CABLING BURAU REPRESENTATION

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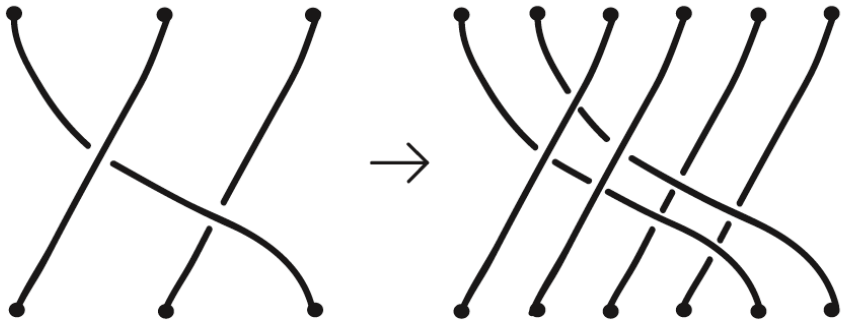
Abstract. The Burau representation enables to define many other representations of the braid group B_n by the topological operation of "cabling braids". We show here that these representations split into copies of the Burau representation itself and of a representation of $B_n/(P_n, P_n)$. In particular, we show that there is no gain in terms of faithfulness by cabling the Burau representation.

Introduction

The Burau representation is the oldest and most natural non-trivial representation of the braid group B_n. Debates over its faithfulness were a central concern in the past decades, until it has been shown to be non faithful for $n \geq 5$, the case $n = 4$ remaining open.

A natural question, brought to us by T. Fiedler and S. Orevkov, is whether it is possible to reduce the size of the kernel of the Burau representation by the operation of "cabling braids". Here we answer this question for the most natural cabling, sometimes called parallel cabling.

The answer is negative. More precisely, letting $\Delta_{n,r} : B_n \to B_{nr}$ denote the morphism of cabling where each strand is replaced by r parallel strands, as in the diagram below,



and letting $R_{\mathrm{bur}} : B_{nr} \to GL(V)$ denote the Burau representation of B_{nr}, we decompose the representation $R_{\mathrm{bur}} \circ \Delta_{nr}$ of B_n. Our main theorem states that this representation is semisimple and splits into a variant of the Burau representation of B_{nr} and copies of an irreducible representation of $B_n/(P_n, P_n)$, where P_n denotes the pure braid group. As a

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consequence, we show that the kernel of $R_{\mathrm{bur}} \circ \Delta_{nr}$ coincides with the kernel of the Burau representation of B_n.

Our proof has two ingredients, both based on the use of Drinfeld associators. These associators define morphisms from the (group algebra of) the braid group B_n to some "infinitesimal braids algebra" $\mathfrak{B}_n$. In the first part we recall the classical extension of this construction, originally due to Kontsevich, in terms of monoidal categories. In particular, we insist on the compatibility with cabling, for which we give a self-contained proof.

The second part is based on a correspondance between representations of B_n and representations of its infinitesimal couterpart $\mathfrak{B}_n$, for which one can find a detailed account in [Ma2]. We use the fact that the Burau representation corresponds to a very simple representation of $\mathfrak{B}_n$, and that cabling on the infinitesimal braids is also an easy (additive) operation, to get the decomposition that we are interested in.

1. Drinfeld-Kontsevich functor for parenthesized braids

1.1. **Braids and chord diagrams.** The n strands braid group B_n is the fundamental group of the configuration space Y_n of n points in the complex plane. It may be convenient to vary the base point, and to consider the fundamental groupoid; parenthesized braids amount to consider limit configurations in a convenient compactification of Y_n.

A parenthesized braid [BN2] is a braid whose ends are points on the real line, together with a parenthesization of its bottom end (the domain) and its top end (the range). We obtain a groupoid **PaB** which is a subcategory in the parenthesized tangle category also called the q-tangle category [LM1, LM2].

Let $\mathcal{A}_n = \mathcal{A}_n(\Bbbk)$ be the algebra of chords diagrams for n-strands pure braids, over the scalar field $\Bbbk$ of characteristic 0. As a unital $\Bbbk$-algebra $\mathcal{A}_n$ is generated by the $t_{ij} = t_{ji}$, $0 \leq i < j \leq n$, represented by a chord between strands numbered i and j, with relations (infinitesimal braid relations):

$$[t_{ij}, t_{kl}] = 0 \text{ if the 4 indices } i, j, k, l \text{ are distinct,}$$

$$[t_{jk}, t_{ij} + t_{ik}] = 0 \text{ if the 3 indices } i, j, k, \text{ are distinct.}$$

The number of chords provides a grading on the algebra $\mathcal{A}_n$, and we denote by $\widehat{\mathcal{A}}_n$ the completion with respect to this grading.

We use the notation $(\sigma, D) \mapsto {}^{\sigma}D$ for the natural left action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{A}_n$ (resp. $\widehat{\mathcal{A}}_n$); for a generator t_{ij}, we have ${}^{\sigma}(t_{ij}) = t_{\sigma(i)\sigma(j)}$. The algebra $\mathfrak{B}_n$ (resp. $\widehat{\mathfrak{B}}_n$) is defined as the crossed product of $\mathcal{A}_n$ (resp. $\widehat{\mathcal{A}}_n$) with the symmetric group $\mathfrak{S}_n$. As a module, $\mathfrak{B}_n$ (resp. $\widehat{\mathfrak{B}}_n$) is a free $\mathcal{A}_n$-module (resp. $\widehat{\mathcal{A}}_n$)-module), with basis $\mathfrak{S}_n$. In Vassiliev finite type invariants theory, $\mathfrak{B}_n$ is the algebra of chords diagram for braids. Its completion $\widehat{\mathfrak{B}}_n$ is the natural target for the Drinfeld-Kontsevich functor which we want to consider now. We will need extra structures on our categories, namely cabling operations and strand removal operations [BN2].

Cabling braids: For a parenthesized braid B, $d_i(B)$ is the parenthesized braid obtained from B by doubling the ith strand (counting at the bottom).

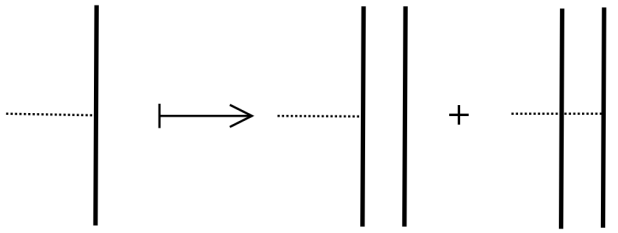


FIGURE 1.

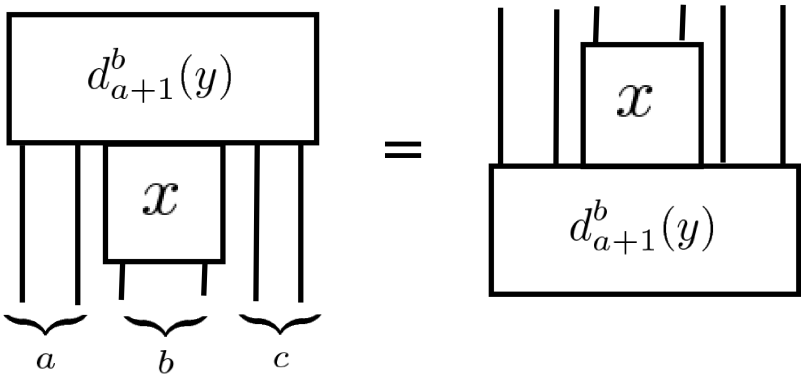


FIGURE 2.

Cabling chord diagram: For a chord diagram D, $d_i(D)$ is the chord diagram obtained form D by doubling the ith strand (counting at the bottom). Each chord incident to the ith strand is expanded as depicted in figure 1. Note that $d_i \circ d_i = d_{i+1} \circ d_i$.

Strand removal on braids: For a parenthesized braid B, $s_i(B)$ is obtained by removing the ith strand (counting at the bottom).

Strand removal on chord diagrams: For a chord diagram D, $s_i(B)$ is obtained by removing the ith strand (counting at the bottom) if no chord is incident to this ith strand, and is zero otherwise. The following lemma [BN1], which is an immediate consequence of the infinitesimal braid relations, will play a key role.

Lemma 1 (Naturality of cabling)**.** *For any positive integers a, b, c, and any $x \in \widehat{\mathcal{A}}_b$, $y \in \widehat{\mathcal{A}}_{a+1+c}$, one has the equality in figure 2.*

We will also need the coproduct map

$$\square : \widehat{\mathcal{A}}_n \to \widehat{\mathcal{A}}_n \widehat{\otimes} \widehat{\mathcal{A}}_n$$

defined so that it is continuous and each generator t_{ij} is primitive. An element $\Psi \in \widehat{\mathcal{A}}_n$ is group-like if and only if $\square\Psi = \Psi \otimes \Psi$; this implies invertibility. We denote by $\mathcal{T}_n$ the Lie algebra of primitive elements in $\mathcal{A}_n$, and by $\widehat{\mathcal{T}}_n$ its closure in $\widehat{\mathcal{A}}_n$. The Lie algebra $\mathcal{T}_n$ is known as the Lie algebra of infinitesimal braids.

Definition. *An associator is a group-like element $\Phi \in \widehat{\mathcal{A}}_3$ satisfying*

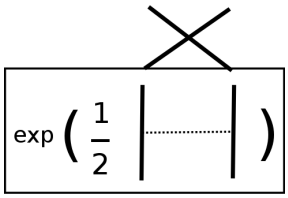
- *the pentagon identity*
$$(\,\big|\otimes\Phi).d_2\Phi.(\Phi\otimes\big|\,) = d_3\Phi.d_1\Phi$$
- *the hexagon identities*
$$d_1\exp\left(\pm\frac{1}{2}t_{12}\right) = {}^{312}\Phi\exp\left(\pm\frac{1}{2}t_{13}\right){}^{132}\big(\Phi^{-1}\big)\exp\left(\pm\frac{1}{2}t_{23}\right)\Phi$$
(The notation ijk is used for the permutation $(1,2,3)\mapsto(i,j,k)$.)
- Φ *is non-degenerate:* $s_1\Phi = s_2\Phi = s_3\Phi = \mathbf{1}_{(\bullet\bullet)}$*;*

Remarks. 1. The group-like element Φ can be written $\Phi = \exp(\phi)$, with $\phi\in\widehat{\mathcal{T}}_3$. From the non degeneracy condition, we get that ϕ belongs to $[\widehat{\mathcal{T}}_3,\widehat{\mathcal{T}}_3]$.
2. The hexagon identity with negative sign could be replaced by ${}^{321}\Phi = \Phi^{-1}$. 3. The non degeneracy condition can be deduced from group-like and pentagone properties [Ma2, Proposition 1]

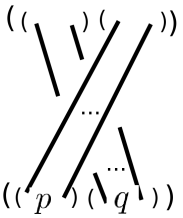
The notion of associator is due to Drinfeld [D], who also obtained an associator with complex coefficients using monodromy of KZ system, showed that associators with rational coefficients exist, and emphasized the role of associators in constructing monoidal functors from braids to the universal envelopping algebras of infinitesimal braids. This functor was extended to the tangle category by Kontsevich [K] and further developped by Bar-Natan, Le-Murakami and others. Considering parenthesized braids and tangles converts Drinfeld-Kontsevich functor into a strictly monoidal functor. Drinfeld-Kontsevich functor on the whole parenthesized tangle category is constructed in [LM1, BN1]. We consider as a target category the (strict) monoidal category $\widehat{\mathfrak{B}}$ whose objects are integers and morphisms are defined by
$$\mathrm{End}_{\widehat{\mathfrak{B}}}(n) = \widehat{\mathfrak{B}}_n\ ,$$
$$\mathrm{Hom}_{\widehat{\mathfrak{B}}}(n,m) = \{0\}\quad \text{for } n\neq m\ .$$

Cabling associators and braidings. For $p\in\mathrm{Obj}(\mathbf{PaB})$, we denote by $|p|$ the number of points of the parenthesization p. Let Φ be an associator and let $p,q,r\in\mathrm{Obj}(\mathbf{PaB})$, we define the cabled associator $\Phi_{p,q,r}$ by
$$\Phi_{p,q,r} = (d_1^{|p|}\circ d_2^{|q|}\circ d_3^{|r|})\Phi\ .$$
The identity braid, viewed as an element in $\mathrm{Hom}_{\mathbf{PaB}}\left((pq)r,p(qr)\right)$, is denoted by $a_{p,q,r}$.

Let $\mathrm{R}\in\widehat{\mathcal{B}}_2$ be the element depicted below. Here a product from right to left is depicted from the bottom.



For parenthesizations p and q, $\mathrm{R}_{p,q}$ denote $d_1^{|p|}d_2^{|q|}\mathrm{R}$, and $c_{p,q}\in\mathrm{Hom}_{\mathbf{PaB}}\left(pq,qp\right)$ denote the element depicted below.



From $d_i^2 = d_{i+1}d_i$, we get that $(p,q,r) \mapsto \Phi_{p,q,r}$ and $(p,q) \mapsto \mathrm{R}_{p,q}$ commutes with the cabling operation. We quote that $\Phi_{p,q,r}$ and $\mathrm{R}_{p,q}$ only depend on the integers $|p|$, $|q|$, $|r|$.

Theorem 1. *If Φ is an associator, then there exists a unique strictly monoidal functor*

$$Z : \mathbf{PaB} \to \widehat{\mathfrak{B}}$$

such that, for a parenthesization p, $Z(p) = |p|$, and for all parenthesizations p, q, r

$$Z(c_{p,q}) = \mathrm{R}_{p,q} \quad , \quad Z(a_{p,q,r}) = \Phi_{p,q,r} \ .$$

From the definition of the functor Z in the above theorem, we obtain the following result, first proved by Le-Murakami [LM2].

Corollary. *The functor Z commutes with cabling operations.*

Proof. The elements $c_{p,q}$ and $a_{p,q,r}$ generate **PaB** as a monoidal category; unicity of the functor Z follows.

As already said, the proof of existence can be found in [LM1, BN1] and rests on Mac-Lane coherence theorem for braided monoidal categories [JS, Section 2]. We like to give here a rather self contained argument. From the pentagon identity, we get that for any two parenthesizations P, Q with the same length $|P| = |Q| = n$ we have a canonical invertible element $\Phi_P^Q \in \hat{\mathcal{A}}_n$ obtained by composing associators corresponding to a decomposition of the identity braid, considered as an element in $\mathrm{Hom}_{\mathbf{PaB}}(P,Q)$, into tensor products and composition of elements $a_{p,q,r}^{\pm 1}$.

Now consider a parenthesized braid $\tau \in \mathrm{Hom}_{\mathbf{PaB}}(P,Q)$, and represent it as a diagram corresponding to a word w in Artin generators $w = \sigma_{i_k}^{\pm 1} \dots \sigma_{i_1}^{\pm 1}$. For each integer n we denote by $p(n)$ the parenthesization *from the left*, i.e. $p(n+1) = (p(n)\bullet)$, and $p_i = p_i(n) = d_i(p(n-1))$. We decompose τ as follows

$$\tau = a^Q_{p_{i_k}} \sigma_{i_k}^{\pm 1} a^{p_{i_k}}_{p_{i_{k-1}}} \dots \sigma_{i_1}^{\pm 1} a_P^{p_{i_1}} \ .$$

For $0 < i < n$, let $\mathrm{R}_i = s_i \exp(\frac{1}{2} t_{i,i+1}) \in \widehat{\mathcal{B}}_n$ (R is inserted along strands i, $i+1$). Here $s_i \in \mathfrak{S}_n$ is the transposition $(i, i+1)$.

We associate to the representative of τ written above

$$Z(\tau, w) = \Phi^Q_{p_{i_k}} \mathrm{R}_{i_k}^{\pm 1} \Phi^{p_{i_k}}_{p_{i_{k-1}}} \dots \mathrm{R}_{i_1}^{\pm 1} \Phi_P^{p_{i_1}} \ .$$

We will show that the assignement $t \mapsto Z(t) = Z(t,w)$ constructs a well defined functor Z with the required properties. This follows from the statements below whose proof is left to the reader. This is just playing with the definition, hexagon and pentagon relations,

together with the naturality property of the cabling operation on chord diagrams (lemma 1).

(1) Functoriality: $Z(\tau\tau', ww') = Z(\tau, w)Z(\tau', w')$, for any τ, τ' for which $\tau\tau'$ makes sense.
(2) Braid relations: if w and w' are equivalent as braids, then we have $Z(\tau, w) = Z(\tau, w')$. Hence we get that Z is a well defined functor on **PaB**.
It is sufficient to show that for $0 < i+1 < j < n$,
$$Z(\tau, w_1\sigma_i\sigma_j w_2) = Z(\tau, w_1\sigma_j\sigma_i w_2) \ ;$$
and for $0 < i < n$,
$$Z(\tau, w_1\sigma_i\sigma_{i+1}\sigma_i w_2) = Z(\tau, w_1\sigma_{i+1}\sigma_i\sigma_{i+1} w_2) \ .$$
(3) Z is strictly monoidal, i.e. Z is compatible with juxtaposition.
(4) $Z(c_{p,q}) = \mathrm{R}_{p,q}$, and $Z(a_{p,q,r}) = \Phi_{p,q,r}$.

□

1.2. **Commutation property.** We already considered the cabling operator which duplicates one strand. We want now to duplicate all strands. We denote by Δ the functor $\mathbf{PaB} \to \mathbf{PaB}$ which duplicate all strands (and put a parenthesis around each pair). More generally, we denote by Δ_r the functor $\mathbf{PaB} \to \mathbf{PaB}$ which replace each strand by r parallel copies (for each r-uple put parenthesis from the left). The functors for chord diagrams similar to Δ and Δ_r are denoted respectively by ∂ and ∂_r. We saw that the functor Z commutes with the cabling operation. This shows the following proposition.

Proposition 1. *The following square of functors is commutative.*

$$\begin{array}{ccc} \mathbf{PaB} & \xrightarrow{Z} & \widehat{\mathfrak{B}} \\ \downarrow{\scriptstyle \Delta_r} & & \downarrow{\scriptstyle \partial_r} \\ \mathbf{PaB} & \xrightarrow{Z} & \widehat{\mathfrak{B}} \end{array}$$

For each parenthesization p with $|p| = n$, by extending linearly the map, we have an algebra morphism $x \mapsto Z(x)$:
$$Z_p : \Bbbk B_n \approx \Bbbk End_{\mathbf{PaB}}(p) \to \widehat{\mathfrak{B}}_n \ .$$
If q is another parenthesization with $|q| = n$, then Z_q is obtained by intertwining with Z(identity braid), where identity braid is considered as an element in $\mathrm{Hom}_{\mathbf{PaB}}(p, q)$.

Definition. *Let σ_i, $i \in [1, n-1]$, be the Artin generators of B_n. A morphism $\varphi : \Bbbk B_n \to \widehat{\mathfrak{B}}_n$ for some n is said to satisfy property (*) if, for all $i \in [1, n-1]$,*
$$(*) \quad \varphi(\sigma_i) \text{ is conjugated to } \mathrm{R}_i = s_i \exp(\frac{1}{2} t_{i,i+1}) \text{ by some } \psi_i \in \exp[\widehat{\mathcal{T}_n}, \widehat{\mathcal{T}_n}]$$
Here $s_i \in \mathfrak{S}_n$ is the transposition $(i, i+1)$.

This implies that $\varphi(\sigma_i)$ equals $s_i(1+\frac{1}{2}t_{i,i+1})$ plus higher terms.

Recall that for an integer n, we denote by $p(n)$ the object in **PaB** equal to the left parenthesization $(\dots((\bullet\bullet)\bullet)\cdots\bullet)$. Observe that the object $\Delta_r(p(n))$ is not equal to the left parenthesization $p(rn)$, but rather: $\Delta_r p(n) = (\dots((p(r))p(r))\dots p(r))$. We denote by $\Delta_{r,n}$ the linear extension of $\Delta_r p(n)$:

$$\Delta_{r,n} : \Bbbk B_n \approx \Bbbk\mathrm{End}_{\mathbf{PaB}}(p(n)) \to \Bbbk\mathrm{End}_{\mathbf{PaB}}(\Delta_r p(n)) \approx \Bbbk B_{rn} \ .$$

The morphism for chord diagrams similar to $\Delta_{r,n}$ is denoted by $\partial_{r,n}$.

Proposition 2. *For all $n \geq 1$, $r \geq 2$, the morphisms*

$$\varphi_n = Z_{n(p)} : \Bbbk B_n \approx \Bbbk\mathrm{End}_{\mathbf{PaB}}(p(n)) \to \widehat{\mathfrak{B}}_n$$

and

$$\varphi_{r,n} = Z_{\Delta_r(n(p))} : \Bbbk B_{nr} \approx \Bbbk\mathrm{End}_{\mathbf{PaB}}(\Delta_r(n(p))) \to \widehat{\mathfrak{B}}_{nr}$$

satisfy (*) and the following diagram commutes :

$$\begin{array}{ccc} \Bbbk B_n & \xrightarrow{\varphi_n} & \widehat{\mathfrak{B}}_n \\ \downarrow{\scriptstyle \Delta_{r,n}} & & \downarrow{\scriptstyle \partial_{r,n}} \\ \Bbbk B_{nr} & \xrightarrow{\varphi_{r,n}} & \widehat{\mathfrak{B}}_{nr} \end{array}$$

Proof. We have property (*), with $\psi_i = \Phi^{p_i}_{n(p)}$. Commutativity of the diagram follows from proposition 1. □

2. Representations of the braid groups from an infinitesimal view point

For the decomposition of representations of the braid groups B_n with generic parameters it is much more easier, if possible, to deal with the representations of $\mathfrak{B}_n$ from which they arise. Indeed, there is a dictionnary between properties of representations of $\mathfrak{B}_n$ and the representations of B_n obtained through a morphism φ such as the ones discussed above.

This dictionnary is discussed at length in [Ma2]. In the following sections we show how to get the representations involved here from " infinitesimal representations ".

2.1. **General facts.** Let $\mathbb{K} = \Bbbk((h))$ and $\mathbb{L} = \Bbbk(q)$. We view $\mathbb{L}$ as embedded in $\mathbb{K}$ by $q \mapsto \exp(h/2)$. Let $n \geq 2$ and $\varphi : \Bbbk B_n \to \widehat{\mathfrak{B}}_n$ satisfying (*). To every representation $\rho : \mathfrak{B}_n \to \mathrm{End}(\Bbbk^n)$ one associates a representation $\tilde{\rho} : \widehat{\mathfrak{B}}_n \to \mathrm{End}(\mathbb{K}^n)$ defined by

$$\left\{ \begin{array}{lcll} \tilde{\rho}(t_{ij}) & = & h\rho(t_{ij}) & \\ \tilde{\rho}(s) & = & \rho(s) & \text{if } s \in \mathfrak{S}_n. \end{array} \right.$$

and a representation $\widehat{\varphi}(\rho) = \tilde{\rho} \circ \varphi$. of B_n. The condition (*) implies that ρ is (absolutely) irreducible iff $\widehat{\varphi}(\rho)$ is (absolutely) irreductible (cf. [Ma2], proposition 7 and proposition 8).

In general, for any representation ρ of $\mathfrak{B}_n$ and $R = \widehat{\varphi}(\rho)$ with φ satisfying (*), one has $R(\sigma_i) \equiv \rho(s_i)$ modulo h. It follows that $\mathrm{Ker}\, R \subset P_n$ as soon as the restriction of ρ to $\mathfrak{S}_n$ is faithful. Let us assume that this is the case. Recall that P_n has standard generators ξ_{ij} with $1 \leq i < j \leq n$ whose images generate $P_n/(P_n, P_n)$. Under condition (*) one has

$R(\xi_{ij}) = 1 + h\rho(t_{ij})$ modulo h^2, hence $\operatorname{Ker} R \subset (P_n, P_n)$ as soon as the endomorphisms $\rho(t_{ij})$ are linearly independant.

2.1.1. *Framing.* For any given representation $R : B_n \to GL_N(\mathbb{K})$ and $a \in \mathbb{K}^\times$ one may define another representation, denoted aR and defined by $(aR)(\sigma_i) = aR(\sigma_i)$ for all $i \in [1, n-1]$. It is clear that, if $\operatorname{Ker} R \subset (B_n, B_n)$, then $\operatorname{Ker} R \subset \operatorname{Ker} aR$.

At the infinitesimal level, to any $\rho : \mathfrak{B}_n \to \operatorname{End}(\Bbbk^N)$ and $v \in \mathbb{K}$ one can associate the representation $v + \rho$ defined by

$$\left\{ \begin{array}{lcll} (v+\rho)(t_{ij}) & = & \rho(t_{ij}) + v\mathrm{Id} & \\ (v+\rho)(s) & = & \rho(s) & \text{if } s \in \mathfrak{S}_n. \end{array} \right.$$

It is easily checked that, under condition (*), one has

$$\widehat{\varphi}(v+\rho) = \exp(hv)\widehat{\varphi}(\rho).$$

Let us assume that $R = \widehat{\varphi}(\rho)$ with φ satisfying (*). Then $\det R(x)$ is a formal series in h with constant term 1. Such series form a group with unique roots. When $\operatorname{Ker} R \subset P_n$, a sufficient condition for the converse inclusion $\operatorname{Ker} aR \subset \operatorname{Ker} R$ to hold is then that $\det R(\sigma_1)^2 \neq a^{-2\dim R}$. Indeed, if $l = l(x)$ denotes the length of $x \in B_n$ with respect to the set of Artin generators, $x \in \operatorname{Ker} aR$ means that $R(x) = a^{-l}$, hence $\det R(x) = a^{-l\dim R}$. If $l = 0$, then $x \in \operatorname{Ker} R$ and we are done. Otherwise, since the Artin generators are conjugated one to the other, then $\det R(x) = \det(R(\sigma_1))^l$, thus $\det(R(\sigma_1^2))^l = (a^{-2\dim R})^l$ hence $\det R(\sigma_1)^2 = a^{-2\dim R}$, which has been ruled out.

2.1.2. *Twisting.* For any given representation $R : B_n \to GL_N(\mathbb{L})$ and $r \in \mathbb{Z} \setminus \{0\}$ one may also define another representation, denoted R^{q^r} and defined by twisting R by the field morphism $q \mapsto q^r$. It is clear that R and R^{q^r} have the same kernel.

In a similar manner, to any $b \in \Bbbk^*$ and any representation ρ of $\mathfrak{B}_n$ one can associate the representation $b\rho$ defined by

$$\left\{ \begin{array}{lcll} (b\rho)(t_{ij}) & = & b\rho(t_{ij}) & \\ (b\rho)(s) & = & \rho(s) & \text{if } s \in \mathfrak{S}_n. \end{array} \right.$$

In the situations described below, these two operations are intimately related.

2.2. **The Burau representation.** Let $H_n(q) = \mathbb{L}B_n/I$, where I is the ideal generated by the elements $(\sigma_i - q)(\sigma_i + q^{-1})$ for $i \in [1, n]$. Note that, since the Artin generators are conjugated one to the other, this is the same as the ideal generated by $(\sigma_1 - q)(\sigma_1 + q^{-1})$. This algebra is the (generic) Iwahori-Hecke algebra of type A. This is a well-known finite-dimensional algebra, isomorphic to the group algebra over $\mathbb{L}$ of the symmetric group on n letters. Its representations are explicitely described in [Ho, Wz].

Let V be a n-dimensional $\mathbb{L}$-vector space with basis $e_1, \ldots, e_n$. The Burau representation of the braid group $R_{\mathrm{bur}} : B_n \to GL(V)$ is defined in matrix block-diagonal form on this basis by

$$R_{\mathrm{bur}}(\sigma_i) = qI_{k-1} \oplus \begin{pmatrix} q - q^{-1} & q \\ q^{-1} & 0 \end{pmatrix} \oplus qI_{n-k+1}.$$

It is easily checked that $(R_{\mathrm{bur}}(\sigma_1) - q)(R_{\mathrm{bur}}(\sigma_1) + q^{-1}) = 0$. From the classical representation theory of the Iwahori-Hecke algebra, one checks that the Burau representation is characterized among its representations by the following properties

(1) It is n-dimensional, with a $(n-1)$-dimensional irreducible subspace
(2) $-q^{-1}$ has multiplicity 1 in the spectrum of σ_1.

In particular, the image of σ_1 has determinant $-q^{n-2}$. The Burau representation R_{bur} can be deduced from the representation ρ_{bur} of $\mathfrak{B}_n$ defined by

$$\left\{\begin{array}{rcll} \rho_{\mathrm{bur}}(t_{ij}).e_i & = & e_j & \\ \rho_{\mathrm{bur}}(t_{ij}).e_k & = & e_k & \text{if } k \notin \{i,j\} \\ \rho_{\mathrm{bur}}(s).e_i & = & e_{s(i)} & \text{if } s \in \mathfrak{S}_n \end{array}\right.$$

Indeed, given any $\varphi : \Bbbk B_n \to \widehat{\mathfrak{B}}_n$ satisfying (*), the representation $\widehat{\varphi}(\rho_{\mathrm{bur}})$ extended to $\mathbb{L}$ factorizes through $H_n(q)$ because the eigenvalues of s_1 are $1, -1$ and $\widehat{\varphi}(\rho_{\mathrm{bur}})(\sigma_1)$ is conjugated to $\rho_{\mathrm{bur}}(s_1)\exp(h\rho_{\mathrm{bur}}(s_1)/2)$. Moreover, -1 has multiplicity 1 in the spectrum of $\rho_{\mathrm{bur}}(s_1)$, hence $-q^{-1}$ has multiplicity 1 in the spectrum of $\widehat{\varphi}(\rho_{\mathrm{bur}})(\sigma_1)$. Finally, the kernel of the linear map $\alpha : V \to L$ defined by $\alpha(e_i) = 1$ is stable under ρ_{bur}, and it is a standard fact from the representation theory of the symmetric group that it is irreducible under the action of $\mathfrak{S}_n$. It follows that $\widehat{\varphi}(\rho_{\mathrm{bur}})$ admits an irreducible $(n-1)$-dimensional subspace, hence the following proposition.

Proposition 3. *For all $\varphi : \Bbbk B_n \to \widehat{\mathfrak{B}}_n$ satisfying (*), the Burau representation is isomorphic to $\widehat{\varphi}(\rho_{\mathrm{bur}})$.*

Moreover, by the same characterization, one gets that, once extended over $\mathbb{K}$, the twisted representation $R_{\mathrm{bur}}^{q^r}$ is isomorphic to $\widehat{\varphi}(r\rho_{\mathrm{bur}})$ for all $r \in \mathbb{Z} \setminus \{0\}$.

It is readily checked that the restriction of R_{bur} to the center is faithful for $n \geq 3$. For $n \geq 5$ it is known to be unfaithful. Indeed, Bigelow found in [Bi], improving results of Moody [Mo], Long and Paton [LP], that the element $\beta = (\psi_2\psi_1^{-1}\sigma_4\psi_1\psi_2^{-1}, \delta_5)$ is non trivial and lies in the kernel of R_{bur} for $n \geq 5$, where

$$\left\{\begin{array}{rcl} \psi_1 & = & \sigma_3^{-1}\sigma_2\sigma_1^2\sigma_2\sigma_4^3\sigma_3\sigma_2 \\ \psi_2 & = & \sigma_4^{-1}\sigma_3\sigma_2\sigma_1^{-2}\sigma_2\sigma_1^2\sigma_2^2\sigma_1\sigma_4^5 \\ \delta_5 & = & \sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_4. \end{array}\right.$$

At the present time, very few things are known about this kernel. However, from the basic observations above, the faithfulness of the permutation representation of the symmetric group implies that $\operatorname{Ker} R_{\mathrm{bur}} \subset P_n$, and the fact that the $\rho(t_{ij})$ are linearly independent implies $\operatorname{Ker} R_{\mathrm{bur}} \subset (P_n, P_n)$.

2.3. **The extended permutation representation.** The *extended symmetric group* (from the french "groupe symétrique étendu") can be defined as $\widetilde{\mathfrak{S}}_n = B_n/(P_n, P_n)$ and has been introduced by J. Tits in [Ti]. If ρ is a representation of $\mathfrak{B}_n$ and $\varphi : \mathbb{K}B_n \to \widehat{\mathfrak{B}}_n$ satisfies (*), then $\widehat{\varphi}(\rho)$ factorizes through $\widetilde{\mathfrak{S}}_n$ if and only if $\rho([t_{ij}, t_{kl}]) = 0$ for all i, j, k, l, which is

equivalent to saying that $\rho([t_{12},t_{23}])=0$ (see [Ma2] lemma 5). Moreover, in that case, for all $i\in[1,n]$,

$$\widehat{\varphi}(\rho)(\sigma_i)=\rho(s_i)\exp(h\rho(t_{i,i+1})/2).$$

The natural representation $R_{\rm sym}$ of $\widetilde{\mathfrak{S}}_n$ over $\mathbb{L}$ can be defined over a n-dimensional vector space $\mathbb{L}^n$ in block-diagonal form by

$$R_{\rm sym}(\sigma_k)=I_{k-1}\oplus\begin{pmatrix}0&q\\q&0\end{pmatrix}\oplus I_{n-k+1}.$$

It is readily checked that, if φ satisfies (*) then $R_{\rm sym}=\widehat{\varphi}(\rho_{\rm sym})$ where the restriction of $\rho_{\rm sym}$ to $\mathfrak{S}_n$ is defined in block-diagonal form by

$$\begin{aligned}\rho_{\rm sym}(s_k)&=I_{k-1}\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus I_{n-k+1}\\ \rho_{\rm sym}(t_{k,k+1})&=0_{k-1}\oplus\begin{pmatrix}1&0\\0&1\end{pmatrix}\oplus 0_{n-k+1}.\end{aligned}$$

More generally, $\widehat{\varphi}(r\rho_{\rm sym})=R_{\rm sym}^{q^r}$. It is easily checked that $\rho_{\rm sym}$, thus $R_{\rm sym}$, is (absolutely) irreducible (cf. [Ma1] **II** 2.3.1 lemme 8). Again by faithfulness of the permutation representation of the symmetric group one gets that $\operatorname{Ker}R_{\rm sym}\subset P_n$. Moreover by definition of the extended symmetric group one has $(P_n,P_n)\subset\operatorname{Ker}R_{\rm sym}$, hence $\operatorname{Ker}R_{\rm bur}\subset\operatorname{Ker}R_{\rm sym}$.

3. Cabling the Burau representation

3.1. **Infinitesimal result.** Let us consider the infinitesimal Burau representation of $\mathfrak{B}_{nr}$. We can index the basis elements e_s as e_i^j for $1\leq i\leq r$ and $1\leq j\leq n$ with $s=r(j-1)+i$. Using $\partial_{n,r}:\mathfrak{B}_n\to\mathfrak{B}_{nr}$ one gets a representation $\rho_{\rm bur}\circ\partial_{n,r}$ defined by $s.e_i^j=e_i^{s(j)}$ for $s\in\mathfrak{S}_n$ and

$$\begin{cases}t_{ij}.e_s^i&=\sum_{t\in[1,r]}e_t^j+r(r-1)e_s^i\\ t_{ij}.e_s^j&=\sum_{t\in[1,r]}e_t^i+r(r-1)e_s^j\\ t_{ij}.e_s^k&=r^2e_s^k\qquad\text{if } k\notin\{i,j\}.\end{cases}$$

For all $1\leq i\leq n$, Let us introduce $u_i=\sum_{s\in[1,r]}e_s^i$. The subspace U generated by $u_1,\dots,u_n$ has dimension n. From the above formulas one gets

$$\begin{cases}t_{ij}.u_i&=ru_j+r(r-1)u_i\\ t_{ij}.u_j&=ru_i+r(r-1)u_j\\ t_{ij}.u_k&=r^2u_k\qquad\text{if } k\notin\{i,j\}.\end{cases}$$

This means that, on U, t_{ij} acts in the same way that $r(i\ j)+r(r-1)$, hence the action of $\widehat{\mathfrak{B}}_n$ is isomorphic to $r\rho_{\rm bur}+r(r-1)$. Using φ one thus gets a representation of B_n isomorphic to $r(r-1)R_{\rm bur}^{q^r}$.

Let E_i for $1\leq i\leq n$ denote the subspace spanned by e_s^i pour $s\in[1,r]$, and α_i be the linear form on E_i^* defined by $\alpha_i(e_s^i)=1$. Let $K_i=\operatorname{Ker}\alpha_i\subset E_i$, which is spanned by the $e_s^i-e_t^i$, and let K be the (direct) sum of these subspaces.

If $x\in K_i$ or $x\in K_j$, one has $t_{ij}.x=r(r-1)x$ and, if $x\in K_k$ for $k\notin\{i,j\}$, then $t_{ij}.x=r^2$. It follows that K is stable under the action of $\mathfrak{B}_n$ and that the action of $\mathcal{A}_n$ on

K is commutative. More precisely, on the subspaces $F_{s,t}$ spanned by the $v^i_{s,t} = e^i_s - e^i_t$ for $i \in [1, n]$, when $s \neq t$, the action of $\mathfrak{B}_n$ is isomorphic to $r^2 - r\rho_{\text{sym}}$. It follows that we have the following result.

Proposition 4. *The representation $\rho_{\text{bur}} \circ \partial_{n,r}$ is isomorphic to the direct sum of $r\rho_{\text{bur}} + r(r-1)$ and $r-1$ copies of $r^2 - r\rho_{\text{sym}}$.*

3.2. **Global result.** Since there exist morphisms φ and $\varphi_{n,r}$ satisfying the conclusions of proposition 2, the above proposition implies our main theorem.

Theorem 2. *The representation $R_{\text{bur}} \circ \Delta_{n,r}$ is isomorphic to the direct sum of $q^{r(r-1)}R^{q^r}_{\text{bur}}$ and $r-1$ copies of $q^{r^2}R^{q^{-r}}_{\text{sym}}$.*

As far as faithfulness questions are concerned, this theorem implies that the kernel of $R_{\text{bur}} \circ \Delta_{n,r}$ in B_n is the intersection of those of $q^{r(r-1)}R_{\text{bur}}$ and $q^{r^2}R_{\text{sym}}$. Because $q^{2n-4} \neq q^{-2r(r-1)n}$ for all values of $r \geq 1$, and $\operatorname{Ker} R_{\text{bur}} \subset (B_n, B_n) \cap P_n$, then $\operatorname{Ker} q^{r(r-1)}R_{\text{bur}} = \operatorname{Ker} R_{\text{bur}}$. Moreover $\operatorname{Ker} R_{\text{bur}} \subset (P_n, P_n) \subset \operatorname{Ker} q^{r^2}R_{\text{sym}}$. It follows that cabling does not improve faithfulness of the Burau representation.

Corollary. *For all $n \geq 3$ and $r \geq 1$, the kernel of $R_{\text{bur}} \circ \Delta_{n,r}$ coincides with the kernel of R_{bur}. In particular $R_{\text{bur}} \circ \Delta_{n,r}$ is not faithful for $n \geq 5$.*

References

[BN1] Bar-Natan, Non-associative tangles, in "Geometric topology (Athens, GA, 1993)" 139-183 AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI (1997).

[BN2] D. Bar Natan, *On associators and the Grothendieck-Teichmuller group*, Sel. math., New ser. 4, 183–212 (1998).

[Bi] S. Bigelow, *The Burau representation is not faithful for $n = 5$*, Geom. Topol. 3 397–404 (1999).

[D] V. G. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. 2, 829–860 (1991).

[Ho] P.N. Hoefsmit, *Representations of Hecke algebras of finite groups with BN-pairs of classical type*, Ph.D. Thesis, University of British Columbia, 1974.

[JS] A. Joyal and R. Street, *Braided tensor categories*, Advances in Math. 102, 20–78 (1993).

[K] M. Kontsevich, *Vassiliev's knot invariants.* Adv. Sov. Math. 16(2), 137-150 (1993).

[LM1] T. T. Q. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Compositio Math. 102, 42–64 (1996).

[LM2] T. T. Q. Le and J. Murakami, *Parallel version of the universal Vassiliev-Kontsevich invariant*, J. Pure and Applied Algebra 121, 271–291 (1997).

[L] C. Lescop, *Introduction to the Kontsevich integral of framed tangles*, Summer School lectures, Grenoble 1999, `www-fourier.ujf-grenoble.fr/ECOLETE`

[LP] D.D. Long, M. Paton, *The Burau representation is not faithful for $n \geq 6$*, Topology **32**, no. 2, 439–447 (1993).

[Mo] J.A. Moody, *The Burau representation of the braid group B_n is unfaithful for large n*, Bull. Amer. Math. Soc. (N.S.) **25**, no. 2, 379–384 (1991).

[Ma1] I. Marin, *Représentations linéaires des tresses infinitésimales*, Ph. D. Thesis, Orsay, 2001.

[Ma2] I. Marin, *On the representation theory of braid groups*, preprint `arXiv:math.RT/0502118 (v3)`.

[Ti] J. Tits, *Normalisateurs de tores I. Groupes de Coxeter étendus* J. Algebra 4, 96–116 (1966).

[Wz] H. Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math. 92 349–383 (1988).