

CRYSTALLOGRAPHIC GROUPS AND FLAT MANIFOLDS FROM COMPLEX REFLECTION GROUPS

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ABSTRACT. Following an idea of Gonçalves, Guaschi and Ocampo on the usual braid group we construct crystallographic and Bieberbach groups as (sub)quotients of the generalized braid group associated to an arbitrary complex reflection group.

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1. INTRODUCTION

In the paper [6], Gonçalves, Guaschi and Ocampo notice that the quotient of the (usual) braid group B_n by the commutator subgroup (P_n, P_n) of the pure braid is a crystallographic group. They then prove the remarkable fact that this quotient has no 2-torsion. From this, they can build Bieberbach groups, namely torsion-free crystallographic groups (characterizing compact flat manifolds), by taking the preimage under the natural projection of a 2-subgroup of the symmetric group \mathfrak{S}_n . In the same paper, they prove a number of results on the finite-order elements and finite subgroups of this quotient $B_n/(P_n, P_n)$.

We prove here that *all* the results of [6] can be generalized to the following more general setting, and we provide at the same time possibly simpler proofs. Let W be a (finite) complex reflection group, and B the corresponding generalized braid group in the sense of [2]. If W is a real reflection group (a.k.a. finite Coxeter group), then B is an Artin group of finite Coxeter type. The quotient $B/(P, P)$ of B by the commutator subgroup of the pure braid group P has been studied in the ‘real’ case by J. Tits (under the name ‘ V ’) in his seminal 1966 paper [14] and by F. Digne in the unpublished paper [4]. In the general ‘complex’ case, it has been studied by the author in [10] and by V. Beck in [1].

In the present paper, we prove that this quotient is always a crystallographic group, and that it never contains elements of order 2. This provides a way to construct Bieberbach groups, by taking the preimage of the 2-Sylow subgroup of the quotient $W/Z(W)$ of W by its center. This is

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done in section 2 of the paper, in which we also prove that parabolic inclusions between reflection groups induce inclusions between the corresponding crystallographic groups.

In section 3, we describe a general way to construct elements of finite order inside $B/(P, P)$, generalizing the elements constructed in [6]. It is based on Springer's theory of regular elements and regular numbers. We connect the possible orders of these elements, that we call the *freely regular numbers*, with a previously introduced integer $\kappa(W)$, which is naturally associated with the extension $1 \rightarrow P^{ab} \rightarrow B/(P, P) \rightarrow W \rightarrow 1$ (where $P^{ab} = P/(P, P)$ denotes the abelianization of P). Conversely, we prove a criterion ensuring that $B/(P, P)$ cannot contain elements of certain orders, in addition to the powers of 2. This enables us to prove that a few groups of the form $B/(P, P)$ actually *are* Bieberbach groups. For this study, we need to provide a detailed description of the regular elements in complex reflection groups, for which we could not find an adequate reference. Finally, we explain and generalize the construction given in [6] of a non-abelian finite subgroup of order 21 inside $B_7/(P_7, P_7)$.

In the final section 4 we explore, focusing on low-dimensional examples, to which extent the constructed manifolds can be endowed with a Kähler structure.

As a concluding remark, we notice that *none* of our proofs need to use the Shephard-Todd classification of irreducible complex reflection groups, *except* the result that freely regular numbers are necessarily coprime to $\kappa(W)$. It would be nice to have a proof which does not use the classification for this result, too.

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2. MAIN CONSTRUCTION

2.1. Reminder on Bieberbach groups. Our reference on the subject will be [3]. Recall from there that every compact flat manifold can be obtained as a quotient \mathbf{R}^N/Γ , where Γ is any torsion-free cocompact (a.k.a. uniform) discrete subgroup of the group of affine isometries $\mathbf{R}^N \rtimes O_N(\mathbf{R})$ of the Euclidean space \mathbf{R}^N . Moreover, such groups, considered up to isomorphisms, completely characterize the manifold. This is the content of the famous Bieberbach theorems, and these groups are called Bieberbach groups. Removing the 'torsion-free' assumption defines the larger class of so-called crystallographic groups.

A classical result (see [3], theorem 2.1.4) states that an abstract group Γ is crystallographic if and only if there exists a short exact sequence

$$1 \rightarrow \mathbf{Z}^N \rightarrow \Gamma \rightarrow \Phi \rightarrow 1$$

where Φ is finite group and \mathbf{Z}^N is maximal abelian inside Γ . This is clearly equivalent to saying that Φ is finite and the natural action $\Phi \rightarrow \text{Out}(\mathbf{Z}^N) = \text{Aut}(\mathbf{Z}^N) = \text{GL}_N(\mathbf{Z})$ is faithful. If Γ is torsion-free, then Φ is the holonomy group of the corresponding manifold. In the general case, Φ is still canonically determined by Γ and we call it the holonomy group of Γ . Finally, if the group Γ has no p -torsion, then the preimage of any p -subgroup S of Φ is torsion-free, and therefore is a Bieberbach group providing a flat manifold with holonomy S .

Additional geometric properties of the manifold can be checked from the associated representation $\rho : \Phi \rightarrow \text{GL}_N(\mathbf{C})$ of the group Φ . For instance it is Kähler if and only if N is an even integer and each of the irreducible constituents of ρ which have orthogonal (or real) representation type appear an even number of times (see [8]).

2.2. Complex braid groups. We refer the reader to [2] for the construction of complex braid groups. Let $W < \text{GL}_n(\mathbf{C})$ be a complex reflection group. We let \mathcal{A} denote the set of hyperplanes fixed by the reflections of W (so-called reflecting hyperplanes), and denote $X = \mathbf{C}^n \setminus \bigcup \mathcal{A}$ their complement. We denote \mathcal{R} the set of (pseudo-)reflections in W . We set \mathcal{R}^* its subset of distinguished reflections, that is (pseudo-)reflections s with eigenvalues $\{1, e^{2\pi i/m}\}$ where m is the order of the cyclic subgroup of W fixing $\text{Ker}(s - 1) \in \mathcal{A}$. There is a natural correspondance $\mathcal{R}^* \leftrightarrow \mathcal{A}$ given by $s \mapsto \text{Ker}(s - 1)$. It is W -equivariant w.r.t. the conjugation action on \mathcal{R}^* and the permutation action on \mathcal{A} .

By definition, neglecting base points, we have $P = \pi_1(X)$ and $B = \pi_1(X/W)$. The short exact sequence $1 \rightarrow P^{ab} \rightarrow B/(P, P) \rightarrow W \rightarrow 1$ is exact, and $P^{ab} = \pi_1(X)^{ab} = H_1(X, \mathbf{Z})$ is a free \mathbf{Z} -module of finite rank, $H_1(X, \mathbf{Z}) \simeq \mathbf{Z}^N$ where $N = |\mathcal{A}|$ is the number of reflecting hyperplanes. It admits a basis $(c_H)_{H \in \mathcal{A}}$ uniquely defined by the condition $\int_{c_{H_1}} \omega_{H_2} = \delta_{H_1, H_2}$, where $\omega_H = \frac{1}{2\pi i} d\varphi_H / \varphi_H$ is the logarithmic 1-form associated to the hyperplane $H = \text{Ker} \varphi_H$, and δ_{H_1, H_2} is the Kronecker symbol. Moreover, the conjugation action of $B/(P, P)$ on P^{ab} factorizes through W and coincides with the permutation action of W on \mathcal{A} under the correspondance $H \leftrightarrow c_H$. The kernel of this permutation action is exactly the set of all elements of W which commute with all the distinguished pseudo-reflections. Since these reflections generate W , this kernel is thus exactly the centre of W .

Let $\overline{W} = W/Z(W)$, and P_0 the subgroup of $B/(P, P)$ generated by P^{ab} and by the (image of the) element $\mathbf{z} \in B$ defined as the homotopy class of $t \mapsto e^{2\pi i t/d} x_0$, where x_0 is the chosen base point in X and $d = |Z(W)|$. This element is central and its image in W generates $Z(W)$ when W is irreducible (see [2]). Moreover $\mathbf{z}^d = \pi : t \mapsto e^{2\pi i t} x_0$ is a central element in P , that can be written $\sum_{H \in \mathcal{A}} c_H$ inside P^{ab} . If $W = W_1 \times \cdots \times W_r$ is a decomposition of W in irreducible components, and $B = B_1 \times \cdots \times B_r$ is the corresponding decomposition of B , let $\mathbf{z}_1, \dots, \mathbf{z}_r$ denote the associated central elements. We let $Z_0(B)$ denote the central subgroup of B generated by $\mathbf{z}_1, \dots, \mathbf{z}_r$. Although we will not use it here it can be shown that $Z_0(B) = Z(B)$ and that the projection $B \rightarrow W$ induces a short exact sequence $1 \rightarrow Z(P) \rightarrow Z(B) \rightarrow Z(W) \rightarrow 1$ (see [5]). However, we will keep the notation $Z_0(B)$ in order to emphasize that our proofs do not make use of the classification of complex reflection groups, as opposed to the proof that $Z(B) = Z_0(B)$.

Theorem 2.1. *For every complex reflection group W , the group $B/(P, P)$ is crystallographic with holonomy group $W/Z(W)$ of dimension $N = |\mathcal{A}|$. The kernel of the projection map $B/(P, P) \rightarrow W/Z(W)$ is the subgroup P_0 generated by P^{ab} and $Z_0(B)$. We have $P_0 \simeq \mathbf{Z}^N$.*

Proof. Decomposing W into irreducibles $W_1 \times \cdots \times W_r$, we get, with obvious notations, that $P \simeq \prod_{1 \leq k \leq r} P_k$, $B/(P, P) \simeq \prod_{1 \leq k \leq r} B_k/(P_k, P_k)$, $Z_0(B) \simeq \prod_{1 \leq k \leq r} Z_0(B_k)$ and from this one easily checks that we can assume w.l.o.g. that W is irreducible.

Clearly the kernel of the composite map $B/(P, P) \rightarrow W \rightarrow \overline{W}$ contains P^{ab} and \mathbf{z} , and therefore P_0 . Conversely, if $b \in B/(P, P)$ has trivial image inside \overline{W} , then its image $\bar{b} \in W$ belongs to $Z(W)$. There exists $c \in \langle \mathbf{z} \rangle \subset P_0$ whose image \bar{c} inside W is equal to \bar{b} hence $c^{-1}b \in P^{ab}$ and $b \in P_0$. Therefore we have a short exact sequence $1 \rightarrow P_0 \rightarrow B/(P, P) \rightarrow \overline{W} \rightarrow 1$. Since P_0 is abelian we have an action $\overline{W} \rightarrow \text{Aut}(P_0)$. Since $w \in W$ acts trivially on $P^{ab} \subset P_0$ if and only if $w \in Z(W)$, this action is faithful. It remains to prove $P_0 \simeq \mathbf{Z}^N$.

We identify P^{ab} with \mathbf{Z}^N by using the basis $(c_H)_{\mathcal{A}}$ and an arbitrary total ordering on \mathcal{A} , and we let $d = |Z(W)|$. Then $\mathbf{z}^d = \pi \in P^{ab}$ corresponds to the vector $(1, \dots, 1) \in \mathbf{Z}^N \simeq P^{ab}$. By definition P_0 is a quotient of $P^{ab} \times \langle \mathbf{z} \rangle \simeq \mathbf{Z}^N \times \mathbf{Z} \simeq \mathbf{Z}^{N+1}$ and the kernel of $\mathbf{Z}^{N+1} \simeq P^{ab} \times \langle \mathbf{z} \rangle \rightarrow P_0$ can be identified with the line spanned by the vector $v = (1, 1, \dots, 1, -d)$, because $\mathbf{z}^m \in P^{ab}$ iff d divides m . Therefore, $P_0 \simeq \mathbf{Z}^{N+1}/\mathbf{Z}v \simeq \mathbf{Z}^N$ and this concludes the proof. \square

In particular we get the following characterization of when $B/(P, P)$ is crystallographic with holonomy group W . This of course includes the case of the ordinary braid group.

Corollary 2.2. *The group $B/(P, P)$ is crystallographic with holonomy group W if and only if $Z(W) = 1$.*

Of course we only need to analyse this condition $Z(W) = 1$ for *irreducible* groups. We refer the reader to [9] for the Shephard-Todd classification of such groups (see also table 1). Among irreducible *Coxeter* groups and since $Z(W) \subset \mathbf{R} \cap \mu_\infty(\mathbf{C}) = \{-1, 1\}$, this condition is equivalent to the condition $-1 \notin W$. It is well-known that this happens exactly in Coxeter types $A_n, n \geq 2$, $D_{2n+1}, n \geq 1$, $I_2(2m+1), m \geq 1$ and E_6 . Among non-real irreducible complex reflection groups, we have $Z(W) \neq 1$ for all exceptional ones. The irreducible groups inside the family $G(de, e, n)$ have center of order $d(e \wedge n)$. Therefore, the only possibility for $Z(W) = 1$ is $d = 1$ and $e \wedge n = 1$.

Thus, the non-real irreducible complex reflection groups with $Z(W) = 1$ are the $G(e, e, n)$ with $e \wedge n = 1$. Note that this includes the real groups $A_n, n \geq 2$, $D_{2n+1}, n \geq 1$ and $I_2(2m+1), m \geq 1$.

Corollary 2.3. *If the irreducible components of W are of type $G(e, e, n)$ with $e \wedge n = 1$, or of type E_6 , then $B/(P, P)$ is crystallographic with holonomy group W .*

We now prove that this construction is compatible with parabolic inclusions.

Proposition 2.4. *Let W' be a parabolic subgroup of W , and B' (resp. P') the corresponding (pure) braid group. There is an embedding $B'/(P', P') \hookrightarrow B/(P, P)$, canonical up to P^{ab} -conjugacy.*

Proof. We consider an embedding $\iota : B' \hookrightarrow B$ as defined in [2]. Such embeddings are canonical up to P -conjugacy. By composition with the canonical projection $B \rightarrow B/(P, P)$ we get a morphism $\bar{\iota} : B' \rightarrow B/(P, P)$. We want to show $\text{Ker } \bar{\iota} = (P', P')$. Since $\iota(P') \subset P$ we get $\text{Ker } \bar{\iota} \supset (P', P')$.

Let $x \in \text{Ker } \bar{\iota}$. We have $\iota(x) \in (P, P)$ hence $\iota(x) \in (P, P) \cap \iota(B')$ and we need to show $(P, P) \cap \iota(B') = (P', P')$. By commutation of the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & P' & \longrightarrow & B' & \longrightarrow & W' \longrightarrow 1 \\ & & \downarrow & & \downarrow \iota & & \downarrow \\ 1 & \longrightarrow & P & \longrightarrow & B & \longrightarrow & W \longrightarrow 1 \end{array}$$

we have $\iota(B') \cap P = \iota(P')$, and we thus need to show $(P, P) \cap \iota(P') = (P', P')$. For this we need to recall that $j = \iota|_{P'}$ is constructed from

- (1) the choice of an open ball Ω in \mathbf{C}^n
- (2) a base point $x_2 \in \Omega \cap \tilde{X}'$

where $\tilde{X}' \simeq X' \times \mathbf{C}^m$ is the complement in \mathbf{C}^n of the reflecting hyperplanes of $W' \subset W$. From this there is an isomorphism $\pi_1(X \cap \Omega, x_2) \rightarrow \pi_1(\tilde{X}', x_2)$. Letting \bar{x}_2 denote the natural projection of x_2 on X' , we have an isomorphism $\pi_1(\tilde{X}', x_2) = \pi_1(X' \times \mathbf{C}^m, x_2) \rightarrow \pi_1(X', \bar{x}_2)$. These isomorphisms identify $P' = \pi_1(X', \bar{x}_2)$ and $\pi_1(X \cap \Omega, x_2)$. The embedding of P' into $P = \pi_1(X, x_2)$ is then induced by the inclusion $X \cap \Omega \subset X$. Now, the commutator subgroup (P, P) is the kernel of the Hurewicz morphism $\pi_1(X, x_2) \rightarrow H_1(X, \mathbf{Z})$. Functoriality of this morphism applied to the inclusion $(X \cap \Omega, x_2) \subset (X, x_2)$ yields a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & (P', P') & \longrightarrow & P' & \longrightarrow & H_1(X', \mathbf{Z}) \longrightarrow 1 \\ & & \downarrow & & \downarrow \iota & & \downarrow \\ 1 & \longrightarrow & (P, P) & \longrightarrow & P & \longrightarrow & H_1(X, \mathbf{Z}) \longrightarrow 1. \end{array}$$

Finally, it is known that $H_1(X', \mathbf{Z}) \simeq H_1(X' \times \mathbf{C}^m, \mathbf{Z}) = H_1(\tilde{X}', \mathbf{Z}) \rightarrow H_1(X, \mathbf{Z})$ is an embedding, by the basic homological theory of hyperplane arrangements (see e.g. [11]). This injectivity implies $(P, P) \cap \iota(P') = (P', P')$, and this concludes the proof of the proposition. \square

2.3. Bieberbach subgroups from 2-subgroups. If G is a subgroup of W , we let $\mathfrak{P}_W(G)$ denote its inverse image under the map $B/(P, P) \rightarrow W$. If \bar{G} denote the image of G inside $\bar{W} = W/Z(W)$, we have a short exact sequence $1 \rightarrow P_0 \rightarrow \mathfrak{P}_W(G) \rightarrow \bar{G} \rightarrow 1$, and the group $\mathfrak{P}_W(G)$ is again crystallographic, of dimension $|\mathcal{A}|$ and holonomy group \bar{G} .

Theorem 2.5. *For every complex reflection group W , the group $B/(P, P)$ has no element of order 2.*

Proof. Decomposing W into irreducibles $W_1 \times \cdots \times W_r$, we get $B/(P, P) \simeq \prod_{1 \leq k \leq r} B_k/(P_k, P_k)$ and thus we can assume w.l.o.g. that W is irreducible.

Assume by contradiction that there exists $\beta \in B$ such that $\beta^2 \in (P, P)$ and $\beta \notin (P, P)$. Since P^{ab} is torsion-free we have $\beta \notin P$. Let g denote the image of β in W . Then $g^2 = 1$ and $g \neq 1$. Therefore $\mathbf{C}^n = \text{Ker}(g - 1) \oplus \text{Ker}(g + 1)$ with $E = \text{Ker}(g - 1)$, and $E^\perp = \text{Ker}(g + 1) \neq 0$. By

Steinberg's theorem, the parabolic subgroup $W_0 = \{w \in W; w|_E = 1\}$ is generated by $\mathcal{R} \cap W_0$. Since $g \in W_0$, we have $W_0 \neq \{1\}$ and therefore there exists $s \in \mathcal{R}^* \cap W_0$. We denote by $H_0 = \text{Ker}(s - 1)$ its reflecting hyperplane. We associate to each $H \in \mathcal{A}$ a linear form $\varphi_H \in (\mathbf{C}^N)^*$ with kernel H . We have $E \subset H_0$. We denote by $x_0 \in X = \mathbf{C}^N \setminus \bigcup \mathcal{A}$ the chosen basepoint, that we write as $x_0 = x_E + x_E^\perp$ according to the decomposition $\mathbf{C}^N = E \oplus E^\perp$. We let $a(t) = x_E + \exp(\pi i t)x_E^\perp$. Changing the base point amounts to conjugating all the elements of B we are interested in by an element of P , and therefore this affects neither our conditions on β – namely $\beta^2 \in (P, P)$ and $\beta \notin (P, P)$ – nor g . Therefore, up to replacing x_E by some multiple of it, we can assume that $\varphi_H(x_E) \neq 0 \Rightarrow |\varphi_H(x_E)| > |\varphi_H(x_E^\perp)|$. This implies $\varphi_H(a(t)) \neq 0$ for all t and all H , and therefore a defines an element α of B . Its square $\alpha^2 \in P$ is (the class of) the loop $t \mapsto x_E + \exp(2\pi i t)x_E^\perp$. Finally, the class $\bar{\alpha}$ of α in W is equal to g . Therefore, we can write $\beta = x\alpha$ for some $x \in P$. Then $1 = \beta^2 = x\alpha x\alpha = x\alpha x\alpha^{-1}\alpha^2 = x(g.x)\alpha^2$. We have $\alpha^2 = \sum_{H \in \mathcal{A}} a_H c_H$ with $a_H = (1/2\pi i) \int_{\alpha^2} \frac{d\varphi_H}{\varphi_H}$. If $E \subset H$, we have $\varphi_H(\alpha^2(t)) = \exp(2\pi i t)\varphi_H(x_E^\perp) \neq 0$ and

$$a_H = \frac{1}{2\pi i} \int_{\alpha^2} \frac{d\varphi_H}{\varphi_H} = \frac{1}{2\pi i} \int_0^1 2\pi i dt = 1.$$

In particular, we get $a_{H_0} = 1$. Writing $x = \sum_H u_H c_H$ the equation $1 = x(g.x)\alpha^2$ yields

$$0 = \sum_H u_H c_H + \sum_H u_H c_{g(H)} + \sum_{H \in \mathcal{A}} a_H c_H.$$

Since $g(H_0) = H_0$, and $g(H) = H_0 \Rightarrow H = g^2(H) = g(H_0) = H_0$ the coefficient of c_{H_0} is $0 = 2u_{H_0} + 1$ with $u_{H_0} \in \mathbf{Z}$, a contradiction. \square

Remark 2.6. In case W is a real reflection group, hence B is an Artin group of finite Coxeter type, a partly combinatorial variation of this proof can be given, using Richardson's classification of involutions in Coxeter groups. Richardson's theorem (see [12]) indeed states that, if W has I for set of Coxeter generators, then g is up to conjugation equal to w_J , for w_J the longest element of the standard parabolic subgroup W_J generated by $J \subset I$. Moreover, this element has to be central inside W_J . Choosing for α an Artin generator of B corresponding to some element of J , and writing $\beta = x\alpha$ we get the same contradiction.

As a corollary of the above theorem, and following the idea of [6], one gets Bieberbach groups associated to each complex reflection group.

Corollary 2.7. If G is a 2-subgroup of W , then $\mathfrak{P}_W(G)$ is a Bieberbach group, of dimension $|\mathcal{A}|$ and holonomy group \bar{G} .

Proof. We already noticed that $\mathfrak{P}_W(G)$ is crystallographic. If $x \in \mathfrak{P}_W(G)$ had finite order, then $x^{|G|} \in P^{ab}$ would have finite order, whence $x^{|G|} = 1$ since P^{ab} is torsion-free. Therefore the order of x is a power of 2, hence $\mathfrak{P}_W(G) \subset B/(P, P)$ would contain an element of order 2, contradiction. \square

Note that this corollary is (almost) always non-void, because the order of an irreducible reflection group of rank at least 2 is always even. Indeed, the classification easily implies that they all contain a reflection of order 2, except for a few exceptional groups, which all happen to have finite order (see table 1 below).

3. POSSIBLE TORSION INSIDE $B/(P, P)$

3.1. Finite order elements and freely regular numbers. We recall that a vector $x \in X$ is called *regular* (in the sense of Springer) if there exists $w \in W$ for which x is an eigenvector. The order d of the corresponding eigenvalue is called a regular integer (w.r.t. W). Such a w is called a regular element. It has necessarily order d . For basic properties of regular elements we refer the reader to [13] or [9].

We say that such x, w, d are *freely regular* if in addition $\langle w \rangle$ acts freely on \mathcal{A} . In that case, since the action of $\langle w \rangle$ has in particular to be faithful, d coincides with the order of w inside $W/Z(W)$.

Proposition 3.1. *Let d be a freely regular element with respect to W . Then there exists $b \in B/(P, P)$ of order d .*

Proof. Let $w \in W$, $x \in X$ and $\zeta \in \mathbf{C}^\times$ of order d such that $w.x = \zeta x$ and there is no $H \in \mathcal{A}$ such that $w(H) = H$. Up to raising w to some power coprime to d , we can assume $\zeta = \exp(2\pi i/d)$. Let $\gamma(t) = \exp(2\pi i t/d)x$. Since $x \in X$ and X is defined by linear (in)equations it defines a path $\gamma : [0, 1] \rightarrow X$ joining x and $\exp(2\pi i/d)x = w.x$. Therefore it defines a class $[\gamma] \in B = \pi_1(X/W, \bar{x})$. It is straightforward to check that $[\gamma]^d = \pi$, and we recall that the image w of $[\gamma]$ in $B/P = W$ has order d . Let us consider a collection $(u_H)_{H \in \mathcal{A}}$ of integers, $p = \sum_H u_H c_H \in P/(P, P)$. Let $\tilde{p} \in B$ designate a lift of p , and set $\tilde{q} = [\gamma]\tilde{p} \in B$. We denote q the image of \tilde{q} inside $B/(P, P)$. The image of \tilde{q} in $B/P = W$ is w and therefore has order d . It follows that the order of q is at least d . Since

$$\tilde{q}^d = ([\gamma]\tilde{p})^d = [\gamma]\tilde{p} \cdot [\gamma]^2\tilde{p} \cdot [\gamma]^3\tilde{p} \cdot \dots \cdot [\gamma]^d\tilde{p} \cdot [\gamma]^d$$

we get $q^d = (w.p)(w^2.p) \dots (w^d.p)\pi = \sum_{H \in \mathcal{A}} c_H + \sum_{k=1}^d \sum_{H \in \mathcal{A}} u_H c_{w^k(H)}$. Let $G = \langle w \rangle \subset W$. We have $G \simeq \mathbf{Z}/d\mathbf{Z}$ and G acts freely on \mathcal{A} . Let $\mathcal{B} \subset \mathcal{A}$ be a system of representatives of \mathcal{A}/G . We associate to each H its orbit $\mathcal{O}(H) \in \mathcal{A}/W$. Then, the above equation can be reformulated as

$$q^d = \sum_{H \in \mathcal{A}} c_H + \sum_{H \in \mathcal{A}} \sum_{g \in G} u_{g(H)} c_H = \sum_{H \in \mathcal{B}} \sum_{J \in \mathcal{O}(H)} c_J + \sum_{H \in \mathcal{B}} \sum_{J \in \mathcal{O}(H)} \sum_{g \in G} u_{g(J)} c_J.$$

Since the action is free, this is equal to

$$\sum_{H \in \mathcal{B}} \sum_{J \in \mathcal{O}(H)} c_J + \sum_{H \in \mathcal{B}} \sum_{J \in \mathcal{O}(H)} \left(\sum_{K \in \mathcal{O}(H)} u_K \right) c_J = \sum_{H \in \mathcal{B}} \left(\left(1 + \sum_{K \in \mathcal{O}(H)} u_K \right) \left(\sum_{J \in \mathcal{O}(H)} c_J \right) \right)$$

and thus $q^d = 1$ if and only if, for every orbit $\mathcal{O} \in \mathcal{A}/G$, we have $\sum_{K \in \mathcal{O}} u_K = -1$. Since this condition is easy to fulfill (e.g. take $u_H = -1$ for $H \in \mathcal{B}$ and $u_H = 0$ for $H \notin \mathcal{B}$) there exists $q \in B/(P, P)$ of order d . □

In particular, the symmetric group \mathfrak{S}_n contains a regular element $(1 \ 2 \ \dots \ n)$ of order n , which is freely regular if and only if n is odd. From this we recover proposition 19 of [6], namely that $B_n/(P_n, P_n)$ has elements of order n if n is odd, and therefore of order k for $k \leq n$ odd. Note also that, if W is decomposed into a sum of irreducibles $W_1 \times \dots \times W_r$, then every collection (b_1, \dots, b_r) of elements of $B_k/(P_k, P_k)$, $k = 1, \dots, r$ of finite orders m_1, \dots, m_r , defines an element of order $m = \text{lcm}(m_1, \dots, m_r)$ of the group $B_1/(P_1, P_1) \times \dots \times B_r/(P_r, P_r) \simeq B/(P, P)$. It follows that, if W admits a parabolic subgroup W' which is a sum $W_1 \times \dots \times W_r$ of such irreducibles, then by proposition 2.4 we know $B/(P, P)$ also contains elements of order $m = \text{lcm}(m_1, \dots, m_r)$, and even a subgroup isomorphic to $(\mathbf{Z}/m_1\mathbf{Z}) \times \dots \times (\mathbf{Z}/m_r\mathbf{Z})$.

In particular, the results above reprove theorem 3 and theorem 6 of [6]. This is clear for theorem 3. For theorem 6, since a transitive abelian permutation group has the same order as its degree, we know that every abelian subgroup $(\mathbf{Z}/m_1\mathbf{Z}) \times \dots \times (\mathbf{Z}/m_r\mathbf{Z})$ possibly embedding in \mathfrak{S}_n must satisfy $m_1 + \dots + m_r \leq n$ (see e.g. [7], prop. 2), and in particular, since we know how to build such a group from a parabolic $\mathfrak{S}_{m_1} \times \dots \times \mathfrak{S}_{m_r}$ this reproves that the isomorphism types of abelian subgroups that can be embedded into $B_n/(P_n, P_n)$ are exactly the same that can be embedded into \mathfrak{S}_n .

Propositions 3.1 and 2.4, combined together, produce lots of elements of finite order inside $B/(P, P)$. Note that proposition 3.1 alone is not enough to produce all the possible orders : there are parabolic subgroups $W_0 \subset W$ where W_0 admits a freely regular degree d which is not a freely regular degree for W (for example W_0 of type $E_7 = G_{36}$ as a parabolic subgroup of W of type $E_8 = G_{37}$, see section 3.5 and table 1), and by proposition 2.4 this provides an element of degree d inside B not directly produced by proposition 3.1.

Nevertheless, proposition 3.1 admits a partial converse.

Proposition 3.2. *Let d be a regular number for W . If $B/(P, P)$ has an element of order d whose image in W is regular, then d is freely regular.*

Proof. Let x be such an element, and let $g_0 \in W$ denote its image. We can choose the base-point so that it is a regular eigenvector for g_0 , and therefore we get as before a lift $g \in B/(P, P)$ with $g^d = \pi = \sum_{H \in \mathcal{A}} c_H$. Now $x = gy$ for some $y = \sum_H y_H c_H$ in P^{ab} , with $y_H \in \mathbf{Z}$. Then, $x^d = (g_0 \cdot y)(g_0^2 \cdot y) \dots (g_0^{d-1} \cdot y)yg^d$ is equal to

$$\sum_{H \in \mathcal{A}} \left(\frac{|G|}{|G_H|} \left(\sum_{J \in G.H} y_J \right) + 1 \right) c_H$$

where $G = \langle g_0 \rangle$, $G_H \subset G$ is the fixer of $H \in \mathcal{A}$, and $G.H$ denote its orbit. Thus $x^d = 1$ implies $|G| = |G_H|$ for all H , thereby proving that d is freely regular. \square

Corollary 3.3. *Let G is a subgroup of W such that*

- (1) *no freely regular prime number divides $|G|$*
- (2) *for every odd prime number p dividing $|G|$, all elements of order p are regular*

Then, $\mathfrak{P}_W(G)$ is a Bieberbach group.

Proof. Assume that x is a finite-order element in $\mathfrak{P}_W(G)$. Up to raising it to some power, we can assume this order p is a prime number. By theorem 2.5 it is odd. Let \bar{x} denote its image in $G \subset W$. Since P^{ab} is torsion-free, the order of \bar{x} is equal to p . Since it divides $|G|$, it is a regular number by (2), and also \bar{x} is a regular element. By proposition 3.2 this implies that p is freely regular, contradicting (1). This proves the claim. \square

Of course these conditions are quite strong. They are nevertheless satisfied in some cases, which enables us at least to construct some flat manifolds whose holonomy groups exceed the class of 2-groups.

Corollary 3.4. *If $W = G_4$, then $B/(P, P)$ is a Bieberbach group, of dimension 4. If $W = G_6$, then $B/(P, P)$ is a Bieberbach group, of dimension 10. Both have for holonomy group the alternating group \mathfrak{A}_4 .*

Proof. We first assume $W = G_4$. One has $|W| = 24 = 2^3 \cdot 3$, the regular numbers are 1, 2, 3, 4, 6 (see section 3.5 and table 1), and one checks easily that there are no freely regular numbers. The pseudo-reflections of G_4 have order 3, and by the Sylow theorems every element of order 3 is conjugated to one of them. Since 3 is a regular number they are regular and condition (2) of the previous corollary is satisfied. Since $|\mathcal{A}| = 4$ and $\overline{W} = \mathfrak{A}_4$ the conclusion follows. Now assume $W = G_6$. Then $|W| = 2^4 \cdot 3$, 3 is regular, again there are no freely regular numbers, and the same argument applies since one class of reflections for W has order 3. Since $|\mathcal{A}| = 10$ and $\overline{W} = \mathfrak{A}_4$ the conclusion follows. \square

It can be checked that there are no other exceptional group on which the above criterion can be applied, with $G = W$.

3.2. Determination of the freely regular numbers. In case W is a real reflection group, the freely regular numbers are fairly easy to determine, because of the next proposition.

Proposition 3.5. *If W is a real reflection group, then its freely regular numbers are exactly its odd regular numbers.*

Proof. Let $g \in W$ be regular of order d . If d is freely regular, it has to be odd by theorem 2.5 and proposition 3.1. Assume conversely that d is odd, and choose a root system $R = \{\pm v_H, H \in \mathcal{A}\}$. Assume by contradiction that the action of g on \mathcal{A} is not free. This means that $g.v_H = \pm v_H$ for some H , and thus $g^2.v_H = v_H$. Since g^2 is still regular of order d , this contradicts the freeness of its action on R , known by [13] proposition 4.10 (i) – note that, although this freeness statement is stated there only for rational reflection groups, the proof is valid for an arbitrary real reflection group. \square

In case of complex reflection groups, the situation is more complicated. Let us introduce the order $\kappa(W)$ of the extension $1 \rightarrow P/(P, P) \rightarrow B/(P, P) \rightarrow W \rightarrow 1$ as an element of $H^2(W, P^{ab})$. This integer, which originally appeared in [10] as the periodicity of a monodromy representation of

B , and was subsequently identified by V. Beck in [1] as the order of this extension, can be described as follows. First introduce, for every reflecting hyperplane $H \in \mathcal{A}$, the parabolic subgroups $W_H = \{w \in W \mid w|_H = \text{Id}_H\}$ and $C(H) = \{w \in W \mid w|_{H^\perp} = \text{Id}_{H^\perp}\}$. They are normal subgroups of $N(H) = \{w \in W \mid w(H) = H\}$ and we have $W_H \cap C(H) = \{\text{Id}\}$. We set $e_H = |W_H|$, $f_H = |N(H)/C(H)|$. Clearly e_H divides f_H . Then $\kappa(W) = \text{lcm}\{f_H; H \in \mathcal{A}\}$ ([1], corollary 1). It is a general fact that $|Z(W)|$ divides $\kappa(W)$ (see [10], cor. 5.10).

In case W is a Coxeter group, we have $\kappa(W) = 2$ (see [10], theorem 6.4 or [1] remark 2). If $W = G(de, e, n)$ with $n \geq 2$, then (see [10] proposition 6.1, corrected in [1])

- $\kappa(W) = 2de$ if de is odd and $n \geq 3$.
- $\kappa(W) = de$ if $d \neq 1$, $n = 2$, de even.
- $\kappa(W) = de$ if $n \geq 3$, de even.
- $\kappa(W) = 2de$ if $d \neq 1$, $n = 2$ and de is odd.
- $\kappa(W) = 2$ if $d = 1$ and $n = 2$.

Note that $\kappa(W)$ is an even integer for every complex reflection group. Moreover, when $W = G(de, e, n)$ and W is not of the form $G(e, e, 2)$, we have $\kappa(W) = \text{lcm}(2, de)$. In particular this formula is always valid in rank at least 3, and also when W is *not* a real reflection group.

Theorem 3.6. *Let d be a regular number. The integer d is coprime to $\kappa(W)$ if and only if d is freely regular.*

Proof. Let $w \in W$ be regular of order d , and $v \in X$ such that $w.v = \zeta v$, ζ a primitive d -th root of 1. If we assume by contradiction that w is not freely regular, then there exists $H \in \mathcal{A}$, $k \in \{1, \dots, d-1\}$ such that $w^k(H) = H$. Then $w^{kf_H} \in C(H)$. We now borrow an argument which is apparently due to Kostant (see [13], proof of proposition 4.10) for the case of Coxeter groups. Let v_2 be a spanning vector for H^\perp , and $(\cdot | \cdot)$ a W -invariant hermitian scalar product on V . Then $(v|v_2) = (w^{kf_H}v|w^{kf_H}v_2) = (\zeta^{kf_H}v|v_2)$. Since $v \in X$ we have $v \notin H$, that is $(v|v_2) \neq 0$. This implies $\zeta^{kf_H} = 1$. But since d is coprime to $\kappa(W)$, it is in particular coprime to f_H , hence $\zeta^{kf_H} = 1 \Rightarrow \zeta^k = 1$. This means $w^k.v = v$. But since $w^k \neq 1$ and $v \in X$, this contradicts Steinberg's theorem and concludes the direct part of the proof.

We have no direct proof of the converse implication, but we can check it by using the classification. We check it case by case on the exceptional groups (by computer means), see the table below. For the general series, we use an explicit description of the regular elements, that we postpone to section 3.3 below. The conclusion of the theorem is then consequence of the case of real reflection groups (proposition 3.5) together with lemmas 3.7 and 3.8 proved there. \square

3.3. Freely regular elements in types $G(de, e, n)$. We first consider the case $W = G(d, 1, n)$, $n \geq 2$ and $d > 1$. Let $x_0 = (1, \zeta_{dn}^{-1}, \zeta_{dn}^{-2}, \dots, \zeta_{dn}^{-(n-1)})$, where $\zeta_k = \exp(2\pi i/k)$. Since $n \geq 2$ we have $x_0 \in X$. Let now $g_0 \in \text{GL}(V)$ be defined by $g_0.e_i = e_{i+1}$ if $i < n$, $g_0.e_n = \zeta_d e_1$. It is clear that $g_0 \in W$, and that $g_0.x_0 = \zeta_{dn}x_0$. Therefore g_0 is a regular element of W , associated to the primitive dn -th root of unity ζ_{dn} . Now, the degrees of W as a reflection group are $d, 2d, 3d, \dots, nd$, while its codegrees are $0, d, \dots, (n-1)d$ (see e.g. [9] table D.5). By a well-known (and useful) criterion (see [9], 11.28), the regular numbers being the numbers which divide an equal number of degrees and codegrees, it follows that the regular numbers for W are the divisors of nd . Therefore, the regular elements of W are the conjugates of the powers of g_0 . We write $g_0 = \delta\sigma$ where $\sigma = (1 \ 2 \ \dots \ n) \in \mathfrak{S}_n$ and $\delta = \text{diag}(\zeta_d, 1, \dots, 1)$. We have $\sigma = p_\Sigma(g_0)$ where $p_\Sigma : G(d, 1, n) \rightarrow \mathfrak{S}_n$ is the natural projection. Now note that the orthogonal of a reflecting hyperplane is spanned by a vector having at most 2 non-zero entries. Therefore, g_0^k may stabilize a reflecting hyperplane only if σ^k has a 1-cycle or a 2-cycle in its decomposition into disjoint cycles. We write $k = nq + k_0$, $k_0 < n$, $q \geq 0$. Since $\sigma^k = \sigma^{k_0}$, this is possible only if $k_0 = 0$ or $k_0 = n/2$. Note that $g_0^n = \zeta_d \text{Id} \in Z(W)$ stabilizes every hyperplane and therefore so do the g_0^k when $k_0 = 0$. Similarly, if $k_0 = n/2$, then $g_0^k = \zeta_d^q g_0^{n/2}$ stabilizes the same hyperplanes as $g_0^{n/2}$. But since $\sigma^{n/2}$ is a disjoint product of transpositions, these hyperplanes should be of the form $z_{r+(n/2)} = \alpha z_r$ for some α, r . It is straightforward to check that such an hyperplane is stable if and only if $\alpha^2 = \zeta_d^{-1}$ and there exists such an α exactly when d is odd.

Let now $n \geq 2$, $e > 1$ and $d > 1$. We set $W = G(de, e, n)$. The reflecting hyperplanes of W are the same as for $G(de, 1, n)$, and therefore the (freely acting) regular elements for $G(de, e, n)$ are exactly the (freely acting) regular elements for $G(de, 1, n)$ which belong to $G(de, e, n)$. Recall that $G(de, e, n)$ is the subgroup of the $g \in G(de, 1, n)$ such that $\Pi(g) \in \mu_d$, where $\Pi : G(de, e, n) \rightarrow \mu_{de}$ is the morphism obtained by multiplying together all the non-zero entries. So it is sufficient to check when $g_0^k \in G(de, e, n)$. Since $\Pi(g_0^k) = \Pi(g_0)^k = (\zeta_{de})^k$, we have $g_0^k \in G(de, e, n)$ iff $(\zeta_{de})^{kd} = 1$ iff $\zeta_e^k = 1$ iff e divides k .

From this study we get the following characterization of freely regular numbers.

Lemma 3.7. *Let $W = G(de, e, n)$, with $d > 1$ and $n \geq 2$.*

- (1) *The regular numbers of W are the divisors m of dn .*
- (2) *The freely regular numbers of W are the regular numbers coprime to $\text{lcm}(2, de)$.*

Proof. We already proved (1), as the condition e divides k above means that e divides den/m , that is m divides dn . For (2) let $g \in W$ be regular of order m with m freely regular. Up to conjugating g , we can assume that $g \in \langle g_0 \rangle \simeq \mathbf{Z}/(den\mathbf{Z})$. Since $\langle g_0^n \rangle$ has order de we have $\langle g \rangle \cap \langle g_0^n \rangle = \{1\}$ if and only if m is coprime to de . When de is even, we have $de = \text{lcm}(2, de)$ and $\langle g \rangle$ acts freely if and only if $\langle g \rangle \cap \langle g_0^n \rangle = \{1\}$, so this proves (2) in this case. When de is odd, we have $\text{lcm}(2, de) = 2de$ and we proved that $\langle g \rangle$ acts freely if and only if,

- $\langle g \rangle \cap \langle g_0^n \rangle = \{1\}$ when n is odd
- $\langle g \rangle \cap \langle g_0^{\frac{n}{2}} \rangle = \{1\}$ when n is even.

The first condition is equivalent to saying that m is coprime to de . Since m divides den it implies that m divides n and therefore m is odd, whence m is coprime to $\text{lcm}(2, de) = 2de$. The second condition is equivalent to saying that m is coprime to the order $2de$ of $\langle g_0^{\frac{n}{2}} \rangle$. This proves the claim. \square

We now consider the case $W = G(e, e, n)$, $n \geq 3$, $e > 1$. The degrees of W are $e, 2e, \dots, (n-1)e, n$ and its codegrees are $0, e, \dots, (n-2)e, (n-1)e - n$. By the preceedingly mentioned criterion, a regular number is then either a divisor of $(n-1)e$ or a divisor of n . It is thus sufficient to describe a regular element of order $(n-1)e$ and one of order n . We get the first one by embedding $G(e, 1, n-1)$ into $G(e, e, n)$ through $g \mapsto (g, \Pi(g)^{-1})$. Indeed, letting $\tilde{x}_0 = (x_0, 0)$ and $\tilde{g}_0 = g_0 \oplus \zeta_e^{-1} \text{Id}_1$, where x_0, g_0 are as defined above for $W = G(e, 1, n-1)$, we get $\tilde{g}_0 \cdot \tilde{x}_0 = \zeta_{(n-1)e} \tilde{x}_0$ and $\tilde{x}_0 \in X$. Therefore \tilde{g}_0 is a regular element of order $(n-1)e$, associated to $\zeta_{(n-1)e}$.

If n divides e , the divisors of n are divisors of $(n-1)e$ and we are done. We now assume otherwise. We first describe a regular element of order n .

Let us write $n = \delta n'$, $e = \delta e'$, with $\delta = \gcd(n, e)$ (hence $\gcd(e', n') = 1$). We set $x_2 = (1, \zeta_n^{-1}, \zeta_n^{-2}, \dots, \zeta_n^{-(n'-1)}) \in \mathbf{C}^{n'}$, and $g_2 \in \text{GL}_{n'}(\mathbf{C})$ defined by $g_2 \cdot e_i = e_{i+1}$ for $i < n'$, $g_2 \cdot e_{n'} = \zeta_{\delta e'} e_{n'}$, where $e_1, \dots, e_{n'}$ denotes the canonical basis of $\mathbf{C}^{n'}$. We have $g_2 \cdot x_2 = \zeta_n x_2$. We choose now δ positive reals $0 < \lambda_1 < \lambda_2 < \dots < \lambda_\delta$, and let $x_1 = (\lambda_1 x_2) \oplus (\lambda_2 x_2) \oplus \dots \oplus (\lambda_\delta x_2) \in \mathbf{C}^n$. We have $x_1 \notin X$ if and only if $\zeta_n^k \in \mu_e$ for some $1 \leq k < n'$. But this means $1 = \zeta_n^{ke} = \zeta_{n'}^{ke'}$ that is n' divides ke' . Since n' and e' are coprime this implies n' divides k , a contradiction. Therefore $x_1 \in X$. Setting $g_1 = g_2 \oplus g_2 \oplus \dots \oplus g_2 \in \text{GL}_n(\mathbf{C})$ we have $g_1 \cdot x_1 = \zeta_n x_1$. Since $g_1 \in G(e, 1, n)$ and $\Pi(g_1) = (\Pi(g_2))^{n/n'} = (\Pi(g_2))^\delta = \zeta_\delta^\delta = 1$, we have $g_1 \in G(e, e, n)$. Therefore g_1 is a regular element of W , attached to ζ_n .

Now $g_0^{n-1} = g_0^{n-1} \oplus \zeta_e^{-(n-1)} \text{Id}_1 = \zeta_e \text{Id}_{n-1} \oplus \zeta_e^{-(n-1)} \text{Id}_1$ stabilizes the hyperplane $z_1 = z_2$, and no hyperplane is stabilized by \tilde{g}_0^k if $(n-1)$ does not divide k , unless n and e are odd, in which case the condition is that k is not a multiple of $(n-1)/2$. Likewise, g_1^k stabilizes a reflecting hyperplane if and only if

- k is a multiple of n' , or
- n' is even, k is a multiple of $n'/2$ and δ is odd.

This yields a determination of the freely regular numbers in this case:

Lemma 3.8. *Let $W = G(e, e, n)$, with $e > 1$ and $n \geq 3$.*

- (1) *The regular numbers of W are the divisors of n or $(n-1)e$.*

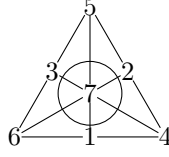


FIGURE 1. Fano plane.

(2) *The freely regular numbers of W are the regular numbers coprime to $\text{lcm}(2, e)$.*

Proof. We already proved (1). For (2) let $g \in W$ be regular of order m with m freely regular. Up to conjugating g , we can assume that $g \in \langle \tilde{g}_0 \rangle$ or $g \in \langle g_1 \rangle$. In the first case the proof that the order of g is freely regular if and only if it is coprime to $\text{lcm}(2, e)$ is the same as in the previous lemma. We assume now $g \in \langle g_1 \rangle \simeq \mathbf{Z}/n\mathbf{Z}$ and write as before $n = \delta n'$, $e = \delta e'$, with $\delta = \gcd(n, e)$. When δ is even, $\langle g \rangle$ acts freely if and only if $\langle g \rangle \cap \langle g_1^{n'} \rangle = \{1\}$. This condition means that m is coprime to $\delta = \gcd(n, e)$. Since m divides n and δ is even this is equivalent to saying that m is coprime to 2 and e , which proves the claim.

When δ is odd and n' is odd, again $\langle g \rangle$ acts freely if and only if $\langle g \rangle \cap \langle g_1^{n'} \rangle = \{1\}$ and this condition still means that m is coprime to $\delta = \gcd(n, e)$. Since m divides n this is equivalent to saying that m is coprime to e . But in this case, since m divides n' it has to be odd, and therefore this is equivalent to saying that m is coprime to $\text{lcm}(2, e)$.

Finally, assume that δ is odd and n' is even. Then $\langle g \rangle$ acts freely if and only if $\langle g \rangle \cap \langle g_1^{n'/2} \rangle = \{1\}$. This condition means that m is coprime to $2\delta = 2\gcd(n, e)$. This is equivalent to saying that m is coprime to 2 and to δ , and therefore to 2 and e . This concludes the proof. \square

Since the groups $G(1, 1, n)$ are the symmetric groups and the groups $G(e, e, 2)$ are the dihedral groups, which are both real reflection groups, for which we know the freely regular numbers by proposition 3.5, this concludes our study.

3.4. Some non-abelian finite subgroups of $B/(P, P)$. Let G be a finite subgroup of $B/(P, P)$. Since P_0 is torsion-free, the projection map $\pi : B/(P, P) \rightarrow \overline{W}$ identifies G with a subgroup of \overline{W} . Conversely, if Q is a subgroup of \overline{W} , it is the projection of a finite subgroup of $B/(P, P)$ if and only if the induced short exact sequence $1 \rightarrow P_0 \rightarrow \pi^{-1}(Q) \rightarrow Q \rightarrow 1$ is split. In cohomological terms, it means that the image of the cohomology class $c \in H^2(\overline{W}, P_0)$ inside $H^2(Q, P_0)$ is zero.

Proposition 3.9. *Assume that $Z(W) = 1$, and that G is a finite subgroup of W acting freely by conjugation on \mathcal{R}^* . Then there is a finite subgroup of $B/(P, P)$ isomorphic to G through π .*

Proof. Since $Z(W) = 1$, we have $\overline{W} = W$, $P_0 = P/(P, P) = \mathbf{Z}\mathcal{A}$, and the action of W on $\mathbf{Z}\mathcal{A} = \mathbf{Z}\mathcal{R}^*$ is defined by the conjugation action on \mathcal{R}^* . Since it is G -free, we can decompose \mathcal{R}^* into G -orbits $C_1 \sqcup \dots \sqcup C_m$ such that the G -action on each C_k is free and transitive. Then, considering the action by translation of G on $\mathbf{Z}G$, we have $H^2(G, \mathbf{Z}C_k) \simeq H^2(G, \mathbf{Z}G) \simeq H^2(G, \text{Ind}_{\{1\}}^G \mathbf{1}) \simeq H^2(\{1\}, \mathbf{Z}) = 0$ and therefore $H^2(G, \mathbf{Z}\mathcal{A}) = 0$. Therefore the sequence $1 \rightarrow P^{ab} \rightarrow \pi^{-1}(G) \rightarrow G \rightarrow 1$ splits and this concludes the proof. \square

From this we recover theorem 7 of [6].

Corollary 3.10. ([6], theorem 7) *The group $B_7/(P_7, P_7)$ contains a finite non-abelian subgroup of order 21.*

Proof. Let G be a subgroup of order 21 of the collineation group of the Fano plane generated by a 3-fold symmetry s fixing a point and a cyclic permutation t of all 7 points, satisfying $st = t^2s$, for instance $s = (1, 2, 3)(4, 5, 6)$ and $t = (1, 3, 4, 2, 5, 6, 7)$ (see figure 1). Then it is straightforward to check that no element of G can commute with a transposition, so we can apply the proposition to get the result. \square

More generally, we have the following result.

Corollary 3.11. *If p is a prime number with $p \equiv 3 \pmod{4}$, then $B_p/(P_p, P_p)$ contains a Frobenius group of order $p(p-1)/2$, non-abelian if $p > 3$.*

Proof. Let x denote a generator of $\mathbf{F}_p^\times \simeq \mathbf{Z}/(p-1)\mathbf{Z}$. We let G denote the group of affine transformations of \mathbf{F}_p of the form $z \mapsto x^{2k}z + \beta$, $\beta \in \mathbf{F}_p$. Since x^2 has order $(p-1)/2$, G has order $p(p-1)/2$, and is non-abelian when $p > 3$. Now G acts faithfully on \mathbf{F}_p and can therefore be considered as a subgroup of \mathfrak{S}_p . We now prove that it acts freely on transpositions. Let us choose $g \in G \setminus \{1\}$, that is $g : z \mapsto x^{2k}z + \beta$ with either $\beta \neq 0$ or $x^{2k} \neq 1$. If g fixes a transposition, there exists $i \neq j$ such that $g(\{i, j\}) = \{i, j\}$. If $g(i) = i$ then $(x^2 - 1)i = \beta$ hence $i = \beta(x^2 - 1)^{-1}\beta$, and similarly $j = \beta(x^2 - 1)^{-1}\beta = i$, contradicting $i \neq j$ (alternatively : this would contradict the fact that G is a Frobenius group !). If $g(i) = j$ and $g(j) = i$, then we get $(x^{2k} + 1)(i - j) = 0$. Since $i - j \neq 0$ this implies $x^{2k} = -1$, hence -1 is a square inside \mathbf{F}_p . This is not possible when $p \equiv 3 \pmod{4}$, and this concludes the proof. \square

3.5. Table for exceptional groups. In table 1 we gather the datas on the exceptional complex reflection groups which are relevant for our studies. The column S indicates the 2-Sylow subgroup of $W/Z = W/Z(W)$. Here D_n is the dihedral group of order n , Q_8 is the quaternion group of order 8, and $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$. In our description of $W/Z(W)$ we have taken care of choosing the description where the 2-Sylow subgroup is the most apparent, when possible. The gap from the family of all groups of a certain rank to the next one is realized by a horizontal double line separating them. The column ‘regular numbers’ contains the regular numbers for W but 1 (which is regular for every group), where the freely regular numbers have been circled. Recall that the list of regular numbers are characterized among all positive integer numbers by the fact that they divide exactly the same number of degrees and codegrees for W . Therefore, this list can be deduced from the tables for degrees and codegrees, such as the one in [9], table D.3, p. 275.

4. KÄHLER MANIFOLDS

We notice in this section that our construction sometimes provides Kähler manifolds. More precisely, letting S denote the Sylow 2-subgroup of $W/Z(W)$, it happens that the corresponding N -manifold can be endowed with a flat Kähler structure which, by [8], is equivalent to saying that the corresponding Bieberbach group is a discrete cocompact torsion-free subgroup of $U_{\frac{N}{2}} \times \mathbf{C}^{N/2}$. As mentioned above, to check this it is enough to compute the character table of the Sylow subgroup and the character of its permutation action on \mathcal{A} . We get that this phenomenon happens exactly, as far the exceptional groups are concerned, for $G_5, G_6, G_9, G_{13}, G_{17}, G_{18}, G_{21}, G_{25}$, yielding Kähler manifolds of (real) dimension 8, 10, 18, 18, 42, 32, 50 and 12, respectively, with the holonomy group given by the table. Of course, by considering smaller 2-groups, one might obtain Kähler manifolds from other groups as well. For instance, it can be checked that each subgroup of order 2 of $W/Z(W)$ for W of types G_4, G_7, G_{16} provides a Kähler manifold, of complex dimensions 2, 7 and 6, respectively. It is also the case in types G_8, G_{12} for *some* of the subgroups of order 2.

Moreover, we checked that the flat manifold of real dimension 10 and holonomy group \mathfrak{A}_4 built in corollary 3.4 from the whole complex reflection group G_6 , is a Kähler manifold. This is not the case for the one afforded by G_4 , since it is not even the case for the manifold afforded by its 2-Sylow subgroup, as noticed above.

We now describe, as an illustrative example, a kind of Kähler manifold arising systematically. Let $m \geq 3$ be an odd integer and let W be the dihedral group of order $4m$. Then $Z(W) = \mathbf{Z}_2$, and $|\mathcal{A}| = 2m$. Let $\overline{W} = W/Z$. As $\mathbf{Q}\overline{W}$ -modules, $P_0 \otimes_{\mathbf{Z}} \mathbf{Q} \simeq P^{ab} \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathbf{Q}\mathcal{A}$ and therefore the Kähler condition, for an arbitrary 2-subgroup Q of \overline{W} , can be checked on the $\mathbf{Q}Q$ -module $\mathbf{Q}\mathcal{A}$. Let us set $Q = \{\bar{1}, \bar{s}\}$ where s is a reflection in W and \bar{s} its class modulo $Z(W)$. There are exactly 2 hyperplanes fixed by the action of \bar{s} (the hyperplane fixed by s and its orthogonal), and this is enough to ensure that the Kähler condition is satisfied. For a concrete description of the group, note that B is an Artin group of Coxeter type $I_2(2m)$, with generators σ, τ and relation $(\sigma\tau)^m = (\tau\sigma)^m$. Our group can be constructed as the inverse image inside the group B/P_0 of

W	$ \mathcal{A} $	W/Z	$ W/Z $	$ Z $	regular numbers	$\kappa(W)$	S
G_4	4	\mathfrak{A}_4	12	2	2, 3, 4, 6	6	\mathbf{Z}_2^2
G_5	8	\mathfrak{A}_4	12	6	2, 3, 4, 6, 12	6	\mathbf{Z}_2^2
G_6	10	\mathfrak{A}_4	12	4	2, 3, 4, 6, 12	12	\mathbf{Z}_2^2
G_7	14	\mathfrak{A}_4	12	12	2, 3, 4, 6, 12	12	\mathbf{Z}_2^2
G_8	6	\mathfrak{S}_4	24	4	2, $\textcircled{3}$, 4, 6, 8, 12	4	D_8
G_9	18	\mathfrak{S}_4	24	8	2, $\textcircled{3}$, 4, 6, 8, 12, 24	8	D_8
G_{10}	14	\mathfrak{S}_4	24	12	2, 3, 4, 6, 8, 12, 24	12	D_8
G_{11}	26	\mathfrak{S}_4	24	24	2, 3, 4, 6, 8, 12, 24	24	D_8
G_{12}	12	\mathfrak{S}_4	24	2	2, $\textcircled{3}$, 4, 6, 8	2	D_8
G_{13}	18	\mathfrak{S}_4	24	4	2, $\textcircled{3}$, 4, 6, 12	8	D_8
G_{14}	20	\mathfrak{S}_4	24	6	2, 3, 4, 6, 8, 12, 24	6	D_8
G_{15}	26	\mathfrak{S}_4	24	12	2, 3, 4, 6, 12	24	D_8
G_{16}	12	\mathfrak{A}_5	60	10	2, $\textcircled{3}$, 4, 5, 6, 10, 15, 20, 30	10	\mathbf{Z}_2^2
G_{17}	42	\mathfrak{A}_5	60	20	2, $\textcircled{3}$, 4, 5, 6, 10, 12, 15, 20, 30, 60	20	\mathbf{Z}_2^2
G_{18}	32	\mathfrak{A}_5	60	30	2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60	30	\mathbf{Z}_2^2
G_{19}	62	\mathfrak{A}_5	60	60	2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60	60	\mathbf{Z}_2^2
G_{20}	20	\mathfrak{A}_5	60	6	2, 3, 4, $\textcircled{5}$, 6, 10, 12, 15, 30	6	\mathbf{Z}_2^2
G_{21}	50	\mathfrak{A}_5	60	12	2, 3, 4, $\textcircled{5}$, 6, 10, 12, 15, 20, 30, 60	12	\mathbf{Z}_2^2
G_{22}	30	\mathfrak{A}_5	60	4	2, $\textcircled{3}$, 4, $\textcircled{5}$, 6, 10, 12, 20	4	\mathbf{Z}_2^2
G_{23}	15	\mathfrak{A}_5	60	2	2, $\textcircled{3}$, $\textcircled{5}$, 6, 10	2	\mathbf{Z}_2^2
G_{24}	21	$\text{SL}_3(\mathbf{F}_2)$	168	2	2, $\textcircled{3}$, 6, $\textcircled{7}$, 14	2	D_8
G_{25}	12	$\text{SU}_3(\mathbf{F}_2)$	216	3	2, 3, 4, 6, 9, 12	6	Q_8
G_{26}	21	$\text{SU}_3(\mathbf{F}_2)$	216	6	2, 3, 6, 9, 18	6	Q_8
G_{27}	45	$\mathfrak{A}_6 = \text{Sp}_4(\mathbf{F}_2)'$	360	6	2, 3, $\textcircled{5}$, 6, 10, 15, 30	6	D_8
G_{28}	24	$2^4 \rtimes (\mathfrak{S}_3)^2$	576	2	2, $\textcircled{3}$, 4, 6, 8, 12	2	
G_{29}	40	$2^4 \rtimes \mathfrak{S}_5$	1920	4	2, 4, $\textcircled{5}$, 10, 20	4	
G_{30}	60	$\mathfrak{A}_5 \wr 2$	7200	2	2, $\textcircled{3}$, 4, $\textcircled{5}$, 6, 10, 12, $\textcircled{15}$, 20, 30	2	
G_{31}	60	$2^4 \cdot \text{Sp}_4(\mathbf{F}_2)$	11520	4	2, $\textcircled{3}$, 4, $\textcircled{5}$, 6, 8, 10, 12, 20, 24	4	
G_{32}	40	$\text{SU}_4(\mathbf{F}_2)$	25920	6	2, 3, 4, $\textcircled{5}$, 6, 8, 10, 12, 15, 24, 30	6	
G_{33}	45	$\text{SU}_4(\mathbf{F}_2)$	25920	2	2, 3, $\textcircled{5}$, 6, 9, 10, 18	6	
G_{34}	126	$\text{PSU}_4(\mathbf{F}_3) \rtimes \mathbf{Z}_2$	6531840	6	2, 3, 6, $\textcircled{7}$, 14, 21, 42	6	
G_{35}	36	$O_6^-(\mathbf{F}_2)$	51840	1	2, $\textcircled{3}$, 4, 6, 8, $\textcircled{9}$, 12	2	
G_{36}	63	$\text{SO}_7(\mathbf{F}_2)$	1451520	2	2, $\textcircled{3}$, 6, $\textcircled{7}$, $\textcircled{9}$, 14, 18	2	
G_{37}	120	$\text{SO}_8^+(\mathbf{F}_2)$	348364800	2	2, $\textcircled{3}$, 4, $\textcircled{5}$, 6, 8, 10 12, $\textcircled{15}$, 20, 24, 30	2	

TABLE 1. Table for exceptional groups.

the subgroup generated by a reflection. Therefore, one gets that it admits a presentation with generators σ, x_k with $k \in \mathbf{Z}/2m\mathbf{Z}$, and relations $\sigma^2 = x_0$, $\sigma x_k \sigma^{-1} = x_{-k}$.

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