Infinitesimal Hecke algebras

Algèbres de Hecke infinitésimales

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Abstract

In this Note, we define infinitesimal analogues of the Iwahori-Hecke algebras associated with finite Coxeter groups. These are reductive Lie algebras for which we announce several decomposition results. These decompositions yield irreducibility results for representations of the corresponding (pure) generalized braid groups deduced from Hecke algebra representations through tensor constructions. *To cite this article: I. Marin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Dans cette Note, nous définissons des analogues infinitésimaux des algèbres d'Iwahori-Hecke associées aux groupes de Coxeter finis. Ce sont des algèbres de Lie réductives, pour lesquelles nous obtenons des décompositions partielles. Ces décompositions entrainent des résultats d'irréductibilité pour certaines représentations du groupe de tresses (pures) généralisé correspondant, qui sont déduites des représentations de l'algèbre de Hecke par constructions tensorielles. *Pour citer cet article : I. Marin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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On associe classiquement à tout groupe de Coxeter fini G, défini sur un ensemble S de générateurs élémentaires par un ensemble $\mathcal{R}$ de relations de Coxeter, les structures suivantes : une algèbre de Hecke $H(q)$ dépendant d'un paramètre $q \in \mathbf{C} \setminus \{0\}$, et un groupe de tresses généralisé B. Le groupe B est défini sur le même ensemble S de générateurs, avec relations $\mathcal{R}$ auxquelles on a soustrait les relations $s^2 = 1$ pour $s \in S$. L'algèbre de Hecke $H(q)$ est une $\mathbf{C}$-algèbre définie sur S, avec relations $\mathcal{R}$ dans lesquelles on a remplacé $s^2 = 1$ pour $s \in S$, c'est-à-dire $(s-1)(s+1) = 0$, par $(s-q)(s+q^{-1}) = 0$. Pour tout $q \in \mathbf{C} \setminus \{0\}$, $H(q)$ est ainsi un quotient de l'algèbre de groupe $\mathbf{C}B$.

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Il est classique depuis un théorème de Tits que, pour des valeurs génériques de q, l'identité $H(1) = \mathbf{C}G$ se déforme en un isomorphisme $H(q) \simeq \mathbf{C}G$. Nous nous cantonnons ici, pour G fixé, à ces valeurs génériques de q. Ainsi, à toute représentation (irréductible) U de G correspond une représentation U^q de $H(q)$, donc de B. Le produit tensoriel de deux représentations de $H(q)$ n'est pas muni en général d'une action de $H(q)$, mais néanmoins toujours d'une action de B. Dans cette Note, nous étudions la décomposition en irréductibles de ces produits tensoriels.

Si $U_1, \dots, U_r$ sont des représentations irréductibles de G, l'action de B sur $U_1^{q_1} \otimes \dots \otimes U_r^{q_r}$ est irréductible pour des valeurs génériques de $(q_1, \dots, q_r) \in \mathbf{C}$, au moins pour un grand nombre de groupes de Coxeter (proposition 2.3). Néanmoins, pour des valeurs génériques de $q \in \mathbf{C}$, $U_1^q \otimes \dots \otimes U_r^q$ n'est pas irréductible en général, comme le montre immédiatement le cas $U_1 = \dots = U_r$. Pour étudier sa décomposition, nous introduisons une version infinitésimale, notée $\mathcal{H}$, de l'algèbre de Hecke $H(q)$. Cette algèbre de Lie infinitésimale est une algèbre de Lie réductive canoniquement associée à G, dont nous déterminons d'abord le centre (proposition 3.1). La description de ses facteurs simples permettrait de résoudre complètement le problème posé, pour des valeurs génériques de $q \in \mathbf{C}$. Nous annonçons les progrès effectués dans l'accomplissement de ce programme, en supposant désormais que G est réduit (c'est-à-dire que son diagramme de Coxeter est connexe). En type A, une décomposition partielle (théorème 5.1) permet de conclure pour les représentations de l'algèbre de Temperley-Lieb, facteur semi-simple classique de $H(q)$. Une décomposition complète est obtenue pour les groupes diédraux (proposition 3.3) et pour d'autres groupes de faible rang. Enfin, pour tous les types, une décomposition partielle de $\mathcal{H}$ (théorème 4.1) permet de décomposer $(V^q)^{\otimes n}$ pour V la représentation de réflexion de G, c'est-à-dire la puissance tensorielle n^{e} de la représentation de Burau (réduite) généralisée : pour tout foncteur de Schur F, $F(V^q)$ est irréductible pour des valeurs génériques de q. On généralise ainsi un résultat de Kilmoyer sur l'irréductibilité des puissances alternées de V^q – qui présentent la particularité d'être encore des représentations de $H(q)$, et étaient étudiées pour cette raison dès [4].

1. Introduction

Let G denote a finite Coxeter group of rank n considered as a finite reflexion subgroup in $GL(V)$, $V = \mathbf{C}^n$, $\mathcal{A}$ its set of reflecting hyperplanes, R its set of reflections, and $S \subset R$ its set of elementary reflections. Let B (resp. P) denote its braid group (resp. pure braid group) in the sense of [1], and $X = V \setminus \cup_{H \in \mathcal{A}} H$ its configuration space, endowed with the usual action of G. A base point for X, hence for X/G, is chosen once and for all. For every $H \in \mathcal{A}$ we denote by $\tau_H \in P = \pi_1(X)$ the twist around the hyperplane H – the collection of all these twists is known to generate P by general arguments (see e.g. [2] appendix 1, also for a detailed definition of these twists). We denote by K the minimal common field of definition for the irreducible (linear, finite-dimensional, complex) representations of G. Recall that, if $(m_{s,t})_{s,t \in S}$ denotes the Cartan matrix of G, then K is the number field generated by the family $\cos(\frac{2\pi}{m_{s,t}})$ for $s, t \in S$. The natural bijection $\mathcal{A} \to R$ is denoted by $H \mapsto s_H$.

To every $H \in \mathcal{A}$ let us associate the 1-form $\omega_H \in \Omega^1(X)$ defined by $\omega_H = \frac{\mathrm{d}\alpha_H}{\alpha_H}$ where α_H denotes any linear form on V with kernel H. We then associate to every representation ρ of G on $\mathbf{C}^m$ the $\mathfrak{gl}_m(\mathbf{C})$-valued 1-form $\omega_\rho = \sum_{H \in \mathcal{A}} \rho(s_H)\omega_H \in \Omega_1(X) \otimes \mathfrak{gl}_m(\mathbf{C})$. Since ω_ρ is integrable, $h\omega_\rho$ defines for every $h \in \mathbf{C}$ a flat connection on the trivial vector bundle $X \times \mathbf{C}^m \to X$ and, since these forms are G-invariant, also a flat connection on the quotient bundle X_ρ, by making G act diagonally on $X \times \mathbf{C}^m$. It is well-known that, for almost all $h \in \mathbf{C}$, the monodromy action of $B = \pi_1(X/G)$ is (isomorphic to) the Hecke algebra representation associated with ρ, with parameter $q = e^{\mathrm{i}\pi h}$. Its restriction to $P = \pi_1(X)$ is the monodromy of ω_ρ over X.

2. Holonomy Lie algebras

Let us denote by $\mathfrak{g}_X$ the holonomy Lie algebra of X, and recall briefly its construction. It is defined, over any subfield $\mathbf{k}$ of $\mathbf{C}$, from a set $\{t_H \mid H \in \mathcal{A}\}$ of generators by the fact that a linear map $\rho : \mathfrak{g}_X \to \mathfrak{gl}_m(\mathbf{C})$ is a Lie algebra morphism iff $\sum \rho(t_H)\omega_H$ is integrable. Because such a 1-form is always closed, this integrability condition is homogeneous, and so are the defining relations for $\mathfrak{g}_X$. Every linear representation ρ of G on $\mathbf{C}^m$ then defines a 1-parameter family ρ_h of $\mathfrak{g}_X$ (resp. $\mathbf{C}G \ltimes \mathsf{U}\mathfrak{g}_X$)-modules by $\rho_h(t_H) = h\rho(t_H)$, hence $\mathfrak{gl}_m(\mathbf{C})$-valued integrable 1-forms on X (resp. X/G). Here the action of G on $\mathfrak{g}_X$ is deduced from its action on $\mathcal{A}$, and $\mathsf{U}\mathfrak{g}_X$ denotes the universal envelopping algebra of $\mathfrak{g}_X$. We denote by $\int \rho$ the monodromy representation of P (resp. B) corresponding to a given representation of $\mathfrak{g}_X$ (resp. $\mathbf{C}G \ltimes \mathsf{U}\mathfrak{g}_X$). We will make use of the following two facts.

Lemma 2.1 *For ρ^1, ρ^2 two representations of $\mathfrak{g}_X$ (resp. $\mathbf{C}G \ltimes \mathsf{U}\mathfrak{g}_X$), we have $\int \rho^1 \otimes \rho^2 = (\int \rho^1) \otimes (\int \rho^2)$*

The proof of this lemma is obvious. The proof of the next one is based on the first order approximation of the monodromy :

$$\left(\int \rho_h\right)(\tau_H) = 1 + 2\mathrm{i}\pi h \rho(t_H) + o(h).$$

One gets from this formula that

Lemma 2.2 *Let A be an associative finite-dimensional $\mathbf{C}$-algebra and $\rho : \mathsf{U}\mathfrak{g}_X \to A$ be a surjective morphism. Then, for generic values of $h \in \mathbf{C}$, the map $\int \rho_h : \mathbf{C}P \to A$ is surjective. If ρ is an irreductible representation of $\mathfrak{g}_X$, then $\int \rho_h$ is irreducible for generic $h \in \mathbf{C}$.*

Indeed, the fact that ρ is surjective and that $\mathfrak{g}_X$ is graded enables one to pick a finite family of homogeneous elements in $\mathfrak{g}_X$ whose image form a basis of the finite-dimensional $\mathbf{C}$-vector space A. The above first order formula enables one to express each generator $\rho(t_H)$ of $\mathsf{U}\mathfrak{g}_X$, hence any of these homogeneous elements, as the images of an element in $\mathbf{C}P$ plus higher terms. The analyticity in h of these expressions shows that they span A for h outside the vanishing set of a meromorphic function, hence for h outside some locally finite subset of $\mathbf{C}$. See the proof of proposition 2 in [7] for more details. In particular, consider an irreducible representation ρ of $\mathfrak{g}_X$; by Burnside theorem, $\rho(\mathsf{U}\mathfrak{g}_X)$ is a full matrix algebra. In this case, the lemma implies that $\int_h \rho$ will be irreducible for almost all h.

Since G is generated by R, it follows that every Hecke algebra representation of G is irreducible under P, and also that for generic q the composite morphism $\mathbf{C}P \hookrightarrow \mathbf{C}B \twoheadrightarrow H(q)$ is surjective. Note that, by the same argument, these representations are already irreducible under the action of certain free subgroups of P – for instance by the subgroup of P generated by the squares of the Artin generators of B, which is a free group after a conjecture of Tits proved in [3]. The corresponding maps to $H(q)$ are also generically surjective by the same lemma.

Let us now consider r irreducible representations $\rho^1, \ldots, \rho^r$ of G defined over $\mathbf{k}$, corresponding to r Hecke algebra representations $\int \rho^1_h, \ldots, \int \rho^r_h$. We proved in [7] the following two facts.

Proposition 2.3 *For generic $(h_1, \ldots, h_r) \in \mathbf{k}^r$ endowed with the Zariski topology, the $\mathfrak{g}_X$-representation $\rho^1_{h_1} \otimes \ldots \otimes \rho^r_{h_r}$ is irreducible, except perhaps if G contains parabolic subgroups of type D_4.*

We think that this result should also be valid for the finite Coxeter groups which contain D_4. This result does not extend to the case $h_1 = \ldots = h_r$, for every $\rho_h \otimes \rho_h$ admits symmetric and antisymmetric parts. However, the fact that the stable subspaces for P are also stable under B remains true :

Proposition 2.4 *If $h \neq 0$, the subspaces of $\rho^1_h \otimes \ldots \otimes \rho^r_h$ which are stable under $\mathfrak{g}_X$ are also stable under $\mathbf{C}G \ltimes \mathsf{U}\mathfrak{g}_X$.*

It follows from this, that in the generic case, the problem of decomposing tensor products or Schur functors of Hecke algebra representations reduces to the analogous problem for the corresponding $\mathfrak{g}_X$-representations.

Let us notice that, depending on the type of G, the Hecke algebra of G may admit several parameters. Since we are only interested in the generic case here, and since the irreducibility property is topologically open, we can (and will) restrict ourselves to a 1-parameter description.

3. Infinitesimal Hecke Algebras

We denote by $\mathcal{H}$ (resp. $\mathcal{H}_\rho$) the image of $\mathfrak{g}_X$ in $\mathbf{k}G$ (resp. $\mathrm{End}_{\mathbf{k}}(\rho)$, for ρ a representation of G) considered as a Lie algebra with bracket $ab-ba$, and call it the infinitesimal Hecke algebra (associated with ρ). Again because R generates both $\mathcal{H}$ (as a Lie algebra) and G (as a group), every irreducible representation of G is irreducible for $\mathcal{H}$. Since $\mathcal{H}$ has then, by definition, a faithful semi-simple linear representation, it is a *reductive* Lie algebra. Its study then boils down to the description of its center, Cartan subalgebra, and simple factors.

For any conjugacy class c in G, we denote by T_c the sum of its elements in $\mathbf{Z}G$. Notice that these elements belong to the center of $\mathbf{Z}G$.

Proposition 3.1 *The center of $\mathcal{H}$ is generated by $\{T_c \mid c \in R/G\}$, where R/G denotes the set of conjugacy classes of reflections.*

A good candidate for a Cartan subalgebra comes from analogues of the so-called Jucys-Murphy elements of the symmetric group. These elements, introduced by Jucys [5] for G of type A_n, satisfy the following properties : (A) they are sums of reflections in $\mathbf{Z}G$, (B) they generate a maximal commutative algebra in $\mathbf{C}G$, (C) they are diagonal over K. We prove in [6] that analogous elements which satisfy (A), (B) and (C) exist at least for every finite Coxeter group which does not contain parabolic subgroups of type D_4, F_4 or H_4 – that is, for finite Coxeter groups whose irreducible components are of type A_n, B_n, $I_2(m)$ and H_3. As a consequence of this, we prove in [7] the following :

Proposition 3.2 *Except perhaps if G contains parabolic subgroups of type D_4, F_4 or H_4, $\mathcal{H}$ is split as soon as $K \subset \mathbf{k}$.*

From now on, we concentrate on the study of simple factors. We notice that, if $G = G_1 \times G_2$ is a decomposition of G *as a Coxeter group*, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\mathcal{H}$ (resp. $\mathcal{H}_1$, $\mathcal{H}_2$) the infinitesimal Hecke algebra associated with G (resp. G_1, G_2). Hence it is sufficient to handle the case of the reduced (finite) Coxeter groups. In the case of the dihedral groups $I_2(m)$, we get in [7] an explicit description :

Proposition 3.3 *If G is a dihedral group of type $I_2(m)$ for $m \geq 3$, then the derived Lie algebra $\mathcal{H}'$ is of type A_1^n for $n = \lfloor \frac{m-1}{2} \rfloor$, and is split if $K \subset \mathbf{k}$.*

Moreover, these n factors are in natural 1-1 correspondence with the irreducible representations $\rho_1, \dots, \rho_n$ of G : ρ_i corresponds to the standard representation of $\mathcal{H}'_{\rho_i}$, which is a simple Lie algebra of type A_1, and $\rho_1 \otimes \dots \otimes \rho_n$ is irreducible under $\mathcal{H}$.

4. Burau representation

We now assume that G is *reduced*, i.e. its Coxeter diagram is connected. For every irreducible representation ρ of G, a decomposition of the reductive Lie algebra $\mathcal{H}_\rho$ solves the problem of decomposing Schur functors of the corresponding Hecke algebra representation, under the action of P and in the generic case. A case of particular interest occurs when ρ is the reflection representation of G, which corresponds to the generalized (reduced) Burau representation. We prove in [7] the following :

Theorem 4.1 *If G is reduced and has rank n, $K \subset \mathbf{k}$, and ρ is the reflection representation, then $\mathcal{H}_\rho \simeq \mathfrak{gl}_n(\mathbf{k})$ for $n > 2$, and $\mathcal{H}_\rho \simeq \mathfrak{sl}_2(\mathbf{k})$ for $n = 2$.*

This leads to a generalization of Kilmoyer's classical result (see [4]) on the irreducibility of the alternating powers of the generalized Burau representations. Indeed, this result implies that *every* Schur functor of the generalized Burau representation is irreducible in the generic case.

5. Infinitesimal Temperley-Lieb Algebra

In case G is of type A_{n-1}, it is isomorphic to the symmetric group on n letters, and its irreducible representations, which can all be defined over $\mathbf{Q}$, are indexed by Young diagrams or partitions of size n. There exists a well-known semisimple factor of its Hecke algebra : the Temperley-Lieb algebra, which corresponds to the collection of all Young diagrams having at most two rows. Let us denote by $\rho_{n,p}$ the irreducible representation of G indexed by the partition $[n-p,p]$ of n with $0 \leq p \leq \frac{n}{2}$, $V_{n,p}$ the corresponding vector space , and ρ_n the direct sum of all these representations. A decomposition of the infinitesimal Temperley-Lieb algebra $\mathcal{TL}_n = \mathcal{H}_{\rho_n}$ would solve our problem for the Temperley-Lieb algebra representations. We denote by $\tilde{\rho}_{n,p}$ and $\tilde{\rho}_n$ the representations of $\mathcal{TL}_n$ corresponding respectively to $\rho_{n,p}$ and ρ_n. It happens that the decomposition of $\mathcal{TL}_n$ admits a very simple form, as shown in [8] :

Theorem 5.1 *$\mathcal{TL}_n$ is isomorphic to*

$$\mathbf{Q} \oplus \bigoplus_{p=0}^{\frac{n}{2}} \mathfrak{sl}(V_{n,p})$$

and $\tilde{\rho}_{n,p}$ corresponds to the standard representation of the p-th simple factor of $\mathcal{TL}_n$ in this decomposition.

In particular, every Schur functor applied to an irreducible representation of the Temperley-Lieb algebra, as well as the tensor product of any family of two-by-two non isomorphic representations of this kind, is irreducible under P in the generic case.

6. Miscellaneous

With the help of a computer, a full decomposition of $\mathcal{H}$ was found for G of small rank. In case G is of type A_3, A_4, A_5, B_3, B_4, D_4, H_3, then the derived Lie algebra $\mathcal{H}'$ is of Cartan type A_1A_2, A_3A_4, $A_4^2A_8D_8$, $A_1A_2^2$, $A_1A_2A_3^2A_5A_7$, $A_1A_2^3A_3C_4$, $A_4A_3A_2^2$ respectively.

In case of type A_n, a similar construction can be established (see [8]) for representations of B which factorize through the more general Birman-Wenzl-Murakami algebra – including the famous Krammer representation.

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