

# CHARACTERS OF THE GROTHENDIECK-TEICHMÜLLER GROUP THROUGH RIGIDITY OF THE BURAU REPRESENTATION

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**Abstract.** We present examples of characters of absolute Galois groups of number fields that can be recovered through their action by automorphisms on the profinite completion of the braid groups, using a “rigidity” approach. The way we use to recover them is through classical representations of the braid groups, and in particular through the Burau representation. This enables one to extend these characters to Grothendieck-Teichmüller groups.

**MSC 2000 :** 14G32.

## 1. INTRODUCTION

**1.1. Braid groups and free groups.** Let  $n \geq 3$ . We define the braid group  $B_n$  as follows. Let  $D_n$  be the disk with boundary with  $n$  points removed (see figure 1). Then  $B_n$  is the group  $\text{Diff}^+(D_n)/\text{Diff}_0^+(D_n)$  of diffeomorphisms which preserve orientation and fix the boundary pointwise modulo isotopy. We note that the fundamental group of  $D_n$  is the free group  $F_n$  on  $n$  generators, hence we get from the definition a natural action by automorphisms of  $B_n$  on  $F_n$ . This action is called the Artin action and is known to be faithful, hence  $B_n$  can be considered as a subgroup of  $\text{Aut}(F_n)$ .

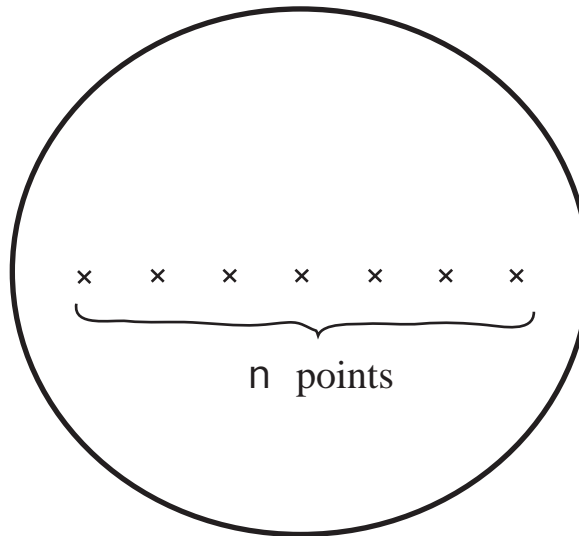
**1.2. Analogy : rigid local systems.** For motivation, let us say that an irreducible representation  $R : F_n \rightarrow \text{GL}(V)$  is B-rigid if, for all  $g \in B_n$ , we have  $R \circ g \simeq R$  as representations of  $F_n$ . In that case, we get a well-defined projective representation  $Q_R : B_n \rightarrow \text{PGL}(V)$  of the braid group  $B_n$ . It turns out that many B-rigid representations of the free group are known. They are constructed by the way of so-called *rigid local systems*. We recall what this is, referring to [Ka] for further reading.

Call a  $(n+1)$ -tuple  $\underline{A} = (A_1, \dots, A_{n+1}) \in \text{GL}(V)^{n+1}$  a local system if  $A_1 A_2 \dots A_{n+1} = 1$ . Another local system  $\underline{B}$  is considered equivalent to  $\underline{A}$  if there exists  $P \in \text{GL}(V)$  such that  $PA_i = B_i P$  for all  $i \in [1, n+1]$ . Geometrically, this is a local system on  $\mathbb{P}^1(\mathbb{C})$  minus  $n+1$  points, which is homotopically equivalent to  $D_n$ . We call it irreducible if  $V$  is irreducible under the action of  $A_1, \dots, A_{n+1}$ . Note that this is the same as to ask it to be irreducible under the action of  $A_1, \dots, A_n$ , as  $A_{n+1} = (A_1 \dots A_n)^{-1}$ . We call such a local system *rigid* if, for any irreducible local system  $\underline{B}$  such that  $A_i$  and  $B_i$  are conjugated in  $\text{GL}(V)$  for all  $i \in [1, n+1]$ , then  $\underline{B}$  is equivalent to  $\underline{A}$ .

An explicit description of the Artin action enables one to prove the following : if  $\underline{A}$  is a rigid (irreducible) local system such that  $A_i$  is conjugated to  $A_j$  in  $\text{GL}(V)$  for all  $i, j$ , then the associated irreducible representation  $R : F_n \rightarrow \text{GL}(V)$  which sends the  $i$ -th generator of  $F_n$  to  $A_i$  is B-rigid, hence defines a projective representation of  $B_n$  over  $V$ .

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FIGURE 1. Disk with  $n$  points removed

**1.3. Rigidity of Braid group representations.** In this paper, we show how one can apply the same approach to the Grothendieck-Teichmüller group  $\widehat{\text{GT}}$ , being defined as a group of automorphism of some completion of the braid group  $B_n$ . In this way we get characters of natural subgroups of  $\widehat{\text{GT}}$ .

One of the main interest of the group  $\widehat{\text{GT}}$  is that it contains the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ . Here we pay special attention to this aspect. We show in particular how some of the classical characters of absolute Galois groups of number fields, namely the Kummer character  $\rho_2$  and the Soulé characters  $\kappa_3$  and  $\kappa_5$  can be extended to the Grothendieck-Teichmüller group by using rigidity of representations of the braid groups.

In §2 and §3 we recall classical facts about completions of groups and about the various groups involved here. Section 4 recovers the Kummer character of roots of 2 by rigidity of a natural representation  $B_3 \rightarrow \text{SL}_2(\mathbb{Z})$ , which can be seen as a degeneration of the Burau representation of  $B_3$ . Section 5 defines the notion of GT-rigidity, and section 6 applies this mechanism to the classical Burau representation.

Part of the results exposed here have been proved in [Ma]. This paper is roughly based on a talk given in Toulouse on January 26th, 2007, during the “Séminaire Méditerranéen d’Algèbre et de Topologie” dedicated to the Grothendieck-Teichmüller group, which was organised by C. Kapoudjian and I. Potemine.

## 2. COMPLETIONS

**2.1. General facts.** If  $G$  is an arbitrary group and  $\mathcal{Q}$  is a class of groups satisfying certain well-known conditions, the completion  $G_{\mathcal{Q}}$  of  $G$  with respect to  $\mathcal{Q}$  is the limit of the direct system formed by the quotients of  $G$  which belong to  $\mathcal{Q}$ . We consider here the following classes : the finite groups, the nilpotent groups, the finite nilpotent groups and the finite

$\ell$ -groups for some fixed prime number  $\ell$ . We denote the corresponding completions  $\widehat{G}$ ,  $G_{nil}$ ,  $G_{f-nil}$ ,  $G^\ell$ .

Completions with respect to subclasses of the class of finite groups are so-called profinite : with respect to the topology inherited from the discrete finite quotients, they are compact and totally discontinuous. It follows that the natural (continuous) morphism  $\widehat{G} \rightarrow G_{f-nil}$  is always surjective. Another general feature is that finite nilpotent groups are direct products of their  $\ell$ -Sylows. A consequence here is that  $G_{f-nil}$  is canonically isomorphic to the direct product of the groups  $G^\ell$ , for  $\ell$  a prime number. Finally, nilpotent groups are residually finite. Recall that a group is said to be residually  $\mathcal{Q}$  if every non-trivial element admit a non-trivial image in some group in  $\mathcal{Q}$ . A consequence here is that the natural morphism  $G_{nil} \rightarrow G_{f-nil}$  is always injective.

In case  $G$  is a free group  $F$ , it is known that  $F$  is residually nilpotent, and in particular residually finite. It follows that the morphisms  $F \rightarrow F_{nil}$  and  $F \rightarrow \widehat{F}$  are injective. The whole picture is illustrated by the following commutative diagram.

$$\begin{array}{ccccc}
 & & \widehat{F} & & \\
 & \nearrow & & \searrow & \\
 F & & & & F_{f-nil} \xrightarrow{\simeq} \prod_{\ell} F^{\ell} \\
 & \searrow & & \nearrow & \\
 & & F_{nil} & & 
 \end{array}$$

Also note that the universal property of free groups extends to their completions. Assume for simplicity that  $F$  is free on two generators  $x, y$ . Choosing two elements  $\bar{x}, \bar{y}$  in an arbitrary group  $G$  amounts to defining a morphism  $\varphi : F \rightarrow G$ , such that  $\varphi(f) = f(\bar{x}, \bar{y})$ . If  $G$  is finite (respectively nilpotent, finite and nilpotent, or a finite  $\ell$ -group), this morphism extends to the profinite (respectively pro-nilpotent, pro-finite-nilpotent, pro- $\ell$ ) completion of  $F$ . In the general case it extends to a morphism between the corresponding completions of  $F$  and  $G$ . If  $f$  lies in one these completions, we will denote  $f(x, y)$  the image of  $x, y$  under this morphism.

The braid groups are residually finite, so  $B_n$  embeds into  $\widehat{B_n}$ . This can be deduced from the same statement for the free groups, as automorphism groups of residually finite groups are residually finite by a classical result of Baumslag.

However,  $B_n$  does not embed in  $(B_n)_{nil}$  for  $n \geq 3$ . This comes from the fact that its abelianization  $B_n/C^1 B_n$  is isomorphic to  $\mathbb{Z}$ . Indeed, if  $G/C^1 G \simeq \mathbb{Z}$  it is known (see [Bo] §4 ex. 7) that  $C^{r+1} G = C^r G$  for all  $r \geq 1$ , whence  $G_{nil} = G/C^1 G = \mathbb{Z}$  (and  $G^\ell \simeq \mathbb{Z}_\ell$ ).

**2.2. Maltsev Hull.** Let  $N$  be a torsion-free nilpotent group, assumed to be finitely generated for simplicity. Recall that a group is called divisible if, for all  $r \in \mathbb{Z} \setminus \{0\}$ , the map  $x \mapsto x^r$  is a bijection. The Maltsev hull of  $N$  is defined by the following classical result.

**Theorem 2.1.** (*Maltsev*) *There exists a unique embedding  $j : N \hookrightarrow N \otimes \mathbb{Q}$  where  $N \otimes \mathbb{Q}$  is a divisible nilpotent group such that  $\forall x \in N \otimes \mathbb{Q} \ \exists r \in \mathbb{Z} \setminus \{0\} \ x^r \in j(N)$ .*

It is also known that these divisible nilpotent groups are unipotent algebraic groups over  $\mathbb{Q}$  (see the [Bo] for the full details and [De] §9 for a crash course on this correspondence). It follows that we can define  $N \otimes \mathbb{k}$ , for an arbitrary commutative  $\mathbb{Q}$ -algebra  $\mathbb{k}$  with unit, by the  $\mathbb{k}$  points of the unipotent algebraic group  $N \otimes \mathbb{Q}$ . Schematically,  $N \otimes \mathbb{k} = (N \otimes \mathbb{Q})(\mathbb{k})$ .

**Example 1.** (see [Bo] §4 ex. 15 and ex. 16.) Assume that  $N < T_n(\mathbb{Z})$  where  $T_n(A)$  denote the set of unipotent upper-triangular matrices with coefficients in  $A$ . Then  $N < T_n(\mathbb{Q})$  and  $N \otimes \mathbb{Q}$  can be chosen to be the subgroup of  $T_n(\mathbb{Q})$  defined by  $\{g \in T_n(\mathbb{Q}) \mid \exists r \in \mathbb{Z} \setminus \{0\} \ g^r \in N\}$ . This example is universal in the sense that any finitely generated torsion-free unipotent group can be embedded in some  $T_n(\mathbb{Z})$ . Moreover  $N \otimes \mathbb{k}$  can be seen inside  $T_n(\mathbb{k})$  in the same way.

**Example 2.** Let  $H$  be the torsion-free nilpotent group defined by the presentation  $\langle A, B, C \mid (A, B) = C, (A, C) = (B, C) = 1 \rangle$ . It is known and easily checked that  $H$  admits a normal form of the following type : every  $h \in H$  can be expressed uniquely as  $A^\alpha B^\beta C^\gamma$  for some  $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ . In other words, the group  $(H, \cdot)$  can be identified to some group structure  $(\mathbb{Z}^3, \star)$  on  $\mathbb{Z}^3$ . This group law on  $\mathbb{Z}^3$  is clearly polynomial, more precisely given by

$$(u, v, w) \star (x, y, z) = (u + x, v + y, w - xv + z)$$

and thus it extends to a group structure on  $\mathbb{Q}^3$ . Again, the same construction yields  $H \otimes \mathbb{k}$  by replacing  $\mathbb{Q}^3$  by  $\mathbb{k}^3$ .

**2.3. Maltsev completion of the free groups.** A free group  $F$  is not torsion-free nilpotent, however it has the nice property that each nilpotent quotient  $F/C^r F$  is torsion-free. It follows that  $F$  embeds into

$$F(\mathbb{k}) = \varprojlim_r F/C^r F \otimes \mathbb{k}$$

which is called the Maltsev or the pro- $\mathbb{k}$ -unipotent completion of  $F$ . A non-canonical but explicit description of  $F(\mathbb{k})$  can be obtained as follows. Assume for convenience that  $F$  is a free group on two generators  $x, y$  and consider two non-commutative variables  $A$  and  $B$ . We form

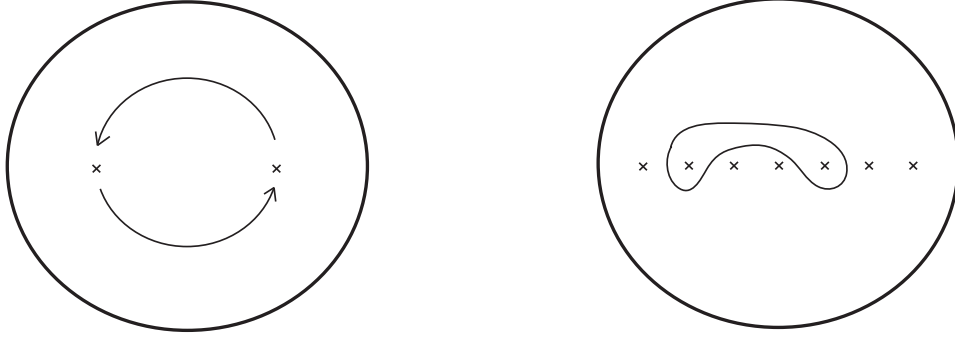
- the free associative  $\mathbb{k}$ -algebra with unit on  $A, B$ , denoted  $\mathbb{k} \langle A, B \rangle$  (“non-commutative polynomials in  $A, B$ ”)
- the completion  $\mathbb{k} \ll A, B \gg$  of the former, with respect to the grading  $\deg A = \deg B = 1$  (“non-commutative formal series in  $A, B$ ”)
- the free Lie  $\mathbb{k}$ -algebra on  $A, B$ , denoted  $\mathfrak{F}$  (“Lie polynomials in  $A, B$ ”)

The Lie algebra  $\mathfrak{F}$  is also graded by  $\deg A = \deg B = 1$  and it is a classical fact that the completion  $\widehat{\mathfrak{U}\mathfrak{F}}$  of its universal enveloping algebra is canonically isomorphic to  $\mathbb{k} \ll A, B \gg$ . In particular, the group  $\exp \widehat{\mathfrak{F}}$  can be seen as a group of invertible elements in  $\mathbb{k} \ll A, B \gg$ . By sending  $x$  to  $\exp A$  and  $y$  to  $\exp B$  we define a morphism  $F \rightarrow \exp \widehat{\mathfrak{F}}$ . This morphism extends to the pro- $\mathbb{k}$ -unipotent completion  $F(\mathbb{k})$  and induces an isomorphism  $F(\mathbb{k}) \simeq \exp \widehat{\mathfrak{F}}$ :

$$\begin{array}{ccc} F & \xrightarrow{\quad} & \exp \widehat{\mathfrak{F}} \subset \mathbb{k} \ll A, B \gg \\ & \searrow & \nearrow \simeq \\ & F(\mathbb{k}) & \end{array}$$

This description makes it easier to compute with the pro- $\ell$  completion of free groups, for the natural map  $F \rightarrow F(\mathbb{Q}_\ell)$  factorizes through  $F^\ell$ . Moreover, it is a classical theorem of Lazard that the induced map  $F^\ell \rightarrow F(\mathbb{Q}_\ell)$  is injective.

$$\begin{array}{ccc} F & \xrightarrow{\quad} & F(\mathbb{Q}_\ell) \\ & \searrow & \nearrow \\ & F^\ell & \end{array}$$

FIGURE 2. General braid twist and  $\xi_{25}$  as a Dehn twist.

### 3. THE GROTHENDIECK-TEICHMÜLLER GROUPS

**3.1. Reminder on braid groups.** In order to define the Grothendieck-Teichmüller group we need the classical presentation of  $B_n$ , that we recall. In a small disk surrounding the points  $i$  and  $i+1$  in  $D_n$  one can define a diffeomorphism which fixes the boundary and sends  $i$  to  $i+1$  by a half-turn rotation. More precisely, a diffeomorphism of the unit disk in  $\mathbb{C}$  exchanging  $\frac{1}{2}$  and  $\frac{-1}{2}$  while fixing the boundary is easily defined by  $z \mapsto \exp^{2i\pi|z|} z$ , and we apply this to the chosen disk and extend it on  $D_n$  by the identity.

We denote  $\sigma_i$  the class of this diffeomorphism in  $B_n$ . These elements for  $1 \leq i \leq n-1$  generate  $B_n$  with presentation  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for  $|j-i| \geq 2$  and  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ .

The braid group  $B_n$  acts on the set of punctures, thus defining a morphism  $B_n \rightarrow \mathfrak{S}_n$ . The kernel of this map is the pure braid group  $P_n$ . It can be seen as a mapping class group and is generated by Dehn twists. Recall that a Dehn twist along a simple closed curve inside  $D_n$  is defined by applying the diffeomorphism  $(t, z) \rightarrow (t, e^{2i\pi t} z)$  of the annulus  $[0, 1] \times \{z \in \mathbb{C} \mid |z| = 1\}$  to a tubular neighbourhood of the curve and extending it trivially outside this neighbourhood. Note that all Dehn twists of  $D_n$  obviously belong to  $P_n$ .

For  $1 \leq i < j \leq n$  we define the usual generators  $\xi_{i,j} \in P_n$  to be the Dehn twists along a curve surrounding  $i$  and  $j$  while passing above the points in-between (see figure 2). They are all conjugated in  $B_n$ .

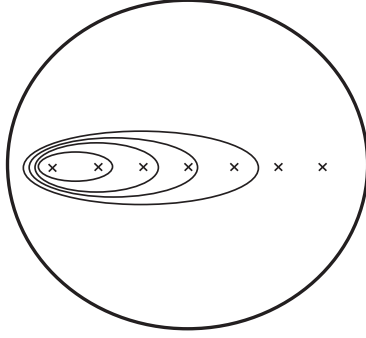
**3.2. The Grothendieck-Teichmüller group.** The Grothendieck-Teichmüller group  $\widehat{\text{GT}}$  is defined as a set of couples  $(\lambda, f) \in \widehat{\mathbb{Z}}^\times \times C^1 \widehat{F}$  satisfying the following relations :

- (I)  $f(x, y)f(y, x) = 1$
- (II)  $y^\mu f(x, y)x^\mu f(z, x)z^\mu f(y, z) = 1$ , where  $\mu = (\lambda - 1)/2$ ,  $z = (xy)^{-1}$
- (III)  $f(\xi_{12}, \xi_{23}\xi_{24})f(\xi_{13}\xi_{23}, \xi_{34}) = f(\xi_{23}, \xi_{34})f(\xi_{12}\xi_{13}, \xi_{24}\xi_{34})f(\xi_{12}, \xi_{23})$  in  $\widehat{P}_4$

Note that (II) implicitly states that  $\lambda \in 1 + 2\widehat{\mathbb{Z}}$ . Drinfeld defined in [Dr] an associative composition law on the set of couples satisfying (I)-(II)-(III), namely

$$(\lambda_1, f_1)(\lambda_2, f_2) = (\lambda_1 \lambda_2, f_1 \left( f_2 x^{\lambda_2} f_2^{-1}, y^{\lambda_2} \right) f_2)$$

for which  $(1, 1)$  is the neutral element. He then defined the group  $\widehat{\text{GT}}$  as the subset of invertible elements for this law.

FIGURE 3. The Dehn twists  $\delta_2, \delta_2\delta_3, \delta_2\delta_3\delta_4, \delta_2\delta_3\delta_4\delta_5$ 

This (non-obvious) group structure satisfies that the map  $(\lambda, f) \rightarrow \lambda$  extends to an homomorphism  $\widehat{\text{GT}} \rightarrow \widehat{\mathbb{Z}}^\times$ . One of the main feature of  $\widehat{\text{GT}}$  is that it contains  $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$  and that the composite map  $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \rightarrow \widehat{\mathbb{Z}}$  is the cyclotomic character, whose kernel is the absolute Galois group of the field  $\mathbb{Q}(\mu_\infty)$  of cyclotomic numbers. Denoting  $\widehat{\text{GT}}_1$  the kernel of the projection  $\widehat{\text{GT}} \rightarrow \widehat{\mathbb{Z}}^\times$  the following natural diagram commutes.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \widehat{\text{GT}}_1 & \longrightarrow & \widehat{\text{GT}} & \longrightarrow & \widehat{\mathbb{Z}}^\times \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \parallel \\
 1 & \longrightarrow & \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}(\mu_\infty)) & \longrightarrow & \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) & \longrightarrow & \widehat{\mathbb{Z}}^\times \longrightarrow 1
 \end{array}$$

The group  $\widehat{\text{GT}}$  can be seen as an automorphism group of the profinite completion  $\widehat{B}_n$  of  $B_n$ , which gives a natural description of its composition law. For this, we need to define a collection of elements  $\delta_r$  in  $B_n$ , such that  $\delta_2 = \sigma_1^2$ , and such that  $\delta_2\delta_3 \dots \delta_r$  is the Dehn twist associated to a curve surrounding the first  $r$  points (see figure 3). Note that these curves can be chosen so that they do not intereseect each other. This implies that the elements  $\delta_2, \dots, \delta_r$  commute one to the other. The collection of these curves can be considered as a pants decomposition of  $D_n$ .

The action of  $(\lambda, f) \in \widehat{\text{GT}}$  on  $\widehat{B}_n$  is defined by  $\sigma_1 \mapsto \sigma_1^\lambda$  and

$$(\lambda, f) \cdot \sigma_r = f(\sigma_r^2, \delta_r) \sigma_r^\lambda f(\sigma_r^2, \delta_r)^{-1}$$

for  $r \geq 2$ . This identifies  $\widehat{\text{GT}}$  to a subgroup of  $\widehat{B}_n$ .

**3.3. Its pro- $\ell$  and  $\mathbb{k}$ -prounipotent versions.** The relations defining  $\widehat{\text{GT}}$  as well as the composition law make sense if  $(\lambda, f)$  are taken in the other previously described completions of  $\mathbb{Z}$  and  $F_2$ , respectively. This in particular defines variations  $\text{GT}^\ell$ , and  $\text{GT}(\mathbb{k})$ , by considering the pro- $\ell$  and pro- $\mathbb{k}$ -unipotent completions of  $F_2$ , respectively. If we denote  $\underline{\text{GT}}(\mathbb{k})$  the semigroup of elements  $(\lambda, f) \in \mathbb{k}^\times \times C^1 F_2(\mathbb{k})$  satisfying the analogues of (I)-(II)-(III), then  $\text{GT}(\mathbb{k})$  is simply the subset of  $\underline{\text{GT}}(\mathbb{k})$  defined by  $\lambda \neq 0$ . We define  $\text{GT}_1^\ell$ , and  $\text{GT}_1(\mathbb{k})$  in the obvious way.

We have natural morphisms  $\widehat{\text{GT}} \twoheadrightarrow \text{GT}^\ell \hookrightarrow \text{GT}(\mathbb{Q}_\ell)$ . They are compatible with the extensions of the cyclotomic character, namely the following diagram commutes.

$$\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\mathrm{GT}}_1 & \twoheadrightarrow & \mathrm{GT}_1^\ell & \hookrightarrow & \mathrm{GT}_1(\mathbb{Q}_\ell) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\mathrm{GT}} & \twoheadrightarrow & \mathrm{GT}^\ell & \hookrightarrow & \mathrm{GT}(\mathbb{Q}_\ell) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\mathbb{Z}}^\times & \twoheadrightarrow & \mathbb{Z}_\ell^\times & \hookrightarrow & \mathbb{Q}_\ell^\times \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}$$

Defining an action of  $\mathrm{GT}^\ell$  on the pro- $\ell$  completion of  $B_n$  would have little meaning, as this latter completion is isomorphic to  $\mathbb{Z}_\ell$ . The same holds for  $\mathrm{GT}(\mathbb{k})$  and the classical pro- $\mathbb{k}$ -unipotent completion of  $B_n$ . The trick used by Drinfeld in [Dr] is to define a twisted completion  $B_n(\mathbb{k})$ , loosely speaking, as a semi-direct product of  $\mathfrak{S}_n$  and

$$P_n(\mathbb{k}) = \varprojlim_r P_n/C^r P_n \otimes \mathbb{k}.$$

This group  $P_n(\mathbb{k})$  is well-defined because the groups  $P_n/C^r P_n$  can be proved to be torsion-free, however it does not admit an action of  $\mathfrak{S}_n$ , thus one has to use a refinement of the semi-direct product construction instead. The Drinfeld completion  $B_n(\mathbb{k})$  of  $B_n$  is defined as the quotient of  $B_n \rtimes P_n(\mathbb{k})$  by the subgroup generated by the elements  $\sigma \cdot \sigma^{-1}$  for  $\sigma \in P_n$ , considered on the left as elements of  $B_n$  and on the right as elements of  $P_n(\mathbb{k})$ .

Contrary to what happens for a genuine pro- $\mathbb{k}$ -unipotent completion, it is not possible to define  $x^\lambda$  for an arbitrary  $x \in B_n(\mathbb{k})$  and  $\lambda \in \mathbb{k}^\times$ , nor  $f(x, y)$  for  $f \in F_2(\mathbb{k})$ . This however makes sense if  $x \in P_n(\mathbb{k})$ . If  $x^2 \in P_n(\mathbb{k})$ , denoting  $x^{<\lambda>}$  the element  $x(x^2)^{\frac{\lambda-1}{2}}$ , Drinfeld defined an action by automorphisms of  $\mathrm{GT}(\mathbb{k})$  on  $B_n(\mathbb{k})$  which sends  $\sigma_1$  to  $\sigma_1^{<\lambda>}$  and  $\sigma_r$  to  $f(\sigma_r^2, \delta_r) \sigma_r^{<\lambda>} f(\sigma_r^2, \delta_r)^{-1}$  for  $r \geq 2$ . This action sends  $\delta_r$  to  $\delta_r^\lambda$ .

#### 4. AN EXAMPLE OF PROFINITE RIGIDITY : THE KUMMER CHARACTER $\rho_2$

Recall that  $B_3$  with generators  $\sigma_1, \sigma_2$  is presented by the unique relation  $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ . We consider the representations  $R : B_3 \rightarrow \mathrm{SL}_2(A)$ , where  $A$  is a commutative integral ring with 1, such that  $R(\sigma_1) = a$ , where

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We have the following lemma, that we admit for now, as it will be a special case of a more general one.

**Lemma 4.1.** *If  $R(\sigma_1) = a$ , then  $R(\sigma_2) = R(\sigma_1)$  or there exists  $u \in A$  such that  $R(\sigma_2) = b(u)$ , where*

$$b(u) = \begin{pmatrix} 1-u & u^2 \\ -1 & 1+u \end{pmatrix}$$

On the other hand, the centralizer of  $a$  in  $\mathrm{PGL}_2(A)$  is easily computed. A complete system of representatives in  $\mathrm{GL}_2(A)$  of this centralizer is given by the matrices  $c(v) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$  for  $v \in A$ . It is easily checked that, if  $b = b(1)$ , then  $c(v)bc(v)^{-1} = b(v)$ . Hence we have

**Lemma 4.2.** *If  $R_1, R_2 : B_3 \rightarrow \mathrm{SL}_2(A)$  satisfy  $R_1(\sigma_1) = R_2(\sigma_2) = a$  and  $R_i(\sigma_1) \neq R_i(\sigma_2)$  for  $i \in \{1, 2\}$ , then they are conjugated by an element of  $\mathrm{PGL}_2(A)$ .*

Choose  $A = \mathbb{Z}_\ell$  and define  $R : B_3 \rightarrow \mathrm{SL}_2(\mathbb{Z}_\ell)$  by  $R(\sigma_1) = a$  and  $R(\sigma_2) = b$ . Since  $\mathrm{SL}_2(\mathbb{Z}_\ell)$  has natural structure of a profinite group,  $R$  extends to a representation of  $\widehat{B}_3$ . Let  $g \in \widehat{\mathrm{GT}}_1$ . Then  $R' = R \circ g$  defines a representation of  $\widehat{B}_3$  such that  $R'(\sigma_1) = a$ . We have  $R'(\sigma_2) \neq R'(\sigma_1)$ ; indeed, since  $R = R' \circ g^{-1}$ ,  $R(\sigma_2)$  would otherwise be conjugated to  $R(\sigma_1) = R'(\sigma_1)$  by an element of the commutator subgroup of  $\langle R'(\sigma_1), R'(\sigma_2) \rangle$  which is trivial, contradicting  $R(\sigma_1) \neq R(\sigma_2)$ .

It follows from lemma 4.2 that there exists a well-defined  $\rho(g) \in \mathbb{Z}_\ell$  such that  $R'(\sigma_2) = c_g^{-1}bc_g$  with  $c_g = c(\rho(g))$ . This  $\rho : \widehat{\mathrm{GT}}_1 \rightarrow \mathbb{Z}_\ell$  is a continuous character of  $\widehat{\mathrm{GT}}_1$ , arising from the rigidity of the representation  $R$ .

Since, in the profinite group  $\mathrm{SL}_2(\mathbb{Z}_\ell)$ , we have, for all  $x \in \widehat{\mathbb{Z}}$ ,

$$a^x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

which is conjugated to  $a$ , we can extend  $\rho$  to a 1-cocycle of  $\widehat{\mathrm{GT}}$  in the same spirit. More precisely, let  $g = (\lambda, f) \in \widehat{\mathrm{GT}}$ . The above calculation shows that  $R \circ g(\sigma_1) = R(\sigma_1)^\lambda$ . On the other hand, lemma 4.1 can be strenghtend as follows. Let

$$a_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

**Lemma 4.3.** *Let  $\lambda \in A$ . If  $R(\sigma_1) = a_\lambda$ , then  $R(\sigma_2) = R(\sigma_1)$  or  $\lambda \in A^\times$  and there exists  $u \in A$  such that  $R(\sigma_2) = b_\lambda(u)$ , where*

$$b_\lambda(u) = \begin{pmatrix} 1 - u & \lambda u^2 \\ -1/\lambda & 1 + u \end{pmatrix}$$

*Proof.* Since  $a$  and  $b$  have to be conjugate, we are looking for an invertible matrix of trace 2, of the form

$$b = \begin{pmatrix} 1 - u & v \\ w & 1 + u \end{pmatrix}$$

such that  $aba = bab$ . Since the determinant of  $b$  has to be 1, we get  $vw = -u^2$ . The equation  $aba = bab$  yields to the system

$$\begin{cases} u(1 + \lambda w) & = 0 \\ \lambda + \lambda^2 w - v + \lambda u^2 & = 0 \\ -w(1 + \lambda w) & = 0 \\ -u(1 + \lambda w) & = 0 \end{cases}$$

Since  $A$  is integral, the third equation means  $w = 0$  or  $\lambda w = -1$ . But  $w = 0$  implies  $u = 0$  and  $v = 1$  hence  $a = b$ . The other possibility is that  $\lambda \in A^\times$  and  $w = -1/\lambda$ . It follows that the system of equations boils down to  $v = \lambda u^2$  which is the form required.  $\square$



We have  $c(v)b_\lambda c(v)^{-1} = b_\lambda(v)$ . Introduce the right action of  $A^\times$  on  $\mathrm{SL}_2(A)$  induced by the morphism  $A^\times = \mathrm{GL}_1(A) \rightarrow \mathrm{GL}_2(A)$  which sends  $\lambda$  to

$$d_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

and by the action by conjugation of  $\mathrm{GL}_2(A)$  on  $\mathrm{SL}_2(A)$ , that is  $x.\lambda = d_\lambda^{-1}x d_\lambda$ . Then  $b_\lambda(u) = d_\lambda^{-1}b(u)d_\lambda = b(u).\lambda$ . It follows that  $R \circ g(\sigma) = c_g^{-1}(R(\sigma).\lambda)c_g$  for all  $\sigma \in \widehat{B}_3$  and some well-defined  $c_g = c_{\rho(g)}$  with  $\rho(g) \in A$ , and it is easily checked that  $c_{g_1 g_2} = (c_{g_1}.\lambda_2)c_{g_2}$  hence  $\rho(g_1 g_2) = \lambda_2 \rho(g_1) + \rho(g_2)$ . This shows that  $\rho$  can be extended to a continuous 1-cocycle of  $\widehat{\mathrm{GT}}$  with values in  $\mathbb{Z}_\ell$ , endowed with the action by multiplication induced by  $\widehat{\mathrm{GT}} \rightarrow \mathbb{Z}_\ell^\times$ .

For  $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ ,  $\rho$  coincides with the  $\ell$ -part of  $-8\rho_2$ , where  $\rho_2 : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \rightarrow \widehat{\mathbb{Z}}$  is the Kummer character of (positive) roots of 2, defined by

$$\sigma(\sqrt[n]{2}) = \zeta_n^{\rho_2(\sigma)} \sqrt[n]{2}$$

where  $\zeta_n = \exp(2i\pi/n)$ . This can be inferred from [Na], corollary 4.13, where  $f(a^2, b^2)$  is calculated in  $\mathrm{SL}_2(\widehat{\mathbb{Z}})$  for  $(\lambda, f)$  the image in  $\widehat{\mathrm{GT}}$  of  $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ . Using this calculation we find in this situation

$$R \circ g(\sigma_2) = f(a^2, b^2)^{-1} b^\lambda f(a^2, b^2) = b_\lambda(8\rho_2 \lambda^{-1}) = c(-8\rho_2)^{-1}(b.\lambda)c(-8\rho_2)$$

where  $\rho_2 = \rho_2(\sigma)$ .

It has been shown in [NS] that  $\rho_2$  can be extended to  $\widehat{\mathrm{GT}}$ . This construction thus gives an alternative simple way to do so.

Note that the morphism  $B_3 \rightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$  used here can be seen as a degeneration for  $q = -1$  of the classical Burau representation

$$\sigma_1 \mapsto \begin{pmatrix} -q & 1 \\ 0 & 1 \end{pmatrix} \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ q & -q \end{pmatrix}$$

## 5. GT-RIGIDITY

Let  $\mathbb{k}$  be a field of characteristic 0,  $\mathcal{A} = \mathbb{k}[[h]]$  the ring of formal power series and  $K = \mathbb{k}((h))$  the field of formal Laurent series. We let  $\mathrm{GL}_N^0(K) = \{x \in \mathrm{GL}_N(K) \mid x \equiv 1 \pmod{h}\} \subset \mathrm{GL}_N(\mathcal{A})$ .

We consider representations  $R : B_n \rightarrow \mathrm{GL}_N(K)$  such that  $R(P_n) \subset \mathrm{GL}_N^0(K)$ . This implies that  $R(B_n) \subset \mathrm{GL}_N(\mathcal{A})$ , as is readily checked. Such a representation can be naturally extended to  $B_n(\mathbb{k})$ .

A natural idea would be to call an irreducible representation  $R : B_n \rightarrow \mathrm{GL}(V)$  GT-rigid if  $R \circ g \simeq R$  for all  $g \in \mathrm{GT}(\mathbb{k})$ . However, this notion has little interest, as shows the following proposition.

**Proposition 5.1.** *If  $R \circ g \simeq R$  for all  $g \in \mathrm{GT}(\mathbb{k})$ , then  $R(P_n) = \{1\}$  and  $R \circ g = R$  for all  $g \in \mathrm{GT}(\mathbb{k})$ .*

*Proof.* We know that the morphism  $\mathrm{GT}(\mathbb{k}) \rightarrow \mathbb{k}^\times$  is surjective. Let  $\lambda \mapsto g_\lambda$  be a set-theoretical section of this morphism. By assumption we have

$$R(\sigma_1^2)^\lambda = R((\sigma_1^2)^\lambda) = R(g_\lambda.\sigma_1^2) \sim R(\sigma_1^2)$$

for all  $\lambda \in \mathbb{k}^\times$ . By comparing spectrums in some algebraic closure of  $K$  it follows that, since  $\mathbb{k}$  is infinite,  $R(\sigma_1^2)$  is unipotent. Thanks to a result of Drinfeld (see [Dr] propositions 5.2 and 5.3), we can choose  $\lambda \mapsto g_\lambda$  to be an algebraic section of  $\mathrm{GT}(\mathbb{k}) \rightarrow \mathbb{k}^\times$ . Writing  $R(\sigma_1^2)^\lambda = (1 + N)^\lambda$  with  $N$  nilpotent we show that  $R(\sigma_1^2)^\lambda$  tends to 1 when  $\lambda \rightarrow 0$  namely that the 1-parameter subgroup  $\lambda \mapsto R \circ g_\lambda(\sigma_1^2)$  can be algebraically extended to a morphism from  $\mathbb{k}$  which sends 0 to  $1 \in \mathrm{GL}_N(K)$ .

More generally, the map  $\lambda \mapsto R \circ g_\lambda \in \mathrm{Hom}(B_n(\mathbb{k}), \mathrm{GL}_N(K))$  can be chosen to be a polynomial mapping that can be specialized in  $\lambda = 0$ . Indeed, the map  $\lambda \mapsto g_\lambda$  as constructed in [Dr] extends to a polynomial mapping  $\mathbb{k} \rightarrow \underline{GT}(\mathbb{k})$  (see [Dr] §5 remark 2 after proposition 5.9) and  $\underline{GT}(\mathbb{k})$  acts by endomorphisms on  $B_n(\mathbb{k})$ . It follows that the image of any generator  $\xi_{i,j}$  of  $P_n$  under  $R \circ g_\lambda$  can be written as  $1 + N_{i,j}(\lambda)$  where  $N_{i,j}(\lambda)$  is a nilpotent matrix with values in  $K[\lambda]$ .

Let us consider the character  $\chi_\lambda$  of the restriction of  $R \circ g_\lambda$  to  $P_n$ . For all  $\sigma \in P_n$  we know that  $R \circ g_\lambda$  has coefficients in  $K[\lambda]$  hence  $\chi_\lambda(g) \in K[\lambda]$ . By assumption it is independant of  $\lambda \in \mathbb{k}^\times$  and equals  $\dim V$  for  $\lambda = 0$ , whence we get that it is the character of the trivial representation. Since  $K$  has characteristic 0, characters determine isomorphism classes of the semisimplification of representations. It follows that the restriction of  $R$  to  $P_n$  is a unipotent representation. On the other hand, it is the restriction to a normal subgroup of a (semi)simple representation of  $B_n$ , hence it is semisimple. It follows that  $R(P_n) = \{1\}$ . By the formulas defining the action of  $\mathrm{GT}(\mathbb{k})$  it also follows that  $R \circ g = R$  for all  $g \in \mathrm{GT}(\mathbb{k})$ .  $\square$

This proof shows that an obstacle comes from the quotient map  $\pi : \mathrm{GT}(\mathbb{k}) \rightarrow \mathbb{k}^\times$ . It is the only one, namely there exist representations  $R$  such that  $R \circ g \simeq R$  for all  $g$  in the kernel  $\mathrm{GT}_1(\mathbb{k})$  of this map.

**Definition 5.2.**  *$R$  is called  $\mathrm{GT}_1$ -rigid if  $\forall g \in \mathrm{GT}_1(\mathbb{k}) \quad R \circ g \simeq R$ .*

If  $R$  is such a representation we thus get a projective representation  $Q_R : \mathrm{GT}_1(\mathbb{k}) \rightarrow \mathrm{PGL}(V)$ . Consider now the right action of  $\mathbb{k}^\times$  on  $K$  defined by  $(f.\alpha)(h) = f(\alpha h)$  for  $f(h) \in K$  and  $\alpha \in \mathbb{k}^\times$ . It induces an action of  $\mathrm{GT}(\mathbb{k})$  on  $K$  hence on  $\mathrm{PGL}(V)$  by the morphism  $\pi : \mathrm{GT}(\mathbb{k}) \rightarrow \mathbb{k}^\times$ .

**Definition 5.3.**  *$R$  is called  $\mathrm{GT}$ -rigid if  $\forall g \in \mathrm{GT}(\mathbb{k}) \quad R \circ g \simeq (\sigma \mapsto R(\sigma).\pi(g))$ .*

In particular, if  $R$  is  $\mathrm{GT}$ -rigid then it is  $\mathrm{GT}_1$ -rigid. It turns out that such representations exist. They provide (non-commutative) cocycles in  $Z^1(\mathrm{GT}(\mathbb{k}), \mathrm{PGL}_N(K))$ .

## 6. BURAU REPRESENTATION

**6.1. The Burau characters.** Let  $q = e^h \in K$ . We define the classical (reduced) Burau representation by the following formulas. Let  $e_1, \dots, e_{n-1}$  be the canonical basis of  $K^{n-1}$ . We let  $\sigma_1.e_1 = qe_1$  and  $\sigma_1.e_m = -q^{-1}e_m$  for all  $m \neq 1$ . For  $2 \leq d \leq r-1$  we let  $\sigma_d.e_m = -q^{-1}e_m$  if  $m \notin \{d, d+1\}$ , and we make  $\sigma_d$  act on the plane with basis  $(e_{d-1}, e_d)$  by the matrix

$$\frac{1}{[d]_q} \begin{pmatrix} -q^{-d} & * \\ * & q^d \end{pmatrix}$$

with

$$* = \sqrt{[d+1]_q[d-1]_q} \quad \text{and} \quad [\alpha]_q = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.$$

This defines an irreducible  $(n - 1)$ -dimensional representation of  $B_n$ , called the Burau representation. It can be shown (see [Ma]) that this representation is GT-rigid, hence we get a projective representation  $Q_R : \text{GT}_1(\mathbb{k}) \rightarrow \text{PGL}_{n-1}(K)$ .

$$\begin{cases} \delta_r e_{r-1} &= q^2 e_{r-1} \\ \delta_r e_s &= q^{-2(r-2)} e_s \text{ if } s < r-1 \\ \delta_r e_s &= q^{-2(r-1)} e_s \text{ if } s > r-1 \end{cases}$$

It is easily checked that the algebraic hull of the subgroup generated by these matrices is the set of diagonal matrices, that is a maximal torus of  $\text{GL}_{n-1}(K)$ . Since the  $\delta_r$ 's are fixed by  $\text{GT}_1(\mathbb{k})$ , it follows that the image of  $Q_R$  centralizes this torus, hence is part of it. We thus can write  $Q_R(g)$  as the image in  $\text{PGL}_{n-1}(K)$  of a well-defined matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & & \\ 0 & \chi_2(g) & 0 & & \\ 0 & 0 & \chi_2 \chi_3(g) & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}$$

where  $\chi_d : \text{GT}_1(\mathbb{k}) \rightarrow K^\times$  is a character that can be extended to a 1-cocycle  $\text{GT}(\mathbb{k}) \rightarrow K^\times$ . This decomposition of  $Q_R$  is compatible with the usual embedding  $B_n \subset B_{n+1}$ , so we do not need to specify  $n$  and, as  $n$  varies, this defines an infinite family  $\chi_d, d \geq 2$  of characters of  $\text{GT}_1(\mathbb{k})$ .

**6.2. Transcendental aspects.** Recall from [Dr] that  $\Phi_{KZ}$  is defined as  $G_1^{-1}G_0$ , where  $G_1, G_0$  are the solutions over  $]0, 1[$  of the differential equation

$$G'(x) = \frac{1}{2i\pi} \left( \frac{A}{x} + \frac{B}{x-1} \right) G(x)$$

with values in  $\mathbb{C} \ll A, B \gg$ , defined by the asymptotic conditions  $G_0(x) \sim x^{\frac{A}{2i\pi}}$  when  $x \rightarrow 0$  and  $G_1(x) \sim (1-x)^{\frac{B}{2i\pi}}$  when  $x \rightarrow 1$ . Moreover,  $\Phi_{KZ}$  as well as  $\overline{\Phi_{KZ}}(A, B) = \Phi_{KZ}(-A, -B)$  belong to the  $\mathbb{C}$ -points of an affine scheme  $\text{ASSOC}_1$  over  $\mathbb{Q}$  which is a torsor under  $\text{GT}_1$ . In particular there exists a well-defined  $g_0 \in \text{GT}_1(\mathbb{C})$  such that  $\overline{\Phi_{KZ}} = g_0 \cdot \Phi_{KZ}$ . Using the work of J. González-Lorca in [Go], it was shown in [Ma] that

$$\chi_d(g_0) = \exp \left( 2 \sum_{n=1}^{+\infty} \frac{\zeta(2n+1)}{2n+1} \hbar Q_n(d) \right)$$

where  $Q_n(d) = (d+1)^{2n+1} + (d-1)^{2n+1} - 2d^{2n+1}$  and  $\hbar = h/i\pi$ .

**6.3. Arithmetic aspects.** On the arithmetic side, if we fix  $\ell$  and denote  $\bar{\sigma}$  the image of  $\sigma \in \text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}(\mu_{\ell^\infty}))$  in  $\text{GT}_1(\mathbb{Q}_\ell)$ , we have

$$\chi_d(\bar{\sigma}) = 1 - 8\kappa_3^*(\sigma)dh^3 - \frac{8}{3}\kappa_5^*(\sigma)d(1+2d^2)h^5 + \dots$$

where  $\kappa_m^*(\sigma) = \kappa_m(\sigma)/(\ell^{m-1}-1)$  and  $\kappa_m$  are  $\mathbb{Z}_\ell$ -valued characters called the Soulé characters. They can be elementary defined as follows (see [IS]). Let  $\zeta_n$  denote a primitive  $\ell^n$ -th root of 1, and introduce

$$\epsilon_{m,n} = \prod_{\substack{0 < a < \ell^n \\ a \wedge \ell = 1}} (\zeta_n^a - 1)^{[a^{m-1}]}$$

where  $[a^{m-1}]$  denotes the (non-negative) euclidean remainder of the division of  $a^{m-1}$  by  $\ell^n$ . It can be proved that these elements are totally real and totally positive, and  $\kappa_m$  is defined by

$$\sigma(\sqrt[n]{\epsilon_{m,n}}) = \zeta_n^{\kappa_m(\sigma)} \sqrt[n]{\sigma(\epsilon_{m,n})} \zeta_n^{\kappa_m(\sigma)}.$$

Taking the logarithm of  $\chi_d$ , this proves that the Soulé characters, like the Kummer character  $\rho_2$ , can also be extended to  $\widehat{\text{GT}}$ , at least for  $m \in \{3, 5\}$ .

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