INFINITESIMAL HECKE ALGEBRAS IV

IVAN MARIN

Abstract. We refine the infinitesimal Hecke algebra associated to a 2-reflection group into a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra, as a first step towards a global understanding of a natural $\mathbb{N}$-graded object. We provide an interpretation of this Lie algebra in terms of the image of the braid group into the Hecke algebra, in connection with unitary structures. This connection is particularly strong when W is a Coxeter group, and when in addition we can use generalizations of Drinfeld's rational even associators. Finally, in the Coxeter ADE case, we provide a generating set of the even part of this Lie algebra, which originates from the rotation subgroup of the Coxeter group.

Contents

Date: December 5, 2012.

1. Introduction

This paper is a continuation of [M2, M5, M6]. Let W be a complex reflection group, that is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by reflections (of order 2), and $\Bbbk$ a field of characteristic 0. We denote $\mathcal{R}$ the set of reflections in W. In [M2, M5] we defined the infinitesimal Hecke algebra $\mathcal{H}$ as the Lie sub-algebra of the group algebra $\Bbbk W$ generated by $\mathcal{R}$. We proved there that it can be identified with the Lie algebra of the Zariski closure of the braid group of W inside the Hecke algebra (with generic parameter). Here we first define a graded version of this Lie algebra. Let us introduce the group algebra $\Bbbk[[h]]W$ of W over the ring of formal series $\Bbbk[[h]]$, and consider it as a $\mathbb{Z}_{\geq 0}$-graded $\Bbbk$-algebra.

Definition 1.1. *The graded infinitesimal Hecke algebra $\mathcal{H}_{\mathrm{gr}}$ is the Lie subalgebra of $\Bbbk[[h]]W$ generated by the $hs, s \in \mathcal{R}$.*

An approximation of this still mysterious graded Lie algebra is given by $\mathcal{H}_{\mathrm{gr}} \otimes_{\Bbbk[[h]]} \Bbbk((h)) = \mathcal{H} \otimes_{\Bbbk} \Bbbk((h))$, whose structure is known from [M5]. A better approximation is given by the following object, which is the central focus of this paper.

Definition 1.2. *The 'tail' infinitesimal Hecke algebra $\mathcal{H}_{\mathrm{tail}}$ is the Lie $\Bbbk(h^2)$-subalgebra of $\Bbbk(h)W$ generated by the $hs, s \in \mathcal{R}$.*

Note that $\mathcal{H}_{\mathrm{tail}}$ has a natural structure of $\mathbb{Z}/2$-graded Lie algebra over $\Bbbk(h^2)$. Let $\mathcal{H}_{\mathrm{gr}}^{(r)}$ denote the homogeneous part of degree r of $\mathcal{H}_{\mathrm{gr}}$. It is a $\Bbbk$-vector space which can be clearly identified to a subspace of $\mathcal{H}$, that is $\mathcal{H}_{\mathrm{gr}}^{(r)} = h^r \mathcal{H}_{\mathrm{gr}}^r$ with $\mathcal{H}_{\mathrm{gr}}^r \subset \mathcal{H}$.

Proposition 1.3. *(see proposition 2.1) As subspaces of $\mathcal{H}$, for all $r \geq 1$, $\mathcal{H}_{\mathrm{gr}}^r \subset \mathcal{H}_{\mathrm{gr}}^{r+2}$.*

Let $\mathcal{H}_{\mathrm{gr}}^{2\infty}$ denote the union of the $\mathcal{H}_{\mathrm{gr}}^{2r}$, and $\mathcal{H}_{\mathrm{gr}}^{2\infty+1}$ denote the union of the $\mathcal{H}_{\mathrm{gr}}^{2r+1}$ for $r \geq 0$. We let $\mathcal{H}_{\mathrm{tail}}^0$ and $\mathcal{H}_{\mathrm{tail}}^1$ denote the homogeneous components of $\mathcal{H}_{\mathrm{tail}}$. We adopt the following convention : whenever V is a $\Bbbk$-subspace of the group algebra $\Bbbk((h))W$, and R is a $\Bbbk$-subalgebra of $\Bbbk((h))$, we let RV denote the image in $\Bbbk((h))W$ of the natural morphism $V \otimes_{\Bbbk} R \to \Bbbk((h))W$. The following is a straightforward consequence of proposition 2.1.

Proposition 1.4. *$\mathcal{H}_{\mathrm{tail}}^0 = \Bbbk(h^2)\mathcal{H}_{\mathrm{gr}}^{2\infty}$ and $\mathcal{H}_{\mathrm{tail}}^1 = h\Bbbk(h^2)\left(Z(\mathcal{H}) \oplus \mathcal{H}_{\mathrm{gr}}^{2\infty+1}\right)$.*

This means that $\mathcal{H}_{\mathrm{tail}}$ indeed captures the 'tail' of the sequence $(\mathcal{H}_{\mathrm{gr}}^r)_{r\geq 0}$. The Lie algebra $\mathcal{H}_{\mathrm{tail}}$ has a meaning in its own right, that we explain now. Recall from [M5, M6] the existence of morphisms $\Psi_H : H(q) \to KW$, a priori only for $\Bbbk = \mathbb{C}$ but also conjecturally for $\Bbbk = \mathbb{Q}$, where $H(q)$ is the Hecke algebra associated to W ; more precisely, $H(q)$ is defined over $K = \Bbbk((h))$, $q = e^h$, and $H(q)$ is the quotient of the group algebra KB of the braid group B associated to W by relations $(\mathfrak{s} - q)(\mathfrak{s} + q^{-1}) = 0$, where $\mathfrak{s}$ runs among the braided reflections of B.

When W is a Coxeter group, it is known that these morphisms provide isomorphisms $H(q) \simeq KW$. In the general case, it is also conjectured to be the case. We call this statement the weak BMR conjecture, and refer to [M9] for a detailed discussion of it. It is known to hold true for the general series $G(de, e, n)$ of irreducible complex reflection groups as well as for all the groups of rank 2, plus some other cases.

We let $\sigma \in \mathrm{Aut}(K)$ be defined by $f(h) \mapsto f(-h)$, and we let $K^\sigma = \Bbbk((h^2))$. We prove the following theorem

Theorem 1.5. *Under the weak BMR conjecture for W, the Zariski closure of $\Psi_H(B)$ inside $H(q)^\times$ considered as an algebraic group over $\Bbbk((h^2))$ has for Lie algebra $\mathcal{H}_{\mathrm{tail}} \otimes_{\Bbbk(h^2)} \Bbbk((h^2))$.*

Recall from [M2], [M5] that $\mathcal{H} \subset \mathcal{L}_\varepsilon(W)$, where $\mathcal{L}_\varepsilon(W)$ is the Lie subalgebra of $\mathbb{k}W$ spanned by the $g - \varepsilon(g)g^{-1}$ for $g \in W$, where $\varepsilon : W \to \{\pm 1\}$ is the sign morphism, induced by the determinant. We denote $\mathrm{O} = \mathrm{Ker}\,\varepsilon$ the rotation subgroup of W. In section 4 we endow $\mathcal{L}_\varepsilon(W)$ with the structure of a $\mathbb{Z}/2$-graded Lie algebra $\mathcal{L}_\varepsilon(W) = \mathcal{L}_1(\mathrm{O}) \oplus \mathcal{L}_1(\mathrm{O}^\dagger)$, such that

$$\mathcal{H}^0_{\mathrm{tail}} = \mathcal{H}_{\mathrm{tail}} \cap \mathbb{k}(h^2)\mathcal{L}_1(\mathrm{O}) \text{ and } h^{-1}\mathcal{H}^1_{\mathrm{tail}} = \mathcal{H}_{\mathrm{tail}} \cap \mathbb{k}(h^2)\mathcal{L}_1(\mathrm{O}^\dagger)$$

In the same section 4 we determine the structure as $\mathbb{Z}/2$-graded Lie algebras of $\mathcal{L}_\varepsilon(W)$ and of the $\rho(\mathcal{L}_\varepsilon(W))$, for ρ an irreducible representation of W, notably when W is a Coxeter group.

We then prove the following theorem.

Theorem 1.6. *(see propositions 5.4 and 4.10) Let W be a reflection group, $\rho : W \to \mathfrak{gl}_N(\mathbb{k})$ an irreducible representation, and $\Psi : B \to \mathrm{GL}_N(K)$ the representation of $H(q)$ associated to ρ. The closure of $\Psi(B)$ in the K^σ-group $\mathrm{GL}_N(K)$ has for Lie algebra $K^\sigma\rho(\mathcal{H}^0_{\mathrm{tail}}) \oplus hK^\sigma\rho(\mathcal{H}^1_{\mathrm{tail}})$. If $\rho^* \simeq \rho$ (e.g. if W is a Coxeter group) then*

$$\begin{array}{rcl} K^\sigma\rho(\mathcal{H}^0_{\mathrm{tail}}) & = & K^\sigma\left(\rho(\mathcal{H}) \cap \rho(\mathcal{L}_1(\mathrm{O}))\right) \\ K^\sigma\rho(\mathcal{H}^1_{\mathrm{tail}}) & = & hK^\sigma\left(\rho(\mathcal{H}) \cap \rho(\mathcal{L}_1(\mathrm{O}^\dagger))\right) \end{array}$$

In the latter case, we moreover prove that this closure is the intersection of the closure over K with a suitable unitary group (see proposition 5.8).

A natural question is whether $\mathcal{H}^0_{\mathrm{tail}} \subset \mathbb{k}(h^2)\mathrm{O}$ can be defined intrinsically from O, or needs instead to be defined inside $\mathbb{k}(h^2)W$. Let $\mathcal{A}$ denote the Lie subalgebra of $\mathbb{k}\mathrm{O}$ generated by elements $[s,u] = su - us = su - (su)^{-1}$ for $s,u \in \mathcal{R}$. In sections 7, 8 and 9, we prove the following.

Theorem 1.7. *$\mathbb{k}(h^2)\mathcal{A}$ is contained in $\mathcal{H}^0_{\mathrm{tail}}$, and $\mathbb{k}(h^2)\mathcal{A} = \mathcal{H}^0_{\mathrm{tail}}$ when W is a Coxeter group of type ADE.*

As opposed to the non-graded setting, where all the results could be stated and prove (sometimes to the expense of admitting a couple of natural or well-established conjectures) in the most general setting of complex pseudo-reflection groups (see [M6]) it seems that this $\mathbb{Z}/2\mathbb{Z}$-graded version behaves more naturally in the case of Coxeter groups. In §6 we detail some special aspects of Coxeter groups that are relevant here, and state a refinement of the conjectures of [M5, M6] which is specific to the Coxeter setting. Loosely speaking, this refinement is connected to the existence of analogues of Drinfeld's *even* associators. Under this conjecture, which imply special cases of a classical result of Lusztig and is shown to hold in Coxeter types $A, B_n/C_n$ and $I_2(m)$, $\mathcal{H}^0_{\mathrm{tail}}$ can be identified with the 'orthogonal part' of the image inside the Hecke algebra of the pro-unipotent completion of the pure braid group. Ironically, one can prove that the image of the pure braid group itself has no 'orthogonal part' (see remark 6.7).

The next natural goal would be to understand the full graded version $\mathcal{H}_{\mathrm{gr}}$. Another purpose is to use this broader knowledge provided by the $\mathbb{Z}/2\mathbb{Z}$-graded version in the case of specialized values of the parameter q. We already showed in [BM], in the case $W = \mathfrak{S}_n$ and over a finite field, that the image of the braid group may fill in either full linear groups, or instead preserve a unitary form, depending on whether the unitary form existing at the generic level admits an avatar in the finite field and for the parameter q under consideration.

In order to check some representation-theoretic facts on specific complex reflection groups, we made use of the development version of the CHEVIE package for GAP3. This software is maintained by Jean Michel and can be found at `http://www.math.jussieu.fr:/~jmichel/chevie/index.html`. It will simply be referred to as 'CHEVIE' in the sequel.

Acknowledgements. I thank the University of Tokyo and T. Kohno for a one-month stay in 2009 where I started this work, M. Cabanes for help in proving lemma 9.1, J. Michel and C. Bonnafé for the reference [Ca], C. Cornut for useful discussions about unitary groups. For the tables below, and also for the determination of $\mathcal{A}$ in small rank, this work made heavy use of a fast computer funded by the ANR grant ANR-09-JCJC-0102-01.

2. Infinitesimal graded Hecke algebras

Since $\mathcal{H}_{\mathrm{gr}}$ is generated as a Lie algebra by a conjugacy class in W, each of the homogeneous components $\mathcal{H}_{\mathrm{gr}}^r$ is a W-submodule of $\mathcal{H}$. We first prove the following

Proposition 2.1. (i) $\mathcal{H}_{\mathrm{gr}}^1 = Z(\mathcal{H}) \oplus \left(\mathcal{H}' \cap \mathcal{H}_{\mathrm{gr}}^1\right)$ *and* $\mathcal{H}' \cap \mathcal{H}_{\mathrm{gr}}^1 \subset \mathcal{H}_{\mathrm{gr}}^3$
(ii) *For all* $r \geq 2$, $\mathcal{H}_{\mathrm{gr}}^r \subset \mathcal{H}' = [\mathcal{H},\mathcal{H}]$ *and* $\mathcal{H}_{\mathrm{gr}}^r \subset \mathcal{H}_{\mathrm{gr}}^{r+2}$.

Proof. (i) is clear from the general properties of $\mathcal{H}$. We prove (ii) and assume $r \geq 2$. Again clearly $\mathcal{H}_{\mathrm{gr}}^r \subset \mathcal{H}' = [\mathcal{H},\mathcal{H}]$. Let $x \in \mathcal{H}_{\mathrm{gr}}^r$ and $s \in \mathcal{R}$. One has $[s,[s,x]] = 2(sxs - x) \in \mathcal{H}_{\mathrm{gr}}^{r+2}$, hence $s.x - x \in \mathcal{H}_{\mathrm{gr}}^{r+2}$ where $g.x$ for $g \in W$ and $x \in \mathcal{H}$ denotes the conjugation action. We introduce $M = \sum_{s\in\mathcal{R}}(s-1).\mathcal{H}_{\mathrm{gr}}^r$. By the above $M \subset \mathcal{H}_{\mathrm{gr}}^{r+2}$, and also $M \subset \mathcal{H}_{\mathrm{gr}}^r$ because $\mathcal{H}_{\mathrm{gr}}^r$ is W-stable. We prove that, when $r \geq 2$, then $M = \mathcal{H}_{\mathrm{gr}}^r$. For this, to each W-submodule U of $\mathcal{H}_{\mathrm{gr}}^r$ we associate the subspace $M_U = \sum_{s\in\mathcal{R}}(s-1).U$. Since U is a W-submodule we get $M_U \subset U$, and moreover M_U is stabilized by W : if $w \in W$, then

$$w.\left(\sum_{s\in\mathcal{R}}(s-1).U\right) = \sum_{s\in\mathcal{R}}(ws-w).U = \sum_{s\in\mathcal{R}}(wsw^{-1}-1).(w.U) = \sum_{s\in\mathcal{R}}(s-1).U = M_U.$$

When $U \neq 0$, $M_U = 0$ if and only if $\forall s \in \mathcal{R} \forall x \in U\ s.x = x$, that is $\forall w \in W \forall x \in U\ \ w.x = x$, hence $\mathbb{1} \hookrightarrow U$. But when $r \geq 2$, $\mathcal{H}_{\mathrm{gr}}^r \subset \mathcal{H}' \subset (\Bbbk W)'$, which as a W-module (under the conjugation action) is a linear complement of $Z(W)$ in $\Bbbk W$. From this we deduce, when $r \geq 2$, that $U \neq 0 \Rightarrow M_U \neq 0$, hence if U is irreductible then $M_U = U$. Since $\mathcal{H}_{\mathrm{gr}}^r$ is a sum of irreducible submodules this proves $\mathcal{H}_{\mathrm{gr}}^r = M \subset \mathcal{H}_{\mathrm{gr}}^{r+2}$. □

For a given group W, the computation of the grading can be made as follows. Let $\mathcal{M}_r \subset \mathcal{H}$ denote the subspace spanned by the brackets of at most r reflections, that is $\mathcal{M}_1 = \Bbbk\mathcal{R}$ and $\mathcal{M}_{r+1} = \mathcal{M}_r + [\mathcal{R},\mathcal{M}_r]$. The datas of $\dim \mathcal{M}_r$ determines $\dim \mathcal{H}_{\mathrm{gr}}^r$ as follows. First of all, $\mathcal{M}_1 = \Bbbk\mathcal{R} = \mathcal{H}_{\mathrm{gr}}^1$. Then $\mathcal{M}_2 = \mathcal{M}_1 + \mathcal{H}_{\mathrm{gr}}^2$ hence $\dim \mathcal{H}_{\mathrm{gr}}^2 = \dim \mathcal{M}_2 - \dim \mathcal{M}_1$; $\mathcal{M}_3 = \mathcal{M}_2 + \mathcal{H}_{\mathrm{gr}}^3 + \mathcal{H}_{\mathrm{gr}}^1 = \mathcal{H}_{\mathrm{gr}}^2 + \mathcal{H}_{\mathrm{gr}}^3 + (\mathcal{H}_{\mathrm{gr}}^1 \cap \mathcal{H}') + Z(\mathcal{H})$, thus $\mathcal{M}_3 = Z(\mathcal{H}) \oplus \mathcal{H}_{\mathrm{gr}}^2 \oplus \mathcal{H}_{\mathrm{gr}}^3$ and $\dim \mathcal{H}_{\mathrm{gr}}^3 = \dim \mathcal{M}_3 - \dim \mathcal{H}_{\mathrm{gr}}^2 - \dim Z(\mathcal{H})$.

For $n \geq 4$, $\mathcal{M}_n = \mathcal{M}_{n-1} + \mathcal{H}_{\mathrm{gr}}^n$, and $\mathcal{M}_{n-1} \cap \mathcal{H}_{\mathrm{gr}}^n = \mathcal{H}_{\mathrm{gr}}^{n-2}$ hence $\dim \mathcal{M}_n = \dim \mathcal{M}_{n-1} + \dim \mathcal{H}_{\mathrm{gr}}^n - \dim \mathcal{H}_{\mathrm{gr}}^{n-2}$ and $\dim \mathcal{H}_{\mathrm{gr}}^n = \dim \mathcal{M}_n - \dim \mathcal{M}_{n-1} + \dim \mathcal{H}_{\mathrm{gr}}^{n-2}$. Summing up all these equalities yields to

$$\dim \mathcal{H}_{\mathrm{gr}}^n = \dim \mathcal{M}_n - \dim \mathcal{H}_{\mathrm{gr}}^{n-1} - \dim Z(\mathcal{H})$$

whenever $n \geq 3$. A tabulation of the dimensions for some of the irreducible groups is given in tables 1 and 2 (the figures in bold constitute the repeating pattern).

For the record, we recall the following result for Coxeter groups (see [Ca], lemma 2).

Lemma 2.2. *If* $W < \mathrm{GL}_n(\mathbb{R})$ *is a real reflection group and* $g \in W$ *with* $k = \mathrm{rk}\,(g-1)$*, then* g *is a product of* k *reflections.*

Proof. The main point is to show that, if $w \in W$ has no (nonzero) fixed vector, then there is $s \in \mathcal{R}$ such that ws has a fixed vector. For this one notes that $w-1$ is invertible. Choosing $s \in \mathcal{R}$ and $v \in \mathbb{R}^n \setminus \{0\}$ with $s.v = -v$, let $\tilde{v}$ such that $(w-1).\tilde{v} = v$. Then $w.\tilde{v} = \tilde{v} + v$, hence $\langle \tilde{v}, \tilde{v} \rangle = \langle w.\tilde{v}, w.\tilde{v} \rangle = \langle \tilde{v}, \tilde{v} \rangle + 2\langle v, \tilde{v} \rangle + \langle v, v \rangle$ hence $2\langle v, \tilde{v} \rangle / \langle v, v \rangle = -1$. This implies $s.\tilde{v} = \tilde{v} + v$, hence ws has $v + \tilde{v} \neq 0$ for fixed vector. □

Remark 2.3. *This statement is not true for complex reflection groups. Actually, we checked that for* every *non-Coxeter irreducible 2-reflection group W of exceptional type, the maximal length $m(W)$ of an element with respect to the generating set of all reflexions is always greater than the rank of W. More precisely, we get the following table*

W	$\operatorname{rk} W$	$m(W)$	W	$\operatorname{rk} W$	$m(W)$	W	$\operatorname{rk} W$	$m(W)$
G_{12}	2	3	G_{24}	3	4	G_{31}	4	6
G_{13}	2	3	G_{27}	3	5	G_{33}	5	7
G_{22}	2	3	G_{29}	4	6	G_{34}	6	> 6

As far as the infinite series $G(e,e,r)$ and $G(2e,e,r)$ are concerned, there seems to be a similar phenomenon, as illustrated by the tables below. This strongly suggests that this property is an elementary characterization of Coxeter groups among irreducible 2-reflection groups.

W	$m(W)$	W	$m(W)$	W	$m(W)$	W	$m(W)$	W	$m(W)$
$G(3,3,3)$	4	$G(4,4,3)$	4	$G(5,5,3)$	4	$G(6,6,3)$	4	$G(7,7,3)$	4
$G(3,3,4)$	5	$G(4,4,4)$	6	$G(5,5,4)$	6				
$G(3,3,5)$	6	$G(4,4,5)$	7						

W	$m(W)$	W	$m(W)$	W	$m(W)$	W	$m(W)$	W	$m(W)$
$G(4,2,2)$	3	$G(6,3,2)$	3	$G(8,4,2)$	3	$G(10,5,2)$	3	$G(12,6,2)$	3
$G(4,2,3)$	4	$G(6,3,3)$	5	$G(8,4,3)$	5	$G(10,5,3)$	5	$G(12,6,3)$	5
$G(4,2,4)$	6	$G(6,3,4)$	6	$G(8,4,4)$	7	$G(10,5,4)$	7	$G(12,6,4)$	7
$G(4,2,5)$	7								

Example 2.4. *(Dihedral groups) Let W be a dihedral group of order $2m$, with generators s, ω, relations $s^2 = \omega^m = 1$, $s\omega s = \omega^{-1}$. We have $\mathcal{R} = \{s\omega^k; 0 \leq k \leq m-1\}$, and $\mathcal{H}^1_{\mathrm{gr}} = \Bbbk\mathcal{R}$, has dimension m. $\mathcal{H}^2_{\mathrm{gr}}$ is spanned by the $[s, s\omega^k]$, that is by the $\omega^k - \omega^{-k}$ for $0 \leq k \leq m-1$, hence $\dim \mathcal{H}^2_{\mathrm{gr}} = \lfloor \frac{m-1}{2} \rfloor$. $\mathcal{H}^3_{\mathrm{gr}}$ is spanned by the $[s\omega^i, \omega^k - \omega^{-k}]$, that is (as car.$\Bbbk \neq 2$) by the $s\omega^{i+k} - s\omega^{i-k}$, thus $\dim \mathcal{H}^3_{\mathrm{gr}} = m - \dim \mathbb{Z}(\mathcal{H})$, as $\dim \mathbb{Z}(\mathcal{H}) = 1$ if m is odd, $\dim \mathbb{Z}(\mathcal{H}) = 1$ if m is even. Since $\dim \mathcal{H}' = 3\lfloor \frac{m-1}{2} \rfloor$, we have $\mathcal{H}^3_{\mathrm{gr}} = \mathcal{H}^5_{\mathrm{gr}} = \cdots = \mathcal{H}^{2\infty+1}_{\mathrm{gr}}$ and $\mathcal{H}^2_{\mathrm{gr}} = \mathcal{H}^4_{\mathrm{gr}} = \cdots = \mathcal{H}^{2\infty}_{\mathrm{gr}}$.*

3. Preliminaries on $\mathbb{Z}/2$-algebras

Let $\Bbbk$ denote a field of characteristic 0. Recall that a $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, also called a symmetric pair $(\mathfrak{g}_0, \mathfrak{g}_1)$, is the same thing as a Lie algebra $\mathfrak{g}$ endowed with an involution in $\mathrm{Aut}(\mathfrak{g})$. A trivial kind of $\mathbb{Z}/2$-graded Lie algebra is the *double* of the Lie algebra $\mathfrak{g}$, that is $\mathfrak{g} \oplus \mathfrak{g}$.

We describe a list of $\mathbb{Z}/2$-graded Lie algebras, that we will call the classical $\mathbb{Z}/2$-graded Lie algebras. More precisely, in the sequel we will say that a $\mathbb{Z}/2$-graded Lie algebra over $\Bbbk$ is classical of type (x) if *up to some extension of scalars* it is isomorphic to the $\mathbb{Z}/2$-graded Lie

	$\mathcal{H}^1_{\rm gr}$	$\mathcal{H}^2_{\rm gr}$	$\mathcal{H}^3_{\rm gr}$	$\mathcal{H}^4_{\rm gr}$	$\mathcal{H}^5_{\rm gr}$	$\mathcal{H}^6_{\rm gr}$	$\mathcal{H}^7_{\rm gr}$	$\mathcal{H}^8_{\rm gr}$	$\mathcal{H}^9_{\rm gr}$	$\mathcal{H}^{10}_{\rm gr}$
G_{12}	12	**11**	**16**	11	16	...				
G_{13}	18	**23**	**28**	23	28	23	...			
G_{22}	30	**59**	**69**	59	69	59	...			
$G_{23} = H_3$	15	**22**	**33**	22	33	22	...			
G_{24}	21	49	**90**	**72**	90	72	90	...		
G_{27}	45	237	**551**	**512**	551	512	551	...		
$G_{28} = F_4$	24	50	142	196	**280**	**228**	280	228	...	

TABLE 1. Grading for some exceptional groups

	$\mathcal{H}^1_{\rm gr}$	$\mathcal{H}^2_{\rm gr}$	$\mathcal{H}^3_{\rm gr}$	$\mathcal{H}^4_{\rm gr}$	$\mathcal{H}^5_{\rm gr}$	$\mathcal{H}^6_{\rm gr}$	$\mathcal{H}^7_{\rm gr}$	$\mathcal{H}^8_{\rm gr}$	$\mathcal{H}^9_{\rm gr}$	$\mathcal{H}^{10}_{\rm gr}$	$\mathcal{H}^{11}_{\rm gr}$	$\mathcal{H}^{12}_{\rm gr}$	$\mathcal{H}^{13}_{\rm gr}$	$\mathcal{H}^{14}_{\rm gr}$
$\mathfrak{S}_3$	3	**1**	**2**	1	2	...								
$\mathfrak{S}_4$	6	**4**	**7**	4	7	...								
$\mathfrak{S}_5$	10	10	19	**16**	**23**	16	23	...						
$\mathfrak{S}_6$	15	20	44	56	92	92	122	**112**	**136**	112	136	...		
$\mathfrak{S}_7$	21	35	90	161	342	533	838	987	1081	**1002**	**1087**	1002	1087	...
B_3	9	**7**	**12**	7	12	...								
B_4	16	22	50	53	77	**59**	**80**	59	80	...				
B_5	25	50	153	301	591	701	842	**761**	**869**	761	869	...		
D_4	12	16	35	**32**	**46**	32	46	...						
D_5	20	40	119	216	372	381	445	**391**	**449**	391	449	...		

TABLE 2. Grading for classical Coxeter groups

algebra described at item (x) below. The semisimple part of the cases (a)-(f) below belong to the types (AI), (BDI), (CI) (DIII), (CII), (AII) (in this order) in the Cartan classification of symmetric pairs.

(a) $\mathfrak{gl}_N(\Bbbk)$. Let $< \, , \, >$ a non-degenerate symmetric bilinear form on $\Bbbk^N$, and $x \mapsto x^+$ the adjoint operation on $\mathrm{End}(\Bbbk^N)$. Then $\mathfrak{gl}_N(\Bbbk)^0 = \{x \mid x^+ = -x\} = \mathfrak{so}_N(\Bbbk)$ and $\mathfrak{gl}_N(\Bbbk)^1 = \{x \mid x^+ = x\}$.

(b) $\mathfrak{so}_{2N}(\Bbbk)$ w.r.t. a non-degenerate symmetric bilinear form $< \, , \, >$. Let $U, V \subset \Bbbk^N$ with $\Bbbk^{2N} = U \oplus V$, $\dim U = \dim V = N$ and $U \perp V$ (this implies that the restriction of $< \, , \, >$ to U and V is non-degenerate). Then

$$\mathfrak{so}_{2N}(\Bbbk)^0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \, x^+ = -x, y^+ = -y \right\} \simeq \mathfrak{so}(U) \times \mathfrak{so}(V)$$

and

$$\mathfrak{so}_{2N}(\Bbbk)^1 = \left\{ \begin{pmatrix} 0 & -m^+ \\ m & 0 \end{pmatrix} \middle| \, m \in \mathfrak{gl}_N(\Bbbk) \right\}$$

(where $m \mapsto m^+$ here denotes the adjonction $\mathrm{Hom}(U, V) \to \mathrm{Hom}(V, U)$).

(c) $\mathfrak{sp}_{2N}(\Bbbk)$ w.r.t. a non-degenerate skew-symmetric bilinear form $< \, , \, >$. Let $U, V \subset \Bbbk^{2N}$, $\Bbbk^{2N} = U \oplus V$, $\dim U = \dim V$, and U, V totally isotropic. Assuming $< \, , \, >=$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we can identify U and V. Then

$$\mathfrak{sp}_{2N}(\Bbbk)^0 = \left\{ \begin{pmatrix} m & 0 \\ 0 & -m^+ \end{pmatrix} \middle| m \in \mathfrak{gl}(U) \right\} \simeq \mathfrak{gl}(U)$$

and

$$\mathfrak{sp}_{2N}(\Bbbk)^1 = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \middle| x^+ = x, y^+ = y \right\}$$

(d) $\mathfrak{so}_{2N}(\Bbbk)$ w.r.t. a non-degenerate symmetric bilinear form $< \, , \, >$, written as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $U, V \subset \Bbbk^{2N}$, $\Bbbk^{2N} = U \oplus V$, $\dim U = \dim V$, and U, V totally isotropic. Identifying U with V using $< \, , \, >$, then

$$\mathfrak{so}_{2N}(\Bbbk)^0 = \left\{ \begin{pmatrix} m & 0 \\ 0 & -m^+ \end{pmatrix} \middle| m \in \mathfrak{gl}_N(\Bbbk) \right\} \simeq \mathfrak{gl}_N(\Bbbk)$$

and

$$\mathfrak{so}_{2N}(\Bbbk)^1 = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \middle| x^+ = -x, y^+ = -y \right\}$$

(e) $\mathfrak{sp}_{2N}(\Bbbk)$ for N even w.r.t. a non-degenerate skew-symmetric bilinear form $< \, , \, >$, written as $\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\mathfrak{sp}_{2N}(\Bbbk)^0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathfrak{sp}_N(\Bbbk) \right\} \simeq \mathfrak{sp}_N(\Bbbk) \times \mathfrak{sp}_N(\Bbbk)$$

(remark : here $m^+ = J^{-1} \, {}^t m J$) and

$$\mathfrak{sp}_{2N}(\Bbbk)^1 = \left\{ \begin{pmatrix} 0 & -m^+ \\ m & 0 \end{pmatrix} \middle| m \in \mathfrak{gl}_N(\Bbbk) \right\}$$

(f) $\mathfrak{gl}_N(\Bbbk)$ with N even. Analogous to case (a), with $< \, , \, >$ a non-degenerate *skew-symmetric* bilinear form on $\Bbbk^N$. Then $\mathfrak{gl}_N^0(\Bbbk) \simeq \mathfrak{sp}_N(\Bbbk)$.

We future use, we note that the group

$$U_N^\sigma(K) = \{M \in \mathrm{GL}_N(K) \mid M \, {}^t\sigma(M) = 1\}$$

is a K^σ-subgroup of $\mathrm{GL}_N(K)$, made of the $a + hb \in \mathrm{GL}_N(K)$ such that $a \, {}^t b$ is a symmetric matrix and ${}^t aa = 1 + x {}^t b$. Its Lie algebra with its natural $(\mathbb{Z}/2\mathbb{Z})$-grading, is made of the $\alpha + h\beta$ with ${}^t\alpha = -\alpha$ and $\beta = {}^t\beta$, hence is of type (a).

4. The Lie algebra $\mathcal{L}_\varepsilon(G)$

4.1. **General facts.** Let G be a finite group and $\alpha : G \to \{\pm 1\}$ a character (for instance the trivial character $\alpha = 1$). Then Lie algebra $\mathcal{L}_\alpha(G)$ (see [M8]) is the subspace of $\Bbbk G$ spanned by the $g - \alpha(g) g^{-1}$ for $g \in G$. We assume there is a nontrivial character $\varepsilon : G \twoheadrightarrow \{\pm 1\}$, and let $A = \mathrm{Ker}\, \varepsilon$, $A^\dagger = G \setminus A$. Then $\mathcal{L}_\varepsilon(G)$ has a natural $\mathbb{Z}/2$-grading, given by $\mathcal{L}_\varepsilon(G)^0 = \mathcal{L}_1(A)$ and $\mathcal{L}_\varepsilon(G)^1 = \mathcal{L}_1(A^\dagger) := < b + b^{-1} \mid b \in A^\dagger >$. Let

$\mathcal{E} = \{\rho \in \mathrm{Irr}(G) \mid \rho^* \otimes \varepsilon \not\simeq \rho\}$ $\mathcal{F}^+ = \{\rho \in \mathrm{Irr}(G) \mid \varepsilon \hookrightarrow S^2\rho\}$ $\mathcal{F}^- = \{\rho \in \mathrm{Irr}(G) \mid \varepsilon \hookrightarrow \Lambda^2\rho\}$

and $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^- = \{\rho \in \mathrm{Irr}(G) \mid \varepsilon \hookrightarrow \rho \otimes \rho\}$ (clearly $\mathcal{F}^+ \cap \mathcal{F}^- = \emptyset$). We denote $\sim$ the equivalence relation on $\mathrm{Irr}(G)$ generated by $\rho^* \otimes \varepsilon \sim \rho$ and we identify the set of equivalence classes $\mathcal{E}/\sim$ with a system of representatives in $\mathcal{E}$.

We assume that $\Bbbk$ is a field of realizability for all the irreducible representations of G, and denote V_ρ the underlying $\Bbbk$-vector space of ρ, that is $\rho : G \to \mathrm{GL}(V_\rho)$. Then, we have

$$\Bbbk G = \left(\bigoplus_{\rho \in \mathcal{E}/\sim} (\mathfrak{gl}(V_\rho) \oplus \mathfrak{gl}(V_{\rho^* \otimes \varepsilon})) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{F}} \mathfrak{gl}(V_\rho) \right)$$

and, according to [M8],

$$\mathcal{L}_\varepsilon(G) = \left(\bigoplus_{\rho \in \mathcal{E}/\sim} \mathfrak{gl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{F}} \mathfrak{osp}(V_\rho) \right)$$

where the orthosymplectic Lie algebra $\mathfrak{osp}(V_\rho)$ is defined by the bilinear form induced by $\varepsilon \hookrightarrow \rho \otimes \rho$, and $\mathfrak{gl}(V_\rho) \hookrightarrow \mathfrak{gl}(V_\rho) \oplus \mathfrak{gl}(V_{\rho^* \otimes \varepsilon})$ is given by $x \mapsto (x, -{}^t x)$.

4.2. **Decomposition of the components.** From now on we assume that all irreducible representations of A are also realizable over $\Bbbk$. We will need a 'real version' of Clifford theory. It is provided that the following two lemmas, whose proof is an easy exercise in character theory which is left to the reader.

Lemma 4.1. *Let G be a finite group, $\varepsilon : G \twoheadrightarrow \{\pm 1\}$, $A = \operatorname{Ker} \varepsilon$, $V \in \operatorname{Irr}(G)$ of real type with $V \simeq V \otimes \varepsilon$. One has $\operatorname{Res}_A V = V_+ \oplus V_-$. Then*
1. *V_+ and V_- are of the same type, which is either real or complex.*
2. *$\varepsilon \hookrightarrow S^2 V$ iff V_+ and V_- have real type.*
3. *$\varepsilon \hookrightarrow \Lambda^2 V$ iff V_+ and V_- have complex type.*

Lemma 4.2. *Let G be a finite group, $\varepsilon : G \twoheadrightarrow \{\pm 1\}$, $A = \operatorname{Ker} \varepsilon$, $V \in \operatorname{Irr}(G)$ of quaternionic type with $V \simeq V \otimes \varepsilon$. One has $\operatorname{Res}_A V = V_+ \oplus V_-$. Then*
1. *V_+ and V_- are of the same type, which is either quaternionic or complex.*
2. *$\varepsilon \hookrightarrow S^2 V$ iff V_+ and V_- have complex type.*
3. *$\varepsilon \hookrightarrow \Lambda^2 V$ iff V_+ and V_- have quaternionic type.*

Proposition 4.3. *If ρ has real type, then*
1. *$\rho(\mathcal{L}_1(A)) \cap \rho(\mathcal{L}_1(A^\dagger)) = 0$.*
2. *If $\rho \otimes \varepsilon \not\simeq \rho$ then $\rho(\mathcal{L}_\varepsilon(G))$ is a classical $\mathbb{Z}/2$-graded Lie algebra of type (a).*
3. *If $\varepsilon \hookrightarrow S^2 \rho$ then $\rho(\mathcal{L}_\varepsilon(G))$ is a classical $\mathbb{Z}/2$-graded Lie algebra of type (b).*
4. *If $\varepsilon \hookrightarrow \Lambda^2 \rho$ then $\rho(\mathcal{L}_\varepsilon(G))$ is a classical $\mathbb{Z}/2$-graded Lie algebra of type (c).*

Proof. First consider the nondegenerate symmetric bilinear form induced by $\mathbb{1} \hookrightarrow \rho \otimes \rho$, and denote $m \mapsto m^+$ the adjonction in $\operatorname{End}(V_\rho)$. When $a \in A$ and $b \in A^\dagger$, $\rho(a - a^{-1})^+ = -\rho(a - a^{-1})$ and $\rho(b + b^{-1})^+ = \rho(b + b^{-1})$. Identifying V_ρ with $\Bbbk^N$ with its the canonical bilinear form (this can be done up to an extension of scalar, which is harmless here) this means that $\rho(\mathcal{L}_1(A^\dagger))$ is made of symmetric matrices while $\rho(\mathcal{L}_1(A))$ is made of skew-symmetric matrices. This implies (1) and also (2), since in this case $\rho(\mathcal{L}_\varepsilon(G)) = \mathfrak{gl}(V_\rho)$. In cases (3) and (4), we apply lemma 4.1. We have $\operatorname{Res}_A \rho = \varphi + \xi$, $V_\rho = V_\varphi \oplus V_\xi$. We consider the nondegenerate bilinear form $< , >$ induced by $\varepsilon \hookrightarrow \rho \otimes \rho$. In case (3), it induces a blinear form on $V_\varphi \otimes V_\xi$ which is A-invariant hence zero, as $\xi \not\simeq \varphi^*$ (lemma 4.1). As a consequence $V_\varphi \perp V_\xi$ hence $< , >$ restricts to a nondegenerate A-invariant bilinear form on both V_φ and V_ξ, and we have $\rho(\mathcal{L}_1(A)) \subset \mathfrak{so}(V_\varphi) \times \mathfrak{so}(V_\xi)$. On the other hand, when $b \notin A$, we have $bV_\varphi = V_\xi$ and $bV_\xi = V_\varphi$, as bV_φ is A-stable and $bV_\varphi \neq V_\varphi$ (otherwise V_ρ would be reducible). In matrix form, this implies $\rho(\mathcal{L}_1(A^\dagger)) \subset \left\{ \begin{pmatrix} 0 & -m^+ \\ m & 0 \end{pmatrix} \right\}$, with the notations used in describing type (b). Counting dimensions we get that inclusions are equalities and this implies (3). Case (4) is similar, except that φ and ξ have complex type, hence $\xi \simeq \varphi^*$ (as $\rho \simeq \rho^*$), and this time the induced bilinear forms on $V_\varphi \otimes V_\varphi$ and $V_\xi \otimes V_\xi$ are zero, whereas the ones on $V_\rho \otimes V_\xi$ and $V_\xi \otimes V_\rho$ are nondegenerate. The identification with type (c) is then similar and straightforward. □

Since the representations of a Coxeter group are always of real type, this has the following consequence.

Corollary 4.4. *If W is a finite Coxeter group and $\varepsilon : W \twoheadrightarrow \{\pm 1\}$ is the sign character then, for $i \in \{0, 1\}$,*

$$\mathcal{L}_\varepsilon(W)^i = \left(\bigoplus_{\rho \in \mathcal{E}/\sim} \mathfrak{gl}(V_\rho)^i \right) \oplus \left(\bigoplus_{\rho \in \mathcal{F}} \mathfrak{osp}(V_\rho)^i \right)$$

where the decompositions $\mathfrak{gl}(V_\rho)^i$ (resp. $\mathfrak{osp}(V_\rho)^i$) are given by the description as classical $\mathbb{Z}/2$-graded Lie algebras of types (a),(b),(c).

Proposition 4.5. *If $\rho^* \not\simeq \rho$ and $\rho^* \otimes \varepsilon \simeq \rho$, then*

1. $\rho(\mathcal{L}_1(A)) = \rho(\mathcal{L}_\varepsilon(G)) = \mathfrak{osp}(V_\rho)$
2. $\rho(\mathcal{L}_1(A)) \cap \rho(\mathcal{L}_1(A^\dagger)) \neq 0$ *unless* $\forall b \in A^\dagger \ \rho(b^{-1}) = -\rho(b)$
3. *We have* $(\rho \oplus \rho^*)(\mathcal{L}_\varepsilon(G)) = \mathfrak{osp}(V_\rho) \oplus \mathfrak{osp}(V_{\rho^*}) \simeq \mathfrak{osp}(V_\rho) \oplus \mathfrak{osp}(V_\rho)$ *with* $(\rho \oplus \rho^*)(\rho(\mathcal{L}_1(A)) \simeq \{(x, x) \mid x \in \mathfrak{osp}(V_\rho)\}$ $(\rho \oplus \rho^*)(\rho(\mathcal{L}_1(A^\dagger)) \simeq \{(x, -x) \mid x \in \mathfrak{osp}(V_\rho)\}$
4. *As a $\mathbb{Z}/2$-graded Lie algebra, $(\rho \oplus \rho^*)(\mathcal{L}_\varepsilon(G))$ is the double of $\mathfrak{osp}(V_\rho)$.*

Proof. Since $\rho^* \otimes \varepsilon \simeq \rho$ we have a nondegenerate W-invariant bilinear form on V_ρ afforded by $\varepsilon \hookrightarrow \rho \otimes \rho$, and $\rho(\mathcal{L}_\varepsilon(G)) = \mathfrak{osp}(V_\rho)$. Since $\rho^* \not\simeq \rho$ we have $\rho \not\simeq \rho \otimes \varepsilon$ hence $\mathrm{Res}_A \rho$ is irreducible. Moreover $\mathrm{Res}_A \rho$ is selfdual hence, according to [M8], $\rho(\mathcal{L}_1(A)) = \mathfrak{osp}(V_\rho)$, which proves (1). For all $g \in G$ we have $\varepsilon(w)\rho(w^{-1}) = \rho(w)^+$ hence $\rho(b)^+ = -\rho(b)$ for $b \in A^\dagger$, thus $\rho(\mathcal{L}_1(A)) \cap \rho(\mathcal{L}_1(A^\dagger)) = 0 \Leftrightarrow \rho(\mathcal{L}_1(A^\dagger)) = 0$ means $\forall b \in A^\dagger \ \ \rho(b) = -\rho(b^{-1})$ which proves (2). Now ρ^* can be defined on V_ρ as $g \mapsto \rho(g^{-1})^+$, and then for $x, y \in V_\rho$ we have $< \rho^*(g)x, \rho^*(g)y > = < \varepsilon(g)\rho(g)x, \varepsilon(g)\rho(g)y > = < \rho(g)x, \rho(g)y > = \varepsilon(g) < x, y >$. This proves $(\rho \oplus \rho^*)(\mathcal{L}_\varepsilon(G) = \mathfrak{osp}(V_\rho) \oplus \mathfrak{osp}(V_{\rho^*}) \simeq \mathfrak{osp}(V_\rho) \oplus \mathfrak{osp}(V_\rho$; since $\rho(a)^+ = \rho(a^{-1})$ for $a \in A$ and $\rho(b)^+ = -\rho(b^{-1})$ for $b \in A^\dagger$, (3) follows, and (4) is an immediate consequence of (3). □

Remark 4.6. *When W is a reflection group and ε the determinant, then : (1) one cannot have $\rho(b^{-1}) = -\rho(b)$ for all $b \in A^\dagger = \mathrm{O}^\dagger$, because the reflections provide involutions in $A^\dagger$; (2) one may have $\rho \in \mathrm{Irr}(W)$ with $\rho^* \not\simeq \rho$ and $\rho^* \otimes \varepsilon \simeq \rho$; an example is given by $W = G(3, 3, 4)$, and ρ the restriction of the representation classically denoted $([1, 1], [2], \emptyset)$ of $G(1, 1, 4)$ (it can be checked more precisely, using character tables, that in this example $\varepsilon \hookrightarrow S^2\rho$)*

Remark 4.7. *An example of a triple (G, ε, ρ) as in the proposition with $\forall b \in A^\dagger \ \rho(b^{-1}) = -\rho(b)$is given by $G = \mathbb{Z}/4$, $\rho : G \to \mathrm{GL}_1(\mathbb{C})$, $\varepsilon : 1 \to -1$, $\rho : 1 \mapsto \sqrt{-1}$. There are no example in higher dimension, as proved by the following proposition.*

Proposition 4.8. *Let G be a finite group endowed with $\varepsilon : G \to \{\pm 1\}$. If $\rho \in \mathrm{Irr}(G)$ satisfies $\rho^* \not\simeq \rho$ and $\forall g \in G \ \varepsilon(g) = -1 \Rightarrow \rho(g^{-1}) = -\rho(g)$ then $\dim \rho = 1$.*

Proof. Let $A = \mathrm{Ker}\, \varepsilon \lhd G$, and assume we have such a ρ. Up to considering $G/A \cap \mathrm{Ker}\, \varepsilon$ one can assume that $\rho_{|A}$ is faithful. Since $g^2 \neq 1$ for $g \in G \setminus A$ this implies that ρ is faithful. It follows that there exists $a_0 \in A$ of order 2 with $x^2 = a_0$ for all $x \in G \setminus A$. From $\rho(a_0) = -1$ we also get $a_0 \in Z(G)$. Let $N = \dim V$. Using the Frobenius-Schur indicator, the assumption $\rho^* \not\simeq \rho$ then translates into $\sum_{g \in A} \chi(g^2) = N|A|$ for $\chi = \mathrm{tr}\,\rho$, hence $\chi(g^2) = N$ that is $\rho(g^2) = 1$ hence $g^2 = 1$ for all $g \in A$. In particular A is abelian. For $x, y \in G \setminus A$, $xyx^{-1}y^{-1} = xyxa_0ya_0 = (xy)^2 = 1$ and for $x \in G \setminus A$, $y \in A$, $xy \in G \setminus A$ hence $xyx^{-1}y^{-1} = xyxa_0y = (xy)^2 a_0 = a_0^2 = 1$. This proves that G is abelian hence $\dim \rho = 1$. □

Using lemma 4.2, the proof of the following proposition is analogous to proposition 4.3.

Proposition 4.9. *If ρ has quaternionic type, then*

(1) $\rho(\mathcal{L}_1(A)) \cap \rho(\mathcal{L}_1(A^\dagger)) = 0$.
(2) *If $\rho^* \otimes \varepsilon \not\simeq \rho$ then $\rho(\mathcal{L}_\varepsilon(G))$ is a classical $\mathbb{Z}/2$-graded Lie algebra of type (f).*
(3) *If $\varepsilon \hookrightarrow S^2\rho$ then $\rho(\mathcal{L}_\varepsilon(G))$ is a classical $\mathbb{Z}/2$-graded Lie algebra of type (d).*
(4) *If $\varepsilon \hookrightarrow \Lambda^2\rho$ then $\rho(\mathcal{L}_\varepsilon(G))$ is a classical $\mathbb{Z}/2$-graded Lie algebra of type (e).*

We can apply the above result to the case of a complex reflection group W with rotation subgroup O. We get the following.

Proposition 4.10. *If ρ is a linear representation of W with $\rho^* \simeq \rho$ then, extending ρ into a $\Bbbk(h^2)$-linear map,*

$$\begin{array}{rcl} \rho(\mathcal{H}^0_{\mathrm{tail}}) & = & \Bbbk(h^2)\,(\rho(\mathcal{H}) \cap \rho(\mathcal{L}_1(\mathrm{O}))) \\ \rho(\mathcal{H}^1_{\mathrm{tail}}) & = & h\Bbbk(h^2)\left(\rho(\mathcal{H}) \cap \rho(\mathcal{L}_1(\mathrm{O}^\dagger))\right) \end{array}$$

Proof. This is a consequence of propositions 4.3 (1) and 4.9 (1), as $\Bbbk(h^2)\mathcal{H} = \mathcal{H}^0_{\mathrm{tail}} \oplus h^{-1}\mathcal{H}^1_{\mathrm{tail}} \subset \Bbbk(h^2)W$, and $\rho(\mathcal{H}^0_{\mathrm{tail}}) \subset \Bbbk(h^2)\rho(\mathcal{L}_1(A))$, $\rho(\mathcal{H}^1_{\mathrm{tail}}) \subset h\Bbbk(h^2)\rho(\mathcal{L}_1(A^\dagger))$. □

Let us now assume that W is a Coxeter group. In [M5], a decomposition of $\mathcal{H}$ is obtained, that we recall now, in the special case of a Coxeter group. For this we recall from [M5] the notation

$$\mathrm{X}(\rho) = \{\eta \in \mathrm{Hom}(W, \{\pm 1\}) \mid \forall s \in \mathcal{R}\ \eta(s) = -1 \Rightarrow \rho(s) = \pm 1\}$$

and also that $\mathrm{Ref}(W)$ is the set of all $\rho \in \mathrm{Irr}(W)$ such that, for all $s \in \mathcal{R}$ with $\rho(s) \neq \pm 1$, $\rho(s)$ is a reflection, and

$$\begin{array}{rcl} \mathrm{QRef} & = & \{\eta \otimes \rho \mid \rho \in \mathrm{Ref}, \eta \in \mathrm{Hom}(W, \{\pm 1\})\} \\ \Lambda\mathrm{Ref} & = & \{\eta \otimes \Lambda^k \rho \mid \rho \in \mathrm{Ref}, \eta \in \mathrm{Hom}(W, \{\pm 1\}), k \geq 0\} \end{array}$$

We also recall that there is an equivalence relation $\approx'$ on $\mathrm{Irr}(W)$ which, when W has no component of type H_4, is defined by

$$\rho_1 \approx' \rho_2 \Leftrightarrow \rho_2 \in \{\rho_1 \otimes \eta, \rho_1^* \otimes \eta \otimes \varepsilon \mid \eta \in \mathrm{X}(\rho_1)\}$$

(we refer the reader to [M5] for more details on the H_4 case). Letting $\mathcal{R}/W$ denote the set of conjugacy classes of reflections, and $\mathrm{Irr}'(W) = \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$, theorem 1 of [M5] provides an explicit isomorphism

$$\mathcal{H} \simeq \Bbbk^{\mathcal{R}/W} \oplus \left(\bigoplus_{\rho \in \mathrm{QRef}/\approx'} \mathfrak{sl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathrm{Irr}'(W)/\approx'} \mathfrak{osp}(V_\rho) \right)$$

A consequence of proposition 4.10 and corollary 4.4 is that this isomorphism is $\mathbb{Z}/2\mathbb{Z}$ graded, thus providing the following upgrade of theorem 1 of [M5], in the case of a Coxeter group.

Theorem 4.11. *If W is a Coxeter group, then, as $(\mathbb{Z}/2\mathbb{Z})$-graded Lie algebras, we have an isomorphism, for $L = \Bbbk(h^2)$ and $V_\rho^L = V_\rho \otimes_\Bbbk L$,*

$$\mathcal{H}_{\mathrm{tail}} \simeq L^{\mathcal{R}/W} \oplus \left(\bigoplus_{\rho \in \mathrm{QRef}/\approx'} \mathfrak{sl}(V_\rho^L) \right) \oplus \left(\bigoplus_{\rho \in \mathrm{Irr}'(W)/\approx'} \mathfrak{osp}(V_\rho^L) \right)$$

where the $\mathbb{Z}/2\mathbb{Z}$-grading on the $\mathfrak{sl}(V_\rho^L)$ and the $\mathfrak{osp}(V_\rho^L)$ are given by the description as $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras of types (a),(b),(c), and L is the 1-dimensional commutative $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra of odd degree.

5. Zariski closures

Our goal is to provide interpretations of the $(\mathbb{Z}/2\mathbb{Z})$-grading in terms of Zariski closures of the braid groups inside the Hecke algebra representations. We let $\Bbbk$ denote a field of characteristic 0, $R = \Bbbk[[h]]$, $K = \Bbbk((h))$. The setting of Chevalley in [Ch] is mostly convenient for us, as we are dealing with subgroups of $\mathrm{GL}_m(F)$ for F a field of characteristic 0. In the sequel, the notions that we use constantly refer to this setting. One basic lemma that we will need is the following one.

Lemma 5.1. *Let Γ be a Zariski-closed subgroup of $GL_N(K)$ and $\mathrm{Lie}\,\Gamma$ its Lie algebra over K. For all $x \in M_N(R)$, if $\exp(hx) \in \Gamma$ then*

1. $\exp(Thx) \in \Gamma(K[[T]])$.
2. *for all $u \in \Bbbk$, $\exp(uhx) \in \Gamma(K)$.*
3. $x \in \mathrm{Lie}\,\Gamma$.

Moreover, the algebraic closure of $\exp(hx)$ inside $\mathrm{GL}_N(K)$ is connected. More generally, if L is a subfield of K, then the algebraic closure of $\exp(hx)$ inside $\mathrm{GL}_N(K)$ considered as an L-group is connected.

Proof. Let $a_1, \dots, a_r$ be polynomial functions on $M_N(K)$ with coefficients in R such that $\Gamma = \{m \in GL_N(K) \mid \forall i \in [1,r] \ \ a_i(m) = 0\}$. Since Γ is a subgroup of $GL_N(K)$, one has $a_i(\exp(nhx)) = 0$ for all $i \in [1,r]$ and $n \in \mathbb{Z}$. Let $Q_i = a_i(\exp(Thx)) \in (\Bbbk[T])[[h]]$. Since $\Bbbk$ has caracteristic 0, $\mathbb{Z}$ is Zariski-dense in $\Bbbk$ hence $Q_i = 0$ and $\exp(Thx)$ is a $\Bbbk[[T]]$-point of Γ. This shows (1) and (2). It follows from (1) and [Ch] ch. 2 §12 théorème 7 that $hx \in \mathrm{Lie}\,\Gamma$ hence $x \in \mathrm{Lie}\,\Gamma$, which is (3). We now let L denote a subfield of K, we let Γ denote the algebraic closure of $\exp(hx)$ inside $\mathrm{GL}_N(K)$ considered as an L-group, and show that it is connected. We can assume $x \neq 0$, for otherwise the statement is trivial. Let $G = \{\exp(nhx), n \in \mathbb{Z}\} \subset \Gamma$ be the cyclic subgroup generated by $X = \exp(hx)$, Γ_0 be the connected component of the identity in Γ, and let $\pi : \Gamma \twoheadrightarrow \Gamma/\Gamma_0$. By the same as argument as for (2) we know that Γ contains $G_\Bbbk = \{\exp(uhx), u \in \Bbbk\} \simeq \Bbbk$. Since Γ/Γ_0 is finite the restriction of π to $G_\Bbbk$ is trivial, hence $G \subset \mathrm{Ker}\,\pi = \Gamma_0$. Since Γ is assumed to be minimal this proves $\Gamma = \Gamma_0$. $\square$

Let now $\sigma \in Aut(K)$ be $f(h) \mapsto f(-h)$. Letting $x = h^2$ we have $R^\sigma = \Bbbk[[x]]$, $K^\sigma = \Bbbk((x))$ and $\mathrm{Gal}(K/K^\sigma) \simeq \mathbb{Z}/2\mathbb{Z}$. For $N \geq 1$, $Mat_N(K) = Mat_N(K^\sigma) \oplus hMat_N(K^\sigma)$ can be embedded into $Mat_{2N}(K^\sigma)$ as a (closed) K^σ-subalgebra through $a + hb \mapsto \begin{pmatrix} a & xb \\ b & a \end{pmatrix}$. The group $\mathrm{GL}_N(K)$ can be identified to the subset of elements of $\mathrm{GL}_{2N}(K^\sigma)$ of the above form, which is clearly an algebraic subgroup of $\mathrm{GL}_{2N}(K^\sigma)$.

Lemma 5.2. *Let G be a closed (algebraic) K^σ-subgroup of* $\mathrm{GL}_N(K)$ *and* $X = \exp(a+hb) \in \mathrm{GL}_N(K)$ *for some* $a, b \in \mathfrak{gl}_N(R^\sigma)$ *with* $a \equiv 0 \mod h$. *Then* $a + hb \in \mathrm{Lie}G$.

Proof. First note that, since $a \equiv 0 \mod h$, X is well-defined. By Chevalley's formal exponentiation theory ([Ch], t. 2, ch. 2, §12, thm. 7) this is equivalent to saying $\exp(u(a+hb)) \in G(K^\sigma[[u]])$. We consider G as defined in $\mathrm{GL}_{2N}(K^\sigma)$, and let $\alpha_1, \dots, \alpha_r \in R^\sigma[m_{11}, \dots, m_{2N,2N}]$ be defining equations for G. Inside $\mathrm{GL}_{2N}(K^\sigma[[u]])$, $\exp(u(a+hb))$ is a $2N \times 2N$ matrix which can be written

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \begin{pmatrix} a & xb \\ b & a \end{pmatrix}^n = \sum_{n=0}^{\infty} x^n c_n(u)$$

with $c_n(u) \in Mat_{2N}(\Bbbk[[u]])$. Let $M = \begin{pmatrix} a & xb \\ b & a \end{pmatrix}$. It is easily checked that $M^4 \equiv 0 \mod x$, so we get that, modulo x^r,

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} M^n \equiv \sum_{n \leq 4r} \frac{u^n}{n!} M^n$$

and this shows that $c_n(u) \in Mat_{2N}(\Bbbk[u])$. In particular, $\alpha_i(\exp(u(a+hb))) = \sum_j c_{i,j}(u) x^j \in \Bbbk[u][[x]]$; since $X^n \in G(K^\sigma)$ for all $n \in \mathbb{Z}$ we have $\forall n \in \mathbb{Z}$ $c_{i,j}(n) = 0$, hence $c_{ij} = 0$, which proves the result. □

Remark 5.3. *In the situation above,* $\dim_{K^\sigma} \mathrm{Lie}G = \dim_K K \otimes_{K^\sigma} \mathrm{Lie}G$*, and* $K \otimes_{K^\sigma} \mathrm{Lie}G$ *is the Lie algebra of the Zariski closure of G in K (*[Ch]*, t. 2, ch. 2, §8, prop. 2).*

Let $\Psi : B \to \mathrm{GL}_N(K)$ a representation of the Hecke algebra constructed from $\rho : W \to \mathrm{GL}_N(\Bbbk)$ by monodromy (in which case $\Bbbk = \mathbb{C}$) or through generalizations of Drinfeld associators (see section 6 below), and $\mathfrak{g}$ the Lie subalgebra of $\mathfrak{gl}_N(R)$ generated by the hs for $s \in \mathcal{R}$. Let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{gl}_N(\Bbbk)$ spanned by the brackets of an *even* number of reflections, and $\mathfrak{g}_1$ be the one spanned by the brackets of an *odd* number of reflections. We define $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes_\Bbbk R^\sigma$, $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes_\Bbbk R^\sigma$. We have $\mathfrak{g} \subset \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus h\tilde{\mathfrak{g}}_1$.

Proposition 5.4. *Let $\overline{\Psi(P)}$ denote the Zariski closure of $\Psi(P)$ inside the K^σ-group* $\mathrm{GL}_N(K)$. *It is connected, and* $\mathrm{Lie}\overline{\Psi(P)} = \tilde{\mathfrak{g}} \otimes_{R^\sigma} K^\sigma$. *Moreover* $\mathfrak{g}_0 \cap \mathfrak{g}_1 = \{0\}$.

Proof. First note that, by the monodromy construction, every $g \in P$ is mapped to $\Psi(g) = \exp(hx)$ for some $x \in \mathfrak{gl}_N(R)$. By lemma 5.1 (applied to $L = K^\sigma$) it follows that the Zariski closure of the subgroup generated by each $\Psi(g)$ for $g \in P$ is connected. This implies that $\overline{\Psi(P)}$ is connected (see [Ch] ch. 2, §14, théorème 14). For each $s \in \mathcal{R}$, let $Y_s \in P$ with $\Psi(Y_s) = \exp(hs + \dots)$, that is $\Psi(Y_s) = \exp(a + hb)$ with $a, b \in \mathfrak{gl}_N(R^\sigma)$, $a \equiv 0 \mod x$, $b \equiv s \mod x$. By lemma 5.2 we get $y_s = a + hb \in \mathrm{Lie}\overline{\Psi(P)}$. Note that $y_s \in \mathfrak{gl}_N(R^\sigma) \oplus h\mathfrak{gl}_N(R^\sigma)$ and $y_s \equiv hs \mod x$. Also note that the y_s belong to $\tilde{\mathfrak{g}}_0 \oplus h\tilde{\mathfrak{g}}_1 = \tilde{\mathfrak{g}}$. We let $\mathcal{L}$ denote the subalgebra of $\tilde{\mathfrak{g}} \otimes_{R^\sigma} K^\sigma$ that they generate. As a K^σ-algebra, it admits a basis of elements of the form $h^\eta[s_1, \dots, s_r]$, with $\eta = 1$ if r is odd, $\eta = 0$ if r is even. We have

$$\frac{1}{x^{\lfloor \frac{r}{2} \rfloor}} h^\eta [y_{s_1}, \dots, y_{s_r}] \in \tilde{g} \text{ and } \frac{1}{x^{\lfloor \frac{r}{2} \rfloor}} h^\eta [y_{s_1}, \dots, y_{s_r}] \equiv h^\eta [s_1, \dots, s_r] \mod x$$

where the notation $[a_1, \dots, a_r]$ denotes the iterated Lie bracket of the $a_1, \dots, a_r$. It then follows (e.g. by using the determinant) that these elements are linearly independent over K^σ, hence $\dim_{K^\sigma} \mathcal{L} \geq \dim_{K^\sigma} \tilde{\mathfrak{g}} \otimes_{R^\sigma} K^\sigma$ and $\mathcal{L} = \tilde{\mathfrak{g}} \otimes_{R^\sigma} K^\sigma$. Now the dimension of Lie $\overline{\Psi(P)}$ over

K^σ equals the dimension d of the Zariski closure of $\Psi(P)$ in $\mathrm{GL}_N(K)$ over K ([Ch] t. 2, ch. 2, §6 prop. 5), hence

$$\dim_{K^\sigma} K^\sigma \tilde{\mathfrak{g}}_0 + \dim_{K^\sigma} K^\sigma \tilde{\mathfrak{g}}_1 = \dim_{K^\sigma} \mathcal{L} \leq \dim_{K^\sigma} \mathrm{Lie}\overline{\Psi(P)} = d.$$

On the other hand, we proved in [M5] that d is equal to $\dim_{\mathbb{k}} \rho(\mathcal{H})$. Since $d = \dim_{\mathbb{k}}(\mathfrak{g}_0 + \mathfrak{g}_1) \leq \dim_{\mathbb{k}} \mathfrak{g}_0 + \dim_{\mathbb{k}} \mathfrak{g}_1 = \dim_{K^\sigma} K^\sigma \tilde{\mathfrak{g}}_0 + \dim_{K^\sigma} K^\sigma \tilde{\mathfrak{g}}_1$, we get $\dim_{K^\sigma} \mathrm{Lie}\overline{\Psi(P)} = \dim_{K^\sigma} \tilde{\mathfrak{g}} \otimes_{R^\sigma} K^\sigma$, and $\dim(\mathfrak{g}_0 + \mathfrak{g}_1) = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$, which concludes the proof.

□

Remark 5.5. *The latter statement $\mathfrak{g}_0 \cap \mathfrak{g}_1 = \{0\}$ is a consequence of propositions 4.3 and 4.5 if $\mathrm{tr}\rho$ only has real values. Combined with propositions 4.5 and 4.9, it proves on the other hand that, when $\rho^* \not\simeq \rho$, $\rho^* \otimes \varepsilon \simeq \rho$ and $\dim \rho > 1$, then $\mathfrak{g}_1 \subsetneq \rho(\mathcal{L}_1(\mathrm{O}^\dagger))$.*

This proves the first part of theorem 1.6, the second one being a consequence of proposition 4.10. This also proves theorem 1.5, as follows.

Corollary 5.6. *Under the weak BMR conjecture for W, the Zariski closure of B inside $H(q)^\times$ as an algebraic $\mathbb{k}((h^2))$-group has for Lie algebra $\mathcal{H}_{\mathrm{tail}} \otimes_{\mathbb{k}(h^2)} \mathbb{k}((h^2))$.*

Proof. Since P has finite index in B, we can instead consider the Zariski closure of P. We identify $H(q)^\times$ to a closed subgroup of $\mathrm{GL}_N(K)$ with $N = |W|$ and $K = \mathbb{k}((h))$, and apply proposition 5.4. In this setting $\mathfrak{g}_0 \otimes_{\mathbb{k}} \mathbb{k}(h^2) = \mathcal{H}^0_{\mathrm{tail}}$ and $\mathfrak{g}_1 \otimes_{\mathbb{k}} \mathbb{k}(h^2) = h^{-1}\mathcal{H}^1_{\mathrm{tail}}$ hence $\tilde{\mathfrak{g}} \otimes_{R^\sigma} K^\sigma = \mathcal{H}_{\mathrm{tail}} \otimes_{\mathbb{k}(h^2)} \mathbb{k}((h^2))$ and this proves the statement. □

Let $(C^r P)_{r\geq 0}$ denote the lower central series of P, let $N \otimes \mathbb{k}$ denote the $\mathbb{k}$-Malcev completion of the nilpotent group N, and $P(\mathbb{k})$ the inverse limit of the $(P/C^r P) \otimes \mathbb{k}$, $r \geq 0$. One has $P(\mathbb{k}) \simeq \exp \widehat{\mathcal{T}}$, where $\mathcal{T}$ is the holonomy Lie algebra of the hyperplane complement associated to W (defined over $\mathbb{k}$) generated by the elements $t_s, s \in \mathcal{R}$, whose linear span constitute the homogeneous part of degree 1 of the graded Lie algebra $\mathcal{T}$. This part $\mathcal{T}^1$ can be canonically identified with the first homology group $H_1(X_W, \mathbb{k})$. We let $\widehat{\mathcal{T}}$ denote its completion w.r.t. the natural grading. The morphism $\Psi : P \to \mathrm{GL}_N(K)$ can be extended to $\Psi_+ : P(\mathbb{k}) \to \mathrm{GL}_N(K)$.

However, we have the following, which provides an interpretation of $\mathcal{H}^0_{\mathrm{tail}}$ as an algebraic Lie algebra. We let $\mathcal{H}^{\mathrm{even}}_{\mathrm{gr}} = \bigoplus_r \mathcal{H}^{(2r)}_{\mathrm{gr}} = \bigoplus_r h^{2r}\mathcal{H}^{2r}_{\mathrm{gr}}$, and let $\widehat{\mathcal{H}_{\mathrm{gr}}}$, $\widehat{\mathcal{H}_{\mathrm{gr}}}^{\mathrm{even}}$ denote the completions of $\mathcal{H}_{\mathrm{gr}}$ and $\mathcal{H}^{\mathrm{even}}_{\mathrm{gr}}$ with respect to the grading (or, equivalently : w.r.t. the h-adic topology).

Proposition 5.7. *Assume that $\rho : W \to \mathrm{GL}(\mathbb{k}W) = \mathrm{GL}_N(\mathbb{k})$ is the regular representation of W.*

1. $\mathrm{Im}\,\Psi_+ = \exp \widehat{\mathcal{H}_{\mathrm{gr}}}$
2. $\Psi_+(P(\mathbb{k})) \cap \mathrm{GL}_N(K^\sigma) = \exp \widehat{\mathcal{H}_{\mathrm{gr}}}^{\mathrm{even}}$
3. *The Zariski closure of $\Psi_+(P(\mathbb{k})) \cap \mathrm{GL}_N(K^\sigma)$ inside $\mathrm{GL}_N(K^\sigma)$ is connected, and has $K^\sigma\mathcal{H}^0_{\mathrm{tail}}$ for Lie algebra.*

Proof. By construction of Ψ we have $\Psi_+(P(\mathbb{k})) \subset \exp \mathcal{H}_{\mathrm{gr}}$. Moreover, by construction $P(\mathbb{k}) = \exp \mathfrak{P}(\mathbb{k})$, where $\mathfrak{P}(\mathbb{k})$ is the inverse limit of the $\mathfrak{P}_r$, with $\mathfrak{P}_r$ the nilpotent Lie $\mathbb{k}$-algebra defined by $\exp \mathfrak{P}_r = (P/C^r P) \otimes \mathbb{k}$. Thus $\Psi_+ : P(\mathbb{k}) \to \exp \mathcal{H}_{\mathrm{gr}}$ is surjective if and only if $\mathrm{d}\Psi_+ : \mathfrak{P}(\mathbb{k}) \to \mathcal{H}_{\mathrm{gr}}$ is surjective. The morphism $\mathrm{d}\Psi_+$ respects the natural filtrations of $\mathfrak{P}(\mathbb{k})$ and $\mathcal{H}_{\mathrm{gr}}$, hence it is surjective as soon as the associated morphism between the associated graded Lie algebras $\mathrm{gr.\,d}\,\Psi_+ : \mathrm{gr.}\,\mathfrak{P}(\mathbb{k}) \to \mathrm{gr.}\,\mathcal{H}_{\mathrm{gr}} = \mathcal{H}_{\mathrm{gr}}$ is surjective. Now $\mathrm{gr.}\,\mathfrak{P}(\mathbb{k})$ is generated by $P^{ab} \otimes \mathbb{k} \simeq H_1(X_W, \mathbb{k}) \simeq \mathcal{T}^1$, and $\mathrm{d}\,\Psi_+$ on $\mathcal{T}^1 \to \mathcal{H}^1_{\mathrm{gr}}$ is given by $t_s \mapsto hs$

for $s \in \mathcal{R}$, hence is surjective. This proves (1). We have $\Psi_+(P(\Bbbk)) = \exp \widehat{\mathcal{H}_{\mathrm{gr}}}$ by (1), and clearly $\exp \widehat{\mathcal{H}_{\mathrm{gr}}} \cap \mathrm{GL}_N(K^\sigma) = \exp \widehat{\mathcal{H}_{\mathrm{gr}}}^{\mathrm{even}}$, hence (2). For proving (3) it is thus sufficient to show that the Zariski closure G of $\exp \widehat{\mathcal{H}_{\mathrm{gr}}}^{\mathrm{even}}$ inside $\mathrm{GL}_N(K^\sigma)$ is a connected K^σ-group, whose Lie algebra is $K^\sigma \mathcal{H}^0_{\mathrm{tail}}$. First note that, for each $x \in \mathcal{H}^{2r}_{\mathrm{gr}}$, $\exp(h^{2r}x) \in G(K^\sigma)$, hence $x \in \mathrm{Lie}\, G$ by lemma 5.1. This implies that $\mathrm{Lie}\, G$ contains $K^\sigma \mathcal{H}^0_{\mathrm{tail}}$. Now $\exp \widehat{\mathcal{H}_{\mathrm{gr}}}^{\mathrm{even}}$ is generated by elements of the form $\exp hx$ for $x \in \mathfrak{gl}_N(\Bbbk[[h]])$, which generate cyclic subgroups whose Zariski closures are connected (see lemma 5.1). It follows that G is connected ([Ch] ch. 2, §14, théorème 14). Conversely, since $\Bbbk[[h]]$ is a topological ring w.r.t. its natural h-adic topology, the induced h-adic topology on $\mathfrak{gl}_N(\Bbbk[[h]])$ is thiner than the Zariski topology. Because of this and because of lemma 5.1(2) it follows that $\Psi_+(P(\Bbbk)) = \exp \widehat{\mathcal{H}_{\mathrm{gr}}}$ lies inside $\widehat{\Psi(P)}$, hence $\exp \widehat{\mathcal{H}_{\mathrm{gr}}}^{\mathrm{even}} \subset \widehat{\Psi(P)} \cap \mathrm{GL}_N(K^\sigma)$ and $G \subset \widehat{\Psi(P)} \cap \mathrm{GL}_N(K^\sigma)$. It follows (e.g. [Ch] t.2 ch. 2 §8, corollaire de la proposition 1) that $\mathrm{Lie}\, G \subset (\mathrm{Lie}\, \widehat{\Psi(P)}) \cap \mathrm{Lie}\,(\mathrm{GL}_N(K^\sigma)) = (K^\sigma \mathcal{H}_{\mathrm{tail}}) \cap \mathfrak{gl}_N(K^\sigma) = K^\sigma \mathcal{H}^0_{\mathrm{tail}}$. This concludes the proof.

□

We now establish the connection between the above statements and the existence of unitary structures. For this we first need to recall some well-known facts on unitary groups. In general, assume that K/K_0 is a quadratic extension, whose non-zero automorphism is denoted $x \mapsto \bar{x}$. The corresponding unitary group is $U_N = \{g \in \mathrm{GL}_N(K) \mid {}^t\overline{g}g = 1\}$. It is an algebraic group over K_0. For a K_0-algebra S, its S-points are given by $U_N(S) = \{g \in \mathrm{GL}_N(K \otimes_{K_0} S) \mid {}^t\overline{g}g = 1\}$, where $x \mapsto \bar{x}$ is extended trivially on $K \otimes_{K_0} S$, and in particular $U_N = U_N(K_0)$. For $S = K$, we have a canonical identification $K \otimes_{K_0} K \simeq K \oplus K$ given by $x \otimes \lambda \mapsto (\lambda x, \lambda \overline{x})$, which identifies U_N with the subgroup of $\mathrm{GL}_N(K) \times \mathrm{GL}_N(K)$ made of the couples (g_1, g_2) such that ${}^t g_2 g_1 = 1$. This subgroup being isomorphic (over K_0) with $\mathrm{GL}_N(K)$, we get that an isomorphism $U_N(K) \simeq \mathrm{GL}_N(K)$ which is an isomorphism of algebraic groups over K. In particular, U_N is connected.

Let now $OSP_N(K) = OSP^A_N(K)$ be a closed subgroup of $\mathrm{GL}_N(K)$, described by $\{g \in \mathrm{GL}_N(K) \mid {}^t gAg = A\}$ for some $A \in \mathrm{GL}_N(K_0)$. Under the identification above, we get that $OSP_N(K)$ is isomorphic to the group of the $\{(g_1, g_2) \in \mathrm{GL}_N(K) \mid {}^t g_1 A g_1 = A, {}^t g_2 A g_2 = A\}$, and $OSP_N(K) \cap U_N(K)$ is isomorphic to

$$\begin{array}{rl} & \{(g_1, g_2) \in \mathrm{GL}_N(K) \mid {}^t g_2 g_1 = 1, {}^t g_1 A g_1 = A, {}^t g_2 A g_2 = A\} \\ = & \{(g_1, g_2) \in \mathrm{GL}_N(K) \mid {}^t g_2 g_1 = 1, {}^t g_1 A g_1 = A, g_1^{-1} A {}^t g_1^{-1} = A\} \\ = & \{(g_1, g_2) \in \mathrm{GL}_N(K) \mid {}^t g_2 g_1 = 1, {}^t g_1 A g_1 = A,\ {}^t g_1 A^{-1} g_1 = A^{-1}\} \\ \simeq & OSP^A_N(K) \cap OSP^{A^{-1}}_N(K) \end{array}$$

Finally, if $OSP^A_N(K) \cap U_N(K)$ is known to act irreducibly on K^N, then Schur's lemma implies that there is up to a scalar at most one bilinear form which is preserved, hence $OSP^A_N(K) = OSP^{A^{-1}}_N(K)$. If $OSP^{A,0}_N(K)$ denotes the connected component of $OSP_N(K)$, it follows in this case that $OSP^{A,0}_N(K) \cap U_N(K)$ is connected.

We come back to our setting of a monodromy representation Ψ associated to some representation ρ of W, with notations $\mathfrak{g}_0, \mathfrak{g}_1$ as before.

From now on, we assume that ρ is an orthogonal representation.

This condition is in particular satisfied in two important cases:

- when W is a Coxeter group, or
- when ρ is the regular representation of W.

Under this assumption, we can assume that ${}^t\rho(s)\rho(s) = 1$, hence ${}^t\rho(s) = \rho(s)$ and ${}^t\sigma(\rho(hs)) + \sigma(\rho(hs)) = 0$ for all reflections s. This clearly implies $\rho(\mathcal{H}_{\mathrm{gr}}) \subset \mathfrak{u}_N(K) = \{g \in \mathfrak{gl}_N(K) \mid {}^t\sigma(g) + g = 0\}$. Now, recalling that $R = \Bbbk[[h]]$, we have that $\rho(\mathcal{H}_{\mathrm{gr}}) \subset \tilde{\mathfrak{g}}_0 \oplus h\tilde{\mathfrak{g}}_1 \subset R\rho(\mathcal{H}) = \tilde{\mathfrak{g}}_0 \oplus h\tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus h\tilde{\mathfrak{g}}_1$. Moreover, the elements $a \oplus hb \oplus c \oplus hd \in \tilde{\mathfrak{g}}_0 \oplus h\tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus h\tilde{\mathfrak{g}}_1$ belong to $\mathfrak{u}_N(K)$ if and only if $a +^t a = 0$, $b -^t b = 0$, $c +^t c = 0$, $d -^t d = 0$. Since $m \in \mathfrak{g}_0 \Rightarrow^t m + m = 0$ and $m \in \mathfrak{g}_1 \Rightarrow^t m - m = 0$, this condition is equivalent to $b = c = 0$. This means $(R\rho(\mathcal{H})) \cap \mathfrak{u}_N(K) = \tilde{\mathfrak{g}}_0 \oplus h\tilde{\mathfrak{g}}_1 = R^\sigma(\mathfrak{g}_0 + h\mathfrak{g}_1)$ hence $(K\rho(\mathcal{H})) \cap \mathfrak{u}_N(K) = K^\sigma(\mathfrak{g}_0 \oplus h\mathfrak{g}_1) \simeq \mathcal{H}_{\mathrm{tail}} \otimes_{\Bbbk(h^2)} \Bbbk((h))$.

Proposition 5.8. *Assume that ρ is orthogonal. If $\widehat{\Psi(P)}$ denotes the Zariski closure of $\Psi(P)$ inside the K-group $\mathrm{GL}_N(K)$, then $\overline{\Psi(P)}$ is the connected component of the identity of $\widehat{\Psi(P)} \cap U_N^\sigma(K)$.*

Proof. Clearly $\overline{\Psi(P)} \subset \widehat{\Psi(P)} \cap U_N^\sigma(K)$, and $\overline{\Psi(P)}$ is connected by proposition 5.4. Since $(K\rho(\mathcal{H})) \cap \mathfrak{u}_N(K) \simeq \rho(\mathcal{H}_{\mathrm{tail}}) \otimes_{\Bbbk(h^2)} \Bbbk((h))$ we get by proposition 5.4 that these two K^σ-groups have the same Lie algebra. The conclusion follows. ☐

Remark 5.9. *What happens exactly inside each irreducible representation, in the non-orthogonal case, still requires further investigation.*

Remark 5.10. *In this section, the statements about P also hold for the larger but less usual subgroup of even braids, with the same proofs.*

6. Orthogonal representations and palindromes

Assume W is a Coxeter group, and let $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ standard generators of the corresponding Artin group $\pi_1(X/W, x_0)$, with x_0 in the chosen Weyl chamber. Then complex conjugation induces an outer automorphism of B, known as the mirror image, which maps $\mathfrak{s}_i \mapsto \mathfrak{s}_i^{-1}$.

We state the following conjecture, which is a refinement of conjecture 1 in [M6].

Conjecture 6.1. *Let (W, S) be a finite Coxeter system, $\Bbbk$ a field of characteristic 0. There exists morphisms $\Phi : B \to W \ltimes \exp \widehat{\mathcal{T}}$, with $\mathcal{T}$ defined over $\Bbbk$, such that $\Phi(\mathfrak{s})$ is conjugated to $s \exp t_s$ by some element in $\exp \widehat{\mathcal{T}}_{\mathrm{even}}$ for every $s \in S$ and $\mathfrak{s}$ the generator associated to s.*

Note that this conjecture implies conjecture 1 of [M6], as every braided reflection is conjugated to a simple generator by an element of B that is a product of elements of the form described in this conjecture. Also note that one cannot expect this property to be true for arbitrary braided reflections, as it is not stable under conjugation.

This conjecture implies the following classical property, due to G. Lusztig (see [L] , 1.7 ; see also [G] §4 for a refinement), of the representations of Hecke algebras.

Proposition 6.2. *Let (W, S) be a finite Coxeter system such that 6.1 is true, and that $\rho : W \to \mathrm{GL}_N(\Bbbk)$ be a representation in orthogonal form, meaning $\rho(w)^{-1} = {}^t\rho(w)$ for all $w \in W$. Then the representation Ψ of the generic Hecke algebra deduced from ρ through Φ is symmetric, meaning that ${}^t\Psi(g) = \Psi(\tau(g^{-1}))$ for every $g \in B$.*

Proof. One only needs to show that $R(\mathfrak{s})$ is a symmetric matrix, for $\mathfrak{s}$ a simple generator associated to $s \in S$. Let $H = \mathrm{Ker}\,(s-1)$. Since $\rho(s)$ is orthogonal and has order 2, it is symmetric and so is $\varphi(t_H)$. It follows that $\rho(s) \exp h\varphi(t_H)$ is also symmetric, so it remains to prove that the image of $\exp \widehat{\mathcal{T}}_{\mathrm{even}}$ is made of orthogonal matrices. We let $\sigma \in \mathrm{Aut}(K)$ be

$h \mapsto -h$,$A \in \mathrm{Aut}(\mathfrak{gl}_N(\mathbb{k}))$ be $x \mapsto -\,{}^t x$ and $\tilde{\varphi} : \widehat{\mathcal{T}} \mapsto \mathfrak{gl}_N(K)$ be defined by $t_H \mapsto h\varphi(t_H)$. We have $A \circ \tilde{\varphi}(t_H) = -\,{}^t h\varphi(t_H) = -h\varphi(t_H) = \sigma \circ \varphi(t_H)$, hence $A \circ \tilde{\varphi} = \sigma \circ \tilde{\varphi}$ on $\widehat{\mathcal{T}}$ and $A \circ \tilde{\varphi} = \tilde{\varphi}$ on $\widehat{\mathcal{T}}_{\mathrm{even}}$. This implies that the elements of $\exp \widehat{\mathcal{T}}_{\mathrm{even}}$ are mapped to orthogonal matrices, and this concludes the proof. □

Recall that a consequence of the (in the Coxeter case weaker) conjecture 1 of [M6] is that, under the same conditions, ${}^t\Psi(g) = \sigma(\Psi(g^{-1}))$, hence $\sigma(R(g)) = R(\tau(g))$ for all $g \in B$.

In case $W = \mathfrak{S}_n$ the following lemma is an immediate consequence of Dehornoy's ordering and notions of $\mathfrak{s}_i$-positive braids. In general, it is a straightforward consequence of the injectivity of the palindromization map of [De] (theorem 2.1 there).

Lemma 6.3. *For W a Coxeter group, $\{g \in B \mid \tau(g) = g\} = \{1\}$.*

Theorem 6.4. *Conjecture 6.1 holds when (W, S) has type A_n, $B_n = C_n$ or $I_2(m)$.*

In type A the result is a consequence of the existence of even Drinfeld associators with rational coefficients. Indeed, the isomorphism we need (up to rescaling $t_H \mapsto 2t_H$ in $\mathcal{T}$) is established in [Dr] (see proof of proposition 5.1) provided that the element φ used there belongs to the set denoted $M_1^+(\mathbb{Q})$ of even rational associators. This is proposition 5.4 of [Dr].

In case (W, S) has type B_n, a morphism $B \to W \ltimes \exp \widehat{\mathcal{T}}$ is associated in [En] proposition 2.3, to any element of a set $\mathrm{Pseudo}_{(\bar{1},1)}(2, \mathbb{k})$ of couples inside $(\exp \widehat{\mathcal{T}}^0) \times (\exp \widehat{\mathcal{T}}^1)$ where $\mathcal{T}^0, \mathcal{T}^1$ denote the holonomy Lie algebras for the Coxeter types A_2 and B_2. It satisfies our condition (again up to rescaling) when $(\Phi, \Psi) \in \mathrm{Pseudo}^+_{(\bar{1},1)}(2, \mathbb{k})$ where

$$\mathrm{Pseudo}^+_{(\bar{1},1)}(2, \mathbb{k}) = \mathrm{Pseudo}_{(\bar{1},1)}(2, \mathbb{k}) \cap (\exp \widehat{\mathcal{T}}^0_{\mathrm{even}}) \times (\exp \widehat{\mathcal{T}}^1_{\mathrm{even}}).$$

The conjecture in that case is thus a consequence of the following property.

Proposition 6.5. $\mathrm{Pseudo}^+_{(\bar{1},1)}(2, \mathbb{k}) \neq \emptyset$

Proof. In this proof, we freely use the notations of [En]. The image of the complex conjugation $z \mapsto \bar{z}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ is mapped to $(-1, 1) \in \widehat{\mathrm{GT}} \subset \widehat{\mathrm{GTM}}$. As a consequence we have an element $(-1, -1, 1, 1) \in \widehat{\mathrm{GTM}}$ of order 2 which is mapped to $A_- = (-1, -1, 1, 1) \in \underline{\mathrm{GTM}}(N, \mathbb{Q}) \subset \underline{\mathrm{GTM}}(N, \mathbb{Q}_\ell)$ for an arbitrary prime number ℓ, the map $\widehat{\mathrm{GTM}} \to \underline{\mathrm{GTM}}(N)_\ell \to \underline{\mathrm{GTM}}(N, \mathbb{Q}_\ell)$ being described in §6.4 of [En].

We have $\mathrm{GTM}(N, \mathbb{Q}) \subset \mathbb{Q}^\times \times \mathbb{Q}(N) \times F_2(\mathbb{Q}) \times (\mathrm{Ker}\,\varphi_N)(\mathbb{Q})$, and $\mathbb{Q}(N) \simeq (\mathbb{Z}/N\mathbb{Z}) \times \mathbb{Q}$ with $-1 \mapsto (-\bar{1}, -1)$. It is easily checked that the action of $\mathbb{Q}(N)$ on $\exp \mathcal{T}_0$ and $\exp \mathcal{T}^1$ defined in [En] associated to $-1 \in \mathbb{Q}(N)$ the automorphism $t_H \mapsto -t_H$ exactly when $a \mapsto -a$ is the identity of $\mathbb{Z}/N\mathbb{Z}$, that is when $N = 2$. Identifying $\mathrm{Pseudo}_{(\bar{1},1)}(N, \mathbb{Q})$ with the quotients of $\mathrm{Pseudo}(N, \mathbb{Q})$ by $\mathbb{Q}(N)$, we get an action of $\mathrm{GTM}(N, \mathbb{k})$ on $\mathrm{Pseudo}_{(\bar{1},1)}(N, \mathbb{k})$ that we denote $\star$ (see [En] §7.2 for an explicit formula). From the explicit description of the action of $\mathrm{GTM}(N, k)$ on $\mathrm{Pseudo}(N, \mathbb{k})$ we get that $\mathrm{Pseudo}^+_{(\bar{1},1)}(2, \mathbb{k}) = \{X \in \mathrm{Pseudo}_{(\bar{1},1)}(2, \mathbb{k}) \mid A_- \star X = X\}$. Now, as in the proof of proposition 5.4 in [Dr], we use the fact that there exists a map $\mathrm{Pseudo}^+_{(\bar{1},1)}(N, \mathbb{k}) \to S_{\mathrm{alg}}(\mathbb{k})$, $(\Phi, \Psi) \mapsto \Theta_{\Phi\Psi}$ with $S_{\mathrm{alg}}(\mathbb{k})$ the set of algebraic sections of the split exact sequence of $\mathbb{Q}$-group schemes $1 \to \mathrm{GTM}_{(\bar{1},1)}(N, \mathbb{k}) \to \mathrm{GTM}_{\bar{1}}(N, \mathbb{k}) \to \mathbb{k}^\times \to 1$, uniquely defined by $\Theta_{\Phi\Psi}(\lambda) \star (\Phi, \Psi) = (\Phi, \Psi)$ for all $\lambda \in \mathbb{k}^\times$, and with the property that its composition with the $S_{\mathrm{alg}}(\mathbb{k}) \to \mathcal{S}(\mathbb{k})$, $\Theta \mapsto \mathrm{d}\Theta$, where $\mathcal{S}(\mathbb{k})$ is the set of sections of the corresponding sequence of Lie algebras $0 \to \mathfrak{gtm}_{(\bar{1},1)}(N, \mathbb{k}) \to \mathfrak{gtm}_{\bar{1}}(N, \mathbb{k}) \to \mathbb{k} \to 0$ is a bijection. By conjugation we have an action $\Theta \mapsto \Theta^{A_-} : \lambda \mapsto A_-\Theta(\lambda)A_-^{-1} = A_-\Theta(\lambda)A_-$ of $\mathbb{Z}/2\mathbb{Z} =< A_- >$ on $S_{\mathrm{alg}}(\Bbbk)$, and it is easily checked that $\Theta_{\Phi\Psi}^{A_-} = \Theta_{\tilde{\Phi}\tilde{\Psi}}$ where $\tilde{\Phi}$, $\tilde{\Psi}$ are deduced from Θ, Ψ through $t_H \mapsto -t_H$. It follows that $\mathrm{Pseudo}^+_{(\bar{1},1)}(2,\Bbbk) \neq \emptyset$ iff there exists $\Theta \in S_{\mathrm{alg}}(\Bbbk)$ such that $\Theta^{A_-} = \Theta$, iff there exists some $\mathrm{d}\Theta \in \mathcal{S}(\Bbbk)$ such that $(\mathrm{d}\Theta)^{A_-} = \mathrm{d}\Theta$. Let $c \in \mathrm{Aut} F_2$ be $x \mapsto x^{-1}$, $y \mapsto y^{-1}$ with $F_2 =< x, y >$. Using as before the notations of [En], c induces an action on $\mathrm{Ker}\,\varphi_N$, $F_2(\Bbbk)$ and $(\mathrm{Ker}\,\varphi_N)(\Bbbk)$, for all $N \geq 2$. By explicit computations in $\mathrm{GTM}(2,\Bbbk)$ we get $A_-(\lambda,\mu,f,g)A_- = (\lambda,\mu,c.f,c.g)$. Now, the Lie algebra $\mathfrak{gtm}_{\bar{1}}(N,\Bbbk)$ of $\mathrm{GTM}_{\bar{1}}(N,\Bbbk)$ is made of triples (s,φ,ψ) matching with tangent vectors $\lambda = 1 + \varepsilon s$, $\bar{\mu} = 1$, $f = \exp \varepsilon\varphi$, $g = \exp \varepsilon\psi$ with $\varphi \in \hat{\mathfrak{f}}_2(\Bbbk) = \hat{\mathfrak{f}}(\xi,\eta)$, $\psi \in \hat{\mathfrak{f}}(\Xi, \eta(0), \eta(1), \dots, \eta(N-1))$ with $\xi = \log x$, $\eta = \log y$, $\Xi = \log x^N$, $\eta(\alpha) = \log x^\alpha y x^{-\alpha}$, and the automorphisms induced by c are $\xi \mapsto -\xi$, $\eta \mapsto -\eta$, $\Xi \mapsto -\Xi$, $\eta(\alpha) \mapsto -\eta(-\alpha) = -\eta(\alpha)$ if $N = 2$. Thus c induces $\varphi \mapsto \tilde{\varphi}$, and $\mathrm{Pseudo}^+_{(\bar{1},1)}(2,\Bbbk) \neq \emptyset$ iff there exists $(1,\varphi,\psi) \in \mathfrak{gtm}_{(\bar{1},1)}(2,\Bbbk)$ with $\tilde{\varphi} = \varphi$, $\tilde{\psi} = \psi$. From $(1,\varphi,\psi) \in \mathfrak{gtm}_{(\bar{1},1)}(2,\Bbbk) \neq \emptyset$ one builds $\frac{1}{2}((1,\varphi,\psi), +c.(1,\varphi,\psi))/2 = (1 + (\varphi + \tilde{\varphi})/2, (\psi + \tilde{\psi})/2) \in \mathfrak{gtm}_{(\bar{1},1)}(2,\Bbbk)$, which provides a convenient section in $\mathcal{S}(\Bbbk)$ and concludes the proof. □

In case W is a dihedral group, the conjecture also holds because [M3] (see §6 there) provides the necessary material in order to adapt Drinfeld's proof in a straightforward way : the analogue of proposition 5.2 of [Dr] is proved in §6 for the group $G'(\Bbbk)$ defined there, and even associators are the fixed elements of $\mathbb{A}\mathrm{ss}'_1(\Bbbk)$ under the involution $(-1,1) \in G'(\Bbbk)$, whose action on $B(\Bbbk)$ extends the 'mirror automorphism' (aka complex conjugation) of B.

One actually gets this way an alternate proof of the following result of Lusztig (the original result however do not need the restriction on exceptional types that we need to impose for now, and is thus stronger).

Corollary 6.6. *Let (W,S) be a finite irreducible Coxeter system not of the exceptional types $F_4, H_3, H_4, E_6, E_7, E_8$. Then every representation Ψ of $H(q)$ has a matrix model such that ${}^t\Psi(g) = \Psi(\tau(g))$ for all $g \in B$.*

Proof. Except in type D_n, this corollary is a consequence of proposition 6.2 and of the above results, as all representations of W can be realized over $\mathbb{R}$. Type D can then be reduced to type B by using inclusions between the corresponding Hecke algebras, applying verbatim the arguments of [M6], cor. 6.2.

□

Remark 6.7. *: Assuming that Φ satisfies conjecture 6.1 and that Ψ is the representation of $H(q)$ deduced from Φ and some $\rho \in \mathrm{Irr}(W)$, a consequence of proposition 6.2 and lemma 6.3 is that, if Ψ is faithful, then $\Psi(P) \cap O_N(K^\sigma) = \{1\}$, since $U_N^\sigma(K) \cap O_N(K) = O_N(K^\sigma)$. On the other hand, and using the notations Ψ_+, $P(\Bbbk)$ of §5, we have $\Psi_+(P(\Bbbk)) \subset U_N^\sigma(K)$ and $U_N^\sigma(K) \cap \mathrm{GL}_N(K^\sigma) = O_N(K^\sigma)$. It then follows from proposition 5.7 (3) that $\Psi_+(P(\Bbbk)) \cap O_N(K^\sigma)$ is a large group, since it has for Zariski closure a connected algebraic group whose Lie algebra is $K^\sigma \mathcal{H}^0_{\mathrm{tail}}$.*

7. Rotation algebras : general facts

Let W be an irreducible 2-reflection group, $\varepsilon : W \to \{\pm 1\}$ the sign morphism and $\mathrm{O} = \mathrm{Ker}\,\varepsilon$ the rotation subgroup. We introduce the following Lie algebra, for $\Bbbk$ a commutative unital ring.

Definition 7.1. *The infinitesimal rotation algebra $\mathcal{A}$ is the Lie subalgebra of $\Bbbk\mathrm{O}$ generated by the $[s,u] = su - us$ for $s, u \in \mathcal{R}$ and $su \neq us$.*

7.1. **Preliminaries on finite reflection groups.** We prove the following.

Lemma 7.2. *When W is an irreducible complex 2-reflection group, then O is generated by the su for $s, u \in \mathcal{R}$ and $su \neq us$.*

Proof. Recall that, when H is a subgroup of G with G generated by $a_1, \dots, a_r$, and $\alpha : G/H \to G$ is a section of $G \mapsto G/H, x \mapsto \bar{x}$, then H is generated by the $y a_i \alpha(\overline{y a_i})^{-1}$ for $y \in \alpha(G/H)$ and $1 \leq i \leq r$. In case $G = W$, $H = \mathrm{O}$ and $\{a_1, \dots, a_r\} = \mathcal{R}_0 \subset \mathcal{R}$ an arbitrary generating set for W, we get that O is generated by the su (and $us = (su)^{-1}$) for s a fixed reflection and u running among $\mathcal{R}_0$. Since $W < \mathrm{GL}_n(\mathbb{C})$ is irreducible, there exists $s_1, \dots, s_n$ such that the graph Γ with vertices $s_1, \dots, s_n$ and edges s_i—s_j iff $s_i s_j \neq s_j s_i$ is connected, and such that the subgroup $W_0 < W$ generated by the $s_1, \dots, s_n$ is irreducible (see e.g. [M7], prop. 3.1). We know that A is generated by the $s_1 u$ and $u s_1$ for $u \in \mathcal{R}$. For a given u, because the action of W_0 is irreducible and W_0 is generated by $s_1, \dots, s_n$, there exists $1 \leq j \leq n$ such that $u s_j \neq s_j u$. By connectedness of Γ there exists $i_1, \dots, i_k$ with $i_1 = 1$ and $i_k = j$ such that $s_{i_t} s_{i_{t+1}} \neq s_{i_{t+1}} s_{i_t}$ for $1 \leq t \leq k-1$. Then $s_1 u = (s_{i_1} s_{i_k})(s_{i_k} u) = (s_{i_1} s_{i_2})(s_{i_2} s_{i_3}) \dots (s_{i_{k-2}} s_{i_{k-1}})(s_{i_{k-1}} s_{i_k})(s_j u)$. Using the same argument for $u s_1$, this proves the lemma. ☐

In case of Coxeter groups, we actually have the following stronger version. We do not know whether it holds for an arbitrary irreducible 2-reflection group equipped with an arbitrary generating set of reflections.

Lemma 7.3. *When W is an irreducible Coxeter group, then O is generated by the su for s, u simple reflections with $su \neq us$.*

Proof. Every element of O can be written as a product of an even number of simple reflections, hence it is enough to prove that every su for s, u simple reflections can be written as a product of $s_i s_{i+1}$ for $(s_i s_{i+1})^2 \neq 1$ and s_i, s_{i+1} simple reflections. This is an immediate consequence of the connectedness of the Coxeter graph. ☐

Lemma 7.4. *When W is a complex reflection group of type $G(e,e,n)$ for $n \geq 3$, then O is generated by the su which have order 3, for $s, u \in \mathcal{R}$.*

Proof. By lemma 7.2 we know that W is generated by the su for $s, u \in \mathcal{R}$ and $su \neq us$. We need to express such a su as a product of elements as in the statement. Up to conjugation by an element of $\mathfrak{S}_n$, we can assume that $s = S \oplus \mathrm{Id}_{n-3}$ and $u = U \oplus \mathrm{Id}_{n-3}$ with

- either $S = \begin{pmatrix} 0 & \zeta & 0 \\ \zeta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & \eta^{-1} & 0 \end{pmatrix}$ for some $\zeta, \eta \in \mu_e$, and then SU has order 3
- either $S = \begin{pmatrix} 0 & \zeta & 0 \\ \zeta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 0 & \eta & 0 \\ \eta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and then, letting $\delta = \zeta\mu^{-1}$,

$$SU = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 0 & \delta & 0 \\ \delta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right] \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

is a product of two 'su' of order 3.

□

Lemma 7.5. *When W is an irreducible complex 2-reflection group of exceptional type with a single reflection class, then* O *is generated by the* su *which have order 3, for* $s, u \in \mathcal{R}$.

Proof. Case-by-case check, using CHEVIE. □

Remark 7.6. *The two statements above do* not *hold for the dihedral groups* $G(e, e, 2)$. *For the groups* $G(2e, e, n)$ *with* $n \geq 3$, *as well as for the exceptional groups with two reflection classes, the subgroup generated by the* su *of order* 3 *has order* 4 *in* W. *For the exceptional group this is a simple computer check; in the case of the* $G(2e, e, n)$ *for* $n \geq 3$ *this is because this subgroup* H *is equal to the rotation subgroup of the group* $G(2e, 2e, n)$. *Indeed, it contains this rotation subgroup by lemma 7.4 and conversely, because a reflection* $s \in G(2e, e, n) \setminus G(2e, 2e, n)$ *is necessarily a diagonal matrix, one easily checks that a product* su *or* us *for* u *an arbitrary reflection in* $G(2e, 2e, n)$ *has order* 2 *or* 4, *hence the generators of* H *are all contained in* $G(2e, 2e, n)$.

Using the Shephard-Todd classification, this has for consequence the following.

Proposition 7.7. *If* W *is a 2-reflection group with a single reflection class, then* O *is generated by the* su *of odd order for* $s, u \in \mathcal{R}$.

Proof. When $W = G(e, e, n)$ for $n \geq 3$ or W of exceptional types this is a consequence of the above lemmas. The remaining cases of the dihedral groups $W = G(e, e, 2)$ with e odd is trivial, as O is cyclic and admits as generator a product of two reflections.

□

Lemma 7.8. *If* W *is a Coxeter group of type*

1. A_n *for* $n \geq 4$,
2. D_n *for* $n \geq 5$
3. E_6, E_7, E_8,

then the su *for* $su \neq us$ *constitute a single conjugacy class in* O.

Proof. Considering the elements of the Coxeter group of type A_{n-1} (resp. D_n) as (signed) permutations, we can associate to each element g its support $supp(g) = \{i \mid g(i) \neq i\} \subset \{1, \ldots, n\}$. For $s, u \in \mathcal{R}$, $su \neq us$ implies $supp(s) \cap supp(u) \neq \emptyset$, that is $supp(s) \cup supp(u) = \{i, j, k\} \subset \{1, \ldots, n\}$. Choosing another couple (s', u'), the union of the supports of the 4 reflections has size at most 6, hence up to conjugation by an element of $\mathfrak{S}_n$, we can assume that s, u, s', u' all belong to the Coxeter group of type A_5 (resp.D_6), at least if $n \geq 5$ (resp. $n \geq 6$). This reduces the proof to the check of a few cases which are readily done.

□

Remark 7.9. *There are 2 such conjugacy classes for types* A_3 *and* D_4.

7.2. **Preliminaries on commutative algebras.**

Lemma 7.10. *Let* G *be a finite group,* $g \in G$ *of order* N, *and* $\Bbbk$ *a field of characteristic* 0. *Then*

1. g *is a polynomial of* $g - g^{-1}$ *inside* $\Bbbk G$ *if and only if* N *is odd. In this case, the polynomial can be chosen with coefficients in* $\mathbb{Q}$.
2. $g + g^{-1}$ *is an* even *polynomial in* $g - g^{-1}$ *if and only if* N *is odd. In this case, the polynomial can be chosen with coefficients in* $\mathbb{Q}$.

Proof. We first prove (1). We can assume $\Bbbk \subset \mathbb{C}$. Firstly we assume $\Bbbk = \mathbb{C}$. This amounts to check whether x is a polynomial in $x - x^{-1}$ inside $\mathbb{C}[x]/(x^N - 1)$. Letting $\zeta = \exp(2i\pi/N)$ this means that there exists $P \in \mathbb{C}[X]$ such that $\forall k \ \ \zeta^k = P(\zeta^k - \zeta^{-k})$, which is possible (Lagrange's criterium) exactly when the $\zeta^k - \zeta^{-k}$ are distinct integers for $0 \leq k < N$, that is when N is odd. In that case, P has degree at most $N-1$, and we proved that the family of the $(x-x^{-1})^r$ for $0 \leq r \leq N$ subspace of $\mathbb{C}[x]/(x^N-1)$ spanned by the $(x-x^{-1})^r$, $0 \leq r \leq N$, is a basis.

This proves in the general case that g can be a polynomial in $g - g^{-1}$ only if N is odd. Conversely, we only need to prove that we can choose $P \in \mathbb{Q}[X]$. But we proved that the $(x - x^{-1})^r$ for $0 \leq r < N$ are linearly independent over $\mathbb{C}$ inside $\mathbb{C}[x]/(x^N - 1)$, so they are also linearly independent inside $\mathbb{Q}[x]/(x^N-1)$, hence they form a basis of $\mathbb{Q}[x]/(x^N-1)$ and one can indeed find $P \in \mathbb{Q}[X]$ with $x = P(x - x^{-1})$.

We now prove (2). We can assume $G =< g >$ (and so $G \simeq \mathbb{Z}/N$). If $g + g^{-1} = R(g - g^{-1})$ for some $R \in \Bbbk[X]$, then $g = P(g - g^{-1})$ with $P(X) = (R(X)+X)/2$ hence N is odd by (1). Conversely, we assume N is odd. By (1), we have $g = P(g-g^{-1})$ for some $P \in \mathbb{Q}[X]$. There exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(g) = g^{-1}$. We extend it to an automorphism of $\Bbbk G$, and get $g^{-1} = \varphi(g) = P(\varphi(g - g^{-1})) = P(g^{-1} - g) = Q(g - g^{-1})$ for $Q(X) = P(-X) \in \mathbb{Q}[X]$. Thus $g + g^{-1} = (P + Q)(g - g^{-1})$ with $P + Q \in \mathbb{Q}[X]$ an even polynomial. □

Lemma 7.11. *Let G be a finite group, $g \in G$ of order 3. Then,* $\mathrm{Ad}(g) : x \mapsto gxg^{-1}$ *is a polynomial in* $\mathrm{ad}(g) - \mathrm{ad}(g^{-1}) : x \mapsto gx - xg - (g^{-1}x - xg^{-1})$.

Proof. E.g. by direct computation, a convenient polynomial being $\frac{1}{24}X^4 + \frac{1}{12}X^3 + \frac{5}{8}X^2 + \frac{3}{4}X + 1$, or by using a similar argument as in the previous lemma. □

7.3. **Basic facts.** For every representation $\rho : \mathrm{O} \to \mathrm{GL}(V)$ of O, we let $\rho_{\mathcal{A}} : \mathcal{A} \to \mathfrak{gl}(V)$ denote the induced representation of $\mathcal{A}$.

Proposition 7.12.

Assume W is an irreducible 2-reflection group with a single reflection class.

1. *$\mathcal{A}$ generates $\Bbbk$O as an associative algebra with 1.*
2. *$\forall \rho \in \mathrm{Irr}(\mathrm{O})$, $\rho_{\mathcal{A}}$ is irreducible.*
3. *$\mathcal{A}$ is reductive.*
4. *$\forall \rho^1, \rho^2 \in \mathrm{Irr}(\mathcal{A})$, $\rho^2 \simeq \rho^1 \Leftrightarrow \rho^2_{\mathcal{A}} \simeq \rho^1_{\mathcal{A}}$.*
5. *$\forall \rho^1, \rho^2 \in \mathrm{Irr}(\mathcal{A})$, $\rho^2 \simeq (\rho^1)^* \Leftrightarrow \rho^2_{\mathcal{A}} \simeq (\rho^1_{\mathcal{A}})^*$.*

Proof. By proposition 7.7, O is generated by the su for $s, u \in \mathcal{R}$ of odd order. Since $[s,u] = su - us = (su) - (su)^{-1}$, lemma 7.10 (1) implies $\rho(\mathsf{U}\mathcal{A}) \supset \rho(\mathrm{O})$ hence (1) and (2). Since $\mathbb{C}\mathrm{O} = \bigoplus_{\rho \in \mathrm{Irr}(\mathrm{O})} \mathrm{End}(V_\rho)$ and $\mathcal{A} \subset \mathbb{C}\mathrm{O}$, $\bigoplus_{\rho \in \mathrm{Irr}(\mathrm{O})} \rho_{\mathcal{A}}$ defines a faithful semisimple representation of $\mathcal{A}$, hence (3). Let now $\rho^1, \rho^2 \in \mathrm{Irr}(\mathrm{O})$. Clearly $\rho^1 \simeq \rho^2 \Rightarrow \rho^1_{\mathcal{A}} \simeq \rho^2_{\mathcal{A}}$. Conversely, we can assume $\rho^1, \rho^2 : \mathrm{O} \to \mathrm{GL}(V)$, and let $Q \in \mathrm{End}(V)$ be such that $\rho^2(y) = Q\rho^1(y)Q^{-1}$ for all $y \in \mathcal{A}$. In particular $\rho^2(x - x^{-1}) = Q\rho^1(x - x^{-1})Q^{-1}$ whenever $x = su$ with $s, u \in \mathcal{R}$. If in addition x has odd order there exists by lemma 7.10 (1) a polynomial $P \in \mathbb{Q}[X]$ such that $x = P(x - x^{-1})$, hence $\rho^2(x) = Q\rho^1(x)Q^{-1}$. Since such x generate O we get $\rho^2(g) = Q\rho^1(g)Q^{-1}$ for all $g \in \mathrm{O}$ hence (4). Finally, assuming $\rho^2 \simeq (\rho^1)^*$ means $\rho^2(x) = Q\ {}^t\rho^1(x^{-1})$ for all $x = su \in \mathrm{O}$ we get $\rho^2(x - x^{-1}) = Q\ {}^t\rho^1(x^{-1} - x)Q^{-1} = -Q\ {}^t\rho^1(x - x^{-1})Q^{-1}$ hence $\rho^2_{\mathcal{A}} \simeq (\rho^1_{\mathcal{A}})^*$. Conversely, if $\rho^2_{\mathcal{A}} \simeq (\rho^1_{\mathcal{A}})^*$ then $\rho^2(x - x^{-1}) = Q\ {}^t\rho^1(x^{-1} - x))Q^{-1}$. When x has odd order we can (by lemma 7.10 (2)) choose $P \in \mathbb{Q}[X^2]$ such that $x + x^{-1} = P(x - x^{-1})$

hence $\rho^2(x+x^{-1}) = Q\, {}^t\rho^1(x+x^{-1})Q^{-1}$. Then $2\rho^2(x) = \rho^2(x+x^{-1}+x-x^{-1}) = 2Q\, {}^t\rho^1(-x+x^{-1}+x+x^{-1})Q^{-1} = 2Q\, {}^t\rho^1(x^{-1})Q^{-1}$. This yields $\rho^2 \simeq (\rho^1)^*$ by proposition 7.7 hence (5). □

Proposition 7.13. *Let W be a 2-reflection group. Let $\rho \in \mathrm{Irr}(\mathrm{O})$ such that $\rho^* \simeq \rho$, β the corresponding nondegenerate bilinear form on V_ρ and $y \mapsto y^+$ the adjunction operation w.r.t. β. Then*

$$\rho(\mathcal{A}) \subset \mathfrak{osp}(V_\rho) = \{y \in \mathrm{End}(V_\rho) \mid \mathrm{tr}\, y = 0 \text{ and } y^+ = -y\}$$

Proof. For $x = su$ and $s, u \in \mathcal{R}$ we have $\rho(x)^+ = \rho(x)^{-1} = \rho(x^{-1})$ hence $\rho(x-x^{-1})^+ = \rho(x^{-1}-x) = -\rho(x-x^{-1})$. Moreover $\mathrm{tr}\,\rho(x)^+ = \mathrm{tr}\,\rho(x)$, hence $\mathrm{tr}\,\rho(x-x^{-1}) = 0$. Since $\mathcal{A}$ is generated as a Lie algebra by such $x - x^{-1}$ this concludes the proof. □

Proposition 7.14. *Let W be a 2-reflection group with a single reflection class. If the $su \in \mathrm{O}$ for $su \neq us$ form a single conjugacy class in O, then $Z(\mathcal{A}) = \{0\}$. Thus in that case $\mathcal{A}$ is a semisimple Lie algebra.*

Proof. Let L be the subspace of $\Bbbk\mathrm{O}$ spanned by the $[s,u]$ for $s, u \in \mathcal{R}$. We have $\mathcal{A} \subset L + [\mathcal{A},\mathcal{A}]$. Since $\mathcal{A}$ generates $\Bbbk\mathrm{O}$ as an associative algebra with 1 by proposition 7.12 we have $Z(\mathcal{A}) \subset Z(\Bbbk\mathrm{O})$. For c a conjugacy class in O we let δ_c denote the associated central function. Then $[\mathcal{A},\mathcal{A}] \subset [\Bbbk\mathrm{O}, \Bbbk\mathrm{O}] = \bigcap_c \mathrm{Ker}\,\delta_c$. When $y \in Z(\mathcal{A}) \subset Z(\Bbbk\mathrm{O})$, if $T_c = \sum_{g \in c} g$, then y can be written $y = \sum \lambda_c T_c$ for some $\lambda_c \in \Bbbk$. But $\delta_c(\mathcal{A}) = \delta_c(L)$ and $\delta_c(L) = 0$ if c is not the class c_0 formed by the $x = su$ for $su \neq us$, $s, u \in \mathcal{R}_0$. Thus $y = \lambda_0 T_{c_0}$. Since $x^{-1} \in c_0$ we have $\delta_{c_0}(x - x^{-1}) = 0$ hence $y = 0$.

□

7.4. **Small rank.**

7.4.1. $W = \mathfrak{S}_4$. The irreducible representations of $\mathrm{O} = \mathfrak{A}_4$ are the 1-dimensional $[4]$, $[2,2]^+$, $[2,2]^-$ and the 3-dimensional $[3,1]$. On $[4]$ all the $[s,u]$ act by 0. The 3-cycle $x = (1\ 2\ 3)$ acts by j on $[2,2]^+$ and $j^{-1} = j^2$ on $[2,2]^-$ for j a primitive 3-rd root of 1. Then $x - x^{-1} \in \mathcal{A}$ acts by a non-zero value on $[2,2]^+$ by its complex conjugate on $[2,2]^-$. The same holds true for all of the $x - x^{-1} \in \mathcal{A}$ for x a 3-cycle. On the other hand, $[3,1]$ is a self-dual representation of A, hence the image of $\mathcal{A}$ inside $\mathfrak{gl}([3,1])$ is included in $\mathfrak{so}([3,1]) \simeq \mathfrak{so}_3$. Altogether we get an injective morphism $\mathcal{A} \to \mathbb{C} \times \mathfrak{so}_3(\mathbb{C})$. Since the action of $\mathcal{A}$ on $[3,1]$ is irreducible, the image of $\mathcal{A}$ inside $\mathfrak{so}_3(\mathbb{C})$ is semisimple hence has dimension at least 3, whence is $\mathfrak{so}_3(\mathbb{C})$ and $\mathcal{A}' \simeq \mathfrak{so}_3(\mathbb{C})$. Since the action of $\mathcal{A}$ on $[2,2]^+$ is non-trivial we have $Z(\mathcal{A}) \neq \{0\}$ hence the morphism $\mathcal{A} \to \mathbb{C} \times \mathfrak{so}_3(\mathbb{C})$ is an isomorphism.

7.4.2. $W = G(2,2,4)$. This is a Coxeter group of type D_4. The irreducible representation of the Coxeter group of type B_4 are labelled by couples (λ, μ) of partitions of total size 4, hence the representations of W inherit from Clifford theory the labels $\{\lambda, \mu\}$ for $\lambda \neq \mu$, and $\{2\}^+$, $\{2\}^-$, $\{11\}^+$,$\{11\}^-$. When $\mu = \emptyset$, the representation factors through the parabolic subgroup $\mathfrak{S}_4 = G(1,1,4)$ of type A_3, hence these cases have already been treated. The remaining representations are, up to tensorization by the sign character, $\{3,1\}$, $\{2\}^+$, $\{2\}^-$, $\{2,11\}$, $\{21,1\}$. The last two split, when restricted to O, as $\{21,1\}' + \{21,1\}''$ and $\{2,11\}' + \{2,11\}''$, thus affording 7 irreducible representations of O to care of. Using character theory we get that these 7 representations are selfdual except for $\{21,1\}'$ and $\{21,1\}''$; since $\{21,1\}$ is selfdual this implies $\{21,1\}'' \simeq (\{21,1\}')^*$. The image of $\mathcal{A}$ inside (the endomorphism algebras corresponding to) $\{3,1\}$, $\{2\}^+$, $\{2\}^-$, $\{2,11\}' \oplus \{2,11\}''$ is thus $\mathfrak{so}_4$, $\mathfrak{so}_3$, $\mathfrak{so}_3$, $\mathfrak{so}_3 \times \mathfrak{so}_3$, respectively. Using matrix models for these representations, we can compute the dimension of the image, and we get equality. Recall that $\mathfrak{so}_3 \simeq \mathfrak{sl}_2$ and that $\mathfrak{so}_4 \simeq \mathfrak{so}_3 \times \mathfrak{so}_3$.

Since $\{21,1\}'' = (\{21,1\}')^*$, the image of $\mathcal{A}$ inside $\{21,1\} = \{21,1\}' + \{21,1\}''$ is isomorphic to the image inside $\{21,1\}'$, which is included in $\mathfrak{gl}_4$. By direct computation we get that this image has dimension 16, hence it is all $\mathfrak{gl}_4$. From all this one gets 6 simple ideals : $\mathfrak{so}_3^{(0)}$ from the $\mathfrak{S}_4$-representation $\{31,\emptyset\}$, $\mathfrak{so}_3^{(1)}, \mathfrak{so}_3^{(2)}$ from $\{2\}^+$, $\{2\}^-$, $\mathfrak{so}_3^{(3)}$ from $\{2,11\}'$, $\mathfrak{so}_3^{(4)}$ from $\{2,11\}''$ and $\mathfrak{sl}_4$ from $\{21,1\}'$. The $\mathfrak{so}_3$ ideals are all distinct because $\mathfrak{so}_3$ admits exactly one 3-dimensional irreducible representation. Thus $\mathcal{A}'$ contains $\mathfrak{sl}_4 \times (\mathfrak{so}_3)^5$, of dimension 30. Direct computation shows $\dim \mathcal{A} = 31$ hence $\mathcal{A} \simeq \mathbb{C} \times \mathfrak{sl}_4 \times \mathfrak{so}_4 \times (\mathfrak{so}_3)^3 \simeq \mathbb{C} \times \mathfrak{sl}_4 \times (\mathfrak{so}_3)^5$.

8. Rotation algebras : structure theorem in type A

In this section we let $\mathrm{O} = \mathfrak{A}_n \subset \mathfrak{S}_n = W$ and $\mathcal{A} = \mathcal{A}_n \subset \mathcal{H}_n = \mathcal{H}$ the associated rotation algebra. Recall that $\mathrm{Irr}(\mathfrak{S}_n)$ is parametrized by partitions $\lambda \vdash n$ and that, identifying $\lambda \vdash n$ with the corresponding partition, one has $\lambda \otimes \varepsilon = \lambda'$ the transposed partition. By Clifford theory, irreducible representations of $\mathfrak{A}_n$ are thus parametrized by $\lambda \vdash n$ with $\lambda \neq \lambda'$ up to identification of λ and λ', or in a more convenient way with partitions $\lambda \vdash n$ with $\lambda > \lambda'$ for some arbitrarily chosen total order on partitions (e.g. lexicographic ordering), and by $\lambda^{\pm}$ for $\lambda \vdash n$ with $\lambda = \lambda'$. Recall that a *hook* is a partition of the form $[n-k, 1^k]$. We introduce the following sets : $\Lambda_n = \{\lambda \vdash n \mid \lambda > \lambda' \text{ and } \lambda \text{ not a hook}\}$, $S_n = \{\lambda \vdash n \mid \lambda = \lambda' \text{ and } \lambda \text{ not a hook}\}$, $S_n^+ = \{\lambda \in S_n \mid \lambda^{\pm} \text{ have real type}\}$, $S_n^- = S_n \setminus S_n^+$.

When $\lambda = \lambda'$, it is known that λ^+ and λ^- have real type iff $(n - b(\lambda))/2$ is even, where $b(\lambda)$ is the length of the diagonal in the Young diagram λ (see [M4], lemme 5 and lemme 6).

Theorem 8.1. *For $n \geq 5$,*

$$\mathcal{A}_n \simeq \mathfrak{so}_{n-1} \oplus \left(\bigoplus_{\lambda \in \Lambda_n} \mathfrak{so}(V_\lambda) \right) \oplus \left(\bigoplus_{\lambda \in S_n^+} \mathfrak{so}(V_{\lambda^+}) \oplus \mathfrak{so}(V_{\lambda^-}) \right) \oplus \left(\bigoplus_{\lambda \in S_n^-} \mathfrak{sl}(V_{\lambda^+}) \right)$$

As a corollary, the dimension of $\mathcal{A}_n$ for $n \geq 5$ is $16, 112, 1002, 9115, 86949, 892531, 9924091, \dots$. For $n \geq 5$, we know by lemma 7.8 and proposition 7.14 that $\mathcal{A}_n$ is semisimple. Since $\mathcal{A}_n \subset \mathcal{H}_n$, the decomposition of $\mathcal{H}^0_{\mathrm{tail}}$ obtained in theorem 4.11 provides an embedding of $\mathcal{A}_n$ inside the RHS of the equation. We need to prove that it is surjective. The case of $n = 5, 6$ can be done by a simple computer computation (one only needs to compare the dimension of both sides). The case $n = 7$ is already large enough so that computer calculations need to be done modulo some prime p. The RHS of theorem 8.1 has dimension 1002. Letting $V = V^{(1)} \subset \mathbb{k}\mathfrak{A}_7$ denote the $\mathbb{k}$-linear span of the $su - us$ for $s, u \in \mathcal{R}$, $V^{(i+1)} = V^{(i)} + [V, V^{(i)}]$, we used a C program and encoding of each entry inside one byte to check on the regular representation of $\mathfrak{A}_7$ that, when $\mathbb{k} = \mathbb{F}_{113}$, $\dim V = 35$, $\dim V^{(2)} = 161$, $\dim V^{(3)} = 533$, $\dim V^{(4)} = 987$, $\dim V^{(5)} = 1002$. Since we know that, for $\mathbb{k} = \mathbb{Q}$, $\dim V^{(5)} \leq 1002$, a straightforward application of Nakayama's lemma yields $\dim V^{(5)} = 1002$ for $\mathbb{k} = \mathbb{Q}$, and this settles the case $n = 7$. We thus assume $n \geq 8$ and start a proof by induction (more precisely, we assume by induction that the natural map explicited above is an isomorphism for $n-1$).

We will use the following lemma. Here and in the sequel, $\mathrm{rk}\,\mathfrak{g}$ denotes the semisimple rank of the semisimple Lie algebra $\mathfrak{g}$.

Lemma 8.2. *Let U be a N-dimensional $\mathbb{C}$-vector space endowed with a nondegenerate quadratic form, $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{so}(U) = \mathfrak{so}_N$ two semisimple Lie algebras such that $\mathfrak{g}$ acts irreducibly on*

U, the action of $\mathfrak{h}$ on U is multiplicity-free, and $\mathrm{rk}\,\mathfrak{h} > N/4$. *Then $\mathfrak{g}$ is simple and, if* $\mathrm{rk}\,\mathfrak{h} \geq 5$ *or* $\dim N > 8$, *then* $\mathfrak{g} = \mathfrak{so}_N$.

Proof. We have $\mathrm{rk}\,\mathfrak{g} \leq \mathrm{rk}\,\mathfrak{so}_N \leq N/2$ hence $\mathrm{rk}\,\mathfrak{h} > N/4$ implies $\mathrm{rk}\,\mathfrak{h} > (\mathrm{rk}\,\mathfrak{g})/2$. This implies that $\mathfrak{g}$ is simple ([M5], lemma 3.2). Then $2\mathrm{rk}\,\mathfrak{g} \leq 2(N/2) = \dim U < 4\mathrm{rk}\,\mathfrak{h} \leq 4\mathrm{rk}\,\mathfrak{g}$, hence we can apply the classification of [M5] lemma 3.4. The exceptions are ruled out, as the selfdual ones appear only when $\mathfrak{g}$ has rank at most 4 and U has dimension at most 8, and U is a selfdual representation of $\mathfrak{g}$ as $\mathfrak{g} \subset \mathfrak{so}(U)$. □

We will also need the following combinatorial lemma.

Lemma 8.3. (1) *Let $\lambda \vdash n$ with $n \geq 5$. Then* $\dim \lambda \geq n-1$. *item Let $\lambda \vdash n$ with $n \geq 7$ and $\lambda = \lambda'$. Then either $n = 7$, λ is the hook $[4, 1^3]$ with* $\dim \lambda^{\pm} = 10$, *or* $\dim \lambda^{\pm} \geq 21$.

Proof. (1) is classical and easily deduced from Young's rule (restriction to $\mathfrak{S}_{n-1}$) by induction. For (2), we first consider the case where λ is a hook, $\lambda = [1+m, 1^m]$ hence $n = 2m+1$, and then $\dim \lambda = (2m)!/(m!)^2 \geq (2m-1)2^m$. E.g. by Young's rule this dimension grows in function of m, hence $\dim \lambda^{\pm} \geq 10$ for $n \geq 6$. If λ is not a hook and $\lambda = \lambda'$ with $n \geq 7$, then the diagram of λ contains either the diagram $[4,2,1,1]$, which corresponds to a representation of $\mathfrak{S}_7$ of dimension 70 or the diagram $[3,3,2]$, which corresponds to a representation of $\mathfrak{S}_8$ of dimension 42. It follows that $\dim \lambda^{\pm} \geq \frac{1}{2}\min(70, 42) = 21$. □

For $\lambda \vdash n$, we denote $\rho_\lambda : \mathcal{A}_n \to \mathfrak{gl}(V_\lambda)$ and similarly, when $\lambda = \lambda'$, $\rho_{\lambda^\pm} : \mathcal{A}_n \to \mathfrak{gl}(V_{\lambda^\pm})$. We claim that we only need to prove

$$\begin{cases} \rho_{[n-1,1]}(\mathcal{A}_n) & = & \mathfrak{so}(V_{[n-1,1]}) & \\ \rho_\lambda(\mathcal{A}_n) & = & \mathfrak{so}(V_\lambda) & \text{if } \lambda \in \Lambda_n \\ \rho_{\lambda^\pm}(\mathcal{A}_n) & = & \mathfrak{so}(V_{\lambda^\pm}) & \text{if } \lambda \in S_n^+ \\ \rho_{\lambda^+}(\mathcal{A}_n) & = & \mathfrak{sl}(V_{\lambda^+}) & \text{if } \lambda \in S_n^- \end{cases}$$

Indeed, if this is the case, we get a collection of simple ideals of $\mathcal{A}_n$ from each representation in the above list, by taking the orthogonal of their kernel with respect to the Killing form of $\mathcal{A}$. These ideals are indeed simple, because the exceptional non-simple case $\mathfrak{so}_4$ does not occur by our assumption on n ($n \geq 8$) by lemma 8.3. If they are distinct, then the theorem is proved for this value of n, since $\mathcal{A}$ being semisimple is isomorphic to the direct product of its simple Lie ideals. Finally, they are indeed distinct, as otherwise there would be 2 non-isomorphic representations among the above factoring through the same standard Lie algebra of type $\mathfrak{so}_N$ or $\mathfrak{sl}_N$. But this is not possible, as the exceptional isomorphism $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$ and the exceptional case of $\mathfrak{so}_8$ (which has 3 non-isomorphic 'standard' representations) are ruled out for $n \geq 8$ by lemma 8.3.

In order to prove the above equalities, we will apply repeatedly lemma 8.2 to the following situation : $\mathfrak{g} = \rho(\mathcal{A}_n)$, $\mathfrak{h} = \rho(\mathcal{A}_{n-1})$. Since the restriction to $\mathfrak{A}_{n-1}$ of an irreducible representation of $\mathfrak{A}_n$ is multiplicity free (and since the restriction of a non-hook contains at most one hook) so is the restriction to $\mathfrak{h}$ of ρ by proposition 7.12.

We subdivide our analysis into several cases, and use the notation $\mu \nearrow \lambda$ as an abreviation for saying that $\mu \vdash n-1$ and that the diagram for μ is contained in the diagram of λ (or : μ is deduced from λ by removing one box). We can assume that λ is not a hook, except for the case $\lambda = [n-1, 1]$. We start by doing this case separately. Up to some additional computer calculation, we can assume $n \geq 12$. Then $\mathfrak{h} \simeq \mathfrak{so}_{n-2}$ has rank $\lfloor \frac{n-2}{2} \rfloor \geq \frac{n-2}{2} - 1 > (\dim \lambda)/4 =$

$(n-1)/4$, and $\mathfrak{g}$ is simple by lemma 8.2, and moreover $\mathrm{rk}\, h \geq 5$, $\dim \lambda > 8$, hence $\mathfrak{g} = \mathfrak{so}(V_\lambda)$. From now on, we can thus assume that λ is *not* a hook.

8.1. **Case** $\mu \nearrow \lambda \Rightarrow \mu$ **is not a hook .** Since $n \geq 7$, $\mu \nearrow \lambda \Rightarrow \dim \mu \geq 5$, and $\mu = \mu' \Rightarrow \dim \mu^{\pm} \geq 8$.

8.1.1. *Case* $\lambda \neq \lambda'$, *and* $\mu \nearrow \lambda \Rightarrow \mu \neq \mu'$. Then $\mu \nearrow \lambda \Rightarrow \mu' \not\nearrow \lambda$. Let $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda = \mu_1 + \dots + \mu_d$. Since $\dim \mu_i \geq 5$ we have $\dim \lambda \geq 5d$. By the induction assumption

$$\mathrm{rk}\, \mathfrak{h} = \sum_i \lfloor \frac{\dim \mu_i}{2} \rfloor \geqslant \sum_i \frac{\dim \mu_i}{2} - 1 \geq \frac{\dim \lambda}{2} - d \geq (\frac{1}{2} - \frac{1}{5}) \dim \lambda > \frac{\dim \lambda}{4}$$

hence $\mathfrak{g}$ is simple by lemma 8.2, and moreover $\mathfrak{g} = \mathfrak{so}(V_\lambda)$ unless possibly if $\dim \lambda \leq 8$. But this implies $d = 1$ and then $\mathfrak{h} = \mathfrak{so}(V_{\mu_1}) = \mathfrak{so}(V_\lambda)$.

8.1.2. *Case* $\lambda = \lambda'$, $\lambda^{\pm}$ *of real type and* $\mu \nearrow \lambda \Rightarrow \mu \neq \mu'$. Then, letting $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda^+ = \mu_1 + \dots + \mu_d$, $\mu_i \vdash n-1$, we have

$$\mathrm{rk}\, \mathfrak{h} = \sum_{\mu \nearrow \lambda} \frac{1}{2} \lfloor \frac{\dim \mu}{2} \rfloor \geqslant \frac{1}{2} \sum_{\mu \nearrow \lambda} \frac{\dim \mu}{2} - 1 \geqslant \frac{1}{4} \dim \lambda - d \geqslant \frac{1}{2} \dim \lambda^{\pm} - d.$$

Again $\dim \lambda \geq 5d$ hence $\mathrm{rk}\, \mathfrak{h} > (\dim \lambda)/4$ and $\mathfrak{g}$ is simple and $\mathfrak{g} = \mathfrak{so}(V_{\lambda^{\pm}})$ by lemma 8.2, as $\dim \lambda^{\pm} > 8$ by lemma 8.3.

8.1.3. *Case* $\lambda = \lambda'$, $\lambda^{\pm}$ *of real type and* $\exists \mu_0 \nearrow \lambda \mid \mu_0' = \mu_0$. We recall that we denote $b(\lambda)$ the length of the diagonal in the Young diagram λ. In our situation, μ_0 is uniquely determined by the condition $\mu_0' = \mu_0$, and we have $b(\mu_0) = b(\lambda) - 1$, and $n - 1 - b(\mu_0 = n - b(\lambda)$ hence the $\mu_0^{\pm}$ have real type. Thus

$$\mathrm{rk}\, h = \left(\sum_{\substack{\mu \nearrow \lambda \\ \mu \neq \mu'}} \frac{1}{2} \lfloor \frac{\dim \mu}{2} \rfloor \right) + \lfloor \frac{\dim \mu_0^{\pm}}{2} \rfloor \geqslant \frac{1}{2} \left(\sum_{\substack{\mu \nearrow \lambda \\ \mu \neq \mu'}} \frac{\dim \mu}{2} - 1 \right) + \frac{\dim \mu_0^{\pm}}{2} - 1.$$

Letting $\delta(\lambda) = \#\{i \mid \lambda_i \neq \lambda_{i+1}\} = \#\{\mu \mid \mu \nearrow \lambda\}$, we get

$$\mathrm{rk}\, \mathfrak{h} \geqslant \frac{1}{2} (\frac{1}{2} (\dim \lambda - \dim \mu_0) - \delta(\lambda) + 1) + \frac{\dim \mu_0^{\pm}}{2} - 1 \geqslant \frac{1}{2} \dim \lambda^{\pm} - \frac{1}{2} (\delta(\lambda) - 1) > \frac{\dim \lambda^{\pm}}{4}$$

if and only if $\dim \lambda^{\pm} > 2(\delta(\lambda) + 1)$. On the other hand, $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda^{\pm} = \mu_0^{\pm} + \mu_1 + \dots + \mu_{d-1}$ and $\delta(\lambda) = 1 + 2(d-1) = 2d - 1$, hence $2(\delta(\lambda) + 1) = 4d < \dim \lambda^{\pm}$ because $\dim \mu_0^{\pm} \geq 5$ and $\dim \mu_i \geq 5$ for $i \geq 1$. It then follows from lemma 8.2 that $\mathfrak{g}$ is simple and $\mathfrak{g} = \mathfrak{so}(V_{\lambda^{\pm}})$.

8.1.4. *Case* $\lambda = \lambda'$, $\lambda^{\pm}$ *of complex type and* $\mu \nearrow \lambda \Rightarrow \mu \neq \mu'$. As in subsection 8.1.2 we get $\mathrm{rk}\, \mathfrak{h} > (\dim \lambda^{\pm})/4$. If $\mathrm{rk}\, \mathfrak{g} > (\dim \lambda^{\pm})/2$ then $\mathfrak{g} = \mathfrak{sl}(V_{\lambda^{\pm}})$ ([M5], lemma 3.1). Otherwise $\mathrm{rk}\, \mathfrak{g} \leq (\dim \lambda^{\pm})/2 < 2\mathrm{rk}\, \mathfrak{g}$ and $\mathfrak{g}$ is simple, thanks to [M5], lemma 3.2. Thus $\mathrm{rk}\, \mathfrak{g} \geq \mathrm{rk}\, \mathfrak{h} > (\dim \lambda^{\pm})/4 > (\dim \lambda^{\pm})/5$ hence by [M5] lemma 3.5 this implies $\mathfrak{g} = \mathfrak{sl}(V_{\lambda^{\pm}})$ when $\mathrm{rk}\, \mathfrak{g} \geq 10$, that is $\mathrm{rk}\, \mathfrak{g} > 9$, which holds true as soon as $\dim \lambda^{\pm} > 36$, i.e. $\dim \lambda > 72$. Moreover, for $n \geq 7$, λ contains either $[4,2,1,1]$, or $[3,3,2]$, which have dimension 70 and 42, respectively. Since $\lambda = \lambda'$ and λ is not a hook, this implies that either $\lambda = [3,3,2]$ or λ contains $\mu \neq \mu'$ with $\mu \supset [3,3,2]$ hence $\dim \lambda \geq \dim \mu + \dim \mu' \geq 84$, in which case we are done. If $\lambda = [3,3,2]$, then $\mathfrak{h}$ contains $\mathfrak{so}(V_{[3,3,1]})$ which has rank 10, and then $\mathfrak{g} = \mathfrak{sl}(V_{\lambda^{\pm}})$ by lemma 8.2 in this case, too.

8.1.5. *Case* $\lambda = \lambda'$, $\lambda^{\pm}$ *of complex type and* $\exists \mu_0 \nearrow \lambda \mid \mu_0' = \mu_0$. As in part 8.1.3, the $\mu_0^{\pm}$ have the same type as $\lambda^{\pm}$, so they have complex type here. In the same way we get $\mathrm{rk}\,\mathfrak{h} > (\dim \lambda^{\pm})/4$ and as in part 8.1.4 we deduce from this that $\mathfrak{g}$ is simple, and $\mathfrak{g} = \mathfrak{sl}(V_{\lambda^{\pm}})$ as soon as $\mathrm{rk}\,\mathfrak{g} \geq 10$, which holds true as soon as $\dim \lambda^{\pm} > 36$, except possibly when $\lambda = [4,2,1,1]$, which is ruled out because it is a hook, or $\lambda = [3,3,2]$, which is out of the scope of this case (and has been dealt with earlier).

8.1.6. *Case* $\lambda \neq \lambda'$, *and* $\exists \mu_0 \nearrow \lambda \mid \mu_0' = \mu_0$. Again, such a $\mu_0^{\pm}$ is then uniquely determined, and we can write $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda = \mu_0^+ + \mu_0^- + \mu_1 + \cdots + \mu_{d-2}$. We subdivide in two subcases. Either the $\mu_0^{\pm}$ have real type, in which case

$$\mathrm{rk}\,\mathfrak{h} = \lfloor \frac{\dim \mu_0^+}{2} \rfloor + \lfloor \frac{\dim \mu_0^-}{2} \rfloor + \sum_{i=1}^{d-2} \lfloor \frac{\dim \mu_i}{2} \rfloor \geqslant \frac{\dim \lambda}{2} - 2$$

and we conclude as in 8.1.1, or the $\mu_0^{\pm}$ have complex type,

$$\mathrm{rk}\,\mathfrak{h} = \dim \mu_0^+ - 1 + \sum_{i=1}^{d-2} \lfloor \frac{\dim \mu_i}{2} \rfloor \geqslant \frac{\dim \lambda}{2} - (d-1) \geqslant \frac{\dim \lambda}{2} - \frac{\dim \lambda}{5} + 1 > \frac{\dim \lambda}{4}$$

and we conclude again as in 8.1.1.

8.2. $\exists \mu \nearrow \lambda$, μ **is a hook.** As in [M4] we introduce the partitions/diagrams $D(a,b) = [a+2, 2, 1^b] \vdash n = a+b+4$. Since $n \geq 7$ we can assume $a+b \geq 3$. From [M4] we recall that $\dim D(a,b) = \frac{b+1}{a+2} \begin{pmatrix} a+b+2 \\ a \end{pmatrix} (a+b+4)$. Clearly $D(a,b)' = D(b,a)$, and our assumption means that $\lambda = D(a,b)$ for some a,b.

8.2.1. *Case* $\lambda = [n-2,2]$ *(i.e.* $a=0$ *or* $b=0$*).* $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda = [n-2,1] + [n-3,2]$ and

$$\mathrm{rk}\,\mathfrak{h} = \lfloor \frac{\dim [n-2,1]}{2} \rfloor + \lfloor \frac{\dim [n-3,2]}{2} \rfloor \geqslant \frac{\dim \lambda}{2} - 2 > \frac{\dim \lambda}{4}$$

if and only if $(\dim \lambda)/4 > 2$, that is $\dim \lambda > 8$, which is true for $n \geq 6$. We conclude by lemma 8.2.

8.2.2. *Case* $a, b \geq 1$, $a \neq b$. $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda = \mathrm{Res}_{\mathfrak{A}_{n-1}} D(a,b) = D(a-1,b) + D(a,b-1) + [a+2, 1^{b+1}]$. We first assume $|a-b| > 1$. Then

$$\mathrm{rk}\,\mathfrak{h} = \lfloor \frac{\dim D(a-1,b)}{2} \rfloor + \lfloor \frac{\dim D(a,b-1)}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor \geqslant \frac{\dim D(a,b) - \dim [a+2, 1^{b+1}]}{2} - 2 + \lfloor \frac{n-2}{2} \rfloor$$

hence

$$\mathrm{rk}\,\mathfrak{h} \geqslant \frac{\dim \lambda}{2} - \frac{1}{2} \dim [a+2, 1^{b+1}] + \lfloor \frac{n-2}{2} \rfloor - 2.$$

Since $n \geq 8$ we have $\lfloor \frac{n-2}{2} \rfloor \geq 3$ hence

$$\mathrm{rk}\,\mathfrak{h} > \frac{\dim \lambda}{2} - \frac{1}{2} \dim [a+2, 1^{b+1}] \geqslant \frac{\dim \lambda}{4}$$

as soon as $(\dim \lambda)/4 \geq (\dim [a+2, 1^{b+1}])/2$, that is $\dim D(a,b) \geq 2 \dim [a+2, 1^{b+1}]$. Up to exchanging a, b, we can assume $a \geq 3$, $b \geq 1$. An easy calculation shows $D(a,b)/2 \dim [a+2, 1^{b+1}] \geq \frac{64}{14} > 2$. Lemma 8.2 then implies $\mathfrak{g} = \mathfrak{so}(V_\lambda)$ as soon as $\dim \lambda > 8$, and we know $D(a,b) \supset D(3,1) = [5,2,1]$ which has dimension 64.

8.2.3. *Case* $a = b$, $a, b \geq 1$. That is, $\lambda = D(a,a)$ with $a \geq 1$. Then $\mathrm{Res}_{\mathfrak{S}_{n-1}} \lambda = D(a-1,a) + D(a,a-1) + [a+2, 1^{a+1}]$ and $\mathrm{Res}_{\mathfrak{A}_{n-1}} \lambda^{\pm} = D(a-1,a) + [a+2,1^{a+1}]^{\pm}$. There are two subcases. First assume that $\lambda^{\pm}$ has real type ; this means that $(n-2)/2$ is even, that is a is odd. The case $a = 1$ is $[3,2,1]$ which is ruled out by $n \geq 8$, hence $a \geq 2$. Then

$$\mathrm{rk}\, \mathfrak{h} = \lfloor \frac{\dim D(a-1,a)}{2} \rfloor + \lfloor \frac{n-2}{2} \rfloor \geqslant \frac{\dim D(a-1,a)}{2} - 1 + \lfloor \frac{n-2}{2} \rfloor$$

and

$$\mathrm{rk}\, \mathfrak{h} \geqslant \frac{\dim D(a,a)^{\pm}}{2} - \frac{\dim [a+2, 1^{a+1}]}{2} - 1 + \lfloor \frac{n-2}{2} \rfloor > \frac{\dim \lambda^{\pm}}{4}$$

as soon as $\dim D(a,a)/\dim[a+2,1^{a+1}] \geq 2$. From the explicit formulas for the dimensions this is equivalent to $(1 - \frac{1}{a+2})^2 (2a+4) \geq 2$, which is true. Hence $\mathfrak{g}$ is simple and $\mathfrak{g} = \mathfrak{so}(V_{\lambda^{\pm}})$ as soon as $\dim \lambda^{\pm} > 8$, which is true since $n \geq 8$.

We now assume that $\lambda^{\pm}$ has complex type, that is a is even. Since $n = 2a+1 \geq 8$, this implies $a \geq 4$. As before we get $\mathrm{rk}\, \mathfrak{g} > (\dim \lambda^{\pm})/4 > (\dim \lambda^{\pm}) > 5$, hence either $\mathrm{rk}\, \mathfrak{g} > (\dim \lambda^{\pm})/2$ and we get immediately $\mathfrak{g} = \mathfrak{sl}(V_{\lambda^{\pm}})$ by [M5] lemma 5.1, or $\mathrm{rk}\, \mathfrak{g} < 2 \mathrm{rk}\, \mathfrak{h}$ and $\mathfrak{g}$ is simple by [M5] lemma 3.2. Then $\mathrm{rk}\, \mathfrak{g} > \frac{\dim \lambda^{\pm}}{5}$ implies $\mathfrak{g} = \mathfrak{sl}(V_{\lambda^{\pm}})$ by [M5] lemma 3.5 as soon as $\mathrm{rk}\, \mathfrak{g} > 9$, which is the case as soon as $\dim \lambda^{\pm} > 36$, that is $\dim \lambda > 72$. Since $\dim \lambda \geq \dim D(3,3) = 448$ this concludes this case.

8.2.4. *Case* $|a-b| = 1$, $a, b \geq 1$. We can assume $a = b+1 \geq 2$, that is $\lambda = D(b+1,b)$. Then $\mathrm{Res}_{\mathfrak{A}_{n-1}} D(b+1,b) = D(b,b)^+ + D(b,b)^- + D(b+1,b-1) + [b+3,1^{b+1}]$. Moreover, $D(b,b)^{\pm}$ has real type if b is odd, and complex type if b is even.

We first assume that b is odd. Since $n \geq 8$ this implies $b \geq 3$, hence $n \geq 11$. Then

$$\mathrm{rk}\, \mathfrak{h} \geqslant \frac{\dim D(b,b)^+}{2} + \frac{\dim D(b,b)^-}{2} + \frac{\dim D(b+1,b-1)^+}{2} - 3 + \lfloor \frac{n-2}{2} \rfloor \geqslant \frac{\dim \lambda}{2} - \frac{\dim [b+3, 1^{b+1}]}{2}$$

because $-3 + \lfloor \frac{n-2}{2} \rfloor \geq 0$ for $n \geq 8$. Thus $\mathrm{rk}\, \mathfrak{h} > (\dim \lambda)/4$ as soon as $(\dim D(b+1,b])/(\dim[b+3,1^{b+1}]) > 2$, and this is a consequence of the computation in 8.2.2. Hence $\mathfrak{g}$ is simple and $\mathfrak{g} = \mathfrak{so}(V_\lambda)$ as soon as $\dim \lambda > 8$, which is true as $\dim D(2,1) = 35$.

We now assume that b is even, hence $b \geq 2$. Then

$$\mathrm{rk}\, \mathfrak{h} \geq \dim D(b,b)^+ - 1 + \frac{\dim D(b+1,b-1)}{2} - 1 + \lfloor \frac{n-2}{2} \rfloor > \frac{\dim \lambda}{2} - \frac{\dim [b+3,1^{b+1}]}{2} \geqslant \frac{\dim \lambda}{4}$$

and we conclude as before.

9. Rotation algebras : structure theorem in types D, E

We use the notations of [M5] for the representations of W. In particular, $\Lambda\mathrm{Ref} = \Lambda\mathrm{Ref}(W)$ denotes the irreducible representations deduced from a reflection representation by taking some alternating power of it and tensoring by a linear character. When $W = D_n$ We denote $\mathrm{Ind}\Lambda\mathrm{Ref}$ the set of irreducible representations whose restriction to D_{n-1} have a component in $\Lambda\mathrm{Ref}(W_0)$, for W_0 the standard parabolic subgroup of type D_{n-1}, $\mathrm{Irr}''(W) = \mathrm{Irr}(W) \setminus \mathrm{Ind}\Lambda\mathrm{Ref}$, and $\mathrm{Irr}'(W) = \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$. Recall from [M5] that $\mathrm{Ind}\Lambda\mathrm{Ref} \supset \Lambda\mathrm{Ref}$, hence $\mathrm{Irr}''(W) \subset \mathrm{Irr}'(W) \subset \mathrm{Irr}(W)$. In case W has type E_6, E_7, E_8, we take for W_0 a standard parabolic subgroup of type D_5, E_6, E_7. We let O and O_0 denote the rotation subgroups of W and W_0, respectively. We let $\mathrm{Irr}''(\mathrm{O})$ denote the set of the $\rho \in \mathrm{Irr}(\mathrm{O})$ which are constituents of the restriction of some element of $\mathrm{Irr}''(\mathrm{O})$, and we define similarly $\mathrm{Irr}'(\mathrm{O})$, $\mathrm{Irr}'(\mathrm{O}_0)$.

Here reflection representation means that the reflections of W act by reflections in the representations, but such a representation is not assumed to be faithful. We denote Ref the set of reflections representations. In type D_n, they are labelled $\{[n-1],[1]\}$ (n dimensional) and $\{[n-1,1],\emptyset\}$ ($n-1$ dimensional). Recall that the representations $\{\lambda,\emptyset\}$ factor through $\mathfrak{S}_n$. We parametrize the representations of O as follows.

(1) couples (λ,μ) for $\lambda>\mu,\mu'$ and $\lambda\geq\lambda'$, which originate from representations $\{\lambda,\mu\}$ of W with $\lambda\neq\mu$, $\lambda\neq\mu'$ and $(\lambda,\mu)\neq(\lambda',\mu')$. They have real type.
(2) signed couples $(\lambda,\mu)^{\pm}$ for $\lambda>\mu$, $\lambda=\lambda'$, $\mu=\mu'$, which originate from representations $\{\lambda,\mu\}$ of W.
(3) signed partitions $(\lambda)^{\pm}$ with $\lambda\vdash n/2$ for $\lambda>\lambda'$, which originate from representations $\{\lambda,\lambda'\}$ of W with $\lambda\neq\lambda'$.
(4) signed partitions $((\lambda))^{\pm}$ with $\lambda\vdash n/2$ for $\lambda>\lambda'$, which originate from representations $\{\lambda\}^{\pm}$ of W with $\lambda\neq\lambda'$.
(5) partitions $((\lambda))$ with $\lambda\vdash n/2$, which originate from representations $\{\lambda\}^{\pm}$ of W with $\lambda=\lambda'$ and $n/2$ odd.
(6) doubly signed partitions $((\lambda))^{\pm\pm}$ with $\lambda\vdash n/2$, which originate from representations $\{\lambda\}^{\pm}$ of W with $\lambda=\lambda'$ and $n/2$ even : $\mathrm{Res}_{\mathrm{O}}\{\lambda\}^{\pm}=\{\lambda\}^{\pm+}\oplus\{\lambda\}^{\pm-}$.

The reason for the existence of the last 2 cases is the following lemma.

Lemma 9.1. *Let $\lambda\vdash m$ with $\lambda=\lambda'$. Then $\{\lambda\}^{\pm}\otimes\varepsilon=\{\lambda\}^{\pm}$ if m is even, $\{\lambda\}^{\pm}\otimes\varepsilon=\{\lambda\}^{\mp}$ if m is odd.*

Proof. Let $C=\{\pm1\}^{2m}$, $H<C$ the index two subgroup made of the $2m$-tuples of product 1. Then $\tilde{W}=\mathfrak{S}_{2m}\ltimes C$ and $W=\mathfrak{S}_{2m}\ltimes H$ are the Coxeter groups of type B_n and D_n, respectively, and $A=H\rtimes\mathfrak{A}_n$. The representation (λ,λ) of B_n can be obtained as $\mathrm{Ind}_{\mathfrak{S}_m^2\ltimes C}^{\mathfrak{S}_n\ltimes C}\tilde{\omega}(\lambda\boxtimes\lambda)$ where $\lambda\boxtimes\lambda$ denotes the exterior Kronecker product of two copies of the representation λ of $\mathfrak{S}_m$, extended trivially to $\mathfrak{S}_m^2\ltimes C$, and $\tilde{\omega}$ the trivial extension of $\omega: C\to\{\pm1\}$ to $\mathfrak{S}_m^2\ltimes C$, where $\omega(x_1,\dots,x_{2m})=x_{m+1}\dots x_{2m}$ (note that this character is stabilized by $\mathfrak{S}_m^2$, hence that this 'trivial extension' makes sense). We need to prove that, in case $\lambda=\lambda'$, the $\mathrm{Res}_{H\rtimes\mathfrak{A}_n}^{C\rtimes\mathfrak{S}_n}\mathrm{Ind}_{\mathfrak{S}_m^2\ltimes C}^{\mathfrak{S}_n\ltimes C}\tilde{\omega}(\lambda\boxtimes\lambda)$ has 2 irreducible constituents if m is odd, and 4 otherwise. Note that, by elementary Clifford theory, the number of irreducible constituents is necessary either 2 or 4.

By Mackey formula, and because $H\rtimes\mathfrak{A}_n\backslash\mathfrak{S}_n\ltimes C/\mathfrak{S}_m^2\ltimes C=\{1\}$, we get that this restriction is $\mathrm{Ind}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{H\rtimes\mathfrak{A}_n}\mathrm{Res}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{C\rtimes\mathfrak{S}_m^2}\tilde{\omega}(\lambda\boxtimes\lambda)$. By elementary Clifford theory, since $\lambda'=\lambda$, we have that $\mathrm{Res}_{\mathfrak{A}_n\cap\mathfrak{S}_m^2}^{\mathfrak{S}_m^2}\lambda\boxtimes\lambda$ is the sum of two irreducible constituents $(\lambda\boxtimes\lambda)^+$ and $(\lambda\boxtimes\lambda)^-$, thus $\mathrm{Res}_{C\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{C\rtimes\mathfrak{S}_m^2}\tilde{\omega}(\lambda\boxtimes\lambda)=\tilde{\omega}(\lambda\boxtimes\lambda)^++\tilde{\omega}(\lambda\boxtimes\lambda)^-$. Since the $\mathrm{Res}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{C\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}\tilde{\omega}(\lambda\boxtimes\lambda)^{\pm}$ are clearly irreducible, we get that $\mathrm{Res}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{C\rtimes\mathfrak{S}_m^2}\tilde{\omega}(\lambda\boxtimes\lambda)$ has two irreducible constituents, namely the $\zeta_{\pm}=\tilde{\omega}_H(\lambda\boxtimes\lambda)^{\pm}$, where ω_H denotes the restriction to H of ω, and $\tilde{\omega}_H$ its extension to $H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)$. Let ζ denote one of these constituents. One readily gets that $\{g\in H\rtimes\mathfrak{A}_n\ |\zeta^g\simeq\zeta\}=H\rtimes\{g\in\mathfrak{A}_{2m}\mid\omega_H^g\simeq\omega_H\}=H\rtimes(\mathfrak{A}_n\cap\{g\in\mathfrak{S}_n\mid\omega_H^g\simeq\omega_H\})$, and that $\{g\in\mathfrak{S}_n\mid\omega_H^g\simeq\omega_H\}$ is $\mathfrak{S}_m^2\rtimes<\sigma>$ where $\sigma=(1,2m)(2,2m-1)(3,2m-2)\dots$. If m is odd, $\sigma\notin\mathfrak{A}_n$, $I(\zeta)=\{g\in H\rtimes\mathfrak{A}_n\ |\zeta^g\simeq\zeta\}=H\rtimes(\mathfrak{A}_n\cap S_m^2)$ and $\mathrm{Ind}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{H\rtimes\mathfrak{A}_n}\zeta$ is irreducible by Clifford's theorem ([CR] §11) which proves the claim. If m is even, then $\mathrm{Ind}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{H\rtimes\mathfrak{A}_n}\zeta=\mathrm{Ind}_{I(\zeta)}^{H\rtimes\mathfrak{A}_n}\mathrm{Ind}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{I(\zeta)}\zeta$, $H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)$ has index 2 in $I(\zeta)$, hence by elementary Clifford theory $\mathrm{Ind}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{I(\zeta)}\zeta$ has two irreducible constituents. Thus $\mathrm{Ind}_{H\rtimes(\mathfrak{A}_n\cap\mathfrak{S}_m^2)}^{H\rtimes\mathfrak{A}_n}\zeta$ has at least 2 irreducible constituents, which proves the claim. □

We notice for future use that elementary applications of the branching rules show the following.

Lemma 9.2. *In cases (3-6) above, the restriction to* O_0 *(rotation subgroup of type* D_{n-1}*) is multiplicity free and made of representations of* real *type.*

Theorem 9.3. *If* W *is an irreducible Coxeter group of type ADE, then* $\mathcal{A} = \mathcal{L}_1(\mathrm{O}) \cap \mathcal{H}'$*, unless* W *has type* D_4*,* A_3*,* A_2*.*

According to the theorem, for $n \geq 5$, the dimension of $\mathcal{A}$ in type D_n is 390, 5314, 78758, 1282059, 23189432, 464312278, 10217797426, 245243928461, 6376443304559 . . . For E_6, E_7, E_8 it is 12593, 722403, 174117236.

One clearly has an embedding $\mathcal{A} \hookrightarrow \mathcal{L}_1(\mathrm{O}) \cap \mathcal{H}$, and it falls into the semisimple part of $\mathcal{H}'$ because $\mathcal{A}$ is semisimple for $n \geq 5$ (proposition 7.14 and lemma 7.8). We do a proof by induction, assuming the theorem proved for W_0.

Lemma 9.4. *Assume that the theorem holds for* W_0*, that* $\rho \in \mathrm{Irr}''(\mathrm{O})$ *has real type and dimension* > 8*, that its restriction to* O_0 *is multiplicity free, and that* $\forall \varphi \in \mathrm{Irr}'(\mathrm{O}_0)\ \dim\varphi \geq 5$*. Then* $\rho(\mathcal{A}) = \mathfrak{so}(V_\rho)$*.*

Proof. Letting $\mathfrak{g} = \rho(\mathcal{A})$ and $\mathfrak{h} = \rho(\mathcal{A}_0)$, we can write $\mathrm{Res}_{\mathrm{O}_0}\rho = \rho_1 + \cdots + \rho_r + \varphi_1 + \varphi_1^* + \cdots + \varphi_s + \varphi_s^*$ with the $\rho_i, \varphi_i \in \mathrm{Irr}'(\mathrm{O}_0)$, the ρ_i having real type and the φ_i complex type. Then

$$\mathrm{rk}\,\mathfrak{h} \geqslant \sum_{i=1}^{r} \lfloor \frac{\dim\rho_i}{2} \rfloor + \sum_{i=1}^{s} (\dim\varphi_i - 1) \geqslant \sum_{i=1}^{r} \left(\frac{\dim\rho_i}{2} - 1 \right) + 2\left(\sum_{i=1}^{s} \frac{\dim\varphi_i}{2} \right) - s$$

that is $\mathrm{rk}\,\mathfrak{h} \geq \frac{\dim\rho}{2} - r - 2s$. Since we have $\dim\rho_i > 4$ and $\dim\varphi_i > 4$, then $\mathrm{rk}\,\mathfrak{h} > \frac{\dim\rho}{4}$ and the conclusion follows from lemma 8.2.

□

Lemma 9.5. *Assume that the theorem holds for* W_0*, that* $\rho \in \mathrm{Irr}''(\mathrm{O})$ *has complex type, and* $\mathrm{rk}\,\mathfrak{h} > \dim\rho/4$ *with* $\dim\rho > 21$*. If the restriction of* ρ *to* O_0 *is multiplicity free, then* $\mathfrak{g} = \mathfrak{sl}(V_\rho)$ *.*

Proof. $\mathfrak{g}$ is simple by [M5] lemma 3.3 (I). Since ρ is not selfdual and $\dim\rho > 1$, the conclusion follows by [M5] lemma 3.4. □

The following lemma is similar to [M5], lemma 2.25 :

Lemma 9.6. *Assume* W *has type* $A_n, n \geq 5$*,* $D_n, n \geq 6$*,* E_6*,* E_7*,* E_8 *and* $\rho \in \mathrm{Irr}(\mathrm{O})$ *with* $\dim\rho > 1$*. Then* $\rho(\mathcal{A}) \subset \mathfrak{gl}(V_\rho)$ *do not contain any simple Lie ideal of rank* 1*.*

Proof. By lemma 7.11 we know that $\rho(\mathcal{A}) \subset \mathfrak{sl}(V_\rho) = V_\rho \otimes V_{\rho^*}$ is invariant under A-conjugation, and that so is any simple Lie ideal. As a consequence, a $\mathfrak{sl}_2$ ideal would provide a 3-dimensional sub-representation of $V_\rho \otimes V_{\rho^*}$. It is easily checked that, for the W listed above, the smallest non-linear irreducible character of W has degree 5, hence a 3-dimensional such a representation should be a sum of 1-dimensional ones. But for the same reason and by Clifford theory, there are only one 1-dimensional character of A, and it can can appear only with multiplicity 1 in a tensor product of the form $V_\rho \otimes V_{\rho^*}$, according to Schur's lemma. This contradiction proves the lemma. □

We now concentrate on the case of type D_n, and do a proof by induction. We postpone the case of D_5 to subsection 9.6, so we can assume $n \geq 6$. The case of W of type A_n has been done before. We let $\mathrm{Irr}_0(W)$ the set of irreducible representations of W which do not factor through A_{n-1}, and $\mathrm{Irr}'_0 = \mathrm{Irr}' \cap \mathrm{Irr}_0$, $\mathrm{Irr}''_0 = \mathrm{Irr}'' \cap \mathrm{Irr}_0$.

We let $\mathrm{Irr}''_0(\mathrm{O})$ (resp. $\mathrm{Irr}'_0(\mathrm{O})$) denote the set of irreducible representations of O which appear inside the restriction of an element of $\mathrm{Irr}''_0(W)$ (resp $\mathrm{Irr}'_0(W)$). We have the following combinatorial lemma.

Lemma 9.7.

(1) *If $\rho \in \mathrm{Irr}_0(\mathrm{O})$, in type D_n for $n \geq 5$, has dimension $4, 6$ or 8, then $\rho = ([n-1],[1])$.*
(2) *If $\rho \in \mathrm{Irr}'_0(\mathrm{O})$ in type D_n for $n \geq 5$, then $\dim \rho \geq 5$ (if $n \geq 6$ then $\dim \rho \geq 10$, if $n \geq 7$ then $\dim \rho \geq 21$, if $n \geq 8$ then $\dim \rho \geq 48$).*
(3) *If $\rho \in \mathrm{Irr}''_0(\mathrm{O})$ in type D_n, then $\dim \rho \geq 10$ when $n \geq 6$, and $\dim \rho \geq 35$ when $n \geq 7$.*
(4) *If $\rho \in \mathrm{Irr}'(\mathrm{O})$ in type D_n, then $\dim \rho \geq 5$ when $n \geq 5$.*

Proof. Under the assumptions of (1), by Clifford theory, ρ appears in the restriction of some $\tilde{\rho} \in \mathrm{Irr}_0(W)$ with $\dim \rho \in \{4, 6, 8, 12, 16\}$. For $n = 7$, from the character table we get that all elements in $\mathrm{Irr}_0(W)$ have dimension at least 21, except for $\{6, 1\}$ and $\{1^6, 1\}$. If $n \geq 8$, then the restriction to the obvious parabolic subgroup H of type D_7 has to contain $\{6, 1\}$ or $\{1^6, 1\}$; because of the branching rule, if it is not of the form $\{n-1, 1\}$ or $\{1^{n-1}, 1\}$ then its restriction to H has to contain another constituent, thus of dimension at least 21. A check of the character tables shows that, for $5 \leq n \leq 7$, there are no other choices as well.

□

Note that $\mathrm{Irr}''_0(\mathrm{O}) = \emptyset$ for W of type D_5. As in the case of type A, we are thus reduced to prove, when $n \geq 5$, that

$$\begin{cases} \rho_{\{[n-1],[1]\}}(\mathcal{A}) & = & \mathfrak{so}_n \\ \rho_{\{[n-1],[1]\}}(\mathcal{A}) & = & \mathfrak{so}_n \\ \rho(\mathcal{A}) & = & \mathfrak{so}(V_\rho) & \text{if } \rho \in \mathrm{Irr}'(A) \text{ has real type} \\ \rho(\mathcal{A}) & = & \mathfrak{sl}(V_\rho) & \text{if } \rho \in \mathrm{Irr}'(A) \text{ has complex type} \end{cases}$$

9.1. **Case** $\rho = ([n-1],[1])$**.** Since $\mathfrak{h}$ contains $\mathfrak{so}_{n-1}$, $\mathrm{rk}\, \mathfrak{g} \geq \lfloor \frac{n-1}{2} \rfloor \geq (\dim \rho)/4 = n/4$ as soon as $n > 6$. Moreover $\dim \rho > 8$ when $n > 8$, so we conclude by lemma 8.2, the cases $n \leq 8$ being checked by computer.

9.2. **Case** $\rho \in \mathrm{Ind}\Lambda\mathrm{Ref}$**.**

9.2.1. $\rho = ([n-p],[2,1^{p-2}])$. If $p = 2$, $\mathrm{Res}\rho = ([n-3],[2]) + ([n-2],[1])$. Assuming $n \geq 6$ we have $n - 3 > 2$. By induction we have

$$\mathrm{rk}\, \mathfrak{h} = \lfloor \frac{\dim([n-3],[2])}{2} \rfloor + \lfloor \frac{\dim([n-2],[1])}{2} \rfloor \geqslant \frac{\dim \rho}{2} > \frac{\dim \rho}{4}$$

as soon as $\dim \rho > 8$, which holds true for $n \geq 5$. The case $n = 5$ is checked by computer.

If $p = n-1$, i.e. $\rho = ([1],[2,1^{n-3}])$, we assume again $n \geq 6$. Then by induction we similarly get

$$\mathrm{rk}\, \mathfrak{h} \geqslant \frac{\dim \rho}{2} - 3 - \frac{\dim([1],[1^{n-2}])}{2} + \frac{n-1}{2} > \frac{\dim \rho}{4}$$

as soon as $\dim \rho > 6$, which is true. The case $n = 5$ is checked by computer.

If $p \notin \{2, n-1\}$, that is $n-p \geq 2$ and $p-2 \geq 1$. As before, we get $\mathrm{rk}\,\mathfrak{g} > (\dim \rho)/4$ as soon as $\dim \rho > 6 + 2\dim([n-p],[1^{p-1}]) - 2(n-1)$. Now $\dim \rho = \dim\{[n-p],[2,1^{p-2}]\}$ and the restriction rule to $W(D_{n-2})$ shows that the restriction of $\{[n-p],[2,1^{p-2}]\}$ contains the restriction of $\{[n-p],[1^{p-1}]\}$. As a consequence, $\dim \rho \geq 2\dim([n-p],[1^{p-1}])$ hence $\mathrm{rk}\,\mathfrak{g} > (\dim \rho)/4$ as soon as $n \geq 5$.

In all these cases, $\dim \rho > 8$ for $n \geq 5$, hence lemma 8.2 shows that $\rho(\mathcal{A}) = \mathfrak{so}(V)$.

9.2.2. $\rho = ([1],[n-p,1^{p-1}])$. We can assume $n-p \geq 2$, hence $n \geq p+2 \geq 4$, and $p-1 \geq 1$, hence $p \geq 2$.

We have $\mathrm{Res}\rho = (\emptyset,[n-2,1]) + ([1],[n-p,1^{p-2}]) + ([1],[n-p-1,1^{p-1}])$ as a sum of irreducible constituent unless one of the partitions $[n-p,1^{p-2}]$ and $[n-p-1,1^{p-1}]$ is selftransposed, but also in this case the restriction is multiplicity free. Since ρ has real type and all the constituents have dimension at least 5 for $n \geq 7$, the same proof as in lemma 9.4 yields the conclusion for $n \geq 7$. The case $n = 6$ is dealt with in section 9.6.

9.3. $\rho \in \mathrm{Irr}''_0(\mathrm{O})$, **cases (3)-(6).** In that case, ρ has for restriction $\rho_1 + \cdots + \rho_r$, with the ρ_i distinct representations of real type. Thus

$$\mathrm{rk}\,\mathfrak{h} \geq \sum_i \frac{\dim \rho_i}{2} - 1 = \frac{\dim \rho}{2} - r > \frac{\dim \rho}{4}$$

as soon as $\dim \rho > 4r$. But this holds true if $\dim \rho_i \geq 5$, and this is a consequence of lemma 9.7 (2) because $n \geq 5$ and $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$. Since $\dim \rho > 8$ we get the conclusion when ρ has real type by lemma 8.2. The remaining case is when ρ has complex type and $\dim \rho \leq 21$. But there are no $\rho \in \mathrm{Irr}''_0(A)$ of complex type in type D_6, and $\dim \rho \geq 35$ when $n \geq 7$ by lemma 9.7 (3) so we get the conclusion by lemma 9.5.

9.4. $\rho \in \mathrm{Irr}''_0(\mathrm{O})$, **case (1).** Here $\rho = (\lambda,\mu)$ with $\lambda \neq \mu$, $\lambda \neq \mu'$ and $(\lambda,\mu) \neq (\lambda',\mu')$. If the restriction is multiplicity free, we have the conclusion by lemmas 9.4 and 9.7 (4) at least when $n \geq 6$.

We thus assume it is not. It is easily checked that it can happen only in the following situation : μ is deduced from λ by removing one box at a given row, and $\lambda = \lambda'$; then $\{\lambda,\lambda\} = \{\lambda\}^+ + \{\lambda\}^-$, which restricts to $2((\lambda))$ if in addition $(n-1)/2$ is odd. Since we assumed $n \geq 6$, note that this condition implies $n \geq 9$.

The restriction can be written $2\rho_0 + \rho_1 + \cdots + \rho_r$, and

$$\mathrm{rk}\,\mathfrak{h} \geq \sum_i \frac{\dim \rho_i}{2} - 1 = \frac{\dim \rho - \dim \rho_0}{2} - (r+1) > \frac{\dim \rho}{4}$$

iff $\dim \rho > 2\dim \rho_0 + 4(r+1)$, that is $\dim \rho_1 + \cdots + \dim \rho_r \geq mr > 4(r+1)$ where $m = \min \dim \rho_i$, that is $m > 4(1+\frac{1}{r})$. By lemma 9.7 we know $\dim \rho_i \geq 48$, so this condition is fulfilled. Moreover $48 \leq \dim \rho < 4\mathrm{rk}\,\mathfrak{h}$ implies $\dim \rho < (\mathrm{rk}\,\mathfrak{h} + 1)^2$. We have multiplicities at most 2, and $\mathfrak{g} = \rho(\mathcal{A})$ does not contain any $\mathfrak{sl}_2$ by lemma 9.6, hence $\mathfrak{g}$ is a simple Lie algebra according to [M5], lemma 3.3 (II). Thus $\mathrm{rk}\,\mathfrak{h} > (\dim \rho)/4$ and $\dim \rho > 8$ implies $\mathfrak{g} = \mathfrak{so}(V_\rho)$ by lemma 8.2.

9.5. $\rho \in \mathrm{Irr}''_0(\mathrm{O})$, **case (2).** Here $\rho = (\lambda,\mu)^{\pm}$ with $\lambda = \lambda'$, $\mu = \mu'$ and $\lambda \neq \mu$. Here the restriction is multiplicity-free. If all the constituents in the restriction have real type, then we can conclude as in 9.3. The remaining situation is when there is α with $\alpha \nearrow \lambda$ and $\alpha = \mu$, that is $\mu \nearrow \lambda$. In that case we have a multiplicity-free decomposition $\mathrm{Res}\rho = \{\mu\}^{\pm+} + \{\mu\}^{\pm-} + \rho_1 + \cdots + \rho_r$ with the ρ_i having real type. Note that this implies $n-1 = 4m$.

Since $n \geq 6$, this implies $n \geq 9$, and in particular all the constituents of $\mathrm{Res}\rho$ have dimension at least 48. If ρ has real type, then we are done by lemma 9.4 ; if ρ has complex type, by lemma 9.5 we only need to prove $\mathrm{rk}\, \mathfrak{h} > (\dim \rho)/4$ and this is straightforward.

9.6. **Exceptional types.** First notice that, once the character table of W and W_0 is known, as well as the induction table, the proof of the theorem for W can be partly automatized : if there are no irreducible representations of W susceptible to lead to the the exceptional types $\mathfrak{sl}_4$ and $\mathfrak{so}_8$ and if the induction table is multiplicity free, then one can check systematically whether the conditions of lemma 8.2 (if we are in type $\mathfrak{so}$) or [M5] lemma 3.2 (in type $\mathfrak{sl}$) are satisfied. In addition, in type D_n, representations factoring through $\mathfrak{S}_n$ can be considered tackled. This works for D_5, D_7, D_{11}, and E_7.

When W has type D_6, the only representations remaining to check are

- the restriction to O of the $\{[2,1]\}^{\pm}$, which are irreducible hence of real type. They have dimension 40, their restriction to $\mathrm{O}_0 = \mathrm{Ker}\,(W_0 \to \{\pm 1\})$ are multiplicity free and $\mathfrak{h}$ has rank 10. Since $10 > 40/5$ we get the conclusion by [M5], lemma 3.5 (2).
- the restriction to O of $([1,1],[3,1])$, which is irreducible of real type. It has dimension 45, its restriction to O_0 is multiplicity free and $\mathfrak{h}$ has rank 9. Since $9 = 45/5$, we cannot use the same argument. We compute the linear rank of $\mathfrak{g} = \rho(\mathcal{A})$ by computer, and get $\dim \mathfrak{g} = 990 = \dim \mathfrak{so}(V_\rho)$.

When W has type E_6, the representations that we need to check are

- the restriction to O of the one labelled $\varphi_{15,16}$ in CHEVIE's convention. It has real type and dimension 15. The Lie algebra $\mathfrak{h}$ has rang 2. We compute the linear rank of $\mathfrak{g} = \rho(\mathcal{A})$ by computer : by lemma 7.3 we know that it is generated by the images of the $(su) - (su)^{-1}$ for s, u simple reflections with $su \neq us$, and we have matrix models of this representation over $\mathbb{Q}$. By computer one readily gets $\dim \rho(\mathcal{A}) = 105 = \dim \mathfrak{so}(V_\rho)$ hence $\mathfrak{g} = \mathfrak{so}(V_\rho)$.
- the restriction to O of the ones labelled $\varphi_{30,15}$ in CHEVIE's convention. It has real type and dimension 30. The restriction is multiplicity-free, $\mathfrak{h}$ has rank $7 > 30/5$, and we can use [M5] lemma 3.5 (2) to get the conclusion.
- the restriction to O of the ones labelled $\varphi_{20,20}$ in CHEVIE's convention. It has real type and dimension 20. The restriction is multiplicity-free, $\mathfrak{h}$ has rank 2. We need again to use a computer, and find $\dim \mathfrak{g} = 190 = \dim \mathfrak{so}_{20}$, as wanted.
- the two constituents of the restriction to O of the one labelled $\varphi_{90,8}$ in CHEVIE's convention. They have complex type and dimension 45. The restriction is multiplicity-free, we have $\mathrm{rk}\, \mathfrak{h} = 19$. Since $19 > 45/4$ we can use [M5] lemma 3.5 (2) to get the conclusion.

Finally, when W has type E_8, the remaining ones all have real type. They are the restrictions to O of the representations of W labelled $\phi_{6075,22}, \phi_{3240,31}, \phi_{4536,23}, \phi_{5600,21}$ of dimensions 6075, 3240,4536, 5600, plus the two constituents $\phi^{+}_{7168,17}, \phi^{-}_{7168,17}$ (of dimension 3584) of the restriction of the representation labelled $\phi_{7168,17}$. One easily gets in all these cases that $\mathrm{rk}\, \mathfrak{h} > (\dim \rho)/4$, hence by [M5] lemma 3.4 it is sufficient to prove that $\mathfrak{g}$ is simple. Since the multiplicities are at most 2, it is easily checked to be a consequence of [M5] lemma 3.3 (II), provided we know that $\rho(\mathcal{A})$ has no simple ideal of type $\mathfrak{sl}_2$. This has been proved in lemma 9.6 .

References

[BMR] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998) 127-190.

[BM] O. Brunat, I. Marin, *Image of the braid groups inside the finite Temperley-Lieb algebras*, preprint 2012, arxiv:1203.5210v1.

[Ca] R.W. Carter, *Conjugacy Classes in the Weyl group*, Compositio Math. **25** (1972), 1–59.

[Ch] C. Chevalley, *Théorie des groupes de Lie. Tome II. Groupes algébriques*, Actualités Sci. Ind. no. 1152, Hermann, Paris, 1951.

[CR] C. Curtis, I. Reiner, *Methods of representation theory*, vol. 1, John Wiley, 1981.

[De] F. Deloup, *Palindromes and orderings in Artin groups*, J. Knot Theory Ramifications **19** (2010) 145–162.

[Dr] V. Drinfeld, *On quasitriangular quasi-Hopf algebras and on a group closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, Algebra i Analiz **2** (1990), 149-181, translated in Leningrad Math. J. **2** (1991), 829-860.

[En] B. Enriquez, *Quasi-reflection algebras and cyclotomic associators*, Selecta Math. (N.S.) **13** (2007), 391-463.

[G] M. Geck, *Constructible characters, leading coefficients and left cells for finite Coxeter groups with unequal parameters*, Repr. Theory **6** (2002), 1–30.

[L] G. Lusztig, *Unipotent characters of the symplectic and odd orthogonal groups over a finite field*, Invent. Math. **64** (1981), 263–296.

[M1] I. Marin, *Représentations linéaires des tresses infinitésimales* (doctoral thesis), Université Paris 11-Orsay, 2001.

[M2] I. Marin, *Infinitesimal Hecke Algebras*, C.R.Acad. Sci. Paris Ser. I **337** (2003) 297-302.

[M3] I. Marin, *Monodromie algébrique des groupes d'Artin diédraux*, J. Algebra **303** (2006) 97-132.

[M4] I. Marin, *L'algèbre de Lie des transpositions*, J. Algebra 310 (2007) 742-774.

[M5] I. Marin, *Infinitesimal Hecke Algebras II*, preprint 2009, arxiv:0911.1879v1.

[M6] I. Marin, *Infinitesimal Hecke Algebras III*, preprint 2009, arxiv:1012.4424v1.

[M7] I. Marin, *Reflection groups acting on their hyperplanes*, J. Algebra 322 (2009) , 2848-2860.

[M8] I. Marin, *Group algebras of finite groups as Lie algebras*, Comm. Alg. **38** (2010), 2572-2584.

[M9] I. Marin, *The freeness conjecture for Hecke algebras of complex reflection groups, and the case of the Hessian group G26*, preprint 2012, arxiv:1210.5632v2.

LAMFA, Université de Picardie Jules Verne, Amiens, France
E-mail address: ivan.marin@u-picardie.fr