Cohomology of quasi-abelianized braid groups

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Abstract

We investigate the rational cohomology of the quotient of (generalized) braid groups by the commutator subgroup of the pure braid groups. We provide a combinatorial description of it using isomorphism classes of certain families of graphs. We establish Poincaré dualities for them and prove a stabilization property for the infinite series of reflection groups.

Keywords: braid group, pure braid group, cohomology, representation theory

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1 Introduction

Let Br_n be the braid group on n strands, $\mathrm{P}_n \subset \mathrm{Br}_n$ the pure braid group on n strands, $\mathrm{S}_n = \mathrm{Br}_n/\mathrm{P}_n$ the symmetric group on n letters. The quasi-abelianization of the braid group is the quotient Γ_n of the braid group by the commutator subgroup of the *pure* braid group. More generally, when B is an Artin group of finite Coxeter type or, even more generally, the generalized braid group associated to a (finite) complex reflection group W, its quasi-abelianization is its quotient by the commutator subgroup of the associated pure braid group.

These groups, whose first implicit introduction can arguably be traced back to the same paper where Tits (also implicitly) first defined the now-called Artin groups (see [17]), have gained interest in the past decades. They play a role in several works where linear actions of braids are involved, as some of the linear representations of the braid groups appearing in nature factor through this group, the pure braids being sent to diagonal (and therefore commuting) matrices (see [18],[16],[8] among others). More recently, the group Γ_n has been shown to have a nice presentation and a nice categorical interpretation in [12].

It was observed in [7] that each Γ_n is a crystallographic group, and also proved there that it has no element of order 2, which has the consequence that torsion-free crystallographic groups (also known as Bieberbach groups) can be constructed easily from it. Subsequently, it was proved in [9] that these properties are true for any group Γ similarly associated to an arbitrary complex reflection group, and in [3] that, however, many finite groups of odd order can be embedded in Γ (and actually any of them inside Γ_n for some n).

In this paper, we explore the rational cohomology of these groups and show that it admits a nice combinatorial description in terms of isomorphism classes of certain families of graphs. Moreover, we exhibit a few remarkable phenomena. Among them, for Γ_n and more generally for Γ when W is a complex reflection group of type $G(de, e, n)$, we show that the rational cohomology stabilizes, as in the case of the usual braid group (see [2]) and other related groups (see e.g. [4], [5], [13], [19]). Also, we show that there is a Poincaré duality that can appear in various disguises, and which also admits a simple combinatorial interpretation.

The first part of the paper is devoted to a detailed study of the case of the usual braid group. We then consider the general case in Section 3, establish general properties there, and then provide a combinatorial description in the case of the general series $G(de, e, n)$ of complex reflection groups. This enables us to prove the general stabilization result in Section 3.5.

The scripts used for the computations in this paper are available from the corresponding author upon request.

2 Braid group case

2.1 General properties

Let Br_n be the braid group on n strands, $\mathrm{P}_n \subset \mathrm{Br}_n$ the pure braid group on n strands, $\mathrm{S}_n = \mathrm{Br}_n/\mathrm{P}_n$ the symmetric group on n letters.

Definition 1. We consider *quasi-abelianized braid group*, that is, the quotient

$$\Gamma_n := \mathrm{Br}_n/[\mathrm{P}_n, \mathrm{P}_n].$$

and we write $\mathbb{Z}\mathcal{A}_n$ for the abelianization of the pure braid group, that is, the quotient $\mathrm{P}_n/[\mathrm{P}_n, \mathrm{P}_n] \simeq \mathbb{Z}^{\binom{n}{2}}$.

Remark 1. Since P_n is the fundamental group of the complement in $\mathbb{C}^n$ of the hyperplanes defined by the equations $z_i = z_j$, for $i < j$, this group $\mathbb{Z}\mathcal{A}_n$ can be identified with its first homology groups. Therefore, it is freely generated by one element per hyperplane $z_i = z_j$ (see e.g. [11]), and the symbols ω_{ij}, for $1 \leq i < j \leq n$ denote the elements of the dual basis. Inside the corresponding de Rham first cohomology group it corresponds to (the class of) the 1-form $\mathrm{dlog}(z_i - z_j)$. We order these ω_{ij} lexicographically.

For G a group and M a $\mathbb{Q}G$-module, we denote M^G its G-invariant submodule.

Proposition 1. *For any $\mathbb{Q}\mathrm{S}_n$-module M the following isomorphism holds:*

$$H^q(\Gamma_n; M) = H^q(\mathbb{Z}\mathcal{A}_n; M)^{\mathrm{S}_n}.$$

Proof. The group Γ_n fits in the short exact sequence

$$1 \to \mathbb{Z}\mathcal{A}_n \to \Gamma_n \to \mathrm{S}_n \to 1.$$

Hence the spectral sequence

$$E_2^{p,q} = H^p(\mathrm{S}_n; H^q(\mathbb{Z}\mathcal{A}_n; M))$$

for $p + q = r$ converges to the group $H^r(\Gamma_n; M)$. The group W_n is finite and the module M is divisible: using classical results in group cohomology (see for example [1, Cor. II, 5.4]) this implies that $H^p(\mathrm{S}_n; H^q(\mathbb{Z}\mathcal{A}_n; M)) = 0$ for $p > 0$. Finally, since $H^0(\mathrm{S}_n; H^q(\mathbb{Z}\mathcal{A}_n; M)) = H^q(\mathbb{Z}\mathcal{A}_n; M)^{\mathrm{S}_n}$ we obtain the claim. □

Remark 2. The cohomology ring $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ is the exterior algebra freely generated by

$$H^1(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}) = \mathrm{Hom}(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}) \simeq \mathbb{Q}\mathcal{A}_n \simeq \mathbb{Q}^{\binom{n}{2}}$$

in degree 1. The group S_n acts on the algebra $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ as an automorphism group mapping the elements of degree 1 as follows: a permutation $\sigma \in \mathrm{S}_n$ maps the generator ω_{ij} to

$$\sigma(\omega_{ij}) = \begin{cases} \omega_{\sigma(i),\sigma(j)} & \text{if } \sigma(i) < \sigma(j) \\ \omega_{\sigma(j),\sigma(i)} & \text{otherwise.} \end{cases} \tag{1}$$

Definition 2. We write K_n for the full graph with vertices $\{1, \ldots, n\}$.

Definition 3. For a graph $\Delta \subset K_n$, let $e_1 = (i_1, j_1), \ldots, e_k = (i_k, j_k)$ be the list of edges of Δ ordered lexicographically. We define

$$\mu_\Delta := \omega_{i_1 j_1} \cdots \omega_{i_k j_k} \in H^k(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}). \tag{2}$$

2.2 Combinatorial description of $H^\bullet(\Gamma_n; \mathbb{Q})$

We consider the natural action of S_n on the subgraphs of K_n. Given a graph $\Delta \subset K_n$ we can decompose the group S_n as a union of the cosets $C_1(\Delta), \ldots, C_s(\Delta)$ of $\mathrm{Stab}_{\mathrm{S}_n}(\Delta)$.

Definition 4. We set the following notations:

a) We denote by $\mathcal{G}$ the set of isomorphism classes of finite graphs.
b) For a given graph Δ we write $[\Delta]$ for its isomorphism class in $\mathcal{G}$.
c) We say that Δ is an *invariant graph* if all the automorphisms of the graph induce even permutations on the set of edges.
d) Let $\mathcal{D} \subset \mathcal{G}$ be the set of (isomorphism classes of) invariant graphs.
e) Define $\mathcal{D}_n$ as the subset of $\mathcal{D}$ of graphs with exactly n vertices (allowing isolated vertices).

For every isomorphism class $D \in \mathcal{G}$ of a graph with n vertices and without isolated vertices we choose once and for all a representative $\Delta_D \subset K_n$ of D. This is equivalent to saying that we fix a total ordering of the vertices of D. For an arbitrary isomorphism class D, we consider the isomorphism class of the maximal subgraph D_0 without isolated points and choose the subgraph of $\Pi_{|VD|}$ that coincides with the representative of D_0 on $[1, |VD_0|]$ as a representative Δ_D of D.

We write (i, j) with $i < j$ for the edge with vertices $\{i, j\}$.

Theorem 1. *The set of cohomology classes*

$$\alpha_D := \frac{1}{\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} \sum_{\sigma \in \mathrm{S}_n} \sigma(\mu_{\Delta_D}) \in H^{|ED|}(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}) \tag{3}$$

for $D \in \mathcal{D}_n$ is a basis of $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})^{\mathrm{S}_n}$ over $\mathbb{Q}$.

Proof. Note that if $D \in \mathcal{D}$ then the group $\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)$ acts trivially on μ_{Δ_D}. Hence the average $\pi_{\mathrm{S}_n}(\mu_{\Delta_D}) = \dfrac{1}{|\mathrm{S}_n|} \sum_{\sigma \in \mathrm{S}_n} \sigma(\mu_{\Delta_D}) = \dfrac{|\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)|}{n!} \alpha_D$ is a non-zero S_n-invariant. If $D \notin \mathcal{D}$ then there is a permutation $\sigma \in \mathrm{S}_n$ such that $\sigma(\mu_{\Delta_D}) = -\mu_{\Delta_D}$ and hence the average of μ_{Δ_D} is zero. The Theorem follows since π_{S_n} is a projection from $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ to $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})^{\mathrm{S}_n}$ and the elements α_D for $D \in \mathcal{D}$ are linearly independent. □

We can map a graph of $\mathcal{D}_n$ to graph of $\mathcal{D}_{n+1}$ by adding an isolated vertex. With this identification we can write $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ and $\mathcal{D} = \cup_n \mathcal{D}_n$.

Proposition 2. *The natural morphism $\Gamma_n \hookrightarrow \Gamma_{n+1}$ induces a map $H^\bullet(\Gamma_{n+1}; \mathbb{Q}) \to H^\bullet(\Gamma_n; \mathbb{Q})$. The map send the class $\alpha_D \in H^\bullet(\Gamma_{n+1}; \mathbb{Q})$ to the corresponding class $\alpha_D \in H^\bullet(\Gamma_n; \mathbb{Q})$ when $D \in \mathcal{D}_n$, otherwise to the zero class.*

Since every graph with r edges without isolated vertices has at most $2r$ vertices we get that $H^r(\Gamma_n; \mathbb{Q})$ stabilizes for $r \geq 2n$.

Proof. It is enough to check that if $D \notin \mathcal{D}_n$ then all the summand of α_D contains a factor $\omega_{i,n+1}$ and hence α_D restricts to zero in $H^\bullet(\Gamma_n; \mathbb{Q})$, while if $D \in \mathcal{D}_n$ then $\alpha_D \in H^\bullet(\Gamma_{n+1}; \mathbb{Q})$ restricts to $\alpha_D \in H^\bullet(\Gamma_n; \mathbb{Q})$. □

As a consequence of the Proposition above, the series $\sum \dim H^r(\Gamma_n; \mathbb{Q}) t^r$ has for limit when $n \to \infty$ the Poincaré series of the colimit Γ_∞ of the embeddings $\Gamma_n \hookrightarrow \Gamma_{n+1}$, whose r-th Betti number is then equal to the number of invariant graphs with r edges, without isolated vertices.

We can say something more about the stable Poincaré series. The proof of the following lemma is straightforward.

Lemma 1. *A graph Δ is invariant if and only if the following conditions hold:*

a) the connected components of Δ are invariant graphs;

b) any two distinct connected components of Δ with odd edges are not isomorphic.

Corollary 1. *The cohomology ring $H^\bullet(\Gamma_n; \mathbb{Q})$ is isomorphic, as a group, to the tensor product of a polynomial algebra generated by connected invariant graphs with even edges and an exterior algebra generated by connected invariant graphs with odd edges. The degree of each generator equals the number of edges of the graph.*

The Poincaré series of Γ_∞ decomposes as follows:

$$\sum \dim H^r(\Gamma_\infty;\mathbb{Q})t^r = \left(\prod_{\substack{D\in\mathcal{D} \\ D \text{ connected} \\ |ED| \text{ odd}}} (1+t^{|ED|}) \right) \cdot \left(\prod_{\substack{D\in\mathcal{D} \\ D \text{ connected} \\ |ED| \text{ even}}} (1-t^{|ED|}) \right)^{-1}. \tag{4}$$

This description does not give an account of the actual multiplicative structure of the cohomology ring $H^\bullet(\Gamma_n;\mathbb{Q})$ (which will be investigated in Theorem 3) but only of the multiplicative structure of the graded algebra associated to the filtration defined by the number of connected components of a graph.

2.3 Combinatorial description of the twisted cohomology group $H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon)$

Definition 5. We write $\epsilon(\sigma)$ for the sign of a permutation $\sigma \in \mathrm{S}_n$ and let $\mathbb{Q}_\epsilon$ be the 1-dimensional sign rational representation of S_n extended to Γ_n.

The cohomology group $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$ is naturally identified, as an S_n-representation, with the tensor product of the exterior algebra on the $\mathbb{Q}\mathrm{S}_n$-module $\mathbb{Q}\mathcal{A}_n$ and $\mathbb{Q}_\epsilon$. That is,

$$H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \simeq \mathbb{Q}_\epsilon \otimes H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}). \tag{5}$$

Here we provide a combinatorial description of a basis of the subspace of S_n-invariant elements.

Remark 3. We note that since $\mathbb{Q}_\epsilon$ is a trivial $\mathbb{Z}\mathcal{A}_n$-representation, the cohomology groups $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ and $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$ are isomorphic as graded vector spaces. These groups differ only as S_n-representation and as Γ_n-representations.

Therefore we can extend the notation of Definition 3 to the group $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$: for a graph $\Delta \subset K_n$ we write $\mu_\Delta \in H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$ for the product of the ordered set of generator ω_{ij} corresponding to the edges of Δ.

Remark 4. The group $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ has a natural ring structure given by the cup product, and this structure is compatible with the structure of trivial S_n-module. On the other hand, since $\mathbb{Q}_\epsilon \otimes \mathbb{Q}_\epsilon = \mathbb{Q}$ is the trivial S_n-representation, the ring structure of $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$ given by cup product is not compatible with the structure of S_n-module. Hence it will be natural to consider a different product structure (see Section 2.4) which consists of the maps

$$H^i(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \otimes H^j(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \to H^{i+j}(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}).$$

Definition 6. Let Δ be a graph with n vertices. For any automorphism σ of Δ, we identify σ with the corresponding permutation of the set of vertices of Δ and we write $\sigma_{E\Delta}$ for the induced permutation of the set of edges of Δ.

More generally, let $\Delta, \Delta' \subset K_n$ be two isomorphic graphs. Let $e_1, \ldots, e_k$ be the ordered list of edges of Δ and $e'_1, \ldots, e'_k$ the ordered list of edges of Δ'. Let $\sigma \in \mathrm{S}_n$ be a permutation such that $\sigma(\Delta) = \Delta'$. We write $\sigma_{E\Delta} \in \mathrm{S}_k$ for the permutation induced by σ on the set of the edges for which σ gives $\sigma(e_i) = e'_{\sigma_{E\Delta}(i)}$, for $i \in 1, \ldots, k$.

Definition 7. Given a permutation $\sigma \in \mathrm{S}_n$ that induces an automorphism on the graph Δ we introduce the following notation for the sign of the induced permutation $\sigma_{E\Delta}$ on the set of edges:

$$\epsilon_\Delta(\sigma) := \epsilon(\sigma_{E\Delta}).$$

Definition 8. We define $\mathcal{D}_n^\epsilon$ to be the set of isomorphism classes of graphs with exactly n vertices such that any automorphism σ of the graph satisfies $\epsilon(\sigma)\epsilon_\Delta(\sigma) = 1$. We call those graphs *skew-invariant graphs.*

Remark 5. Note that if $[\Delta] \in \mathcal{D}_n^\epsilon$ then Δ has at most 1 isolated point. In fact, if there are at least 2 isolated points, there is an automorphism of the graph that fixes all the edges, but permutes the isolated points with an odd permutation.

The proof of the following two lemmas is straightforward from the definitions.

Lemma 2. *In $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ the following equality holds:*

$$\sigma(\mu_\Delta) = \epsilon_\Delta(\sigma)\mu_{\sigma(\Delta)}.$$

Lemma 3. *In $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}_\epsilon)$ the following equality holds:*

$$\sigma(\mu_\Delta) = \epsilon(\sigma)\epsilon_\Delta(\sigma)\mu_{\sigma(\Delta)}.$$

Theorem 2. *The set of cohomology classes*

$$\alpha_D^\epsilon := \frac{1}{\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} \sum_{\sigma \in \mathrm{S}_n} \sigma(\mu_{\Delta_D}) \in H^{|ED|}(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}_\epsilon) \tag{6}$$

for $D \in \mathcal{D}_n^\epsilon$ is a basis of $H^\bullet(\Gamma_n; \mathbb{Q}_\epsilon) \subset H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}_\epsilon)$.

The proof of the Theorem above is analogous to the one of Theorem 1 and we omit it.

We note in particular that when $D \notin \mathcal{D}_n^\epsilon$ then the cohomology class $\alpha_D^\epsilon \in H^{|ED|}(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ defined above is trivial.

In the following Proposition, we use the identification $\mathcal{D}_n^\epsilon \subset \mathcal{D}_{n+1}^\epsilon$ given by adding one isolated vertex.

Proposition 3. *The natural morphism $\Gamma_n \hookrightarrow \Gamma_{n+1}$ induces the restriction map $H^\bullet(\Gamma_{n+1}; \mathbb{Q}_\epsilon) \to H^\bullet(\Gamma_n; \mathbb{Q}_\epsilon)$. If $D \in \mathcal{D}_{n+1}^\epsilon$ is a graph with exactly one isolated vertex, the restriction of $\alpha_D^\epsilon \in H^r(\mathbb{Z}\mathcal{A}_{n+1}; \mathbb{Q}_\epsilon)$ is $\alpha_D^\epsilon \in H^r(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}_\epsilon)$. If D is a graph without isolated vertices α_D^ϵ restricts to zero. In particular, the restriction homomorphism $H^\bullet(\Gamma_{n+1}; \mathbb{Q}_\epsilon) \to H^\bullet(\Gamma_n; \mathbb{Q}_\epsilon)$ is injective when restricted to the subspace generated by the classes α_D^ϵ with D a graph with exactly one isolated vertex.*

The proof is analogous to the proof of Proposition 2 and we omit it.

Since a graph with n vertices and r edges contains at least $n - 2r$ isolated vertices we deduce from the Proposition above that $H^r(\Gamma_n; \mathbb{Q}_\epsilon) = 0$ as soon as $n \geq 2r + 2$. Hence we have the following consequence.

Corollary 2. *The stable skew cohomology $H^\bullet(\Gamma_\infty; \mathbb{Q}_\epsilon)$ is zero.*

2.4 Multiplicative structure

We begin describing the multiplicative structure of the cohomology of $\mathbb{Z}\mathcal{A}_n$ with untwisted and twisted rational coefficients and explaining how they interact.

Lemma 4. *The isomorphisms of* S_n*-representations*

$$\mathbb{Q} \otimes \mathbb{Q} \simeq \mathbb{Q}, \tag{7}$$
$$\mathbb{Q}_\epsilon \otimes \mathbb{Q}_\epsilon \simeq \mathbb{Q} \tag{8}$$
$$\mathbb{Q} \otimes \mathbb{Q}_\epsilon \simeq \mathbb{Q}_\epsilon, \tag{9}$$

induce natural bilinear products of S_n*-representations*

$$\begin{aligned}
&\cup : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \otimes H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \to H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}),\\
&\cup^s : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \otimes H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \to H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}),\\
&\cup^t : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \otimes H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \to H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon),
\end{aligned}$$

Proof. Each of the three products is a form of cup product. They are obtained composing the cross products

$$\begin{aligned}
&\times : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \otimes H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \to H^\bullet(\mathbb{Z}\mathcal{A}_n \times \mathbb{Z}\mathcal{A}_n;\mathbb{Q}\otimes\mathbb{Q}),\\
&\times^s : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \otimes H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \to H^\bullet(\mathbb{Z}\mathcal{A}_n \times \mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon\otimes\mathbb{Q}_\epsilon),\\
&\times^t : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \otimes H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \to H^\bullet(\mathbb{Z}\mathcal{A}_n \times \mathbb{Z}\mathcal{A}_n;\mathbb{Q}\otimes\mathbb{Q}_\epsilon),
\end{aligned}$$

with the diagonal map and the homomorphisms induced by the map of coefficients given by equations (7), (8) and (9) (see for example [15, ch. 5.6]). □

Remark 6. We note that the products above can be easily computed in terms of the tensor product and of the usual product of the exterior algebra:

$$\begin{aligned}
&\cup : \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n) \otimes \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n) \to \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n),\\
&\cup^s : (\mathbb{Q}_\epsilon \otimes \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n)) \otimes (\mathbb{Q}_\epsilon \otimes \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n)) \to (\mathbb{Q}_\epsilon \otimes \mathbb{Q}_\epsilon) \otimes \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n) \simeq \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n),\\
&\cup^t : \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n) \otimes (\mathbb{Q}_\epsilon \otimes \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n)) \to \mathbb{Q}_\epsilon \otimes \Lambda^\bullet(\mathbb{Q}\mathcal{A}_n).
\end{aligned}$$

Definition 9. For a given graph Δ, let $S(\Delta, \Delta_1, \Delta_2)$ be the set of pairs (Δ', Δ'') such that

1. Δ', Δ'' are subgraphs of Δ such that $E\Delta' \sqcup E\Delta'' = E\Delta$;
2. Δ' is isomorphic to Δ_1 and Δ'' is isomorphic to Δ_2.

For $(\Delta', \Delta'') \in S(\Delta, \Delta_1, \Delta_2)$ we order the sets vertices $V\Delta'$ and $V\Delta''$ with the ordering induced by the ordering of $V\Delta$. Let $\sigma_1, \sigma_2 \in \mathrm{S}_n$ such that $\sigma_1(\Delta_1) = \Delta'$ and $\sigma_2(\Delta_2) = \Delta''$.

Definition 10. For $[\Delta], [\Delta_1], [\Delta_2] \in \mathcal{D}_n$ and $(\Delta', \Delta'') \in S(\Delta, \Delta_1, \Delta_2)$ we define the coefficient $\epsilon(\Delta, \Delta', \Delta'') \in \{+1, -1\}$ according to the following formula:

$$\mu_\Delta = \epsilon(\Delta, \Delta', \Delta'')\epsilon_{\Delta_1}(\sigma_1)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'} \cup \mu_{\Delta''}.$$

Definition 11. For $[\Delta] \in \mathcal{D}_n, [\Delta_1], [\Delta_2] \in \mathcal{D}_n^\epsilon$ and $(\Delta', \Delta'') \in S(\Delta, \Delta_1, \Delta_2)$ we define the coefficient $\epsilon^s(\Delta, \Delta', \Delta'') \in \{+1, -1\}$ according to the following formula:

$$\mu_\Delta = \epsilon^s(\Delta, \Delta', \Delta'')\epsilon(\sigma_1)\epsilon_{\Delta_1}(\sigma_1)\epsilon(\sigma_2)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'} \cup^s \mu_{\Delta''}.$$

Definition 12. For $[\Delta_1] \in \mathcal{D}_n$, $[\Delta_2], [\Delta] \in \mathcal{D}_n^\epsilon$ and $(\Delta', \Delta'') \in S(\Delta, \Delta_1, \Delta_2)$ we define the coefficient $\epsilon^t(\Delta, \Delta', \Delta'') \in \{+1, -1\}$ according to the following formula:

$$\mu_\Delta = \epsilon^t(\Delta, \Delta', \Delta'')\epsilon_{\Delta_1}(\sigma_1)\epsilon(\sigma_2)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'} \cup^t \mu_{\Delta''}.$$

Lemma 5. *The following relations hold:*

$$\epsilon(\sigma(\Delta), \Delta', \Delta'') = \epsilon_\Delta(\sigma)\epsilon(\Delta, \Delta', \Delta'') \tag{10}$$
$$\epsilon^s(\sigma(\Delta), \Delta', \Delta'') = \epsilon_\Delta(\sigma)\epsilon^s(\Delta, \Delta', \Delta'') \tag{11}$$
$$\epsilon^t(\sigma(\Delta), \Delta', \Delta'') = \epsilon(\sigma)\epsilon_\Delta(\sigma)\epsilon^t(\Delta, \Delta', \Delta'') \tag{12}$$

Proof. We notice that in $H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ we have the equality

$$\mu_\Delta = \epsilon(\Delta, \Delta', \Delta'')\sigma_1(\mu_{\Delta_1}) \cup \sigma_2(\mu_{\Delta_2}) = \epsilon(\Delta, \Delta', \Delta'')\epsilon_{\Delta_1}(\sigma_1)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'} \cup \mu_{\Delta''} \tag{13}$$

and the product $\epsilon(\Delta, \Delta', \Delta'')\epsilon_{\Delta_1}(\sigma_1)\epsilon_{\Delta_2}(\sigma_2)$ is the sign of the shuffle permutation that takes the ordered list of edges of Δ' followed by the ordered list of edges of Δ'' and gives the ordered list of edges of Δ.

Moreover, we have the following similar equalities:

$$\begin{aligned}\mu_\Delta &= \epsilon^s(\Delta, \Delta', \Delta'')\sigma_1(\mu_{\Delta_1}) \cup^s \sigma_2(\mu_{\Delta_2}) = \\ &= \epsilon(\Delta, \Delta', \Delta'')\epsilon(\sigma_1)\epsilon_{\Delta_1}(\sigma_1)\epsilon(\sigma_2)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'} \cup^s \mu_{\Delta''}.\end{aligned}$$

and

$$\begin{aligned}\mu_\Delta &= \epsilon^t(\Delta, \Delta', \Delta'')\sigma_1(\mu_{\Delta_1}) \cup^t \sigma_2(\mu_{\Delta_2}) = \\ &= \epsilon(\Delta, \Delta', \Delta'')\epsilon_{\Delta_1}(\sigma_1)\epsilon(\sigma_2)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'} \cup^t \mu_{\Delta''}.\end{aligned}$$

Equation (10) follows since $\mu_\Delta = \epsilon(\Delta, \Delta', \Delta'')\sigma_1(\mu_{\Delta_1}) \cup \sigma_2(\mu_{\Delta_2}) \in H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$ and hence, applying σ we get

$$\mu_{\sigma(\Delta)} = \epsilon_\Delta(\sigma)\sigma(\mu_\Delta) = \epsilon_\Delta(\sigma)\epsilon(\Delta, \Delta', \Delta'')(\sigma \circ \sigma_1)(\mu_{\Delta_1}) \cup (\sigma \circ \sigma_2)(\mu_{\Delta_2}).$$

For Equation (11) we can apply σ to

$$\mu_\Delta = \epsilon^s(\Delta, \Delta', \Delta'')\sigma_1(\mu_{\Delta_1}) \cup^s \sigma_2(\mu_{\Delta_2}) \in H^\bullet(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})$$

and the equality follows from the same argument since we have

$$\mu_{\sigma(\Delta)} = \epsilon_\Delta(\sigma)\sigma(\mu_\Delta) = \epsilon_\Delta(\sigma)\epsilon^s(\Delta, \Delta', \Delta'')(\sigma \circ \sigma_1)(\mu_{\Delta_1}) \cup^s (\sigma \circ \sigma_2)(\mu_{\Delta_2}).$$

For Equation (12) we apply σ to $\mu_\Delta = \epsilon^t(\Delta,\Delta',\Delta'')\epsilon_{\Delta_1}(\sigma_1)\epsilon(\sigma_2)\epsilon_{\Delta_2}(\sigma_2)\mu_{\Delta'}\cup^t\mu_{\Delta''}$, hence we get

$$\mu_{\sigma(\Delta)} = \epsilon(\sigma)\epsilon_\Delta(\sigma)\sigma(\mu_\Delta) = \epsilon(\sigma)\epsilon_\Delta(\sigma)\epsilon^t(\Delta,\Delta',\Delta'')(\sigma\circ\sigma_1)(\mu_{\Delta_1})\cup^t(\sigma\circ\sigma_2)(\mu_{\Delta_2}). \quad \square$$

Definition 13. We define the coefficients $a(\Delta,\Delta_1,\Delta_2)$, $a^s(\Delta,\Delta_1,\Delta_2)$ and $a^t(\Delta,\Delta_1,\Delta_2)$ as follows:

$$a(\Delta,\Delta_1,\Delta_2) := \sum_{(\Delta',\Delta'')\in S(\Delta,\Delta_1,\Delta_2)} \epsilon(\Delta,\Delta',\Delta''); \tag{14}$$

$$a^s(\Delta,\Delta_1,\Delta_2) := \sum_{(\Delta',\Delta'')\in S(\Delta,\Delta_1,\Delta_2)} \epsilon^s(\Delta,\Delta',\Delta''); \tag{15}$$

$$a^t(\Delta,\Delta_1,\Delta_2) := \sum_{(\Delta',\Delta'')\in S(\Delta,\Delta_1,\Delta_2)} \epsilon^t(\Delta,\Delta',\Delta''); \tag{16}$$

Lemma 6.

$$a(\sigma(\Delta),\Delta_1,\Delta_2) = \epsilon_\Delta(\sigma)a(\Delta,\Delta_1,\Delta_2); \tag{17}$$

$$a^s(\sigma(\Delta),\Delta_1,\Delta_2) = \epsilon(\sigma)\epsilon_\Delta(\sigma)a^s(\Delta,\Delta_1,\Delta_2); \tag{18}$$

$$a^t(\sigma(\Delta),\Delta_1,\Delta_2) = \epsilon(\sigma)\epsilon_\Delta(\sigma)a^t(\Delta,\Delta_1,\Delta_2). \tag{19}$$

Proof. This follows immediately from Lemma 5. □

Definition 14. We define the direct sum $\mathrm{R} := \mathbb{Q}\oplus\mathbb{Q}_\epsilon$ of the trivial S_n-representation and the sign representation. We endow the S_n-module R with the product structure given by

$$(a,a')\cdot(b,b') \mapsto (ab + a'b', ab' + a'b).$$

The S_n-module R is a $\mathbb{Z}_2$-graded ring with a S_n-covariant product. We can consider the $\mathbb{Z}_2$-graded S_n-representation $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathrm{R}) = H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})\oplus(H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon))$. This is also naturally an algebra, with an S_n-covariant product.

Taking the restriction to the S_n-invariant submodule we get a product that restricts as follows:

$$(H^i(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}))^{\mathrm{S}_n}\otimes(H^j(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}))^{\mathrm{S}_n} \mapsto (H^{i+j}(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}))^{\mathrm{S}_n}; \tag{20}$$

$$(H^i(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon))^{\mathrm{S}_n}\otimes(H^j(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon))^{\mathrm{S}_n} \mapsto (H^{i+j}(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})^{\mathrm{S}_n}; \tag{21}$$

$$(H^i(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon))^{\mathrm{S}_n}\otimes(H^j(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}))^{\mathrm{S}_n} \mapsto (H^{i+j}(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)^{\mathrm{S}_n}; \tag{22}$$

$$(H^i(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}))^{\mathrm{S}_n}\otimes(H^j(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon))^{\mathrm{S}_n} \mapsto (H^{i+j}(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)^{\mathrm{S}_n}. \tag{23}$$

The cohomology groups described previously fit in the uniform setting of a cohomology ring:

$$H^\bullet(\Gamma_n;\mathrm{R}) = H^\bullet(\Gamma_n;\mathbb{Q})\oplus H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon).$$

Definition 15. We consider the projection $\pi_{\mathrm{S}_n} : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \to H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})^{\mathrm{S}_n}$ defined by

$$\pi_{\mathrm{S}_n}(\mu_\Delta) := \frac{1}{|\mathrm{S}_n|}\sum_{\sigma\in\mathrm{S}_n}\sigma(\mu_\Delta) = \frac{1}{|\mathrm{S}_n|}\sum_{\sigma\in\mathrm{S}_n}\epsilon_\Delta(\sigma)\mu_{\sigma(\Delta)} = \frac{|\operatorname{Stab}_{\mathrm{S_n}}(\Delta)|}{n!}\alpha_\Delta$$

and the projection $\pi_{\mathrm{S}_n} : H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \to (H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon))^{\mathrm{S}_n}$ defined by

$$\pi_{\mathrm{S}_n}(\mu_\Delta) := \frac{1}{|\mathrm{S}_n|}\sum_{\sigma\in\mathrm{S}_n}\sigma(\mu_\Delta) = \frac{1}{|\mathrm{S}_n|}\sum_{\sigma\in\mathrm{S}_n}\epsilon(\sigma)\epsilon_\Delta(\sigma)\mu_{\sigma(\Delta)} = \frac{|\operatorname{Stab}_{\mathrm{S_n}}(\Delta)|}{n!}\alpha^\epsilon_\Delta.$$

Remark 7. For $\mu_\Delta \in H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ the projection $\pi_{\mathrm{S}_n}(\mu_\Delta)$ is zero if and only if $\Delta \notin \mathcal{D}_n$ (see the proof of Theorem 1), while for $\mu_\Delta \in H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$ the projection $\pi_{\mathrm{S}_n}(\mu_\Delta)$ is zero if and only if $\operatorname{Stab}_{\mathrm{S_n}}(\Delta)$ contains a permutation σ such that $\epsilon(\sigma)\epsilon_\Delta(\sigma) = -1$, which is equivalent to saying that $\Delta \notin \mathcal{D}^\epsilon_n$.

Remark 8. Since the multiplicative structure of $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ is S_n-covariant, given two classes $\omega, \lambda \in H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ we have that $\pi_{\mathrm{S}_n}(\omega \cup \pi_{\mathrm{S}_n}(\lambda)) = \pi_{\mathrm{S}_n}(\omega) \cup \pi_{\mathrm{S}_n}(\lambda)$. In particular, the product $\pi_{\mathrm{S}_n}(\omega) \cup \pi_{\mathrm{S}_n}(\lambda)$ is S_n-invariant.

We can rewrite the definition of α_Δ given in Theorem 1, Equation (3) as follows:

$$\alpha_\Delta = \sum_{[\sigma]\in\mathrm{S}_n/\operatorname{Stab}_{\mathrm{S_n}}(\Delta)}\sigma(\mu_\Delta) = \frac{n!}{|\operatorname{Stab}_{\mathrm{S_n}}(\Delta)|}\pi_{\mathrm{S}_n}(\mu_\Delta) \tag{24}$$

where for any coset $[\sigma] \in \mathrm{S}_n/\operatorname{Stab}_{\mathrm{S_n}}(\Delta)$ the permutation σ is a representative of $[\sigma]$ and since $\Delta \in \mathcal{D}_n$ the element $\sigma(\mu_\Delta) \in H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ does not depend on the choice of the representative.

In a similar way we can rewrite the definition of α^ϵ_Δ given in Theorem 2, Equation (6):

$$\alpha^\epsilon_\Delta = \sum_{[\sigma]\in\mathrm{S}_n/\operatorname{Stab}_{\mathrm{S_n}}(\Delta)}\sigma(\mu_\Delta) = \frac{n!}{|\operatorname{Stab}_{\mathrm{S_n}}(\Delta)|}\pi_{\mathrm{S}_n}(\mu_\Delta) \tag{25}$$

where we consider $\mu_\Delta \in H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)$.

Theorem 3. *The multiplicative structure of $H^\bullet(\mathbb{Z}\mathcal{A}_n;\mathrm{R})$ can be computed as follows.*

a) For $D_1, D_2 \in \mathcal{D}_n$, the usual cup product in $H^\bullet(\Gamma_n;\mathbb{Q})$ is given by the formula:

$$\alpha_{D_1} \cup \alpha_{D_2} = \sum_{D\in\mathcal{D}_n} a(\Delta_D, \Delta_{D_1}, \Delta_{D_2})\alpha_D \in H^\bullet(\Gamma_n;\mathbb{Q}); \tag{26}$$

b) for $D_1 \in \mathcal{D}_n, D_2 \in \mathcal{D}^\epsilon_n$, the natural bilinear form $H^\bullet(\Gamma_n;\mathbb{Q}) \otimes H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon) \to H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon)$ is given by the formula:

$$\alpha_{D_1} \cup^t \alpha^\epsilon_{D_2} = \sum_{D\in\widetilde{\mathcal{D}}_n} a^t(\Delta_D, \Delta_{D_1}, \Delta_{D_2})\alpha^\epsilon_D \in H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon); \tag{27}$$

c) for $D_1, D_2 \in \mathcal{D}_n^\epsilon$, the natural bilinear form $H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon)\otimes H^\bullet(\Gamma_n;\mathbb{Q}_\epsilon) \to H^\bullet(\Gamma_n;\mathbb{Q})$ is given by the formula:

$$\alpha^\epsilon_{D_1} \cup^s \alpha^\epsilon_{D_2} = \sum_{D\in\mathcal{D}_n^\epsilon} a^s(\Delta_D, \Delta_{D_1}, \Delta_{D_2})\alpha_D \in H^\bullet(\Gamma_n;\mathbb{Q}). \tag{28}$$

Proof. The proof is analogous for the three cases. For conciseness we will write Δ_1 for Δ_{D_1} and Δ_2 for Δ_{D_2}.

a) For $D_1, D_2 \in \mathcal{D}_n$ we have:

$$\begin{aligned}
\alpha_{D_1} \cup \alpha_{D_2} &= \left(\sum_{[\sigma']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_1)} \sigma'(\mu_{\Delta_1})\right) \cup \left(\sum_{[\sigma'']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_2)} \sigma''(\mu_{\Delta_2})\right) = \\
&= \left(\sum_{[\sigma']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_1)} \epsilon_{\Delta_1}(\sigma')\mu_{\sigma'(\Delta_1)}\right) \cup \left(\sum_{[\sigma'']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_2)} \epsilon_{\Delta_2}(\sigma'')\mu_{\sigma''(\Delta_2)}\right) = \\
&= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} \sum_{\substack{[\sigma']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_1)\\ [\sigma'']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_2)\\ E\sigma'(\Delta_1)\sqcup E\sigma''(\Delta_2)=E\sigma(\Delta_D)}} \epsilon_{\Delta_1}(\sigma')\mu_{\sigma'(\Delta_1)}\epsilon_{\Delta_2}(\sigma'')\mu_{\sigma''(\Delta_2)} = \\
&= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} \sum_{\substack{[\sigma']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_1)\\ [\sigma'']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_2)\\ E\sigma'(\Delta_1)\sqcup E\sigma''(\Delta_2)=E\sigma(\Delta_D)}} \epsilon(\sigma(\Delta_D), \sigma'(\Delta_1), \sigma''(\Delta_2))\mu_{\sigma(\Delta_D)} = \\
&= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} a(\sigma(\Delta_D), \Delta_1, \Delta_2)\mu_{\sigma(\Delta_D)} = \\
&= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} a(\Delta_D, \Delta_1, \Delta_2)\epsilon_{\Delta_D}(\sigma)\mu_{\sigma(\Delta_D)} = \\
&= \sum_{D\in\mathcal{D}_n} \sum_{[\sigma]\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_D)} a(\Delta_D, \Delta_1, \Delta_2)\sigma(\mu_{\Delta_D}) = \\
&= \sum_{D\in\mathcal{D}_n} a(\Delta_D, \Delta_1, \Delta_2)\alpha_D.
\end{aligned}$$

b) For $D_1 \in \mathcal{D}_n, D_2 \in \mathcal{D}_n^\epsilon$ we have:

$$\begin{aligned}
\alpha_{D_1} \cup^t \alpha^\epsilon_{D_2} &= \left(\sum_{[\sigma']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_1)} \sigma'(\mu_{\Delta_1})\right) \cup^t \left(\sum_{[\sigma'']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_2)} \sigma''(\mu_{\Delta_2})\right) = \\
&= \left(\sum_{[\sigma']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_1)} \epsilon_{\Delta_1}(\sigma')\mu_{\sigma'(\Delta_1)}\right) \cup^t \left(\sum_{[\sigma'']\in \mathrm{S}_n/\,\mathrm{Stab}_{\mathrm{S_n}}(\Delta_2)} \epsilon(\sigma'')\epsilon_{\Delta_2}(\sigma'')\mu_{\sigma''(\Delta_2)}\right) =
\end{aligned}$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} \sum_{\substack{[\sigma']\in S_n/\operatorname{Stab}_{S_n}(\Delta_1)\\ [\sigma'']\in S_n/\operatorname{Stab}_{S_n}(\Delta_2)\\ E\sigma'(\Delta_1)\sqcup E\sigma''(\Delta_2)=E\sigma(\Delta_D)}} \epsilon_{\Delta_1}(\sigma')\epsilon(\sigma'')\epsilon_{\Delta_2}(\sigma'')\mu_{\sigma'(\Delta_1)}\cup^t\mu_{\sigma''(\Delta_2)} =$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} \sum_{\substack{[\sigma']\in S_n/\operatorname{Stab}_{S_n}(\Delta_1)\\ [\sigma'']\in S_n/\operatorname{Stab}_{S_n}(\Delta_2)\\ E\sigma'(\Delta_1)\sqcup E\sigma''(\Delta_2)=E\sigma(\Delta_D)}} \epsilon^t(\sigma(\Delta_D),\sigma'(\Delta_1),\sigma''(\Delta_2))\mu_{\sigma(\Delta_D)} =$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} a^t(\sigma(\Delta_D),\Delta_1,\Delta_2)\mu_{\sigma(\Delta_D)} =$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} a^t(\Delta_D,\Delta_1,\Delta_2)\epsilon(\sigma)\epsilon_{\Delta_D}(\sigma)\mu_{\sigma(\Delta_D)} =$$

$$= \sum_{D\in\widetilde{\mathcal{D}}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} a^t(\Delta_D,\Delta_1,\Delta_2)\sigma(\mu_{\Delta_D}) =$$

$$= \sum_{D\in\widetilde{\mathcal{D}}_n} a^t(\Delta_D,\Delta_1,\Delta_2)\alpha^\epsilon_D.$$

c) For $D_1, D_2 \in \mathcal{D}^\epsilon_n$ we have:

$$\alpha^\epsilon_{D_1}\cup^s\alpha^\epsilon_{D_2} = \left(\sum_{[\sigma']\in S_n/\operatorname{Stab}_{S_n}(\Delta_1)} \sigma'(\mu_{\Delta_1})\right)\cup^s\left(\sum_{[\sigma'']\in S_n/\operatorname{Stab}_{S_n}(\Delta_2)} \sigma''(\mu_{\Delta_2})\right) =$$

$$= \left(\sum_{[\sigma']\in S_n/\operatorname{Stab}_{S_n}(\Delta_1)} \epsilon(\sigma')\epsilon_{\Delta_1}(\sigma')\mu_{\sigma'(\Delta_1)}\right)\cup^s\left(\sum_{[\sigma'']\in S_n/\operatorname{Stab}_{S_n}(\Delta_2)} \epsilon(\sigma'')\epsilon_{\Delta_2}(\sigma'')\mu_{\sigma''(\Delta_2)}\right) =$$

$$= \sum_{\substack{D\in\mathcal{G}_n\\ [\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)}} \sum_{\substack{[\sigma']\in S_n/\operatorname{Stab}_{S_n}(\Delta_1)\\ [\sigma'']\in S_n/\operatorname{Stab}_{S_n}(\Delta_2)\\ E\sigma'(\Delta_1)\sqcup E\sigma''(\Delta_2)=E\sigma(\Delta_D)}} \epsilon(\sigma')\epsilon_{\Delta_1}(\sigma')\epsilon(\sigma'')\epsilon_{\Delta_2}(\sigma'')\mu_{\sigma'(\Delta_1)}\cup^s\mu_{\sigma''(\Delta_2)} =$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} \sum_{\substack{[\sigma']\in S_n/\operatorname{Stab}_{S_n}(\Delta_1)\\ [\sigma'']\in S_n/\operatorname{Stab}_{S_n}(\Delta_2)\\ E\sigma'(\Delta_1)\sqcup E\sigma''(\Delta_2)=E\sigma(\Delta_D)}} \epsilon^s(\sigma(\Delta_D),\sigma'(\Delta_1),\sigma''(\Delta_2))\mu_{\sigma(\Delta_D)} =$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} a^s(\sigma(\Delta_D),\Delta_1,\Delta_2)\mu_{\sigma(\Delta_D)} =$$

$$= \sum_{D\in\mathcal{G}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} a^s(\Delta_D,\Delta_1,\Delta_2)\epsilon(\sigma)\epsilon_{\Delta_D}(\sigma)\mu_{\sigma(\Delta_D)} =$$

$$= \sum_{D\in\widetilde{\mathcal{D}}_n} \sum_{[\sigma]\in S_n/\operatorname{Stab}_{S_n}(\Delta_D)} a^s(\Delta_D,\Delta_1,\Delta_2)\sigma(\mu_{\Delta_D}) =$$

$$= \sum_{D\in\widetilde{\mathcal{D}}_n} a^s(\Delta_D,\Delta_1,\Delta_2)\alpha_D.$$

□

2.5 Poincaré duality

Recall that we write K_n for the complete graph on n vertices. It has $N := n(n-1)/2$ edges. It is easily checked that any transposition of the vertices acts with parity $(-1)^{n-2}$ on the set of edges, therefore every permutation of the vertices acts with the same parity on the edges when n is odd, and acts as an even permutation on the edges when n is even (in other terms, $\epsilon_{K_n}(\sigma) = \epsilon(\sigma)^n$ for all $\sigma \in \mathrm{S}_n$). In particular, we have the following result.

Proposition 4. *Let K_n be the full graph on n vertices.*

1. *K_n is invariant if and only if n is even;*
2. *K_n is skew-invariant if and only if n is odd.*

Since K_n is the unique graph on n vertices with $n(n-1)/2$ edges, we get the following application.

Corollary 3. *Let $N = n(n-1)/2$.*

1. *$H^N(\Gamma_n;\mathbb{Q}) = \mathbb{Q}$ if and only if n is even, and $H^N(\Gamma_n;\mathbb{Q}) = 0$ otherwise;*
2. *$H^N(\Gamma_n;\mathbb{Q}_\epsilon) = \mathbb{Q}$ if and only if n is odd, and $H^N(\Gamma_n;\mathbb{Q}_\epsilon) = 0$ otherwise.*

Lemma 7. *Let G be a finite group, $\mathbf{k}$ a field of characteristic 0 with trivial G-action, U, V two $\mathbf{k}G$-modules. If $\varphi : U \otimes_{\mathbf{k}} V \to \mathbf{k}$ is a non-degenerate G-equivariant pairing, then it induces a non-degenerate pairing $U^G \otimes_{\mathbf{k}} V^G \to \mathbf{k}$.*

Proof. From the inclusion $U^G \otimes V^G \subset U \otimes V$ we deduce from φ a linear map $\varphi_G : U^G \otimes V^G \to \mathbf{k}$. Let $x \in U^G$ with $x \neq 0$. By assumption, there exists $y \in V$ with $\varphi(x \otimes y) \neq 0$. Then, for every $g \in G$ we have $\varphi(x \otimes g.y) = \varphi(g.(x \otimes y)) = g.\varphi(x \otimes y) = \varphi(x \otimes y)$ hence $\varphi(x \otimes \hat{y}) = \varphi(x \otimes y) \neq 0$ if $\hat{y} = (1/|G|)\sum_{g\in G} g.y$. Since $\hat{y} \in V^G \setminus \{0\}$ and $\varphi_G(x \otimes \hat{y}) \neq 0$, this proves the claim. □

Proposition 5. *If n is even there are nondegenerate pairings*

$$H^r(\Gamma_n;\mathbb{Q}) \otimes H^{N-r}(\Gamma_n;\mathbb{Q}) \to H^N(\Gamma_n;\mathbb{Q}) \simeq \mathbb{Q}; \tag{29}$$
$$H^r(\Gamma_n;\mathbb{Q}_\epsilon) \otimes H^{N-r}(\Gamma_n;\mathbb{Q}_\epsilon) \to H^N(\Gamma_n;\mathbb{Q}) \simeq \mathbb{Q}; \tag{30}$$

while if n is odd there is a nondegenerate pairing

$$H^r(\Gamma_n;\mathbb{Q}) \otimes H^{N-r}(\Gamma_n;\mathbb{Q}_\epsilon) \to H^N(\Gamma_n;\mathbb{Q}_\epsilon) \simeq \mathbb{Q}. \tag{31}$$

Proof. The result is a direct application of Lemma 7 to the case considered in Corollary 3. When the groups $H^N(\Gamma_n;\mathbb{Q})$ and $H^N(\Gamma_n;\mathbb{Q}_\epsilon)$ are non-trivial, the non-degenerate duality pairings

$$\Lambda^r(\mathbb{Q}\mathcal{A}_n) \otimes \Lambda^{N-r}(\mathbb{Q}\mathcal{A}_n) \to \Lambda^N(\mathbb{Q}\mathcal{A}_n) \simeq \mathbb{Q} \tag{32}$$
$$(\mathbb{Q}_\epsilon \otimes \Lambda^r(\mathbb{Q}\mathcal{A}_n)) \otimes (\mathbb{Q}_\epsilon \otimes \Lambda^{N-r}(\mathbb{Q}\mathcal{A}_n)) \to \Lambda^N(\mathbb{Q}\mathcal{A}_n) \simeq \mathbb{Q} \tag{33}$$
$$\Lambda^r(\mathbb{Q}\mathcal{A}_n) \otimes (\mathbb{Q}_\epsilon \otimes \Lambda^{N-r}(\mathbb{Q}\mathcal{A}_n)) \to \mathbb{Q}_\epsilon \otimes \Lambda^N(\mathbb{Q}\mathcal{A}_n) \simeq \mathbb{Q} \tag{34}$$

induces non-degenerate Poincaré duality pairings on the cohomology of Γ_n. $\square$

For Δ a graph on n vertices, we denote by Δ^c its complement inside K_n. This provides a combinatorial description of the corresponding Poincaré duality pairings, which can be checked directly.

Theorem 4. *Let $N = n(n-1)/2$. Then the map $\Delta \mapsto \Delta^c$ induces the following isomorphisms:*

1. *If n is even, $H^r(\Gamma_n; \mathbb{Q}) \to H^{N-r}(\Gamma_n; \mathbb{Q})$;*
2. *If n is even $H^r(\Gamma_n; \mathbb{Q}_\epsilon) \to H^{N-r}(\Gamma_n; \mathbb{Q}_\epsilon)$;*
3. *If n is odd $H^r(\Gamma_n; \mathbb{Q}_\epsilon) \to H^{N-r}(\Gamma_n; \mathbb{Q})$.*

Proof. The results follow applying Lemma 7 to the pairings (32), (33), (34) in the cases where Corollary 3 gives non-trivial groups. The pairings turn out to be a special case of Theorem 3. In particular, we have the following cases:

1. Let Δ be a subgraph of K_n with n vertices, and σ an automorphism of Δ. Then σ is also an automorphism of Δ^c, and we have $\epsilon_\Delta(\sigma)\epsilon_{\Delta^c}(\sigma) = \epsilon_{K_n}(\sigma) = \epsilon(\sigma)^{n-2}$. If n is even, we get $\epsilon_\Delta(\sigma) = \epsilon_{\Delta^c}(\sigma)$ and this yields that $\Delta \mapsto \Delta^c$ maps invariant graphs to invariant graphs, whence (1).
2. If n is even, we get $\epsilon_\Delta(\sigma)\epsilon(\sigma)\epsilon_{\Delta^c}(\sigma)\epsilon(\sigma) = \epsilon_{K_n}(\sigma)\epsilon(\sigma)^2 = 1$. This implies $\epsilon_\Delta(\sigma)\epsilon(\sigma) = \epsilon_{\Delta^c}(\sigma)\epsilon(\sigma)$ and hence $\Delta \mapsto \Delta^c$ maps skew-invariant graphs to skew-invariant graphs, whence (2).
3. If n is odd, $\epsilon(\sigma)\epsilon_\Delta(\sigma)\epsilon_{\Delta^c}(\sigma) = \epsilon(\sigma)\epsilon_{K_n}(\sigma) = \epsilon(\sigma)^{1+n-2} = 1$. This imples that $\epsilon(\sigma)\epsilon_\Delta(\sigma) = \epsilon_{\Delta^c}$ and hence $\Delta \mapsto \Delta^c$ maps skew-invariant graphs to invariant graphs, whence (3). $\square$

Remark 9. We notice that for a skew-invariant graph $\Delta \subset K_n$ with n odd and we can consider the graph $\Sigma\Delta$ given by embedding Δ as a subgraph of K_{n+2} and then adding all possible edges (i, j) with $i \in \{1, \ldots, n\}$ and $j \in \{n+1, n+2\}$. The graph $\Sigma\Delta$ is skew-invariant.

2.6 Representation-theoretic description

In this section, we identify irreducible representations of S_n with partitions of n, with the convention that $[n]$ and $[1^n]$ are (identified with) the $\mathbb{Q}\mathrm{S}_n$-modules previously denoted $\mathbb{Q}$ and $\mathbb{Q}_\epsilon$, respectively.

Proposition 6. *Let us assume $n \geq 4$ and let $b_{n,r} = \dim H^r(\mathbb{Z}\mathcal{A}_n; \mathbb{Q})^{\mathrm{S}_n}$. We have:*

$$b_{n,r} = \sum_{p+q=r} \left\langle [n-p, 1^p], \bigwedge^q [n-2, 2] \right\rangle + \sum_{p+q=r-1} \left\langle [n-p, 1^p], \bigwedge^q [n-2, 2] \right\rangle \quad (35)$$

where $\langle \lambda, \mu \rangle$ denotes the scalar product of characters for S_n.

Proof. As a representation of S_n, $\mathbb{Q}\mathcal{A}_n$ can be decomposed into irreducibles as $[n-2, 2] + [n-1, 1] + [n]$ (see for example [6]). We have

$$H^r(\mathbb{Z}\mathcal{A}_n; \mathbb{Q}) \simeq \bigwedge^r \mathbb{Q}\mathcal{A}_n \simeq \bigwedge^r ([n-2, 2] + [n-1, 1] + [n])$$

$$\simeq \bigoplus_{a+b+c=r} \bigwedge^{a}[n-2,2] \otimes \bigwedge^{b}[n-1,1] \otimes \bigwedge^{c}[n]$$

Since $\bigwedge^{\bullet}[n] = \bigwedge^{\bullet}\mathbb{Q} = \bigwedge^{0}\mathbb{Q} + \bigwedge^{1}\mathbb{Q}$ and $\bigwedge^{p}[n-1,1] = [n-p,1^p]$ this yields

$$H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}) \simeq \left(\bigoplus_{p+q=r}[n-p,1^p] \otimes \bigwedge^{q}[n-2,2]\right) \oplus \left(\bigoplus_{p+q=r-1}[n-p,1^p] \otimes \bigwedge^{q}[n-2,2]\right)$$

with the convention $[n-p,1^p]=0$ if $p \geq n$ or $p<0$. The result follows from the last equation. □

The Proposition above provides a reasonably fast algorithm to compute the first Betti numbers (see Table 1) and the first terms of the Poincaré series for Γ_∞. Computer computations of the formula in Equation (35) give the first terms of the stable Poincaré series.

Theorem 5. *The Poincaré series for Γ_∞ starts with*

$$\begin{aligned} &1+t+t^4+7t^5+17t^6+30t^7+88t^8+335t^9+1143t^{10}+3866t^{11}+ \\ &+14289t^{12}+56557t^{13}+231012t^{14}+971537t^{15}+4238570t^{16}+\dots \end{aligned} \tag{36}$$

One also observes the existence of a factor $(1+t)^m$ of the Poincaré polynomial of $H^\bullet(\Gamma_n;\mathbb{Q})$ for small n, in Table 2. For $n=9,10$, the value of m is $4,5$.

From the isomorphism $H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon) \simeq \mathbb{Q}_\epsilon \otimes H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q})$ we get, using the same argument, the analogous of Proposition 6 for the case of skew coefficients.

Proposition 7. *Let us assume $n \geq 4$ and let $b^\epsilon_{n,r} = \dim H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)^{\mathrm{S}_n}$. We have:*

$$b^\epsilon_{n,r} = \sum_{p+q=r}\left\langle \mathbb{Q}_\epsilon \otimes [n-p,1^p], \bigwedge^{q}[n-2,2]\right\rangle + \sum_{p+q=r-1}\left\langle \mathbb{Q}_\epsilon \otimes [n-p,1^p], \bigwedge^{q}[n-2,2]\right\rangle. \tag{37}$$

As before this enables us to compute the first skew Betti numbers (see Table 3).

Remark 10. The vanishing of $H^r(\mathbb{Z}\mathcal{A}_n;\mathbb{Q}_\epsilon)^{\mathrm{S}_n}$ for n large can also be deduced from Equation (37), as $[n-2,2]$ is a constituent of $U^{\otimes 2}$ where U is the permutation representation $\mathrm{S}_n < \mathrm{GL}_n(\mathbb{Q})$, hence any constituent of $\bigwedge^q[n-2,2] \subset [n-2,2]^{\otimes q}$ is a constituent of $U^{\otimes 2q} = (\mathrm{Ind}_{S_{n-1}}^{\mathrm{S}_n}\mathbb{1})^{\otimes 2q}$, and the Young diagram of such a constituent has, by Young's rule, at most $2q$ rows. Since $\mathbb{Q}_\epsilon \otimes [n-p,1^p]$ has $n-p$ rows, this cohomology group can be nonzero only if $n \leq p+2q$ whenever $p+q \in \{r-1,r\}$.

2.7 Low-dimensional cohomology

With trivial coefficients, natural bases (up to a sign for each basis vector) are thus given, for H^1 by the unique simple graph with 1 edge, for H^4 by the linear graph on 5 vertices and for H^5 by the 7 graphs in Figure 1.

An example of the cup product formula is given in Figure 2. We recall that the coefficients in the formula are defined up to a sign, that can be determined according

Table 1 First Betti numbers for Γ_n

$H^r(\Gamma_n;\mathbb{Q})$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Γ_4	1	1	0	0	0	1	1														
Γ_5	1	1	0	0	1	4	4	1	0	0	0										
Γ_6	1	1	0	0	1	6	10	9	9	10	6	1	0	0	1	1					
Γ_7	1	1	0	0	1	7	15	20	37	72	88	71	48	35	32	22	6	0	0	0	...
Γ_8	1	1	0	0	1	7	17	28	67	182	364	566	767	936	1006	936	767	566	364	182	...
Γ_9[1]	1	1	0	0	1	7	17	30	84	278	738	1673	3499	6619	10855	15464	19862	23572	25458	24285	...

Table 2 Factorized Poincaré polynomials for Γ_n

Γ_3	$t+1$
Γ_4	$(t+1)^2(t^4-t^3+t^2-t+1)$
Γ_5	$(t+1)^2(t^5+2t^4-t^3+t^2-t+1)$
Γ_6	$(t+1)^3(t^{12}-2t^{11}+3t^{10}-4t^9+6t^8-3t^7+5t^6-3t^5+6t^4-4t^3+3t^2-2t+1)$
Γ_7	$(t+1)^3(6t^{13}+4t^{12}+2t^{11}+11t^{10}+5t^9+21t^8-t^7+7t^6-2t^5+6t^4-4t^3+3t^2-2t+1)$
Γ_8	$(t+1)^4(t^{24}-3t^{23}+6t^{22}-10t^{21}+16t^{20}-18t^{19}+27t^{18}-26t^{17}+65t^{16}-12t^{15}+99t^{14}+8t^{13}+124t^{12}+\cdots+16t^4-10t^3+6t^2-3t+1)$

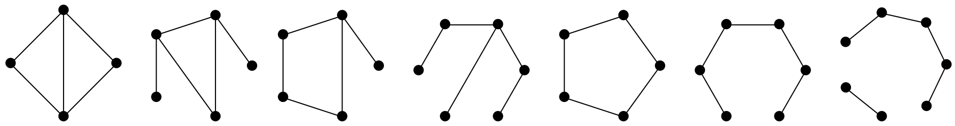
Table 3 First skew Betti numbers for Γ_n

$H^r(\Gamma_n;\mathbb{Q}_\epsilon)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Γ_4	0	0	1	2	1	0	0														
Γ_5	0	0	0	1	4	4	1	0	0	1	1										
Γ_6	0	0	0	0	3	9	10	6	6	10	9	3	0	0	0	0					
Γ_7	0	0	0	0	0	6	22	32	35	48	71	88	72	37	20	15	7	1	0	0	...
Γ_8	0	0	0	0	0	1	16	53	97	160	301	551	815	955	982	955	815	551	301	160	...
Γ_9[2]	0	0	0	0	0	0	3	34	135	327	716	1637	3525	6559	10549	15282	20325	24285	25458	23572	...

[1] The full list of Betti numbers is: 1, 1, 0, 0, 1, 7, 17, 30, 84, 278, 738, 1673, 3499, 6619, 10855, 15464, 19862, 23572, 25458, 24285, 20325, 15282, 10549, 6559, 3525, 1637, 716, 327, 135, 34, 3, 0, 0, 0, 0, 0

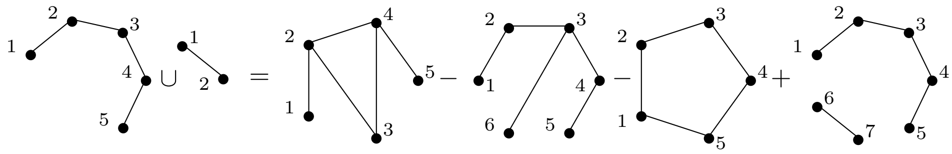
[2] The full list of Betti numbers is: 0, 0, 0, 0, 0, 0, 3, 34, 135, 327, 716, 1637, 3525, 6559, 10549, 15282, 20325, 24285, 25458, 23572, 19862, 15464, 10855, 6619, 3499, 1673, 738, 278, 84, 30, 17, 7, 1, 0, 0, 1, 0.

Fig. 1 Bases for $H^5(\Gamma_n;\mathbb{Q})$

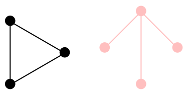


to the choice of the representative of each isomorphism class of graphs as a subgraph of the full graph as in Section 2.2. In the formula in Figure 2 we explicitly choose the representatives providing an ordering of the vertices of the graphs using the labeling and we compute the sign of the coefficients accordingly.

Fig. 2 Example of a cup product formula

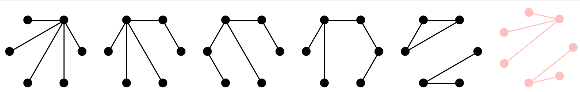


With skew coefficients, a basis for $H^3(\Gamma_4;\mathbb{Q}_\epsilon)$ is given by the following two graphs, the pink one on 4 vertices surviving inside $H^3(\Gamma_5;\mathbb{Q}_\epsilon)$ and providing a basis there.



Bases for $H^5(\Gamma_n;\mathbb{Q}_\epsilon)$ are provided in Table 4, with similar color conventions.

Table 4 Bases for $H^5(\Gamma_n;\mathbb{Q}_\epsilon)$

Γ_5	
Γ_6	
Γ_7	

3 Generalized braid groups

3.1 General setting

Let $W < \mathrm{GL}_\ell(\mathbb{C})$ be a complex reflection group, $\mathcal{A}$ the corresponding hyperplane arrangement. The pure braid group is $P = \pi_1(X)$ with $X = \mathbb{C}^\ell \setminus \bigcup \mathcal{A}$. The Hurewicz identification $P^{ab} \simeq H_1(X;\mathbb{Z})$ induces a W-equivariant isomorphism $P^{ab} \simeq \mathbb{Z}\mathcal{A}$, where $\mathbb{Z}\mathcal{A}$ denotes the free $\mathbb{Z}$-module on $\mathcal{A}$ endowed with the $\mathbb{Z}W$-module structure arising from the natural permutation action of W on $\mathcal{A}$. Let $B = \pi_1(X/W)$.

Definition 16. We consider the *quasi-abelianized complex braid group* $\Gamma = B/[P,P]$ associated to the complex reflection group W.

If W is not irreducible, Γ will be the direct product of the quasi-abelianized braid group associated to its irreducible factors. Therefore, in the sequel we can assume without loss of generality that W is irreducible.

From the previous considerations, we get a short exact sequence

$$1 \to \mathbb{Z}\mathcal{A} \to \Gamma \to W \to 1,$$

and from the Lyndon-Hochschild-Serre spectral sequence, we get the following straightforward generalization of Proposition 1.

Proposition 8. *For any $\mathbb{Q}W$-module M the following isomorphisms hold:*

$$H^\bullet(\Gamma;\mathbb{Q}) = H^\bullet(\mathbb{Z}\mathcal{A};\mathbb{Q})^W \simeq \Lambda^\bullet(\mathbb{Q}\mathcal{A})^W.$$

This provides a way to compute explicitly the Poincaré polynomial for Γ in each of the exceptional cases. We tabulate them in Tables 5 to 9. When there are dots inside the description of the Poincaré polynomial, this implies this is a reciprocal polynomial, so we need to provide only half of the coefficients. The reason why this reciprocity (often) happens is explained in the next section.

Another general fact is the following, where $\mathcal{A}/W$ denotes the set of W-orbits of reflecting hyperplanes.

Proposition 9. *The Poincaré polynomial of Γ admits $(1+t)^{|\mathcal{A}/W|}$ as a factor.*

Proof. Let $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_r$ be the decomposition of $\mathcal{A}$ in W-orbits, with $r = |\mathcal{A}/W|$. For each $i \in [1,r]$ the sum of the hyperplanes inside $\mathcal{A}_i$ spans a W-invariant submodule $\mathbb{Z} \hookrightarrow \mathbb{Z}\mathcal{A}_i$. Considering the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}\mathcal{A}_i \to \mathbb{Z}\mathcal{A}_i/\mathbb{Z} \to 0$, by injectivity of $\mathbb{Q}$ as a $\mathbb{Z}$-module we can apply the functor $H^1(\bullet;\mathbb{Q}) = \mathrm{Hom}(\bullet;\mathbb{Q})$ to it and get a short exact sequence of $\mathbb{Q}W$-modules $0 \to H^1(\mathbb{Z}\mathcal{A}_i/\mathbb{Z};\mathbb{Q}) \to H^1(\mathbb{Z}\mathcal{A}_i;\mathbb{Q}) \to H^1(\mathbb{Z};\mathbb{Q}) \to 0$. By semisimplicity of $\mathbb{Q}W$ this provides a decomposition of $H^1(\mathbb{Z}\mathcal{A}_i;\mathbb{Q})$ as a direct sum of $\mathbb{Q}W$-modules of the form $H^1(\mathbb{Z}\mathcal{A}_i;\mathbb{Q}) \simeq M_i \oplus H^1(\mathbb{Z};\mathbb{Q})$ for some $\mathbb{Q}W$-module M_i. Then, by Künneth formula $H^\bullet(\mathbb{Z}\mathcal{A};\mathbb{Q})$ is isomorphic to $H^\bullet(\mathbb{Z};\mathbb{Q})^{\otimes r} \otimes \Lambda^\bullet(M_1) \otimes \cdots \otimes \Lambda^\bullet(M_r)$ whence

$$H^\bullet(\Gamma;\mathbb{Q}) \simeq H^\bullet(\mathbb{Z};\mathbb{Q})^{\otimes r} \otimes \left(\Lambda^\bullet(M_1) \otimes \cdots \otimes \Lambda^\bullet(M_r)\right)^W$$

which proves the claim, as $1+t$ is the Poincaré polynomial of $H^\bullet(\mathbb{Z};\mathbb{Q})$. □

Table 5 Poincaré polynomial of Γ (1)

G_4	$1+t^1+t^3+t^4$
G_5	$1+2t^1+2t^2+6t^3+10t^4+6t^5+2t^6+2t^7+t^8$
G_6	$1+2t^1+3t^2+10t^3+20t^4+24t^5+20t^6+10t^7+3t^8+2t^9+t^{10}$
G_7	$1+3t^1+7t^2+30t^3+91t^4+177t^5+253t^6+284t^7+253t^8+\cdots+t^{14}$
G_8	$1+t^1+t^3+t^4$
G_9	$1+2t^1+4t^2+30t^3+134t^4+376t^5+767t^6+1284t^7+1830t^8+2088t^9+1830t^{10}$ $+\cdots+t^{18}$
G_{10}	$1+2t^1+2t^2+15t^3+50t^4+87t^5+117t^6+142t^7+136t^8+90t^9+41t^{10}$ $+15t^{11}+5t^{12}+t^{13}$
G_{11}	$1+3t^1+10t^2+102t^3+647t^4+2790t^5+9543t^6+27262t^7+65211t^8+130587t^9$ $+221252t^{10}+321348t^{11}+402472t^{12}+434088t^{13}+402472t^{14}+\cdots+t^{26}$
G_{12}	$1+t^1+t^2+8t^3+23t^4+38t^5+38t^6+28t^7+22t^8+13t^9+3t^{10}$
G_{13}	$1+2t^1+4t^2+30t^3+134t^4+376t^5+767t^6+1284t^7+1830t^8+2088t^9+1830t^{10}$ $+1284t^{11}+\cdots+t^{18}$
G_{14}	$1+2t^1+5t^2+45t^3+217t^4+666t^5+1593t^6+3198t^7+5293t^8+7068t^9+7680t^{10}$ $+6942t^{11}+5267t^{12}+3282t^{13}+1617t^{14}+630t^{15}+206t^{16}+54t^{17}+9t^{18}+t^{19}$
G_{15}	$1+3t^1+10t^2+102t^3+647t^4+2790t^5+9543t^6+27262t^7+65211t^8+130587t^9$ $+221252t^{10}+321348t^{11}+402472t^{12}+434088t^{13}+402472t^{14}+\cdots+t^{26}$
G_{16}	$1+t^1+5t^3+12t^4+14t^5+14t^6+14t^7+12t^8+5t^9+t^{11}+t^{12}$

As in the case of $W=\mathfrak{S}_n$, it is actually quite often the case that the Poincaré polynomial is divisible by a higher power of $1+t$ but, as the quotient polynomial has then negative coefficients, we do not expect any topological reason for this.

3.2 The W-module $H^N(\mathbb{Z}\mathcal{A};\mathbb{Q})$

We set $N=|\mathcal{A}|$. We have $H^N(\Gamma;\mathbb{Q})\simeq H^N(\mathbb{Z}\mathcal{A};\mathbb{Q})^W\simeq\Lambda^N(\mathbb{Q}\mathcal{A})^W$, and $H^N(\mathbb{Z}\mathcal{A};\mathbb{Q})\simeq\Lambda^N(\mathbb{Q}\mathcal{A})$ as a W-module. The W-module structure on the 1-dimensional vector space $\Lambda^N(\mathbb{Q}\mathcal{A})$ is given by the determinant of the permutation representation $\mathbb{Q}\mathcal{A}$, that is, the sign of the permutation action of W on $\mathcal{A}$. Since this abelian representation factorizes through W^{ab}, it is sufficient to determine it on a collection of representatives of the conjugacy classes of reflections.

A uniform description of this module, for instance involving the invariant/coinvariant theory of complex reflection groups, is missing for now. Therefore we explore it by direct computation in each case. Using the computer package CHEVIE (see [10]) one checks the following, which refers to the Shephard-Todd classification of irreducible complex reflection groups ([14]).

Proposition 10. *If W is an irreducible reflection group of exceptional type, then the action of W on $H^N(\mathbb{Z}\mathcal{A};\mathbb{Q})$ is trivial except in the following cases G_8, G_{10}, G_{12}, G_{14}, $G_{28}=F_4$, G_{29}.*

Definition 17. Let $\eta(w)$ denote the sign of the permutation action of w on the set of reflecting hyperplanes and $\mathbb{Q}_\eta$ the corresponding $\mathbb{Q}W$-module.

We have $H^N(\mathbb{Z}\mathcal{A};\mathbb{Q})\simeq\mathbb{Q}_\eta$. Hence, we have a Poincaré duality result in most cases, as observed in the tables.

Proposition 11. *If W is an irreducible reflection group of exceptional type different from those listed in Proposition 10, then $H^r(\Gamma;\mathbb{Q})\simeq H^{N-r}(\Gamma;\mathbb{Q})$.*

If W is an irreducible reflection group of exceptional type listed in Proposition 10, then $H^r(\Gamma;\mathbb{Q}_\eta)\simeq H^{N-r}(\Gamma;\mathbb{Q})$.

Table 6 Poincaré polynomial of Γ (2)

G_{17}	$1 + 2t^1 + 10t^2 + 186t^3 + 1908t^4 + 14276t^5 + 87229t^6 + 449072t^7 + 1968096t^8 + 7434074t^9$ $+24521396t^{10} + 71334960t^{11} + 184308107t^{12} + 425331568t^{13} + 880994128t^{14}$ $+1644502390t^{15} + 2775174184t^{16} + 4244428460t^{17} + 5894942521t^{18} + 7446171200t^{19}$ $+8563197684t^{20} + 8971058152t^{21} + 8563197684t^{22} + \cdots + t^{42}$
G_{18}	$1 + 2t^1 + 5t^2 + 86t^3 + 636t^4 + 3362t^5 + 14983t^6 + 56130t^7 + 175775t^8 + 467520t^9$ $+1074198t^{10} + 2150460t^{11} + 3765292t^{12} + 5789700t^{13} + 7854470t^{14} + 9428804t^{15}$ $+10021408t^{16} + 9428804t^{17} + \cdots + t^{32}$
G_{19}	$1 + 3t^1 + 25t^2 + 622t^3 + 9412t^4 + 108079t^5 + 1023742t^6 + 8194704t^7$ $+56357542t^8 + 338123937t^9 + 1791954240t^{10} + 8471117898t^{11}$ $+36002666511t^{12} + 138471654165t^{13} + 484649380275t^{14} + 1550878152688t^{15}$ $+4555708512956t^{16} + 12327211591453t^{17} + 30818019548700t^{18}$ $+71368043007570t^{19} + 153441311738490t^{20} + 306882630630555t^{21}$ $+571917595672410t^{22} + 994639280131500t^{23}$ $+1616288882564850t^{24} + 2456759132656605t^{25} + 3496157157304842t^{26}$ $+4661542827614878t^{27} + 5826928615825804t^{28} + 6831571545737539t^{29}$ $+7514728617905480t^{30} + 7757139143486168t^{31}$ $+7514728617905480t^{32} + \cdots + t^{62}$
G_{20}	$1 + t^1 + t^2 + 21t^3 + 96t^4 + 262t^5 + 621t^6 + 1302t^7 + 2157t^8 + 2806t^9 + 3032t^{10} + 2806t^{11}$ $+2157t^{12} + 1302t^{13} + 621t^{14} + 262t^{15} + 96t^{16} + 21t^{17} + t^{18} + t^{19} + t^{20}$
G_{21}	$1 + 2t^1 + 15t^2 + 320t^3 + 3912t^4 + 35460t^5 + 264448t^6 + 1663808t^7 + 8950168t^8$ $+41762728t^9 + 171197058t^{10} + 622541248t^{11} + 2023351148t^{12} + 5914410488t^{13}$ $+15630708668t^{14} + 37513654704t^{15} + 82061595180t^{16} + 164123359044t^{17}$ $+300892007724t^{18} + 506765074640t^{19} + 785487036924t^{20} + 1122125085488t^{21}$ $+1479163461648t^{22} + 1800719645168t^{23} + 2025811056898t^{24}$ $+2106844801388t^{25} + 2025811056898t^{26} + \cdots + t^{50}$
G_{22}	$1 + t^1 + 4t^2 + 64t^3 + 476t^4 + 2423t^5 + 9843t^6 + 33748t^7 + 97708t^8 + 238993t^9$ $+500506t^{10} + 909454t^{11} + 1441874t^{12} + 1997499t^{13} + 2423604t^{14}$ $+2583668t^{15} + 2423604t^{16} + \cdots + t^{30}$
G_{23}	$1 + t^1 + t^2 + 5t^3 + 22t^4 + 61t^5 + 93t^6 + 96t^7 + 96t^8 + 93t^9 + \cdots + t^{15}$
G_{24}	$1 + t^1 + t^2 + 6t^3 + 29t^4 + 128t^5 + 355t^6 + 694t^7 + 1168t^8 + 1739t^9 + 2142t^{10}$ $+2142t^{11} + 1739t^{12} + \cdots + t^{21}$
G_{25}	$1 + t^1 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + 5t^7 + 3t^8 + 2t^9 + t^{11} + t^{12}$
G_{26}	$1 + 2t^1 + 2t^2 + 8t^3 + 31t^4 + 106t^5 + 277t^6 + 555t^7 + 951t^8 + 1389t^9 + 1670t^{10}$ $+1670t^{11} + 1389t^{12} + \cdots + t^{21}$
G_{27}	$1 + t^1 + t^2 + 31t^3 + 411t^4 + 3489t^5 + 22736t^6 + 125582t^7 + 598067t^8 + 2463301t^9$ $+8865029t^{10} + 28192301t^{11} + 79877727t^{12} + 202800457t^{13} + 463560616t^{14}$ $+957963894t^{15} + 1796127602t^{16} + 3064066154t^{17} + 4766434844t^{18}$ $+6773270962t^{19} + 8805088666t^{20} + 10482299772t^{21} + 11435425556t^{22}$ $+11435425556t^{23} + 10482299772t^{24} + \cdots + t^{45}$

Proof. The result follows from Lemma 7 applying the same arguments of Theorem 4, cases 1 and 3. Here we are considering the pairings

$$\Lambda^r(\mathbb{Q}\mathcal{A}) \otimes \Lambda^{N-r}(\mathbb{Q}\mathcal{A}) \quad \to \Lambda^N(\mathbb{Q}\mathcal{A}) \simeq \mathbb{Q} \tag{38}$$

$$\Lambda^r(\mathbb{Q}\mathcal{A}) \otimes (\mathbb{Q}_\eta \otimes \Lambda^{N-r}(\mathbb{Q}\mathcal{A})) \quad \to \mathbb{Q}_\eta \otimes \Lambda^N(\mathbb{Q}\mathcal{A}) \simeq \mathbb{Q} \tag{39}$$

where the pairing in Equation (38) applies to the cases not listed in Proposition 10, while the pairing in Equation (39) applies to the cases listed in Proposition 10. □

We note that in the case of the exceptions of Proposition 10, N is even and η maps every distinguished reflection of odd order to 1 and every one of even order to -1.

Table 7 Poincaré polynomial of Γ (3)

G_{28}	$1 + 2t^1 + 2t^2 + 4t^3 + 14t^4 + 64t^5 + 232t^6 + 626t^7 + 1329t^8 + 2308t^9 + 3370t^{10}$
	$+4240t^{11} + 4668t^{12} + 4442t^{13} + 3520t^{14} + 2264t^{15} + 1208t^{16}$
	$+576t^{17} + 260t^{18} + 98t^{19} + 20t^{20}$
G_{29}	$1 + t^1 + 4t^3 + 40t^4 + 350t^5 + 2060t^6 + 9746t^7 + 39959t^8 + 142328t^9 + 441562t^{10}$
	$+1204304t^{11} + 2910343t^{12} + 6267601t^{13} + 12085830t^{14} + 20949016t^{15}$
	$+32736850t^{16} + 46219612t^{17} + 59053222t^{18} + 68369460t^{19} + 71790852t^{20}$
	$+68383196t^{21} + 59059542t^{22} + 46211024t^{23} + 32728563t^{24} + 20951525t^{25}$
	$+12091834t^{26} + 6268424t^{27} + 2907626t^{28} + 1203008t^{29} + 442248t^{30}$
	$+143010t^{31} + 39957t^{32} + 9558t^{33} + 1998t^{34} + 372t^{35} + 57t^{36}$
	$+5t^{37}$
G_{30}	$1 + t^1 + 5t^3 + 74t^4 + 771t^5 + 6872t^6 + 53477t^7 + 356220t^8 + 2055627t^9 + 10468466t^{10}$
	$+47582221t^{11} + 194357924t^{12} + 717683124t^{13} + 2409187294t^{14}$
	$+7387893762t^{15} + 20778746068t^{16} + 53781125768t^{17} + 128476950008t^{18}$
	$+283999914842t^{19} + 582199298729t^{20} + 1108953689280t^{21} + 1965874330196t^{22}$
	$+3247963438620t^{23} + 5007273674460t^{24} + 7210476160546t^{25} + 9706413965476t^{26}$
	$+12222890689810t^{27} + 14405546870106t^{28} + 15895776460298t^{29}$
	$+16425637072516t^{30} + 15895776460298t^{31} + \cdots + t^{60}$
G_{31}	$1 + t^1 + 5t^3 + 49t^4 + 476t^5 + 4276t^6 + 33492t^7 + 222835t^8 + 1284373t^9 + 6541542t^{10}$
	$+29740777t^{11} + 121480400t^{12} + 448546734t^{13} + 1505717800t^{14} + 4617446330t^{15}$
	$+12986790192t^{16} + 33613184414t^{17} + 80297915692t^{18} + 177499968070t^{19}$
	$+363874927936t^{20} + 693096056917t^{21} + 1228670832132t^{22} + 2029977118521t^{23}$
	$+3129546983443t^{24} + 4506547671508t^{25} + 6066507514016t^{26} + 7639306616740t^{27}$
	$+9003468218232t^{28} + 9934860323578t^{29} + 10266021686780t^{30} + 9934860323578t^{31}$
	$+9003468218232t^{32} + 7639306616740t^{33} + 6066507514016t^{34} + 4506547671508t^{35}$
	$+3129546983443t^{36} + \cdots + t^{60}$
G_{32}	$1 + t^1 + 2t^3 + 8t^4 + 34t^5 + 156t^6 + 757t^7 + 3074t^8 + 10633t^9 + 32728t^{10} + 89486t^{11}$
	$+216246t^{12} + 464548t^{13} + 895038t^{14} + 1552859t^{15} + 2427296t^{16}$
	$+3424118t^{17} + 4373170t^{18} + 5066138t^{19} + 5321718t^{20} + 5066138t^{21}$
	$+ \cdots + t^{40}$
G_{33}	$1 + t^1 + t^3 + 7t^4 + 57t^5 + 330t^6 + 1744t^7 + 8320t^8 + 34358t^9 + 123300t^{10} + 391292t^{11}$
	$+1109046t^{12} + 2817848t^{13} + 6440572t^{14} + 13304050t^{15} + 24941988t^{16} + 42557847t^{17}$
	$+66209744t^{18} + 94076221t^{19} + 122282437t^{20} + 145581450t^{21} + 158837210t^{22}$
	$+158837210t^{23} + 145581450t^{24} + 122282437t^{25} + 94076221t^{26} + 66209744t^{27}$
	$+42557847t^{28} + \cdots + t^{45}$

It is also possible to develop 'partial' Poincaré duality when the action of W on $\mathcal{A}$ has several orbits. When W is an irreducible reflection group, its action on $\mathcal{A}$ has at most 3 orbits, and in rank at least 3 it has at most 2, so let us write $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_s$ with $s \leq 3$.

Definition 18. Let $\eta_i : W \to \mathrm{GL}_1(\mathbb{Q}) \simeq \mathrm{GL}(\Lambda^{N_i}\mathbb{Q}\mathcal{A}_i)$ with $N_i = |\mathcal{A}_i|$ denote the sign of the permutation action on the orbit $\mathcal{A}_i$ of reflecting hyperplanes and let $\mathbb{Q}_{\eta_i}$ be the associated $\mathbb{Q}\mathrm{S}_n$-module.

Hence we have $\mathbb{Q}_\eta = \mathbb{Q}_{\eta_1} \otimes \cdots \otimes \mathbb{Q}_{\eta_s}$.

Proposition 12. *We can write* $H^r(\mathbb{Z}\mathcal{A}; \mathbb{Q}) = \Lambda^r(\mathbb{Q}\mathcal{A}) = \bigoplus_{i_1+\cdots+i_s=r} \Lambda^{i_1}(\mathbb{Q}\mathcal{A}_1) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{Q}\mathcal{A}_s)$*. Denoting* $H^{i_1,\ldots,i_s} = \Lambda^{i_1}(\mathbb{Q}\mathcal{A}_1) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{Q}\mathcal{A}_s)$*, whenever* $\mathbb{Q}_{\eta_k} = 1$ *we get a* W*-equivariant isomorphism*

$$H^{i_1,\ldots,i_k,\ldots,i_s} \simeq H^{i_1,\ldots,N_k-i_k,\ldots,i_s}$$

Table 8 Poincaré polynomial of Γ (4)

G_{34}	$1 + t^1 + t^3 + 6t^4 + 74t^5 + 884t^6 + 13116t^7 + 192326t^8 + 2520671t^9 + 2948\ 9979t^{10}$
	$+311014671t^{11} + 2980794710t^{12} + 26139611410t^{13} + 210982251075t^{14}$
	$+1575326752678t^{15} + 10928823475928t^{16} + 70715940374002t^{17} + 428224373356\ 294t^{18}$
	$+2434117544074941t^{19} + 13022528746334731t^{20} + 65732763512909595t^{21}$
	$+313724551754562496t^{22} + 1418580582253661206t^{23} + 6088075007493453094t^{2\ 4}$
	$+24839346043086172191t^{25} + 96491305763321611190t^{26}$
	$+357375206458603492081t^{27} + 1263576622816720686820t^{28}$
	$+427001755315398 62\ 63806t^{29} + 138063900887570905936 96t^{30}$
	$+42755272532737758859696t^{31} + 126929715331036645718996t^{32}$
	$+361557370942693474774788t^{33} + 988965749931371371764614t^{34}$
	$+2599567114105643084316213t^{35} + 6571127982881893678259444t^{36}$
	$+15983824823231350150241083t^{37} + 37435800243874514436429769t^{38}$
	$+84470523627176973820173870t^{39} + 183723388889113441013664706t^{40}$
	$+385371010840651984891616378t^{41} + 779917521939460133077025458t^{42}$
	$+1523559810300218919356528819t^{43} + 2873987823975247126613136662t^{44}$
	$+5237044479243898579045032533t^{45} + 9221752235190675504468316867t^{46}$
	$+15696599549260728754834382628t^{47} + 25833986758157842496665978731t^{48}$
	$+41123489125230657930680765396t^{49} + 63330173252855596855076193586t^{50}$
	$+94374375827785073487122176287t^{51} + 136116888213151322355118222331t^{52}$
	$+190049994863645226542541566541t^{53} + 256919437500854002357709427236t^{54}$
	$+336330900001117489347178011796t^{55} + 426419533929987401697017185243t^{56}$
	$+523673111843844950655299053043t^{57} + 622990426159058822214608054411t^{58}$
	$+718022864047728454391354376763t^{59} + 801792198186627247766309064228t^{60}$
	$+867512870169136852322333424120t^{61} + 909489299370872294431572844472t^{62}$
	$+923925637456126412476131859172t^{63} + 909489299370872294431572844472t^{64}$
	$+ \cdots + t^{126}$

and hence

$$(H^{i_1,\dots,i_k,\dots,i_s})^W \simeq (H^{i_1,\dots,N_k-i_k,\dots,i_s})^W.$$

Proof. The proof follows applying the duality argument to the k-th factor of the tensor product:

$$\begin{aligned} H^{i_1,\dots,i_k,\dots,i_s} &= \Lambda^{i_1}(\mathbb{Q}\mathcal{A}_1) \otimes \cdots \otimes \Lambda^{i_k}(\mathbb{Q}\mathcal{A}_k) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{Q}\mathcal{A}_s) \\ &\simeq \Lambda^{i_1}(\mathbb{Q}\mathcal{A}_1) \otimes \cdots \otimes (\Lambda^{N_k-i_k}(\mathbb{Q}\mathcal{A}_k))^* \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{Q}\mathcal{A}_s) \\ &\simeq \Lambda^{i_1}(\mathbb{Q}\mathcal{A}_1) \otimes \cdots \otimes (\Lambda^{N_k-i_k}(\mathbb{Q}\mathcal{A}_k)^*) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{Q}\mathcal{A}_s) \\ &\simeq \Lambda^{i_1}(\mathbb{Q}\mathcal{A}_1) \otimes \cdots \otimes \Lambda^{N_k-i_k}(\mathbb{Q}\mathcal{A}_k) \otimes \cdots \otimes \Lambda^{i_s}(\mathbb{Q}\mathcal{A}_s) \\ &\simeq H^{i_1,\dots,N_k-i_k,\dots,i_s}. \end{aligned}$$

□

These operations will have a combinatorial interpretation for the groups $G(de, e, n)$.

3.3 Groups $G(e, e, n)$

We now focus on the case of the complex reflection group W of type (e, e, n) and the corresponding quasi-abelianized complex braid group $\Gamma(e, e, n)$.

Proposition 13. *Let us consider a reflection around some hyperplane* $z_i = \zeta z_j$ *for* $i \neq j$, $\zeta \in \mu_e$. *The sign of its action on the hyperplanes is* $(-1)^{(n-2)e+(e-2)/2} = (-1)^{(e-2)/2}$ *if* e *is even, and* $(-1)^{(n-2)e+(e-1)/2} = (-1)^{n+(e-1)/2}$ *if* e *is odd.*

Table 9 Poincaré polynomial of Γ (5)

G_{35}	$1 + t^1 + t^4 + 11t^5 + 44t^6 + 152t^7 + 566t^8 + 1860t^9 + 5004t^{10} + 11572t^{11} + 23972t^{12}$ $+44543t^{13} + 73478t^{14} + 107582t^{15} + 140873t^{16} + 165803t^{17}$ $+175170t^{18} + 165803t^{19} + \cdots + t^{36}$
G_{36}	$1 + t^1 + t^4 + 13t^5 + 78t^6 + 425t^7 + 2660t^8 + 16243t^9 + 87925t^{10} + 423770t^{11}$ $+1838688t^{12} + 7218311t^{13} + 25769943t^{14} + 84136890t^{15} + 252379507t^{16}$ $+697845851t^{17} + 1783538069t^{18} + 4224088466t^{19} + 9292654887t^{20}$ $+19027776867t^{21} + 36326201973t^{22} + 64755423636t^{23} + 107924917344t^{24}$ $+168362842542t^{25} + 246070368091t^{26} + 337208215816t^{27} + 433551008742t^{28}$ $+523248859376t^{29} + 593017264581t^{30} + 631280251719t^{31} + 631280251719t^{32}$ $+593017264581t^{33} + \cdots + t^{63}$
G_{37}	$1 + t^1 + t^4 + 11t^5 + 57t^6 + 374t^7 + 3475t^8 + 35474t^9 + 356059t^{10} + 3406667t^{11}$ $+30454784t^{12} + 251776769t^{13} + 1921850842t^{14} + 13577988054t^{15} + 89107284374t^{16}$ $+545150699683t^{17} + 3119546807020t^{18} + 16747159056781t^{19}$ $+84573155228165t^{20} + 402728709982771t^{21} + 1812277301989416t^{22} + 7721874254027453t^{23}$ $+31209241157562083t^{24} + 119843494614454207t^{25} + 437889713047953430t^{26}$ $+1524504947000485263t^{27} + 5063534269228031737t^{28}$ $+16063625853649334969t^{29} + 48726331566332592976t^{30} + 141463543120705887706t^{31}$ $+393445479558917203986t^{32} + 1049187946405641407641t^{33} + 2684686805077200281859t^{34}$ $+6596659006231560771532t^{35} + 15575444872422490488161t^{36}$ $+35360469435701311971865t^{37} + 77234709556712473520133t^{38}$ $+162390927793545381417225t^{39} + 328841628794206613232301t^{40}$ $+641642202531676306651320t^{41} + 1206898428559421087980171t^{42}$ $+2189257614562845036307157t^{43} + 3831200825453652719183382t^{44}$ $+6470472505226135626407762t^{45} + 10549683432511669927601215t^{46}$ $+16610139872539671530393175t^{47} + 25261254389459174519874612t^{48}$ $+37118577878253099665912691t^{49} + 52708380587001828382184630t^{50}$ $+72344836099829736528394025t^{51} + 95996032517233887861254109t^{52}$ $+123164720965672023324202120t^{53} + 152815487124127820970155769t^{54}$ $+183378584548819686523360133t^{55} + 212850142779608827070753059t^{56}$ $+238989633997983323341287019t^{57} + 259592188653137826945477851t^{58}$ $+272791791466392609895802552t^{59} + 277338321324370157484233484t^{60}$ $+272791791466392609895802552t^{61} + \cdots + t^{120}$

Proof. The non-fixed hyperplanes are the ones of the form $z_a = \xi z_b$ for $a \in \{i, j\}$ and $b \notin \{i, j\}$, and there are $2(n-2)e$ of them, and all the hyperplanes $z_i = \xi z_j$, except for the hyperplane $z_i = \zeta z_j$ itself, and possibly the hyperplane $z_i = -\zeta z_j$ if e is even. Therefore the claim follows. $\square$

In this case, the cohomology classes can be described as follows.

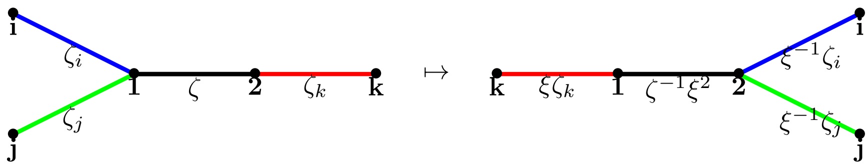
Definition 19. We denote by $K_n(e)$ the complete e-multigraph on the n vertices $\{1, \ldots, n\}$ with e edges between each pair of vertices, labeled by the set μ_e of roots of 1. We fix an ordering of μ_e. We call e-multigraph on n vertices any subgraph of this labeled multigraph.

We associate to an edge between i and j with $i < j$ and label ζ the 1-form $\omega_{i,j}^{\zeta} = \mathrm{dlog}(z_i - \zeta z_j)$, and we associate to a e-multigraph the wedge product of the 1-forms associated to its edges ordered lexicographically from the tuple (i, j, ζ).

The W-action on such forms has a natural combinatorial translation on the collection of e-multigraphs. In particular, the reflection $(z_1, z_2, \ldots, z_n) \mapsto (\xi z_2, \xi^{-1} z_1, z_3, \ldots, z_n)$ acts as follows.

Table 10 Poincaré polynomials for $H^\bullet(\Gamma; \mathbb{Q}_\eta)$

G_8	$t^2 + t^3 + t^5 + t^6$
G_{10}	$t^1 + 5t^2 + 15t^3 + 41t^4 + 90t^5 + 136t^6 + 142t^7 + 117t^8 + 87t^9 + 50t^{10} + 15t^{11} + 2t^{12} + 2t^{13} + t^{14}$
G_{12}	$3t^2 + 13t^3 + 22t^4 + 28t^5 + 38t^6 + 38t^7 + 23t^8 + 8t^9 + t^{10} + t^{11} + t^{12}$
G_{14}	$t^1 + 9t^2 + 54t^3 + 206t^4 + 630t^5 + 1617t^6 + 3282t^7 + 5267t^8 + 6942t^9 + 7680t^{10}$ $+7068t^{11} + 5293t^{12} + 3198t^{13} + 1593t^{14} + 666t^{15} + 217t^{16} + 45t^{17} + 5t^{18} + 2t^{19} + t^{20}$
G_{28}	$20t^4 + 98t^5 + 260t^6 + 576t^7 + 1208t^8 + 2264t^9 + 3520t^{10} + 4442t^{11} + 4668t^{12} + 4240t^{13}$ $+3370t^{14} + 2308t^{15} + 1329t^{16} + 626t^{17} + 232t^{18} + 64t^{19} + 14t^{20} + 4t^{21} + 2t^{22} + 2t^{23} + t^{24}$
G_{29}	$5t^3 + 57t^4 + 372t^5 + 1998t^6 + 9558t^7 + 39957t^8 + 143010t^9 + 442248t^{10} + 1203008t^{11}$ $+2907626t^{12} + 6268424t^{13} + 12091834t^{14} + 20951525t^{15} + 32728563t^{16} + 46211024t^{17}$ $+59059542t^{18} + 68383196t^{19} + 71790852t^{20} + 68369460t^{21} + 59053222t^{22} + 46219612t^{23}$ $+32736850t^{24} + 20949016t^{25} + 12085830t^{26} + 6267601t^{27} + 2910343t^{28} + 1204304t^{29}$ $+441562t^{30} + 142328t^{31} + 39959t^{32} + 9746t^{33} + 2060t^{34} + 350t^{35} + 40t^{36} + 4t^{37} + t^{39}$ $+t^{40}$



Definition 20. Let $\mathrm{Stab}_W(\Delta)$ denote the subgroup of W preserving the multigraph Δ.

Definition 21. A e-multigraph Δ is called *invariant* if every element of $\mathrm{Stab}_W(\Delta)$ induces an even permutation of the edges.

As in Section 2.2 we have the following result. The proof follows from the same argument of the proof of Theorem 1.

Theorem 6. *A basis for the cohomology group $H^\bullet(\Gamma(e, e, n); \mathbb{Q})$ is naturally indexed by the orbits under the action of W of the collection of invariant e-multigraphs.*

As in Section 2.4 and Section 2.5, a similar formula for the cup product and a similar statement for the Poincaré duality hold.

Proposition 14. *Let Γ be the quasi-abelianized complex braid group of type (e, e, n). In the following cases:*

1. *when e even and $(e-2)/2$ is even,*
2. *when e odd and $n + (e-1)/2$ is even*

we have an isomorphism $H^r(\Gamma; \mathbb{Q}) \simeq H^{N-r}(\Gamma; \mathbb{Q})$.

Proof. Because of Proposition 13, the cases above are exactly those when $\mathbb{Q}_\eta$ is the trivial representation, so we can apply Lemma 7. □

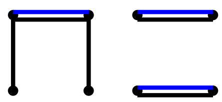
Similar to the classical case, also here the Poincaré duality is combinatorially described by taking the complement e-multigraph.

The hyperplane arrangement in this case contains only one orbit, except possibly when $n = 2$, that is, when W is a dihedral group. In this case, the full multigraph can be identified with $\mu_e(\mathbb{C})$. There are two orbits exactly when e is even, and these are identified with the classes modulo $\mu_2(\mathbb{C}) \subset \mu_e(\mathbb{C})$. We have $\mathbb{Q}_{\eta_1} = \mathbb{Q}_{\eta_2} = \mathbb{Q}$ if and only if $e \equiv 2 \mod 4$, and the partial Poincaré duality is obtained by taking the complement in each of the classes.

Table 11 First Betti numbers for Γ in Coxeter type D_n

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	1	2	1											
3	1	1	0	0	0	1	1							
4	1	1	0	1	2	10	18	10	2	1	0	1	1	
5	1	1	0	0	1	11	27	38	55	90	112	90	55	...
6	1	1	0	0	1	17	64	171	473	1267	2758	4834	7322	...
7	1	1	0	0	1	14	49	122	387	1440	4741	13401	33899	...
8	1	1	0	0	1	14	53	158	630	3030	13848	57350	215531	...

We tabulate in Table 11 the first Betti numbers in Coxeter type D_n corresponding to the case $e = 2$. As an example, the 2-multigraphs appearing for $n = 4$ and $r = 4$ are the ones pictured below, where a black edge means a $+1$ label and a blue one means a -1 label. Both are mapped to 0 in the 1-dimensional cohomology group for $n = 5$, a basis of which is the linear graph on 5 vertices with labels 1, which is inherited from Γ_n. We shall prove later on that this class stabilizes.



3.4 Groups $G(de, e, n)$, $d > 1$

We now focus on the case of the complex reflection group W of type (de, e, n) and the corresponding quasi-abelianized complex braid group $\Gamma(de, e, n)$. In this case, the cohomology classes can be described as follows.

Definition 22. We denote by $\tilde{K}_n(de)$ the full multigraph with loops on the n vertices $\{1, \ldots, n\}$ with de edges between each pair of vertices, labelled by the set μ_{de} of roots of 1, and one unlabelled loop per vertex. We fix an ordering of μ_{de}. We call any subgraph of $\tilde{K}_n(de)$ a de-multigraph on n vertices with loops.

We associate to an edge between i and j with $i < j$ and label ζ the 1-form $\omega_{i,j}^{\zeta} = \mathrm{d}\log(z_i - \zeta z_j)$, we associate to a loop around the vertex i the 1-form $\mathrm{d}\log(z_i)$, and we associate to a e-multigraph the wedge product of the 1-forms associated to its edges and loops, ordered lexicographically from the tuple (i, j, ζ) for the edges, loops ordered by their label, and edges considered smaller than loops.

The W-action on such forms has a natural combinatorial translation on the collection of e-multigraphs with loops. The loops are interverted according to the permutation associated to $w \in G(de, e, n)$ under the natural morphism $G(de, e, n) \to \mathrm{S}_n$. The reflection $(z_1, z_2, \ldots, z_n) \mapsto (\xi z_2, \xi^{-1} z_1, z_3, \ldots, z_n)$ acts as before on the edges. The reflections which are not conjugates of the previous one are conjugates of a reflection of the form $(z_1, z_2, \ldots, z_n) \mapsto (\xi z_1, z_2, \ldots, z_n)$, and this one acts on the edges as follows.

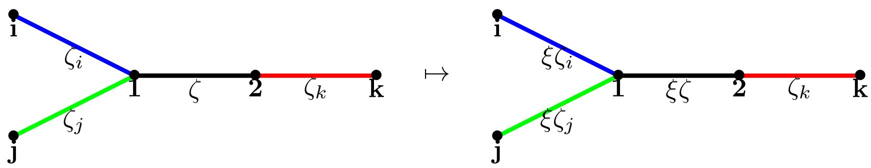


Table 12 First Betti numbers for Γ in Coxeter type B_n

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	1	2	1	0	0									
3	1	2	2	2	2	5	7	3	0	0				
4	1	2	2	3	6	20	46	64	66	59	46	27	9	...
5	1	2	2	3	9	36	109	254	524	1017	1724	2388	2728	...
6	1	2	2	3	9	43	156	467	1383	4081	11027	26065	53897	...
7	1	2	2	3	9	44	175	591	2090	7853	28545	95611	292529	...
8	1	2	2	3	9	44	178	632	2425	10295	44336	184803	735485	...

Definition 23. Let $\mathrm{Stab}_W(\Delta)$ denote the subgroup of W preserving the multigraph with loops Δ.

Definition 24. A de-multigraph with loops Δ is called invariant if every element of $\mathrm{Stab}_W(\Delta)$ induces an even permutation of the edges of Δ.

As in Section 2.2 we have a description of an additive basis of the cohomology. The proof uses the same argument.

Theorem 7. *A basis for the cohomology group $H^\bullet(\Gamma(de, e, n); \mathbb{Q})$ is naturally indexed by the orbits under the action of W of the collection of invariant de-multigraphs with loops.*

A similar formula for the cup product holds as well.

For $n \geq 3$, there are two orbits of hyperplanes, the one corresponding to edges and the one corresponding to loops. We denote η_1, η_2 the corresponding characters, and compute them now.

Proposition 15. *Let ρ be the reflection*

$$(z_1, z_2, \ldots, z_n) \mapsto (\xi z_2, \xi^{-1} z_1, z_3, \ldots, z_n)$$

and let τ be the reflection

$$\tau : (z_1, z_2, \ldots, z_n) \mapsto (\xi z_1, z_2, \ldots, z_n).$$

The value of $\eta_1\eta_2$ on the reflection ρ is given as follows: ρ acts on $H^N(\mathbb{Z}\mathcal{A}; \mathbb{Q})$ as 1 if and only if one of the following conditions is satisfied:

1. *de is even and $(de-2)/2$ is even,*
2. *de is odd and $n + (de-1)/2$ is even.*

The action of ρ on loops is given by a transposition, whence the value of η_2 on ρ is -1. The value of η_1 on ρ is then deduced from the value of $\eta_1\eta_2$ and η_2. The reflection τ fixes the loops, hence the value of η_2 on τ is 1. The reflection τ acts on edges with cycles of length the order $o(\xi)$ of ξ inside $\mathbb{C}^\times$. Since there are $(n-1)de/o(\xi)$ such cycles, the values of η_1 on τ is 1 if and only if one of the following conditions is satisfied

1. *$o(\xi)$ is odd,*
2. *both $o(\xi)$ and $(n-1)de/o(\xi)$ are even.*

When $\mathbb{Q}_{\eta_1} = \mathbb{Q}$, a partial duality, as described in Proposition 12, is obtained by taking the complement inside the collection of edges of the full multigraph $\tilde{K}_n(de)$.

Remark 11. Note that in this case $\mathbb{Q}_\eta = \mathbb{Q}_{\eta_1} \otimes \mathbb{Q}_{\eta_2}$ is never equal to the trivial module.

3.5 Stabilization

We first consider the case of $G(d,1,n)$. The complete d-multigraph with loops on n vertices $\tilde{K}_n(d)$ is naturally identified to a subgraph of $\tilde{K}_{n+1}(d)$, where the vertex $n+1$ has no loops and no incident edge. This enables us to identify any d multigraph Δ with loops on n vertices with a multigraph with loops on $n+1$ vertices.

Proposition 16. *The natural maps $H^r(\Gamma(d,1,n+1);\mathbb{Q}) \to H^r(\Gamma(d,1,n);\mathbb{Q})$ are surjective. Moreover $H^r(\Gamma(d,1,n);\mathbb{Q})$ stabilizes for $n \geq 2r$.*

Proof. We clearly have $\mathrm{Stab}_{G(d,1,n)}(\Delta) = \mathrm{Stab}_{G(d,1,n+1)}(\Delta) \cap G(d,1,n)$ and this implies the first part of the statement. Moreover, since any d-multigraph with loops with at most r edges can be realized as a subgraph of $\tilde{K}_n(d)$ with $n \geq 2r$, in that range the cohomology groups have the same dimension, and therefore we obtain the stability result. $\square$

Theorem 8. *Let $d, e \geq 1$ be integers. The inclusion map $\Gamma(de,e,n) \hookrightarrow \Gamma(de,e,n+1)$ induces stabilization of the rational cohomology groups of degree r for $n \geq 2r+1$ (and even $n \geq 2r$ when $e = 1$).*

Proof. In general, we have the following commutative diagram of groups:

$$\begin{array}{ccc} G(de,e,n) & \longrightarrow & G(de,e,n+1) \\ \cap & \nearrow & \cap \\ G(de,1,n) & \longrightarrow & G(de,1,n+1) \end{array}$$

where the map $\iota : G(de,1,n) \to G(de,e,n+1)$ is given by $M \mapsto \mathrm{diag}(M, m^{-1})$ where m is the product of the non-zero entries in M.

It is readily checked that the groups $\iota(G(de,1,n))$ and $\mathrm{Stab}_{G(de,e,n+1)}(\tilde{K}_n(de))$ have the same image inside the group of permutations of the edges and loops of $\tilde{K}_n(de)$.

This implies that for $d > 1$ the natural map

$$p_n : H^r(\Gamma(de,1,n);\mathbb{Q}) \to H^r(\Gamma(de,e,n+1);\mathbb{Q})$$

is surjective for all r.

Assume $n \geq 2r$. We have the following diagram of cohomology groups.

$$\begin{array}{ccccc} H^r(\Gamma(de,e,n);\mathbb{Q}) & \xleftarrow{a_n} & H^r(\Gamma(de,e,n+1);\mathbb{Q}) & \xleftarrow{a_{n+1}} & H^r(\Gamma(de,e,n+2);\mathbb{Q}) \\ \uparrow c_n & \swarrow p_n & \uparrow c_{n+1} & \swarrow p_{n+1} & \uparrow c_{n+2} \\ H^r(\Gamma(de,1,n);\mathbb{Q}) & \underset{\simeq}{\xleftarrow{b_n}} & H^r(\Gamma(de,1,n+1);\mathbb{Q}) & \underset{\simeq}{\xleftarrow{b_{n+1}}} & H^r(\Gamma(de,1,n+2);\mathbb{Q}) \end{array}$$

Then a_n and a_{n+1} are injective, as all multigraphs with r edges can be realized on $2r$ vertices. From the surjectivity of the maps p_n, p_{n+1}, this readily implies that the vertical maps c_n, c_{n+1} are injective. By elementary diagram chasing this implies that a_{n+1} is surjective. Thus a_{n+1} is an isomorphism, and the sequence $H^r(\Gamma(de,e,n);\mathbb{Q})$ stabilizes for $n \geq 2r+1$.

For $d = 1$ we can define q_n as the composite of p_n with the obvious projection map from $H^r(\Gamma(de,1,n);\mathbb{Q})$ to its submodule $H_0^r(\Gamma(de,1,n);\mathbb{Q})$ spanned by the invariant graphs with no loops. Then, for $n \geq 2r$, the restriction $b_n^0 : H_0^r(\Gamma(de,1,n+1);\mathbb{Q}) \to H_0^r(\Gamma(de,1,n);\mathbb{Q})$ of b_n is an isomorphism and the same argument can be applied to the diagram

$$\begin{array}{ccccc}
H^r(\Gamma(e,e,n);\mathbb{Q}) & \xleftarrow{a_n} & H^r(\Gamma(e,e,n+1);\mathbb{Q}) & \xleftarrow{a_{n+1}} & H^r(\Gamma(e,e,n+2);\mathbb{Q}) \\
\uparrow c_n & \swarrow q_n & \uparrow c_{n+1} & \swarrow q_{n+1} & \uparrow c_{n+2} \\
H_0^r(\Gamma(de,1,n);\mathbb{Q}) & \underset{\simeq}{\xleftarrow{b_n^0}} & H_0^r(\Gamma(de,1,n+1);\mathbb{Q}) & \underset{\simeq}{\xleftarrow{b_{n+1}^0}} & H_0^r(\Gamma(de,1,n+2);\mathbb{Q})
\end{array}$$

thus providing stability again for $n \geq 2r+1$.

This completes the proof of the Theorem. □

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