \mathbb{A}(W)^\times$ iff $a \notin \mu_6$.

For $a^6 \neq 1$ (and $a \neq 0$), we have $\mathcal{H}(\underline{\lambda})' = \mathcal{H}'$. Letting $\Phi \in \mathrm{Aut}(\Bbbk W)$ denote as above the automorphism $s \mapsto js$ for $s \in \mathcal{S}$, we have $\Phi(\mathcal{H}(a)) = \mathcal{H}(j, j^2 a) = j\mathcal{H}(1, ja) = \mathcal{H}(1, ja) = \mathcal{H}(ja)$, so there are only two cases to consider, $a = 1$ for $a \in \mu_3$, and $a = -1$ for $a \in -\mu_3$.

If $W = G_4$, there are three representations in $\overline{\mathrm{QRef}} = \mathrm{QRef} = \Lambda\mathrm{Ref}$ of dimension 2, three 1-dimensional characters and a 3-dimensional one that we denote ρ_3. The 2-dimensional ones are the restrictions to a parabolic subgroup of type G_4 of the representations $U_{\alpha,\beta}$ of G_{25}. In order to avoid confusion, we denote them U_γ for $\{\alpha, \beta, \gamma\} = \mu_3(\mathbb{C})$. The 3-dimensional one is the restriction $\overline{V}$ of $V \in \mathrm{Irr}(G_{25})$. We assume that $\mathcal{S}$ is given by the distinguished pseudo-reflections.

We have $\mathcal{H}' = \mathfrak{sl}(U_1) \times \mathfrak{sl}(U_j) \times \mathfrak{sl}(U_{j^2}) \times \mathfrak{sl}(\overline{V}) \simeq (\mathfrak{sl}_2)^3 \times \mathfrak{sl}_3$. For $a = 1$, we get by computer that $\dim \mathcal{H}(1) = 15$, $\dim \mathcal{H}(1)' = 14$, $\dim Z(\mathcal{H}(1)) = 1$, and that the image in each irreducible representation of W of $\mathcal{H}(1)'$ is the same as $\mathcal{H}'$, except for U_1, where the image is 0. Since $\mathcal{H}(1)'$ is semisimple this implies $\mathcal{H}(1)' = \mathfrak{sl}(U_j) \times \mathfrak{sl}(U_{j^2}) \times \mathfrak{sl}(\overline{V}) \simeq \mathfrak{sl}_2^2 \times \mathfrak{sl}_3$. Moreover, the center of $\mathcal{H}(1)$, having dimension 1, is also spanned by $T_{\mathcal{S}}(1) = \sum_{s \in \mathcal{S}} s + s^2$ in this case. Another argument for this last fact, that will be used for G_{25}, is to notice that the image of $\mathcal{H}(1)$ in U_1 is $\mathbb{C}$ (more precisely, $s + s^2$ acts by -1). It follows that $Z(\mathcal{H}(1)) \subset Z(\mathcal{H}) \subset Z(\mathbb{C}W)$. Since $\mathcal{H}(1)$ is reductive and $\mathcal{H}(1)' \subset \mathrm{Ker}\, p$, we thus recover $Z(\mathcal{H}(1)) = p(\mathcal{H}(1)) = \mathbb{C}T_{\mathcal{S}}(1)$.

For $a = -1$, we get by computer $\dim \mathcal{H}(-1)' = 6$, and the image in each of the irreducible representations of W of $\mathcal{H}(-1)'$ is the same as $\mathcal{H}'$, except for $\overline{V}$, for which the image is of dimension 3. Moreover, since $a \notin \mu_3$, $\mathcal{H}(-1)'$ is semisimple. We check that the image of $\mathcal{H}(-1)'$ in $\mathfrak{sl}(U_j) \times \mathfrak{sl}(U_{j^2}) \times \mathfrak{sl}(\overline{V})$ has dimension 3, hence $\mathcal{H}(-1)' \simeq \mathfrak{sl}(U_1) \times \mathfrak{sl}_2 \simeq (\mathfrak{sl}_2)^2$, the restriction of the three representations $U_j, U_{j^2}, \overline{V}$ to $\mathcal{H}(\underline{\lambda})'$ factorizing through the same ideal $\mathfrak{sl}_2$. More previsely, we can identify $\mathcal{H}(-1)'$ with $\mathfrak{sl}(A_1) \times \mathfrak{sl}(A_2)$ with A_1, A_2 two vector spaces with $\dim A_i = 2$, such that the representations of $\mathcal{H}(-1)'$

corresponding to $U_1, U_j, U_{j^2}, \overline{V}$ are A_1, A_2, A_2, S^2A_2, as $\mathfrak{sl}_2$ admits only one irreducible representation in each dimension.

We now consider $W = G_{25}$. For $a = 1$, we get $\rho(\mathcal{H}(1)') = \rho(\mathcal{H}')$ for each $\rho \in \mathrm{Irr}(W)$ except U_{j,j^2} and U'_{j^2,j^2}, U'_{j,j^2} (recall that $U'_{j,j^2} \approx U'_{j^2,j}$), in which case $\rho(\mathcal{H}(1)') = \{0\}$. Moreover, we check that $\rho(\mathcal{H}(1)) = \mathbb{C}$, and more precisely $\rho(s + s^2) = -1$ in these latter cases. We check by computer that, for any ρ^1, ρ^2 not in these exceptional cases and $\rho^1 \not\approx \rho^2$, then $(\rho^1 \oplus \rho^2)(\mathcal{H}(1)')$ has dimension $\dim \rho^1(\mathcal{H}(1)') + \dim \rho^2(\mathcal{H}(1)')$, hence $(\rho^1 \oplus \rho^2)(\mathcal{H}(1)') = \rho^1(\mathcal{H}(1)') \oplus \rho^2(\mathcal{H}(1)')$. This proves that the simple ideals determined by the non-exceptional $\rho \in \mathrm{Irr}(W)$ (up to $\approx$) never coincide, hence $\mathcal{H}(1)' \simeq \mathfrak{sl}_2^2 \times \mathfrak{sl}_3^3 \times \mathfrak{sl}_6^6 \times \mathfrak{sl}_8^3 \times \mathfrak{sl}_9^2$ is the kernel of the representation $U_{j,j^2} \oplus U'_{j^2,j}$ restricted to $\mathcal{H}'$. We get that $Z(\mathcal{H}(1)) = \mathbb{C}T_{\mathcal{S}}(1,1)$ by the same argument as above.

The situation for $\mathcal{H}(-1)$ is a lot messier. As before, computing the dimensions of $\rho(\mathcal{H}(-1)')$ determines the type of the corresponding ideals (notice that $\mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{sl}_2 \times \mathfrak{sl}_3$ are the only semisimple algebras of dimensions 3,8, and 11 respectively) ; computing the dimensions of the $(\rho^1 \oplus \rho^2)(\mathcal{H}(-1)')$ determines the simple ideals of $\mathcal{H}(-1)$. We get that

$$
\begin{aligned}
\mathcal{H}(-1) &= \mathfrak{sl}(A_1) \times \mathfrak{sl}(A_2) \times \mathfrak{sl}(B_1) \times \mathfrak{sl}(B_2) \times \mathfrak{sl}(C) \times \mathfrak{sl}(D) \times \mathfrak{sl}(E) \\
&\simeq \mathfrak{sl}_2^2 \times \mathfrak{sl}_3^2 \times \mathfrak{sl}_6 \times \mathfrak{sl}_8 \times \mathfrak{sl}_9
\end{aligned}
$$

with $\dim A_i = 2$, $\dim B_i = 3$, $\dim C = 6$, $\dim D = 8$ and $\dim E = 9$. Under this identification, the isomorphism type of the representations $\rho_{\mathcal{H}(-1)'}$ is then determined using $\dim \rho$, the dimension of the invariants in $\rho^1_{\mathcal{H}(-1)'} \otimes \rho^2_{\mathcal{H}(-1)'}$ (which distinguish e.g. between V and V^* for a representation of $\mathfrak{sl}(V)$), and the dimensions of $(\rho^1_{\mathcal{H}(-1)'} \otimes \rho^2_{\mathcal{H}(-1)'})(\mathsf{U}\mathfrak{g})$, which often determines the number of irreducible components in $(\rho^1_{\mathcal{H}(-1)'} \otimes \rho^2_{\mathcal{H}(-1)'})$. The result is tabulated below. In this table, S^2V denote the symmetric square of V, and $F_{[2,1]}$ denotes the Schur functor associated to the partition $[2,1]$, so that $V^{\otimes 3} = S^3V \oplus \Lambda^3 V \oplus 2F_{[2,1]}(V)$.

ρ	U_{j,j^2}	$U_{1,j}$	U_{1,j^2}	V		
$\rho(\mathcal{H}(-1)')$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}_2$		
$\rho_{\mathcal{H}(-1)'}$	A_1	A_2	A_2	S^2A_2		
ρ	$U'_{j,1}$	U'_{1,j^2}	$U'_{j^2,j}$	$U'_{j^2,1}$	$U'_{1,j}$	U'_{j,j^2}
$\rho(\mathcal{H}(-1)')$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$
$\rho_{\mathcal{H}(-1)'}$	B_1	B_1	B_2	B_1^*	B_1^*	B_2^*
ρ	$V_{j^2,1}$	V_{j,j^2}	$V_{1,j}$	$V_{j,1}$	$V_{j^2,j}$	V_{1,j^2}
$\rho(\mathcal{H}(-1)')$	$\mathfrak{sl}_3$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}_3 \times \mathfrak{sl}_2$	$\mathfrak{sl}_3$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}_3 \times \mathfrak{sl}_2$
$\rho_{\mathcal{H}(-1)'}$	$S^2(B_1^*)$	C	$B_1 \otimes A_2$	S^2B_1	C^*	$(B_1^*) \otimes A_2$
ρ	W_1	W_{j^2}	W_j	X	X^*	
$\rho(\mathcal{H}(-1)')$	$\mathfrak{sl}_3$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	$\mathfrak{sl}(V_\rho)$	
$\rho_{\mathcal{H}(-1)'}$	$F_{[2,1]}(B_1)$	D	D^*	E	E^*	

7.3. **The spetsial Hecke algebra.** In view of the Broué-Malle-Michel 'Spetses' program, a specialization of interest is when all the λ_i are equal. We

denote $\mathcal{H}_{st}$ the Lie subalgebra of $\mathcal{H}$ generated by the $\sum_{k=1}^{e_c-1} s^k$ for $s \in c \subset \mathcal{S}$. We have $\mathcal{H}_0 \subset \mathcal{H}_{st} \subset \mathcal{H}$. The decomposition of $\mathcal{H}_{st}$ for the exceptional groups G_4, G_{25} were done in the previous section.

We contend ourselves here to deal with the groups $G(d, 1, r)$.

Proposition 7.5. *Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{d-1})$ be a multipartition of r, $W = G(d, 1, r)$, and $\rho \in \mathrm{Irr}(W)$ the representation of W labelled by $\boldsymbol{\lambda}$. If $\lambda_0 \neq \emptyset$, then $\rho_{\mathcal{H}_{st}}$ is irreducible. If $\lambda_0 = \emptyset$, then $\rho(\mathcal{H}'_{st}) = \rho(\mathcal{H}'_0)$.*

Proof. On note $P_i = (1/d) \sum_{k=0}^{d-1} \rho(t_i)^k$. Letting $\mathbf{T} = (T_0, \ldots, T_{d-1})$ be a multitableau of shape $\boldsymbol{\lambda}$, we have $P_i \mathbf{T} = 0$ if $i \notin T_0$ and $P_i \mathbf{T} = \mathbf{T}$ otherwise. If $\lambda_0 = \emptyset$, then $\rho(\cup \mathcal{H}_{st}) = \rho(\cup \mathcal{H}_0)$ hence $\rho_{\mathcal{H}_{st}}$ is semisimple.

Now assume $\lambda_0 \neq \emptyset$ and U be a subspace setwise invariant under $\mathcal{H}_{st}$. Since $\mathcal{H}_0 \subset \mathcal{H}_{st}$ it is invariant under $W_0 = G(d, d, r)$, hence a sum of the irreducible components. In particular, it is irreducible unless the sequence $\lambda_0, \lambda_1, \ldots$ has a period $0 < u < d$. In that case, the irreducible components for W_0 are the eigenspaces of the endomorphism S of order d/u defined by $S(\mathbf{T}) = (T_u, T_{1+u}, \ldots, T_{d-1+u})$ (see [MM] §2.4), hence U is setwise invariant S. Now let $v = \sum \alpha_{\mathbf{T}} \mathbf{T} \in U$ with $v \neq 0$, that is $\alpha_{\mathbf{T}^0} \neq 0$ for some multitableau $\mathbf{T}^0$. Since $\lambda_0 \neq \emptyset$ there exists some i in $(\mathbf{T}^0)_0$. Then $w = P_i v = \sum_{i \in \mathbf{T}_0} \alpha_{\mathbf{T}} \mathbf{T} \in U \setminus \{0\}$. By definition of S the family $w, S(w), \ldots, S^{d-1}(w) \in U$ is free, hence the $\sum_{k=0}^{d/u-1} \zeta^{sk} S^k(w)$ for $\zeta = \exp(2i\pi d/u)$ afford eigenvectors for all eigenvalues of S. It follows that U meets every irreducible components of the restriction to W_0, hence U is the whole space and $\rho_{\mathcal{H}_{st}}$ is irreducible. $\square$

We assume $r \geq 3$.

Proposition 7.6. *Let $W = G(d, 1, r)$ with $r \geq 3$, and $W_0 = G(d, d, r) < W$. Let $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ associated to multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$. If both $\boldsymbol{\lambda}, \boldsymbol{\mu}$ have a single nonempty part, then $\rho^1_{\mathcal{H}'_{st}} \simeq \rho^2_{\mathcal{H}'_{st}}$ iff this single part is the same. Assuming this is not the case, $\rho^1_{\mathcal{H}'_{st}} \simeq \rho^2_{\mathcal{H}'_{st}}$ iff $\mathrm{Res}_{W_0} \rho^1 = \mathrm{Res}_{W_0} \rho^2$ when $\lambda_0 = \mu_0 = \emptyset$, and $\rho^1_{\mathcal{H}'_{st}} \simeq \rho^2_{\mathcal{H}'_{st}}$ iff $\boldsymbol{\lambda} = \boldsymbol{\mu}$ otherwise.*

Proof. We assume that $\rho^2_{\mathcal{H}'_{st}} = \rho^1_{\mathcal{H}'_{st}}$. Since $\rho^1_{\mathcal{H}'_0} \simeq \rho^2_{\mathcal{H}'_0}$, we know from [Ma09a] that ρ^2 is associated to a multipartition $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_{d-1})$ with $\mu_i = \lambda_{i+k}$ for all i and a given $k \in [1, d-1]$. We thus can assume that $V_{\rho^1} = V_{\rho^2}$ has for basis the collection of multitableaux of shape $\boldsymbol{\lambda}$, with ρ^1 affording the usual action on them, $\rho^1(w) = \rho^2(w)$ for $w \in W_0$, and $\rho^2(t_i) \mathbf{T} = \zeta^j \mathbf{T}$ iff $\rho^1(t_i) \mathbf{T} = \zeta^{j-k}$. Now let $\mathbf{t}_i = \sum_{j=0}^{d-1} t_i^j$. Since $\rho^1_{\mathcal{H}'_{st}} \simeq \rho^2_{\mathcal{H}'_{st}}$ we have $\rho^2(\mathbf{t}_i) = \rho^1(\mathbf{t}_i) + \omega_i$ for some $\omega_i \in \mathbb{C}$. Since $\mathrm{Sp}(\rho^1(\mathbf{t}_i)), \mathrm{Sp}(\rho^2(\mathbf{t}_i)) \subset \{0, d\}$ we get that, either the $\rho^1(\mathbf{t}_i)$ are scalars, or $\omega_i = 0$ for all i. If the $\rho^1(\mathbf{t}_i)$ are not scalars, then $\rho^1(\mathbf{t}_i) = \rho^2(\mathbf{t}_i)$ for all i, and also $\lambda_0 \neq \emptyset$ (otherwise $\rho^1(\mathbf{t}_i) = 0$). Let then $\mathbf{T}$ be a multitableau of shape $\boldsymbol{\lambda}$ with $1 \in T_0$. We have $\rho^1(\mathbf{t}_1) \mathbf{T} = d\mathbf{T}$ and $\rho^2(\mathbf{t}_1) \mathbf{T} = 0$ unless $k = 0$, which implies $\rho^1 = \rho^2$.

We now assume that the $\rho^1(\mathbf{t}_i)$, $\rho^2(\mathbf{t}_i)$ are scalars. Then $\rho^1_{\mathcal{H}'_{st}} \simeq \rho^2_{\mathcal{H}'_{st}}$ iff $\rho^1_{\mathcal{H}'_0} \simeq \rho^2_{\mathcal{H}'_0}$, which is equivalent to $\mathrm{Res}_{W_0} \rho^1 = \mathrm{Res}_{W_0} \rho^2$ by [Ma09a]. This case means that either $\boldsymbol{\lambda}, \boldsymbol{\mu}$ have a single part, that is the $\rho^1(t_i)$ and $\rho^2(t_i)$ themselves are scalars, and then $\rho^1_{\mathcal{H}'_0} \simeq \rho^2_{\mathcal{H}'_0}$ implies that these single parts

are the same, or $\lambda_0 = \mu_0 = \emptyset$. This concludes the proof, as the converse implications are obvious. $\qquad\square$

Proposition 7.7. *Let $W = G(d,1,r)$ with $r \geq 3$. Let $\rho \in \mathrm{Irr}(W)$. If $\rho^2 \in \mathrm{Irr}(W)$ satisfies $(\rho^2)_{\mathcal{H}'_{st}} \simeq (\rho_{\mathcal{H}'_{st}})^*$, then $(\rho^2)_{\mathcal{H}'} \simeq (\rho_{\mathcal{H}'})^*$.*

Proof. Recall from [Ma09a] that, if ρ_0 is the restriction to $W_0 = G(d,d,r)$ of $\rho \in \mathrm{Irr}(W)$ associated to $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{d-1})$, then $\rho_0^* \otimes \epsilon$ is the restriction to W_0 of the representation labelled $\boldsymbol{\lambda}^* := (\lambda'_0, \lambda'_{d-1}, \ldots, \lambda'_1)$.

Assume $\rho, \rho^2 \in \mathrm{Irr}(W)$ with $(\rho_{\mathcal{H}_{st}})^* \simeq \rho^2_{\mathcal{H}'_{st}}$. We have $(\rho_{\mathcal{H}_{st}})^*_{|\mathcal{H}'_0} \simeq (\rho_{\mathcal{H}'_0})^* \simeq ((\rho_{|W_0})^* \otimes \epsilon)_{\mathcal{H}'_0}$. It follows that $\mu_i = \lambda'_{k-i}$ for some $k \in [1, d-1]$, by [Ma09a]. We define a linear isomorphism $V_\rho \to V_{\rho^2}$ by $\sigma(\mathbf{T}) = \mathbf{T}' := (T'_{0+k}, \ldots, T'_{j+k}, \ldots, T'_{1+k})$, where T' denotes the transpose of the (standard) tableau T. Transporting the representation ρ^2 to V_{ρ^1} using σ, this identify $\rho^2(x)$ with $-{}^t\rho(x)$ for any $x \in \mathcal{H}_0$. For $\mathbf{T}$ a multitableau of shape $\boldsymbol{\lambda}$, we have (under σ) $\rho^2(t_i)\mathbf{T} = \zeta^j \mathbf{T}$ iff $\rho^1(t_i)\mathbf{T} = \zeta^{k-j}\mathbf{T}$. Since $\rho^1(\mathbf{t_i})$ and $\rho^2(\mathbf{t_i})$ have spectrum $\{0, d\}$, the only possibility for $\rho^2(\mathbf{t_i})$ to be conjugate to $\omega_i - {}^t\rho^1(\mathbf{t_i})$ for some scalar ω_i is that $\omega_i = d$, and $\rho^1(\mathbf{t_i})\mathbf{T} = d\mathbf{T}$ iff $\rho^2(\mathbf{t_i})\mathbf{T} = 0$. Let $a \in [1, d-1]$ with $\lambda_a \neq \emptyset$, and choose $\mathbf{T}$ with $1 \in T_a$. Then $\rho^1(\mathbf{t_1})\mathbf{T} = 0$, as $a \neq 0$, hence $\rho^2(\mathbf{t_1})\mathbf{T} = d$. This implies $\rho^2(t_1)\mathbf{T} = \mathbf{T} = \zeta^0\mathbf{T}$, whence $\rho^1(t_1)\mathbf{T} = \zeta^k\mathbf{T}$, $a \equiv k$ modulo d and $a = k$. Then $\mu_k = \lambda'_{k-k} = \lambda'_0$ and $\mu_0 = \lambda'_{k-0} = \lambda'_k$. In that case we have $\rho^2 = (\rho^1)^* \otimes \chi$ with $\chi_{|W_0} = \epsilon$ and $\chi(t_1) = \zeta^k$, hence $\rho_{\mathcal{H}'} \simeq (\rho^2)_{\mathcal{H}'}$. $\qquad\square$

Proposition 7.8. *Let $W = G(d,1,r)$ with $r \geq 3$ and $\rho \in \mathrm{Irr}(W)$ associated to a multipartition $\boldsymbol{\lambda}$ with at least 2 parts. If $\lambda_0 = \emptyset$ then $\rho(\mathcal{H}'_{st}) = \rho(\mathcal{H}'_0)$, otherwise $\rho(\mathcal{H}'_{st}) = \rho(\mathcal{H}'_s) = \rho(\mathcal{H}')$.*

Proof. The case $\lambda_0 = \emptyset$ is clear, so we assume $\lambda_0 \neq \emptyset$. In case $\rho \in \mathrm{QRef}(W)$ we can write $\rho(\mathbf{t_i}) = \alpha\rho(t_i) + \beta$ for some α, β with $\alpha \neq 0$, hence $\rho(\mathcal{H}_s) \supset \mathfrak{sl}(V_\rho)$ implies $\rho(\mathcal{H}_{st}) \subset \mathfrak{sl}(V_\rho)$ hence $\rho(\mathcal{H}'_{st}) = \mathfrak{sl}(V_\rho)$, and this implies $\rho(\mathcal{H}'_{st}) = \rho(\mathcal{H}'_s)$ if $\rho \in \Lambda\mathrm{Ref}(W)$ and $\lambda_0 \neq \emptyset$. The rest of the proof then follows verbatim the lines of §3.1, using the irreducibility of $\rho_{\mathcal{H}'_{st}}$ proved by proposition 7.5, and that $\rho_{\mathcal{H}'}$ is selfdual if and only if $\rho_{\mathcal{H}'}$ is so, by proposition 7.7. The only change to make is in section 3.1.3, in case $\boldsymbol{\lambda}$ has the form $(\lambda_0, \ldots, \lambda_0, \ldots)$ and ρ factorizes through $G(d, d/2, r)$. But then $(2/d)\rho(\mathbf{t_i}) - 1 = \rho(t_i)$ and [Ma09a] can also be applied, as $\rho(\mathcal{H}_{st})$ equals the image of the infinitesimal Hecke algebra of $G(d, d/2, r)$ in the corresponding representation. $\qquad\square$

8. Relations between $\mathrm{Ad}(g)$ and $\mathrm{ad}(g)$

8.1. **Preliminaries about cyclotomic fields.** We first need various preliminary results on cyclotomic fields.

Lemma 8.1. $\mathbb{Q}(\mu_n) = \mathbb{Q}(\mu_m)$ *with $m \leq n$ if and only if $n = 2m$ with m odd.*

Proof. Since $\mathbb{Q}(\mu_n) \cap \mathbb{Q}(\mu_m) = \mathbb{Q}(\mu_{\gcd(m,n)})$ we can assume that m divides n. Letting φ denote the Euler function, by taking Galois groups this yiels $\varphi(m) = \varphi(n)$. Since, for p prime, $\varphi(p^r) = p^{r-1}(p-1)$, we get $\varphi(m) = \varphi(n)$ implies $n = ma$ with a prime to m and $\varphi(a) = 1$, that is $a = 2$. $\qquad\square$

We will use the following version of Goursat lemma.

Lemma 8.2. *Let K denote a finite Galois extension of $\mathbb{Q}$ and B a unital $\mathbb{Q}$-subalgebra of K^2 with $p_1(B) = p_2(B) = K$, where p_i is the i-th projection $K^2 \to K$. Then either $B = K^2$ or there exists $\sigma \in \mathrm{Gal}(K|\mathbb{Q})$ with $B = \{(x, \sigma(x)) \mid x \in K\}$.*

Proof. If B is not an integral domain, then it contains some nonzero and noninvertible element of K^2, that can chosen (by symmetry) of the form $(0, x)$ for $x \neq 0$. Since p_2 is onto we habe $b \in B$ with $p_2(b) = x^{-1}$, hence $(0, 1) = (0, x)b \in B$ and similarly $(1, 0) = 1 - (0, 1) \in B$. Then choosing for any $x, y \in K$ preimages $a, b \in B$ with $p_1(a) = x$ and $p_2(b) = y$ we get $(x, y) = a(1, 0) + b(0, 1) \in B$ and $B = K^2$. Now assume that B is an integral domain. Since B has finite dimension over $\mathbb{Q}$ it is a field. Then the $(p_i)_{|B} : B \to K$ are injective, so they are isomorphisms. Letting $\sigma = p_2 \circ (p_1)_{|B}^{-1} \in \mathrm{Gal}(K|\mathbb{Q})$ we thus get $B = \{(x, \sigma(x) \mid x \in K\}$. $\qquad\square$

Lemma 8.3. *Let B be a unital $\mathbb{Q}$-subalgebra of $A = \prod_{d|n} \mathbb{Q}(\mu_d)$ such that $p_d(B) = \mathbb{Q}(\mu_d)$ where $p_d = A \to \mathbb{Q}(\mu_d)$ is the natural projection. We assume that, if $\mathbb{Q}(\mu_d) = \mathbb{Q}(\mu_{d'})$ there exists $b \in B$ such that $p_{d'}(b) \notin \mathrm{Gal}(\mathbb{Q}(\mu_d)|\mathbb{Q})p_d(b)$. Then $B = A$.*

Proof. If $\mathbb{Q}(\mu_d) = \mathbb{Q}(\mu_{d'})$ with d, d' dividing n we can assume $d' = 2d$ with d odd. Then the projection of B to $\mathbb{Q}(\mu_d) \times \mathbb{Q}(\mu_{d'})$ is a unital $\mathbb{Q}$-subalgebra that satisfies the assumptions of the previous lemma, hence $p_{d'}(b) \notin \mathrm{Gal}(\mathbb{Q}(\mu_d)|\mathbb{Q})p_d(b)$ implies that it is equal to $\mathbb{Q}(\mu_d) \times \mathbb{Q}(\mu_{d'})$. Let D the set of divisors of n. For $I \subset D$, we denote $p_I : A \to A_I = \prod_{d \in I} \mathbb{Q}(\mu_d)$ the natural projection, and by contradiction we choose a minimal I with $p_I(B) \neq A_I$. The situation $\mathbb{Q}(\mu_d) = \mathbb{Q}(\mu_{d'})$ for all $d, d' \in I$ implies $|I| \leq 2$ by lemma 8.1, and is excluded for $|I| = 1$ by the hypothesis, and for $|I| = 2$ by lemma 8.2. So there exists $d_1, d_2 \in I$ with $\mathbb{Q}(\mu_{d_1}) \neq \mathbb{Q}(\mu_{d_2})$. Since $\mathbb{Q}(\mu_d) \subset \mathbb{Q}(\mu_\infty)$ is uniquely determined by its Galois group over $\mathbb{Q}$, this implies $\mathbb{Q}(\mu_{d_1}) \not\simeq \mathbb{Q}(\mu_{d_2})$.

If $p_I(B)$ was an integral domain, it would be a field as $\dim_{\mathbb{Q}} p_I(B) \leq \dim B \leq \dim A < \infty$. Then p_{d_1} and p_{d_2} would induce isomorphims $\mathbb{Q}(\mu_{d_1}) \simeq B \simeq \mathbb{Q}(\mu_{d_2})$, a contradiction. So there exists $b \in B$ with $p_I(b) \neq 0$ and $p_d(b) = 0$ for some $d \in I$. Let $J = \{d \in I \mid p_d(b) = 0\} \neq \emptyset$. We have $I \setminus J \neq \emptyset$ as $p_I(b) \neq 0$, and $p_{I \setminus J}(B) = A_{I \setminus J}$, $p_J(B) = A_J$ by the minimality assumption.

Since $p_{J \setminus I}(b)$ is invertible in $A_{I \setminus J}$ it follows that there exists $c \in B$ with $p_{J \setminus I}(cb) = 1$. Since $p_I(b) = 0$ we have $p_I(cb) = 0$. Then, choosing for any $(x, y) \in p_J(B) \times p_{I \setminus J}(B)$ preimages $u, v \in B$ with $x = p_J(u), y = p_{I \setminus J}(v)$, we have $(x, y) = p_I(u(1 - cb) + vcb)$ with $u(1 - cb) + vcb \in B$, hence $p_I(B) = p_J(B) \times p_{I \setminus J}(B) = A_I \times A_{J \setminus I} = A_I$, a contradiction. It follows that, for $I \subset D$ we have $p_I(B) = A_I$ and in particular $B = A_D = A$.

$\qquad\square$

In our situation, we will use the following Galois-theoretic lemma.

Lemma 8.4. *Let ζ be a primitive d-root of 1 and $a \geq 1$. Then $\mathbb{Q}(\zeta)$ is generated as a unital $\mathbb{Q}$-algebra by $u = (\zeta - 1)^{da}$ if d, a are odd. Otherwise it is generated by u and,*

- *if $d \equiv 0 \bmod 4$, by $(\zeta^{d/2} - 1)(\zeta^{d/4} - 1)^2(\zeta^{d/2} - 1)^{2(a-1)} \in i\mathbb{Q}^\times$;*
- *if $d \equiv 2 \bmod 4$, by $(\zeta^{d/2} - 1)(\zeta - 1)^{d/2}(\zeta^{d/2} - 1)^{2(a-1)} \in i\mathbb{Q}^\times$;*
- *if d is odd and a even, by $(\zeta^a - 1)^d$.*

Proof. Since $\mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$ is abelian, any intermediate extension of $\mathbb{Q}$ is Galois. Let K be the one generated by the elements in the statement, depending on d and a. We need to show that $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)|K)$ implies $\sigma = 1$. By contradiction we assume $\sigma \neq 1$. Since $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q})$ we have $\sigma(\zeta) = \zeta^\alpha$ for some $1 \leq \alpha \leq d - 1$ prime to d.

For any $k \geq 1$, $(\zeta - 1)^k$ is invariant under σ iff $(\zeta^\alpha - 1)^k = (\zeta - 1)^k$. We can assume $\zeta = e^{2i\pi/d}$, and then $\zeta^\alpha - 1 = e^{\alpha i\pi/d}(2i)\sin(\alpha\pi/d)$ hence $|\zeta^\alpha - 1|^k = |\zeta - 1|^k$ iff $\sin(\alpha\pi/d) = \sin(\pi/d)$ iff $\alpha = 1$ or $\alpha = d - 1$. For $\alpha = d - 1$, $(\zeta^\alpha - 1)^k = (\zeta - 1)^k$ can be written $e^{k(d-1)i\pi/d} = e^{ki\pi/d}$ that is $k(2 - d) \in 2d\mathbb{Z}$.

Letting $k = da$, it follows that $\sigma(u) = u$ with $\sigma \neq 1$ implies that σ is the complex conjugation. If da is odd, then so are d and $d - 2$, hence $da(2 - d) \notin 2d\mathbb{Z}$ and a contradiction. If d is odd and a even, then ζ^a is a primitive d-th root of 1, and we get from the previous argument that σ fixes $(\zeta^a - 1)^d$ iff $\sigma = 1$. We thus assume that d is even, and σ the complex conjugation. Then K is generated by some $(\zeta - 1)^k$ and any element in $i\mathbb{Q}^\times \cap K$, hence $\sigma = 1$, a contradiction that concludes the proof. $\square$

8.2. On the algebra $\mathbb{Q}[X,Y]/(X^n - 1, Y^n - 1)$.

Proposition 8.5. *Let $n \geq 2$. Then XY^{n-1} belongs to the unital $\mathbb{Q}$-subalgebra of $\mathbb{Q}[X,Y]/(X^n - 1, Y^n - 1)$ generated by the elements $X^k - Y^k$ for $1 \leq k \leq n$. It also belongs to the subalgebra generated by $X - Y$ if n is odd or $n = 2$.*

Proof. Let $C = \mathbb{Q}[X,Y]/(X^n - 1, Y^n - 1)$ and C_0 the subspace spanned (over $\mathbb{Q}$) by the monomials in X, Y of total degree 0 modulo n. It has dimension n and is spanned by the elements $X^k Y^{-k}$ for $0 \leq k < n$. It is also a subalgebra of C, generated by $Z = XY^{n-1} = XY^{-1}$, since $Z^k = X^k Y^{-k} = X^k Y^{n-k}$. We need to show that Z belongs to the subalgebra E of C generated by the $X^k - Y^k$ and, for n odd or $n = 2$, to the subalgebra E_1 of E generated by $X - Y$. The natural $\mathbb{Q}$-algebra morphism $\mathbb{Q}[T]/(T^n - 1) \to C_0$ that maps T to Z is onto because T^k is mapped to $Z^k = X^k Y^{n-k}$ and is into by equality of dimensions, so we can identify C_0 with $\mathbb{Q}[T]/(T^n - 1) = \prod_{d|n} \mathbb{Q}[T]/\Phi_d(T) \simeq \prod_{d|n} \mathbb{Q}(\mu_d) = A$, where Φ_d denotes the d-th cyclotomic polynomial, and $\mathbb{Q}[T]/\Phi_d(T)$ is embedded in $\mathbb{C}$ through $T \mapsto e^{2i\pi/d}$. We let $p_d : A \to \mathbb{Q}(\mu_d)$ denote the natural projections. Then $p_d(Z) = e^{2i\pi/d}$ is a primitive d-th root of 1. Let B, B_1 denote the subalgebras of A corresponding to $E \cap C_0$ and $E_1 \cap C_0$, respectively. For any $d|n$ and $a = n/d$, B contains $(Z-1)^{da}, (Z^a-1)^d$ as well as, for d divisible by 4, $(Z^{d/2} - 1)(Z^{d/4} - 1)^2(Z^{d/2} - 1)^{2(a-1)}$, and, for d even, $(Z^{d/2} - 1)(Z - 1)^{d/2}(Z^{d/2} - 1)^{2(a-1)}$; E_1 contains $(Z - 1)^{da}$. We first assume n odd. By lemma 8.4 we have $p_d(B_1) = \mathbb{Q}(\mu_d)$ for all $d|n$, by lemma 8.1 we have $\mathbb{Q}(\mu_d) = \mathbb{Q}(\mu_{d'}) \Rightarrow d = d'$ for $d, d'|n$, and by lemma 8.3 this implies $B_1 = A$ (hence $B = A$, $C_0 \subset E_1$ and $XY^{n-1} \in E_1$.). The case $n = 2$ follows from $(X - Y)^2 = 2 - 2XY$ in C.

Now assume n is even. If there exists $d_1 < d_2$ with d_1, d_2 dividing n such that $\mathbb{Q}(\mu_{d_1}) = \mathbb{Q}(\mu_{d_2})$, by lemma 8.1 we have d_1 odd and $d_2 = 2d_1$. We let $\zeta_d = p_d(Z) = e^{2i\pi/d}$. Then $(\zeta_{d_1} - 1)^n = (\zeta_{d_2}^2 - 1)^n$ is the image of $(\zeta_{d_2} - 1)^n$ by some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{d_2})|\mathbb{Q})$ iff there exists $1 \leq \alpha \leq d_2 - 1$ prime to d_2 with $(\zeta_{d_2}^2 - 1)^n = (\zeta_{d_2}^\alpha - 1)^n$ hence $\sin(\alpha\pi/d_2) = \sin(2\pi/d_2)$, that is $\alpha = 2$ or $\alpha = d_2 - 2$. Since $d_2 = 2d_1$ is even, this implies α even, contradicting α prime to d_2. Since $(\zeta_{d_1} - 1)^n$ and $(\zeta_{d_2} - 1)^n$ are the images under p_{d_1} and p_{d_2}, respectively, of $(Z - 1)^n \in B$, then $B \subset A$ satisfies the assumptions of lemma 8.3. Then $B = A$, $C_0 \subset E$ and $XY^{n-1} \in E$.

$\square$

We now relate the endomorphisms $\mathrm{Ad}(g)$ and $\mathrm{ad}(g)$. Notice that $\mathrm{ad}(g^k)$ commutes with $\mathrm{ad}(g^l)$ for any k, l.

Theorem 8.6. *Let G be a finite group. For $g \in G$ we let $\mathrm{ad}(g), \mathrm{Ad}(g) \in \mathrm{End}(\mathbb{Q}G)$ defined by $\mathrm{ad}(g) : x \mapsto gx - xg$ and $\mathrm{Ad}(g) : x \mapsto gxg^{-1}$. Let n denote the order of $g \in G$. Then*

(1) $\mathrm{Ad}(g)$ *is a polynomial in the $\mathrm{ad}(g^k), k \geq 1$ that depends only on n.*
(2) *Let n denote the order of $g \in G$. If n is odd or $n = 2$ then $\mathrm{Ad}(g)$ is a polynomial in $\mathrm{ad}(g)$ that depends only on n.*

Proof. Let $\Gamma \simeq \mathbb{Z}/n\mathbb{Z}$ denote the subgroup generated by g. The algebra $\mathbb{Q}G$ is a Γ-bimodule, that is a $\Gamma \times \Gamma$-module. It is thus enough to show that, for any *complex* Γ-bimodule M, then $\mathrm{Ad}(g)$ can be written as a rational polynomial in the $\mathrm{ad}(g^k)$, or in $\mathrm{ad}(g)$ if n is even, that depends only on n. We can take M irreducible, hence of dimension 1 and spanned by some nonzero $v \in M$, for which $g.v = \zeta^r v$ and $v.g = \zeta^s v$, with ζ some fixed primitive n-th root of 1. Then $\mathrm{ad}(g^k)(v) = ((\zeta^r)^k - (\zeta^s)^k)v$ and $\mathrm{Ad}(g)(v) = (\zeta^r \zeta^{-s})v$. By proposition 8.5 we get that XY^{-1} is a rational polynomial in the $X^k - Y^k$, and in $X - Y$ for n odd or $n = 2$, inside $\mathbb{Q}[X, Y]/(X^n - 1, Y^n - 1) = \mathbb{Q}(\Gamma \times \Gamma)$, and the conclusion follows. $\square$

For small n, the polynomials in the statement are easy to find :

$$
\begin{aligned}
n = 2 \qquad \mathrm{Ad}(g) \;&=\; 2\mathrm{Id} - \mathrm{ad}(g)^2 \\
n = 3 \qquad 18\mathrm{Ad}(g) \;&=\; 18\mathrm{Id} + 3\mathrm{ad}(g)^3 + \mathrm{ad}(g)^6 \\
n = 4 \qquad 8\mathrm{Ad}(g) \;&=\; 8\mathrm{Id} - 3\mathrm{ad}(g^2)^2 - \mathrm{ad}(g)^4 + 2\mathrm{ad}(g)^2\mathrm{ad}(g^2) \\
n = 5 \qquad 13750\mathrm{Ad}(g) \;&=\; 13750\mathrm{Id} - 5875\mathrm{ad}(g)^5 + 1900\mathrm{ad}(g)^{10} \\
&\qquad - 10\mathrm{ad}(g)^{15} + 3\mathrm{ad}(g)^{20} \\
n = 6 \qquad 183456\mathrm{Ad}(g) \;&=\; 183456 - 89573\mathrm{ad}(g)^6 - 2210\mathrm{ad}(g)^{12} \\
&\qquad + 55\mathrm{ad}(g)^{18} - 30576\mathrm{ad}(g^2)^3 + 15288\mathrm{ad}(g^2)^3\mathrm{ad}(g^3)^2
\end{aligned}
$$

We remark that it is not possible to express in general $\mathrm{Ad}(g)$ as a polynomial in $\mathrm{ad}(g)$ for $n > 2$ even. Indeed, in $\mathbb{Q}[X, Y]/(X^n - Y^n)$, XY^{-1} does not belong to the subalgebra generated by the $(X - Y)^n$, as $(X - Y)^n$ is symmetric in X, Y and XY^{n-1} is not. Since this algebra is equal to the intersection of the subspace C_0 of homogeneous polynomials with total degree equal to 0 modulo n with the subalgebra generated by $X - Y$, this proves that XY^{n-1} is not a polynomial in $X - Y$. Now there exists groups with a cyclic subgroup $\Gamma \simeq \mathbb{Z}/n\mathbb{Z}$, e.g. $\Gamma \wr \mathfrak{S}_n \simeq G(n, 1, n)$, which admit irreducible representations whose restriction to Γ contains all irreducible representations of Γ. Taking

for g a generator of Γ it follows that $\mathbb{Q}[X,Y]/(X^n-1,Y^n-1) = \mathbb{Q}(\Gamma \times \Gamma)$ embeds in $\mathbb{Q}G$, hence $\mathrm{Ad}(g)$ is not a polynomial in $\mathrm{ad}(g)$ in these cases.

CONTENTS

REFERENCES

[Bd]   M. Benard, *Schur indices and splitting fields of the Unitary reflection groups* J. Algebra **38** (1976), 318–342.

[Bes]   D. Bessis, *Sur le corps de définition d'un groupe de réflexions complexe*, Comm. Algebra **25** (1997), 2703–2716.

[Ar]   S. Ariki, *Representation theory of a Hecke algebra for $G(r,p,n)$*, J. Algebra **177** (1995), 164–185.

[ArKo]   S. Ariki, K. Koike, *A Hecke algebra of $\mathbb{Z}/r\mathbb{Z}\wr\mathfrak{S}_n$ and construction of its irreducible representations*, Adv. Math. **106** (1994), 216–243.

[BMM]   M. Broué, G. Malle, J. Michel, *Toward Spetses I*, Transf. Groups **4** (1999), 157–218.

[BMR]   M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[CHL]   W. Couwenberg, G. Heckman, E. Looijenga, *Geometric structures on the complement of a projective arrangement*, Publ. Math. I.H.E.S. **101** (2005), 69–161.

[Dr]   V.G. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991), 829–860.

[En]   B. Enriquez, *Quasi-reflection algebras and cyclotomic associators*, Selecta Math. (N.S.) **13** (2007), 391–463.

[Ko]   T. Kohno, *Série de Poincaré-Koszul associée aux groupes de tresses pures*, Invent. Math **82** (1985), 57–76.

[Ma01]   I. Marin, *Une caractérisation tensorielle des représentations standard*, Expo. Math. **18** (2000), 245–254.

[Ma03]   I. Marin, *Infinitesimal Hecke Algebras*, Comptes Rendus Mathématiques **337** Série I, 297-302 (2003).

[Ma06]   I. Marin, *Monodromie algébrique des groupes d'Artin diédraux*, J. Algebra **303** (2006), 97–132.

[Ma07a] I. Marin, *L'algèbre de Lie des transpositions*, J. Algebra **310** (2007), 742–774.

[Ma07b] I. Marin, *Krammer representations for complex braid groups*, preprint arxiv:math/0711.3096.

[Ma09a] I. Marin, *Infinitesimal Hecke Algebras II*, preprint arxiv:math/0911.1879.

[Ma09b] I. Marin, *Braids inside the Birman-Wenzl-Murakami algebra*, preprint arxiv:math/0911.2188.

[MM] I. Marin, J. Michel, *Automorphisms of complex reflection groups*, preprint arxiv:math/0701266.

[RR] A. Ram, J. Ramagge, *Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory*, in "A tribute to C.S. Seshadri (Chennai, 2002)", Trends. Math., Birkhäuser, Basel 2003, 428–466.

[St] R. Stanley, *Relative invariants of finite groups generated by pseudoreflections*, J. Algebra **49** (1977), 134–148.