THE FREENESS CONJECTURE FOR HECKE ALGEBRAS OF COMPLEX REFLECTION GROUPS, AND THE CASE OF THE HESSIAN GROUP G_{26}

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ABSTRACT. We review the state-of-the-art concerning the freeness conjecture stated in the 1990's by Broué, Malle and Rouquier for generic Hecke algebras associated to complex reflection groups, and in particular we expose in detail one of the main differences with the ordinary case, namely the lack of 0-Hecke algebras. We end the paper by proving a new case of this conjecture, the exceptional group called G_{26} in Shephard-Todd classification, namely the largest linear group of automorphisms of the Hessian configuration.

CONTENTS

1. INTRODUCTION

Between 1994 and 1998, M. Broué, G. Malle and R. Rouquier introduced a natural generalization of the generic Iwahori-Hecke algebras, associated not only to a Weyl or Coxeter group, but to an arbitrary (finite) complex reflection group W (see [9, 10]). Extending earlier work by Broué and Malle (see [7]), they found an adequate definition involving the generalized braid group B associated to W. They stated a number of conjectures, some of them involving the braid group B, some others involving the Hecke algebra H. All the conjectures concerning the braid group B have apparently been solved now (see [3, 4, 11]). The ones concerning the Hecke algebras, on the other hand, are not solved yet for the finite but rather large number of exceptional groups involved in the Shephard-Todd classification of irreducible reflection groups. Arguably, the most basic one is the so-called *freeness conjecture*, which states that H is free of rank $|W|$ as a module over its ring of definition R.

The many other existing conjectures about these generalized Hecke algebras originate in a program about representation theory of finite groups of Lie type, and involve notably the existence of a canonical trace; this program also suggests a number of other properties, including that the

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center of H should also be a free module, that it behaves well under base changes, and so on. It is also very important to be able to compute matrix models for the irreducible representations of H. However, the reason why the freeness conjecture is more basic than the other ones is that, once it is proved, we can rest on our better knowledge of the world of the associative algebras which have finite type as modules. This better knowledge includes the possibility of putting structure constants for the multiplication into a computer and apply various algorithms in order to improve our understanding of what happens in each case (see [25] for an explanation of how the determination of a canonical trace can be made effective in this way). Also, it is proved in [18] that, provided that the freeness conjecture is true, the category of representations of H (actually defined over a larger ring) is equivalent to a category of representations of a 'Cherednik algebra', and this provides other tools in order to possibly deal with the other conjectures.

The primary goal of this paper was to prove this freeness conjecture, that we call here *the* BMR conjecture in order to emphasize its central role, for the case of the exceptional group G_{26}, which has rank 3 and is the largest of the two complex reflection groups groups that appear as symmetry groups of the classical Hessian configuration (theorem 4.1). In addition, in the first part of the paper, we provide some more scholarly work, that we felt were missing in the literature. This includes the comparison of various versions of the BMR conjecture, and the algebraization of the powerful argument of Etingof and Rains, which proves a weak version of the conjecture for all groups of rank 2. We also explore in detail why it does not seem possible to define an analogue of the 0-Hecke algebra for complex reflection groups, which is a big difference with the usual (Weyl/Coxeter) case.

In this part, we have tried to be as precise and detailed as possible, at the risk of being pedantic or boring. One reason for this is that we felt that previous work on this conjecture, whose difficulty has for a long time been underestimated, has sometimes been sloppy on details. For instance, the proofs given in [7] for the groups G_4, G_5, G_{12} and G_{25} are very sketchy. In addition, one caveat that has repeatedly been overlooked for years is the possibility that H might have *torsion*, a phenomenon that should not happen in view of the conjecture, but which is hard to rule out *a priori* – and this should not be too surprising in view of the example of some torsion elements inside the '0-Hecke algebras' that we describe below (see figure 1 and proposition 3.4). Because of this, one cannot use embeddings of H into $H \otimes F$, for $F = Frac(R)$ the field of fractions of R. This mistake appears in [25]: proposition 2.10 is not correct because of this, and this appears to ruin the strategy explained there of first proving that $H \otimes F$ has finite dimension and deducing from it the freeness conjecture (in the notations of [25], deducing conjecture 2.2 from conjecture 2.1). It had already appeared in [7] §4B, in the few details that are given concerning the proof of the BMR conjecture for G_4, G_5 and G_{12} : the expressions described there have coefficients which are not specified, but are claimed to belong to F, which means that the authors are dealing with $H \otimes F$ instead of H. It also appears in [8], proposition 2.2 where the uniqueness of the trace is not actually proved over R, as it should be in view of [8] 2.1 (2), but over some R_k, $k \supset \mathbf{Z}$ (see below for notations).

Concerning the known cases of the conjecture among the exceptional groups, the situation thus heavily depends on the standard of rigorousness and checkability you are willing to accept: depending on this, you can say that either *almost all* or *almost none* of the exceptional cases have been proved (with the exception of the weak version proved by Etingof and Rains for groups of rank 2). On the lax side, one may say that Broué and Malle proved it for the 4 groups above (Berceanu and Funar independently did the case of G_4 in [16], appendix A), that I proved it for G_{32} (rank 4) in [26], and J. Müller has announced results 10 years ago involving Linton's algorithm called 'vector enumeration', claiming the result for all groups of rank 2 but G_{17}, G_{18}, G_{19}, as well as all the groups of rank 3 (including G_{26}). The missing cases in large rank would then be G_{29}, G_{31}, G_{33} and G_{34}. On the uncompromising side however, we already mentioned possible mistakes in [7], and Müller's program has not been made publicly available and checkable (so far, none of the usual software in computer algebra implements vector enumeration *over* R). This is problematic in view, not only of the possible mistakes mentioned above, not only because the need to trust the scientist's word is something modern science has been trying to avoid for centuries, but also because of the very nature of the vector enumeration algorithm. This algorithm is indeed a (clever) variation of the Todd-Coxeter algorithm, and as such provides no control *a priori* on when it stops if it does. Moreover, the moment it ends depends a lot on a number of heuristic choices that need to be made inside the specific implementation of the algorithm. According to J. Müller (private conversation, Aachen, 2010), it is moreover unclear that the program he wrote would run now on modern computers.

In the current situation, running the welcome risk of being checked and judged with the same severity, we thus stick to the hardline position, of somewhat provocatively claiming that only the cases of G_4, G_{25}, G_{32} and now G_{26} have been fully checked so far. Our hope is to encourage people to treat the other cases 'by hand', which is a way that usually provides more information about the algebras under consideration, and also to encourage authors and editors to provide and publish full details for these computations. This would enable people to check and improve these, or use them in possibly unpredicted ways.

2. General considerations on the generic Hecke algebras

Let W be a (pseudo-)complex reflection group, always assumed to be finite, and let $R = \mathbf{Z}[a_{s,i}, a_{s,0}^{-1}]$ where s runs among a (finite) representative system $\mathcal{P}$ of conjugacy classes of distinguished reflections in W and $0 \leq i \leq o(s)-1$, where $o(s)$ is the order of s in W. We let B denote the braid group of W, as defined in [10], and recall that a reflection s is called distinguished if its only nontrivial eigenvalue is $\exp(2\mathrm{i}\pi/o(s))$, where $\mathrm{i} \in \mathbf{C}$ is the chosen square root of -1.

Definition 2.1. *The generic Hecke algebra is the quotient of the group algebra RB by the relations $\sigma^{o(s)} - a_{s,o(s)-1}\sigma^{o(s)-1} - \dots - a_{s,0} = 0$ for each braided reflection σ associated to s.*

Actually, it is enough to choose one such relation per conjugacy class of distinguished reflection, as all the corresponding braided reflections are conjugated one to the other. In [10] was stated the following conjecture.

Conjecture 2.2. *(BMR conjecture) The generic Hecke algebra H is a free R-module of rank $|W|$.*

2.1. **Root parameters vs. coefficient parameters.** Usually, the Hecke algebra associated to a complex reflection group is defined over the ring $\tilde{R} = \mathbf{Z}[u_{s,i}, u_{s,i}^{-1}]$. More precisely, this Hecke algebra $\tilde{H}$ is defined as the quotient of the group algebra $\tilde{R}B$ of the braid group by the ideal generated by the relations $\prod_{i=0}^{o(s)-1}(\sigma - u_{s,i}) = 0$ where the σ are the braided reflections associated to s. Note that R is the subring $\tilde{R}^{\mathfrak{S}}$ of invariants of $\tilde{R}$ under the natural action of $\mathfrak{S} = \prod_{s \in \mathcal{P}} \mathfrak{S}_{o(s)}$.

The Broué-Malle-Rouquier conjecture for H, namely that H is a free R-module of rank $|W|$, clearly implies that $\tilde{H}$ is a free $\tilde{R}$-module of rank $|W|$. The converse being less obvious, and since most authors including [7] use the Hecke algebra over $\tilde{R}$ instead of R, we prove it here. Note that J. Müller in [29] uses the definition over R for his computer calculations.

Lemma 2.3. *$\tilde{H}$ is a free $\tilde{R}$-module of rank N if and only if H is a free R-module of rank N.*

Proof. Let I denote the two-sided ideal of RB generated by the $\sigma^{o(s)} - a_{s,o(s)-1}\sigma^{o(s)-1} - \dots - a_{s,0}$. By definition $\tilde{H}$ is the quotient of $\tilde{R}B = RB \otimes_R \tilde{R}$ by the image of $I \otimes_R \tilde{R}$ in $RB \otimes_R \tilde{R}$ which, by right-exactness of the tensor product, is $H \otimes_R \tilde{R}$. Thus $\tilde{H} = H \otimes_R \tilde{R}$. If $H = R^N$ then clearly $\tilde{H} = \tilde{R}^N \otimes_R \tilde{R} = \tilde{R}^N$ is free of rank N.

Conversely, we assume that $\tilde{H} = \tilde{R}^N$. First note that $\tilde{R}$ is a free R-module of rank $|\mathfrak{S}|$ (see e.g. [5] chapitre 4, §6, n^o 1, théorème 1), hence $\tilde{H} = R^{N|\mathfrak{S}|}$ as a R-module. Moreover, $\tilde{H} = H \otimes_R \tilde{R} \simeq H \otimes_R R^{|\mathfrak{S}|} \simeq H^{|\mathfrak{S}|}$ as an R-module. In particular H is a direct factor of the free R-module $\tilde{H}$ and is thus projective, hence flat, as an R-module.

Now note that $H = H \otimes_R R = H \otimes_R \tilde{R}^{\mathfrak{S}} = H \otimes_R \mathrm{Hom}_{R\mathfrak{S}}(R, \tilde{R})$ where R is considered as a trivial $R\mathfrak{S}$-module. Since R is a noetherian ring, $R\mathfrak{S}$ is noetherian as an R-module and thus left-noetherian as a ring, hence R admits a finite presentation as an $R\mathfrak{S}$-module. Since R is flat, it follows from general arguments (see [6] ch. 1, §2, numéro 9, prop. 10) that

$$\mathrm{Hom}_{R\mathfrak{S}}(R, H \otimes_R \tilde{R}) \simeq H \otimes_R \mathrm{Hom}_{R\mathfrak{S}}(R, \tilde{R}) = H \otimes_R \tilde{R}^{\mathfrak{S}} = H.$$

But the LHS is $\mathrm{Hom}_{R\mathfrak{S}}(R, \tilde{H}) = \mathrm{Hom}_{R\mathfrak{S}}(R, \tilde{R}^N) = \mathrm{Hom}_{R\mathfrak{S}}(R, \tilde{R})^N = R^N$ and this proves the claim. □

2.2. **The BMR conjecture and Tits' deformation theorem.** Let $F = \mathbf{Q}(a_{s,i})$ denote the field of fractions of R, and $\overline{F}$ an algebraic closure. For k a unital ring, we let $R_k = R \otimes_{\mathbf{Z}} k$, $H_k = H \otimes_R R_k$. We let $\mathbf{C}$ denote the field of complex numbers. Part (1) of the next proposition is in [10] (see the proof of theorem 4.24 there).

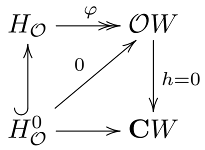
Proposition 2.4.

1. *If H is generated as a module over R by $|W|$ elements, then H is a free R-module of rank $|W|$.*
2. *If H is finitely generated as a module over R, then $H \otimes_R \overline{F} \simeq \overline{F}W$.*
3. *If $H_{\mathbf{C}}$ is finitely generated as a module over $R_{\mathbf{C}}$, then $H \otimes_R \overline{F} \simeq \overline{F}W$.*

Proof. Let $\mathcal{O} = \mathbf{C}[[h]]$, and $K = \mathbf{C}((h))$ its field of fractions. By the Cherednik monodromy construction, one can build a morphism $\varphi : H \otimes \mathcal{O} \to \mathcal{O}W \simeq \mathcal{O}^{|W|}$, where the morphism $R \to \mathcal{O}$ defining the tensor product $H \otimes \mathcal{O}$ depends on the collection of parameters involved in the monodromy construction. We take these parameters to be linearly independent over $\mathbf{Q}$, so that the morphism $R \to \mathcal{O}$ is injective. Modulo h, the morphism φ is the identity of $\mathbf{C}W$, hence the original morphism is surjective by Nakayama's lemma.

Let $N = |W|$. If H is generated by N elements, then there exists a surjective morphism of R-modules $\pi : R^N \twoheadrightarrow H$, which induces $\pi \otimes \mathcal{O} : \mathcal{O}^N \twoheadrightarrow H \otimes \mathcal{O}$. Then $\varphi \circ (\pi \otimes \mathcal{O}) : \mathcal{O}^N \twoheadrightarrow \mathcal{O}W \simeq \mathcal{O}^N$. Such a surjective morphism between two free modules of the same finite rank is necessarily an isomorphism, hence $\pi \circ \mathcal{O}$ is injective, and in particular π is injective. This means $H \simeq R^N$, that is (1). Clearly (3) implies (2), so we prove (3).

For this purpose we consider $H_{\mathcal{O}} = H_{\mathbf{C}} \otimes \mathcal{O}$, and $\pi_{\mathcal{O}} : \mathcal{O}^{N'} \twoheadrightarrow H_{\mathcal{O}}$. Let $H^0_{\mathcal{O}}$ the torsion submodule of $H_{\mathcal{O}}$. Since $H_{\mathcal{O}}$ is finitely generated over the principal ring $\mathcal{O}$, we have that $\hat{H_{\mathcal{O}}} = H_{\mathcal{O}}/H^0_{\mathcal{O}}$ is a finitely generated *free* $\mathcal{O}$-module.



By the above commutative diagram, the specialization morphism $H_{\mathcal{O}} \twoheadrightarrow \mathbf{C}W$ factors through $\hat{H_{\mathcal{O}}}$, hence we can apply Tits' deformation theorem to the free $\mathcal{O}$-module $\hat{H_{\mathcal{O}}}$ and get $\hat{H_{\mathcal{O}}} \otimes \overline{K} \simeq \overline{K}W$. Since $\hat{H_{\mathcal{O}}} \otimes \overline{K} \simeq H_{\mathcal{O}} \otimes \overline{K}$ this means $H_{\mathcal{O}} \otimes \overline{K} \simeq \overline{K}W$, hence $H \otimes \overline{F}$ is a semisimple algebra with the same numerical invariants (because $H_{\mathcal{O}} \otimes \overline{K} = H \otimes \overline{K}$ is an extension of scalars, since $\overline{F} \hookrightarrow \overline{K}$), and this implies $H \otimes \overline{F} \simeq \overline{F}W$.

□

Recall that every finitely generated flat R-module is projective ; moreover, as a consequence of Swan's big rank theorem (see [32], and [20], ch. 5) every finitely generated projective R-module of rank at least 2 is actually free and, because $\mathbf{Z}$ is a regular ring, $K_0(R) \simeq K_0(\mathbf{Z}) = \mathbf{Z}$ which implies that also the rank 1 projective modules are free (see e.g. [20], ch. 5 lemma 4.4 and ch.1 cor. 6.7). We summarize this as follows.

Proposition 2.5. *If H is finitely generated as an R-module, then H is free of rank $|W|$ if and only if it is flat.*

If it is known that H is finitely generated, the BMR conjecture thus becomes a local condition. For a prime ideal $\mathfrak{p}$ of R, and M an R-module, let $R_{\mathfrak{p}}$ denote the localization of R at $\mathfrak{p}$ and $M_{\mathfrak{p}} = M \otimes_R R_{\mathfrak{p}}$.

In the specific neighborhoods of $\operatorname{Spec} R$ corresponding to the specializations $H \to kW$, the following can be proved.

Proposition 2.6. *Let k be a field. Let $\mathfrak{m} = \mathrm{Ker}(R \to k)$ be the maximal ideal defined by $a_{s,i} \mapsto 0$ for $i > 0$ and $a_{s,0} \mapsto 1$. If $H_{\mathfrak{m}}$ is finitely generated as a $R_{\mathfrak{m}}$-module (for instance if H is finitely generated as a R-module), then it is free of rank $|W|$.*

The proposition above is a consequence (see e.g. [6] ch. III §3 no. 5, cor. 2 prop. 9) of the next one, basically deduced from [10] by Etingof and Rouquier (unpublished). This property is essentially what P. Etingof calls 'formal flatness'.

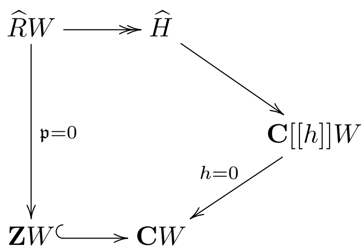
Proposition 2.7. *Let $\mathfrak{p} = \mathrm{Ker}(R \to \mathbf{Z})$ be the prime ideal defined by $a_{s,i} \mapsto 0$ for $i > 0$ and $a_{s,0} \mapsto 1$, and let $\widehat{R}$, $\widehat{H}$, etc., denote the completions w.r.t. the $\mathfrak{p}$-adic topology. Then $\widehat{H}$ is a free $\widehat{R}$-module of rank $|W|$.*

Proof. Let $\tilde{W} = \{\tilde{w} \mid w \in W\}$ be the image of a set-theoretic section of $B \twoheadrightarrow W$, and $\check{W}$ be its image in H. The map $w \mapsto \tilde{w}$ induces using the natural projection $RB \twoheadrightarrow H$ a continuous morphism of R-modules $RW \to \widehat{H}$, hence $\Phi : \widehat{R}W \simeq \widehat{RW} \to \widehat{H}$. We prove that Φ is surjective. We first need a lemma.

Lemma 2.8. *For all $r \geq 1$, $H = R\check{W} + \mathfrak{p}^r H$.*

Proof. Let $P = \mathrm{Ker}(B \to W)$ denote the pure braid group. We have $RB = R\tilde{W} + \tilde{W} \sum_{g \in P}(g-1)$. The image in H of $(g-1)$ for $g \in P$ falls into $\mathfrak{p}H$, hence $H = R\check{W} + \mathfrak{p}H$ which easily implies the conclusion. □

As a consequence, we get that $R\check{W}$ is dense inside $\widehat{H}$, thus proving that Φ is surjective. Now, the KZ-like construction of [10] provides a morphism $\psi : H \to \mathbf{C}[[h]]W$ associated to some morphism $R \to \mathbf{C}[[h]]$, which is continuous for the $(\mathfrak{p}, h)$-topologies, hence extends to a morphism $\widehat{H} \to \mathbf{C}[[h]]W$. We have the following diagram.



Since $\widehat{R}W$ is a free module of finite rank, the injectivity of the natural map $\mathbf{Z}W \to \mathbf{C}W$ implies that $\widehat{R}W \to \widehat{H}$ is injective, whence $\widehat{R}W \simeq \widehat{H}$ and the conclusion.

□

An elementary remark is that, in a number of exceptional cases, the BMR conjecture can be reduced to a problem in 1 variable. Indeed, the following apply to all the exceptional non-Coxeter groups of rank at least 3, except the Shephard group G_{25}, G_{26} and G_{32}, that is to the groups G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}. We let $\mathcal{A}$ denote the collection of all the hyperplanes which are sets of fixed points for reflections in W. It is naturally acted upon by W.

Proposition 2.9. *Assume that $|\mathcal{A}/W| = 1$ and that all reflections have order 2. Let k be a commutative unital ring. Then H_k is a free R_k-module of rank $|W|$ (respectively, a finitely generated R_k-module) if and only if the quotient of $k[x^{\pm 1}]B$ by the ideal generated by $(\sigma-1)(\sigma-x)$ for σ a braided reflection, has the same property.*

Proof. Let H^0 denote the quotient of $k[x^{\pm 1}]B$ by the ideal generated by $(\sigma-1)(\sigma-x)$ for σ one given braided reflection. Since all braided reflections are conjugate in B, this is the algebra involved in the statement. Moreover notice that B is normally generated, as a group, by σ. Let $\tilde{R}_k = k[u_0^{\pm 1}, u_1^{\pm}]$. We already know that the assumptions of the proposition on H_k are equivalent to the same assumptions for $\tilde{H}_k = H_k \otimes_{R_k} \tilde{R}_k$. For every $\beta \in \tilde{R}_k$, there exists an algebra morphism $\varphi_\beta : \tilde{R}_k B \to \tilde{R}_k B$ which maps σ to $\beta\sigma$; it can be defined, using that $B^{ab} \simeq \mathbf{Z}$ is generated by the image $\overline{\sigma}$ of σ, as the composite of the natural algebra morphisms

$$\tilde{R}_k B \xrightarrow[\Delta]{} (\tilde{R}_k B) \otimes (\tilde{R}_k B) \xrightarrow[\mathrm{Id} \otimes Ab]{} (\tilde{R}_k B) \otimes (\tilde{R}_k B^{ab}) \xrightarrow[\overline{\sigma} \mapsto \beta]{} (\tilde{R}_k B) \otimes \tilde{R}_k \xrightarrow[\simeq]{} \tilde{R}_k B$$

where Δ denotes the coproduct. These morphisms are equivariant under the conjugation action of B on itself, thus $\varphi_{\beta_1} \circ \varphi_{\beta_2}(\sigma) = \varphi_{\beta_1\beta_2}(\sigma)$ implies that $\varphi_\beta \circ \varphi_{\beta^{-1}} = \mathrm{Id}$, hence φ_β is an isomorphism when $\beta \in \tilde{R}_k^\times$. Recall now that $\tilde{H}_k$ is the quotient of $\tilde{R}_k B$ by the relation $(\sigma - u_0)(\sigma - u_1)$. Taking $\beta = u_0$, we get that the image of $(\sigma - u_0)(\sigma - u_1)$ under φ_{u_0} is $u_0^2(\sigma - 1)(\sigma - u_1 u_0^{-1})$, whence a $\tilde{R}_k$-isomorphism $\tilde{H}_k \simeq H^0 \otimes_{k[x^{\pm 1}]} \tilde{R}_k$, where $k[x^{\pm 1}] \subset \tilde{R}_k$ is defined by $x \mapsto u_1 u_0^{-1}$. We have $R_k = \bigoplus_{a,b} k u_0^a u_1^b = \bigoplus h(u_1 u_0^{-1})^b u_0^a$ is free hence faithfully flat as a $k[x^{\pm 1}]$-module, and this proves the claim. □

Let $z \in Z(B)$, $\overline{B} = B/\langle z \rangle$, and $s : \overline{B} \to B$ a set-theoretic section of the natural projection $b \mapsto \bar{b}$. We let $R_k^+ = R_k[x, x^{-1}]$. We denote $\sigma_1, \dots, \sigma_r$ a distinguished system of braided reflections (all corresponding to distinguished reflections, and at least 1 for each conjugacy class). We let $P_i \in R_k[X]$ denote polynomials defining H_k, that is H_k is the quotient of $R_k B$ by the relations $P_i(\sigma_i) = 0$.

Proposition 2.10.

(1) *$R_k B$ admits an R_k^+-module structure defined by $x.b = zb$ for $b \in B$. It is a free R_k^+-module, and we have an isomorphism $R_k B \simeq R_k^+ \overline{B}$.*

(2) *Under this isomorphism, the defining ideal of H_k is mapped to the ideal of $R_k^+ \overline{B}$ generated by the $Q_i(\overline{\sigma}_i)$ for $Q_i(X) = P_i(X x^{a_i}) \in R^+[X]$, the $a_i \in \mathbf{Z}$ being defined by $\sigma_i = s(\overline{\sigma}_i) z^{a_i}$.*

Proof. Since $Z(B)$ is torsion-free, $\langle z \rangle \simeq \mathbf{Z}$, and there is a uniquely defined 1-cocycle $\alpha : B \twoheadrightarrow \mathbf{Z}$ such that $\forall b \in B$ $b = z^{\alpha(b)} s(\bar{b})$. Therefore, as R_k-modules, $R_k B = \bigoplus_{b \in B} R_k b = \bigoplus_{b \in B} R_k z^{\alpha(b)} s(\bar{b}) = \bigoplus_{d \in \overline{B}} \bigoplus_{\bar{b} = d} R_k z^{\alpha(b)} d$. Under this identification, for every $d \in \overline{B}$, $\bigoplus_{\bar{b}=d} R_k z^{\alpha(b)} d$ is a free R_k^+-submodule of rank 1, whence (1). Let $I \subset R_k B$ the defining ideal for H_k. It is the R_k-submodule spanned by the $b P_i(\sigma_i) c$ for $i \in \{1, \dots, r\}$ and $b, c \in B$, hence also the R_k^+-submodule spanned by the $s(d) P_i(\sigma_i) s(e)$ for $d, e \in \overline{B}$. We have $P_i(\sigma_i) = Q(s(\overline{\sigma}_i))$, hence I is identified inside $R_k^+ \overline{B}$ with the R_k^+-submodule generated by the $d Q_i(\overline{\sigma}_i) e$ for $d, e \in \overline{B}$, that is to the ideal of $R_k^+ \overline{B}$ generated by the $Q_i(\overline{\sigma}_i)$. □

Note that, if the P_i have been chosen to be monic, one can replace the Q_i by the monic polynomials $x^{-a_i d^\circ P_i} P_i(X x^{a_i}) \in R_k^+[X]$, where $d^\circ P_i$ denotes the degree of P_i. Note also that, being the quotient of $R_k B$ by an R_k^+-submodule, H_k inherits a structure of R_k^+-module.

The following proposition is based on one of the arguments of Etingof-Rains [13].

Proposition 2.11. *If H_k is finitely generated as a R_k^+-module, then it is finitely generated as a R_k-module.*

Proof. By assumption, H_k is generated as a R_k^+-module by a finite set of N_1 elements. Let us choose some $Q \in R_k[x] \subset R_k^+$, and $M = Q H_k \subset H_k$. By assumption it is generated by N_1 elements as a R_k^+-module. We have $M = 0$ as soon as $M_\mathfrak{a} = 0$ for all the maximal ideals $\mathfrak{a}$ of R_k^+, where $(R_k^+)_\mathfrak{a}$ denotes the localization of R_k^+ at $\mathfrak{a}$, and $M_\mathfrak{a} = M \otimes_{R_k^+} (R_k^+)_\mathfrak{a}$ (see e.g. [12] lemma 2.8, p. 67-68). Since $(R_k^+)_\mathfrak{a}$ is a local ring, by Nakayama's lemma we have $M_\mathfrak{a} = 0$ iff $M_\mathfrak{a} = \mathfrak{a} M_\mathfrak{a}$. Since $(R_k^+)_\mathfrak{a}$ is a flat R_k^+-module, this is equivalent to $M = \mathfrak{a} M$, that is $M \otimes_{R_k^+} ((R_k^+)/\mathfrak{a}) = 0$. Let us denote $K = (R_k^+)/\mathfrak{a}$ and $\overline{K}$ the algebraic closure of K, and $\tilde{\lambda} : R_k^+ \to \overline{K}$ the natural morphism. We have $M \otimes_{R_k^+} \overline{K} = Q(H_k \otimes_{R_k^+} \overline{K}) = (Q H_k) \otimes_{R_k^+} \overline{K} = H_k \otimes_{R_k^+} \tilde{\lambda}(Q) \overline{K}$. Let $\tilde{R}_k = \tilde{R} \otimes_{\mathbf{Z}} k$ and $\tilde{R}_k^+ = \tilde{R}_k[x, x^{-1}]$. Since it is clearly an integral extension of R_k^+ and $\overline{K}$ is algebraically closed, $\tilde{\lambda}$ can be extended to $\tilde{\lambda} : \tilde{R}_k^+ \to \overline{K}$.

We know that $H_k \otimes_{R_k^+} \overline{K}$ is a $\overline{K}$-algebra of dimension at most N_1. If $H_k \otimes_{R_k^+} \tilde{\lambda}(Q) \overline{K} \neq 0$, there exists a simple $H_k \otimes_{R_k^+} \overline{K}$-module V of dimension $N_2(V) \leq N_1$ in which $\tilde{\lambda}(Q)$ acts by a nonzero scalar. It defines an irreducible representation $\rho : B \to \mathrm{GL}_{N(V_2)}(\overline{K})$ of B over $\overline{K}$. By definition, z acts on V through $\tilde{\lambda}(x)$, which has determinant $\tilde{\lambda}(x)^{N_2(V)}$. On the other hand, z is the product of N_3 distinguished braided reflections $\sigma_1 \dots \sigma_{N_3}$ with N_3 independent of the previous choices. Since each $\rho(\sigma_i)$ is annihilated by a split polynomial with roots inside $\{\tilde{\lambda}(u_{c,i}) \mid c \in \mathcal{A}/W, 0 \leq i \leq e_c - 1\}$,

$\det \rho(\sigma_i)$ is a monomial of degree N_2 in these variables. It follows that $\tilde{\lambda}(x)^{N_2(V)}$ is a monomial of degree $N_2(V)N_3$ in these variables. Let $\mathcal{M}$ be the set of all such monomials of degree at most N_1N_3. Since $N_2(V) \leq N_1$, we get that $\tilde{\lambda}(x)$ is annihilated by the polynomial

$$\tilde{Q} = \prod_{1 \leq r \leq N_1} \prod_{m \in \mathcal{M}} (X^r - m) \in (\tilde{R}_k[X])^{\mathfrak{S}} = R_k[X]$$

so we set $Q = \tilde{Q}(x) \in R_k^+$. By construction we have that $\tilde{\lambda}(Q) = \tilde{Q}(\tilde{\lambda}(x))$ acts by 0 on $H_k \otimes_{R_k^+} \overline{K}$, hence $M = 0$ and $QH_k = H_k$. It follows that $H_k = H_k \otimes_{R_k^+} (R_k^+/(Q))$ is finitely generated as an $R_k^+/(Q)$-module. Since $R_k^+/(Q)$ is a finitely generated R_k-module, the conclusion follows. □

2.3. **Groups of rank 2.** Assume that W is an irreducible exceptional group, and that it has rank 2. This part is a rewriting of the part of [13, 14, 15] which is relevant here. Let $\overline{B} = B/Z(B)$, and $\overline{W} = W/Z(W)$. A consequence of the classification of the finite subgroups of $\mathrm{SO}_3(\mathbf{R}) \simeq \mathrm{SU}_2/Z(\mathrm{SU}_2)$ is that $\overline{W}$ is the group of rotations of a finite Coxeter group C of rank 3, with Coxeter system y_1, y_2, y_3 and Coxeter matrix (m_{ij}). Let $\tilde{\mathbf{Z}} = \mathbf{Z}[\exp\left(\frac{2i\pi}{m_{ij}}\right)]$.

Etingof and Rains associate to every Coxeter group C with Coxeter system $y_1, \dots, y_n$ the following $\tilde{\mathbf{Z}}$-algebra (for simplicity, we assume $m_{ij} < \infty$, although their construction is more general). Let $a_{ij} = y_i y_j \in \overline{W}$. and define $A(C)$ to be the (associative) algebra with generators $Y_1, \dots, Y_n$, $t_{ij}[k]$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, $k \in \mathbf{Z}/m_{ij}\mathbf{Z}$, and relations $t_{ij}[k]^{-1} = t_{ji}[-k]$, $Y_i^2 = 1$, $\prod_{k=1}^{m_{ij}} (Y_i Y_j - t_{ij}[k]) = 0$, $y_r t_{ij}[k] = t_{ji}[k] y_r$, $t_{ij}[k] t_{i'j'}[k'] = t_{i'j'}[k'] t_{ij}[k]$. The subalgebra $A_+(C)$ generated by the $A_{ij} = Y_i Y_j$ becomes a R^C-algebra, with $R^C = \tilde{\mathbf{Z}}[t_{ij}[k]^{\pm 1}]$. As a R^C-algebra, it admits a presentation by generators A_{ij} and relations $\prod_{k=1}^{m_{ij}} (A_{ij} - t_{ij}[k]) = 0$, $A_{ij} A_{ji} = 1$, $A_{ij} A_{jk} A_{ki} = 1$ whenever $\#\{i, j, k\} = 3$.

Proposition 2.12. *(Etingof-Rains) If C is finite, then $A_+(C)$ is a finitely generated R^C-module.*

Proof. (sketch) Every word in the A_{ij}'s corresponds to a word (of even length) in the y_i's ; if the length of this word is greater than the length of the corresponding element of C, then there is a sequence of braid relations that transforms this word into another one containing y_i^2 for some i. Moreover, it is easily checked that every braid relation can be translated inside $A_+(C)$ into the transformations $A_{ij}^{\frac{m_{ij}}{2}} \rightsquigarrow A_{ji}^{\frac{m_{ij}}{2}} + \dots$ or $A_{ij}^{(m_{ij}-1)/2} A_{i\ell} \rightsquigarrow A_{ji}^{(m_{ij}-1)/2} A_{j\ell} + \dots$ where the dots represent terms of smaller length. Finally, when the word in the y_i's contains a y_j^2, this means that our original word contains a $A_{ij} A_{ik}$, which is either 1 or A_{ik}, hence the length gets reduced. This proves that $A_+(C)$ is generated as a R^C-module by words of bounded length in the A_{ij}'s, hence that it is finitely generated as a R^C-module. □

In order to apply this to our W, Etingof and Rains exhibit case-by-case in [13] explicit lifts $\tilde{a}_{ij} \in \overline{B}$, using which they prove the following, where we use the notations of proposition 2.10.

Proposition 2.13. *There exists a ring morphism $R^C \twoheadrightarrow R^+_{\tilde{\mathbf{Z}}}$ inducing $A_+(C) \otimes_{R^C} R^+_{\tilde{\mathbf{Z}}} \twoheadrightarrow R^+_{\tilde{\mathbf{Z}}} \overline{B}/Q_i(\overline{\sigma}_i)$ through $A_{ij} \mapsto \tilde{a}_{ij}$.*

An immediate consequence of this proposition together with propositions 2.10 and 2.11 is the following, essentially due to Etingof and Rains.

Theorem 2.14. *If W has rank 2, then H is finitely generated over R.*

Proof. By the above proposition and propositions 2.10 and 2.11 we get that $H_{\tilde{\mathbf{Z}}}$ is finitely generated over $R_{\tilde{\mathbf{Z}}}$. Since $\tilde{\mathbf{Z}}$ is a free $\mathbf{Z}$-module of finite rank (being finitely generated and torsion-free) $R_{\tilde{\mathbf{Z}}}$ is also free of finite rank over $R_{\mathbf{Z}}$. This implies that $H \subset H_{\tilde{\mathbf{Z}}}$ and that $H_{\tilde{\mathbf{Z}}}$ is finitely generated over R. Since R is noetherian this implies the conclusion. □

Remark 2.15. *There are exceptional groups of higher rank which are related to Coxeter groups, notably $G_{33}/Z(G_{33})$ and $G_{32}/Z(G_{32})$ are isomorphic to the group of rotations of the Coxeter group of type E_6 (which is a simple group of order 25920). One has $|Z(G_{33})| = 2$, $|Z(G_{32})| = 6$. However, the same method does not readily apply, because of the lack of convenient lifts.*

3. Remarks on the 0-Hecke algebras

In the Coxeter case, there is a notion of a 0-Hecke algebra which, although not being the quotient of the group algebra of B anymore, nevertheless displays many pleasant properties. In particular, it is still a free module of rank the order of the Coxeter group, and it admits an interpretation as an algebra of differential operators. In this section we expose two different kinds of obstructions for such a nice behavior to generalize.

3.1. **Demazure operators.** In [21], problem 5 in appendix C, G. Lehrer and D.E. Taylor ask whether the 'Demazure operators', which provide a description of the 0-Hecke algebra in the Coxeter setting, may provide a satisfactory description of the 0-Hecke algebra in the complex setting. More precisely, they ask whether these operators satisfy the homogeneous relations originating from the usual braid diagrams of the braid group.

In this section we give a negative answer to this problem, by computing precisely these operators in the smallest exceptional case, namely of the exceptional reflection group of type G_4. Recall e.g. from [10] that this group admits a Coxeter-like diagram of the form

$$G_4 \quad \underset{s_1}{③} \text{———} \underset{s_2}{③}$$

meaning that its braid group B is generated by two braided reflections s_1 and s_2 with relations $s_1s_2s_1 = s_2s_1s_2$ (hence B is isomorphic to the usual braid group on 3 strands, or Artin group of type A_2), and that the reflection group itself is the quotient of B by the relations $s_1^3 = s_2^3 = 1$.

The defining embedding $W < \mathrm{GL}_2(\mathbf{C})$ can for instance be described as follows, with $j = \exp(2\mathrm{i}\pi/3)$, $\mathrm{i} = \sqrt{-1}$.

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \quad s_2 = \frac{1}{3}\begin{pmatrix} j-j^2 & 2j+j^2 \\ 4j+2j^2 & -j-2j^2 \end{pmatrix}$$

Let $\delta_i^* \in \mathrm{End}(S(V^*))$ denote the Demazure operator associated to s_i, and $\delta_i \in \mathrm{End}(S(V))$ its dual operator. It is true that $(\delta_1^*)^3 = (\delta_2^*)^3 = 0$ (see e.g. [21], chapter 9, exercises), and the general question specializes to whether $\delta_1^*\delta_2^*\delta_1^* = \delta_2^*\delta_1^*\delta_2^*$ holds, or equivalently whether $\delta_1\delta_2\delta_1 = \delta_2\delta_1\delta_2$ holds, possibly up to a renormalization of the operators by non-zero scalars. We now explain the following computation.

Proposition 3.1. $\delta_1\delta_2\delta_1 \notin \mathbf{Q}(j)^\times\delta_2\delta_1\delta_2$, *hence the Demazure operators associated to the braid diagram of* G_4 *do not satisfy the braid relations up to a scalar.*

Proof. We let $V = \mathbf{C}^2$ with canonical basis $x = e_1$, $y = e_2$, hence $S(V) = \mathbf{C}[x,y]$, and s_2 maps x on $((j-j^2)x + (4j+2j^2)y)/3$, etc. The reflecting hyperplane of s_1 is spanned by x, and its root is a multiple of y ; the reflecting hyperplane of s_2 is spanned by $x-2y$, and its root is a multiple of $x+y$. Thus the corresponding Demazure operators are defined, up to a scalar of degree 0, by $s_1.p - p = y\delta_1 p$, $s_2.p - p = (x+y)\delta_2 p$. The expression of δ_1 is simple, as it maps a monomial of the form x^ay^b to $(j^b-1)x^ay^{b-1}$, as shown by a simple induction. The computation of δ_2 is more intricate. One gets easily

$$\left\{\begin{array}{rcl} 3\delta_2.y &=& 2j+j^2 \\ 3\delta_2.x &=& 4j+2j^2 \end{array}\right. \quad \left\{\begin{array}{rcrcr} 3\delta_2.y^2 &=& -jx &-& (3+j^2)y \\ 3\delta_2.xy &=& j^2x &-& 2y \\ 3\delta_2.x^2 &=& -4x &-& 4y \end{array}\right.$$

and

$$\begin{array}{rcl} 9\delta_2.y^3 &=& (j-j^2)x^2 - (7j+2j^2)xy + (10j+8j^2)y^2 \\ 9\delta_2.y^4 &=& j^2x^3 + (4j-j^2)x^2y - (10j+5j^2)xy^2 - (10+j^2)y^3 \end{array}$$

Starting from $\delta_1.y^4 = (j^4-1)y^3 = (j-1)y^3$ one thus gets

$$3\delta_1\delta_2\delta_1.y^4 = (5j^2-2j)x + (10j+8j^2)y$$

and

$$9\delta_2\delta_1\delta_2.y^4 = (4j-13j^2)x + (2j^2-2j)y.$$

This implies that $\delta_1\delta_2\delta_1.y^4$ and $\delta_2\delta_1\delta_2.y^4$ are linearly independent, which proves the claim. ☐

Of course, this obstruction might *a priori* vanish by taking another kind of diagrams. However, we notice that all the 6 pairs of the form $\{s,t\}$ with s,t among the 4 distinguished pseudo-reflections of the reflection group G_4 are conjugate to each other, whence from the above we get $\delta_s\delta_t\delta_s \notin \mathbf{Q}(j)^\times\delta_t\delta_s\delta_t$ for each of them.

After this example was computed, R. Rouquier told the author that M. Broué had already tried, some twenty years ago, to use Demazure operators for complex reflection groups, and that he had already noticed a similar defect.

3.2. **0-Hecke algebras defined by diagrams.**

3.2.1. *The case of G_4 : finite generation and torsion.* In view of the diagram described above, a natural candidate for the 0-Hecke algebra associated to the reflection group G_4 would be the following algebra.

Proposition 3.2. *Let k be a ring. The unital k-algebra defined by generators s_1, s_2 and relations $s_1s_2s_1 = s_2s_1s_2$, $s_1^3 = s_2^3 = 0$ is not finitely generated as a k-module.*

Proof. Let $\mathcal{W}, \mathcal{W}', \mathcal{Y}, \mathcal{Y}'$ be free k-modules with bases w_r, w'_r, y_r, y'_r, $r \geq 1$, and let $\mathcal{E} = \mathcal{W} \oplus \mathcal{W}' \oplus \mathcal{Y} \oplus \mathcal{Y}'$. We define k-endomorphisms S_1 and S_2 of $\mathcal{E}$ by

$$\left\{\begin{array}{lcl} S_1.w_r & = & 0 \\ S_2.w_r & = & y_{r+1} \end{array}\right. \quad \left\{\begin{array}{lcl} S_1.y_r & = & 0 \\ S_2.y_r & = & w'_r \end{array}\right. \quad \left\{\begin{array}{lcl} S_1.w'_r & = & y'_{r+1} \\ S_2.w'_r & = & 0 \end{array}\right. \quad \left\{\begin{array}{lcl} S_1.y'_r & = & w_r \\ S_2.y'_r & = & 0 \end{array}\right.$$

It is immediately checked that $S_1S_2S_1 = S_2S_1S_2 = S_1^3 = S_2^3 = 0$, and that $S_1^2S_2^2.w_r = w_{r+2}$ for all $r \geq 1$. This proves that S_1, S_2 defines on $\mathcal{E}$ a structure of module over the algebra A that we are considering. If A were finitely generated as a k-module, $A.w_1 \subset E$ would also be finitely generated as a k-module, contradicting the fact that it contains an infinite subset of a basis for E. □

Corollary 3.3. *Let $m \in \mathbf{Z} \setminus \{-1, 1\}$. The unital $\mathbf{Z}$-algebra defined by generators s_1, s_2 and relations $s_1s_2s_1 = s_2s_1s_2$, $s_1^3 = m$, $s_2^3 = m$ is not finitely generated as a $\mathbf{Z}$-module.*

Proof. Choosing a prime p dividing m, we get that this $\mathbf{Z}$-algebra admits for quotient the algebra defined in the proposition for $k = \mathbf{F}_p$. □

An immediate corollary is that we cannot expect the BMR conjecture to hold without invertibility conditions. This is a big difference with the Coxeter case. More precisely we prove the following.

Proposition 3.4. *The algebra defined over $\mathbf{Z}[a,b,c]$ by generators s_1, s_2 and relations $s_1s_2s_1 = s_2s_1s_2$ and $s_i^3 = as_i^2 + bs_i + c$ for $i \in \{1,2\}$ is not finitely generated as a $\mathbf{Z}[a,b,c]$-module. In the specialization $a = b = 0$, a non-zero torsion element of the corresponding $\mathbf{Z}[c]$-module is provided by $(s_1^2s_2^2)^6 - c^8$.*

Proof. Infinite generation follows again from the specialisation $a = b = c = 0$. By the computation described in figure 1, we prove that $c((s_1^2s_2^2)^6 - c^8) = 0$. Specializing to $a = b = c = 0$ we get that, on the $\mathbf{Z}$-module already used above, $s_1^2s_2^2$ is mapped to an endomorphism of infinite order, thus proving $(s_1^2s_2^2)^6 \neq c^8$.

□

3.2.2. *The case of G_{12}.* The example of G_4 might suggest that differences with the Coxeter case may happen only when the reflections have order more than 2. We prove that this is not the case, by considering the reflection group of type G_{12}, whose reflections all have order 2. A suitable monoid for its braid group is given by the presentation $\langle A, B, C \mid ABCA = BCAB = CABC \rangle$. The generators are braided reflections, and the monoid is known to be Garside (see [31]).

Proposition 3.5. *Let k be a ring. The unital k-algebra defined by generators A, B, C and relations $ABCA = BCAB = CABC$, $A^2 = B^2 = C^2 = 0$ is not finitely generated as a k-module. The same holds if the latter relations are replaced by $A^2 = A$, $B^2 = B$, $C^2 = C$.*

$$
\begin{array}{rcl}
c(s_1^2s_2^2)^6 & = & cs_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & s_1cs_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & s_1s_2^3s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & s_1s_2^2(s_2s_1s_2)s_2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & s_1s_2^2s_1(s_2s_1s_2)s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & s_1s_2^2s_1s_1s_2(s_1s_1^2)s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & cs_1s_2^2s_1s_1(s_2s_2^2)s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^2s_1s_2^2s_1(s_1s_1^2)s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^3s_1s_2^2s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^3s_1s_2(s_2s_1s_2)s_2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^3s_1s_2s_1(s_2s_1s_2)s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^3s_1s_2s_1s_1s_2(s_1s_1^2)s_2^2s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^4s_1s_2s_1s_1(s_2s_2^2)s_1^2s_2^2s_1^2s_2^2 \\
 & = & c^5s_1s_2s_1(s_1s_1^2)s_2^2s_1^2s_2^2 \\
 & = & c^6(s_1s_2s_1)s_2^2s_1^2s_2^2 \\
 & = & c^6s_2s_1(s_2s_2^2)s_1^2s_2^2 \\
 & = & c^7s_2(s_1s_1^2)s_2^2 \\
 & = & c^8s_2s_2^2 \\
 & = & c^9
\end{array}
$$

FIGURE 1. Torsion element in type G_4 : $c((s_1^2s_2^2)^6 - c^8) = 0$

Proof. We introduce the free modules $\mathcal{W}^+, \mathcal{W}^-$, with bases w_r^+, w_r^-, for $r \geq 1$, and make A, B, C act through $C.w_r^+ = C.w_r^- = 0$ and

$$
\left\{ \begin{array}{rcl} A.w_r^+ & = & 0 \\ A.w_r^- & = & w_{r+1}^+ \end{array} \right.
\left\{ \begin{array}{rcl} Bw_r^+ & = & w_{r+1}^- \\ Bw_r^- & = & 0 \end{array} \right.
$$

One easily gets that A^2, B^2 et C^2 act by 0, as well as $ABCA$, $BCAB$, $CABC$, thus defining a module structure for the first algebra. Since one can check that AB acts by $w_r^+ \mapsto w_{r+2}^+$ one gets the conclusion. For the second algebra, we make still C act by 0, whereas $A.w_r^+ = w_{r+1}^-$, $A.w_r^- = w_r^-$, $B.w_r^- = w_{r+1}^+$, $B.w_r^+ = w_r^+$. This time BA maps $w_r^+ \mapsto w_{r+2}^+$. ☐

3.2.3. *The case of* $G(d,1,2)$. We finally make a third example, this time inside the infinite series. The usual Hecke algebra has a presentation with generators t, s and relations $stst = tsts$, $t^d = a_0 + a_1 t + \cdots + a_{d-1}t^{d-1}$, $s^2 = \alpha s + \beta$, defined over $\mathbf{Z}[a_i, a_0^{-1}, \alpha, \beta, \beta^{-1}]$, and the BMR conjecture is known for them, by work of Ariki and Koike [2] . However, and somewhat surprisingly in view of the previous examples, it can be proved (see [2]) that it is actually finitely generated over $\mathbf{Z}[a_i, \alpha, \beta, \beta^{-1}]$. This feature is true for the general case of the $G(d,1,r)$. For $r = 2$, an explicit spanning set of $2d^2 = |G(d,1,2)|$ elements is given by the $t^m u^n s^\varepsilon$ for $0 \leq m, n \leq d-1$ and $\varepsilon \in \{0,1\}$, for $u = sts$. The fact that it is a spanning set over $\mathbf{Z}[a_i, \alpha, \beta, \beta^{-1}]$ can be deduced from the easily checked relations $tu = ut$, $us = \beta st + au$, $st = \beta^{-1}us - \beta^{-1}au$; and their consequences $s.u^{n+1} = \beta tsu^n + \alpha u^{n+1}$, $st^{m+1}u^n = \beta^{-1}u.st^m u^n - \beta^{-1}\alpha t^m u^{n+1}$. However, β really needs to be invertible, as we illustrate now.

Proposition 3.6. *Let k be a ring. The unital k-algebra defined by generators t, s and relations $stst = tsts$, $t^3 = 0$, $s^2 = s$, is not finitely generated as a k-module.*

Proof. Let E be the free k-module with basis the elements w_r, w_r', y_r for $r \geq 1$. We make s, t act on E through

$$
\left\{ \begin{array}{rcl} s.w_r & = & w_r \\ t.w_r & = & y_r \end{array} \right.
\left\{ \begin{array}{rcl} s.w_r' & = & w_{r+1} \\ t.w_r' & = & 0 \end{array} \right.
\left\{ \begin{array}{rcl} s.y_r & = & 0 \\ t.y_r & = & w_r' \end{array} \right.
$$

One checks easily than s^2 acts like s and that both t^3 and $stst = tsts$ act by 0. The corresponding module is generated by w_1. Since it is a free k-module of infinite rank this proves that the algebra of the statement is not finitely generated. ☐

Corollary 3.7. *The algebra defined by generators t, s and relations $stst = tsts$, $t^d = a_0 + a_1 t + \cdots + a_{d-1}t^{d-1}$, $s^2 = \alpha s + \beta$, is not finitely generated over $\mathbf{Z}[a_i, \alpha, \beta]$ when $d \geq 3$.*

Proof. The specialization of this algebra at $a_i = 0$, $\alpha = 1, \beta = 0$ admits a quotient (by the ideal generated by t^3) which is not a finitely generated $\mathbf{Z}$-module, whence the conclusion. □

Note that the assumption $d \geq 3$ is necessary, because the case $d = 2$ corresponds to a Coxeter group, for which 0-Hecke algebras *are* finitely generated.

Inside the infinite series, R. Rouquier communicated to us the following other example of the group $G(4, 2, 2)$, for the presentation $\langle A, B, C \mid ABC = BCA = CAB, A^2 = B^2 = C^2 = 0\rangle$. Then, it can be checked that the algebra $\langle A, B | A^2 = B^2 = 0\rangle$ naturally embeds inside the Hecke algebra H, and that it is not finitely generated.

4. The Hecke algebra of G_{26}

According to [29], the BMR conjecture has been checked to hold for G_{26} by J. Müller, using Linton's algorithm of vector enumeration (see [22]) and unpublished software. For completeness, we also note that the Schur elements of G_{26} have been computed in [24], §6C, under the additional assumption of the existence of a suitable trace form.

Theorem 4.1. *The BMR conjecture holds for G_{26}.*

We recall that the group G_{26} is a Shephard group, and a quotient of the braid group of type B_3. It is the largest of the two 'linearizations' of the group of automorphisms of the projective 'Hessian configuration' (see e.g. [30], example 6.30 p. 226), the other one being G_{25}.

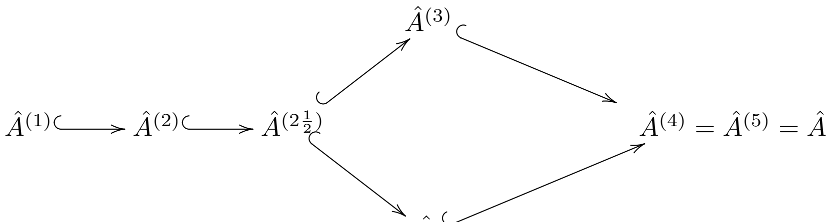
We take for generators of the braid group B of G_{26} the elements t, s_2, s_1 satisfying the braid relations $ts_2ts_2 = s_2ts_2t$, $s_2s_1s_2 = s_1s_2s_1$ and $ts_1 = s_1t$.

The generic Hecke algebra $\hat{A}$ of G_{26} is then defined over the ring $R = \mathbf{Z}[a, b, c^{-1}, d, e^{-1}]$, with generators s_1, s_2, t subject to the above braid relations, and in addition to the relations $s_i^3 = as_1^2 + bs_1 + c$ and $t^2 = dt + e$. The ring $\hat{A}$ admits useful (skew)automorphisms, defined by $s_i \mapsto s_i^{-1}$, $t \mapsto t^{-1}$, $a \mapsto -bc^{-1}$, $b \mapsto -ac^{-1}$, $c \mapsto c^{-1}$, $d \mapsto -de^{-1}$, $e \mapsto e^{-1}$. We let ϕ denote the automorphism, ψ the corresponding skew-automorphism.

Let $A_3 = \langle s_1, s_2\rangle \subset \hat{A}$ and $\hat{A}^{(n+1)} = \hat{A}^{(n)}vA_3$. For technical reasons, we also introduce the following intermediate bimodules

$$\begin{aligned} \hat{A}^{(2\frac{1}{2})} &= \hat{A}^{(2)} + A_3ts_2s_1ts_2tA_3 + A_3ts_2s_1ts_2^{-1}tA_3 + A_3ts_2s_1^{-1}ts_2^{-1}tA_3 + A_3ts_2^{-1}s_1^{-1}ts_2^{-1}tA_3 \\ \hat{B} &= \hat{A}^{(2\frac{1}{2})} + A_3C^2 + A_3C^{-2} \end{aligned}$$

We have the following inclusion/equalities, some of them being obvious from the definitions, the other ones being proved in the sequel.



Let $C = (ts_2s_1)^3$. It is central in $\hat{A}$, as it generates the center of the braid group, and its image in G_{26} has order 6. We let $u_i = R + Rs_i + Rs_i^{-1}$ denote the subalgebra generated by s_i, and $v = R + Rt$ the subalgebra generated by t. We will need the following results on the 'parabolic' subalgebras $A_3 = \langle s_1, s_2\rangle$ and $\langle s_2, t\rangle$, which correspond to the rank 2 parabolic subgroups of Shephard-Todd type G_4 and $G(3, 1, 2)$, respectively.

Proposition 4.2.

(1) $\langle s_1, s_2\rangle = u_1u_2u_1 + u_1s_2s_1^{-1}s_2$
(2) $\langle s_1, s_2\rangle = u_1 + u_1s_2u_1 + u_1s_2^{-1}u_1 + u_1s_2s_1^{-1}s_2$
(3) $\langle s_2, t\rangle = \sum_{a\in\{-1,0,1\}} Rs_2^a + \sum_{a,b\in\{-1,0,1\}} Rs_2^ats_2^b + \sum_{a\in\{-1,0,1\}} Rs_2^ats_2t + \sum_{a\in\{-1,0,1\}} Rs_2^ats_2^{-1}t$

Proof. (1) and (2) are easy and proved in [26]. We prove (2). The RHS clearly contains 1 and is stable under left multiplication by s_2. It thus sufficient to prove that it is stable under left multiplication by t. Let U denote the RHS. Since $\langle s_2 \rangle$ is R-spanned by $1, s_2, s_2^2$, we need to prove $ts_2^{\alpha}ts_2^b \in U$ and $ts_2^{\alpha}ts_2^{\beta}t \in U$ for all $b \in \{0,1,2\}$ and $\alpha, \beta \in \{-1,1\}$. If $\alpha = 1$ we have $ts_2^{\alpha}ts_2^b = ts_2ts_2^b$. If $b = 1$ we get in addition $ts_2ts_2^b = ts_2ts_2 = s_2ts_2t \in U$; If $b = 1$ we get $(ts_2ts_2)s_2 = s_2(ts_2ts_2) = s_2^2ts_2t \in U$; if $b = 0$ we know $ts_2t \in U$. If $\alpha = -1$ we have $ts_2^{-1}ts_2^b \in R^{\times}t^{-1}s_2^{-1}t^{-1}s_2^b + \langle s_2 \rangle t \langle s_2 \rangle \subset R^{\times}t^{-1}s_2^{-1}t^{-1}s_2^b + U$ and the proof of $t^{-1}s_2^{-1}t^{-1}s_2^b \in U$ is similar, taking this time $b \in \{-2,-1,0\}$ and using $t^{-1}s_2^{-1}t^{-1}s_2^{-1} = s_2^{-1}t^{-1}s_2^{-1}t^{-1}$ instead. This in particular implies that $t(s_2^{\alpha}ts_2^{\beta}t) = (ts_2^{\alpha}ts_2^{\beta})t \in Ut$. The same proof, reading from the right, proves that $s_2^{\alpha}ts_2^bt \in U$ for all α, b, which clearly implies $Ut \subset U$, and this concludes the proof. □

4.1. Bimodule decompositions of $\hat{A}^{(k)}$, $1 \leq k \leq 3$.

Proposition 4.3. *(bimodule decomposition of $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$)*

(1) $\hat{A}^{(1)} = A_3 + A_3tA_3$
(2) $\hat{A}^{(2)} = \hat{A}^{(1)} + A_3ts_2tA_3 + A_3ts_2^{-1}tA_3 + A_3ts_2s_1^{-1}s_2tA_3$

Proof. (1) is clear, as $v = \langle t \rangle$ is R-generated by 1 and t. For proving (2) we note that $A_3 = u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ hence $tA_3t \subset tu_1s_2s_1^{-1}s_2t + tu_1u_2u_1t \subset u_1ts_2s_1^{-1}s_2t + u_1tu_2tu_1 \subset u_1ts_2s_1^{-1}s_2t + u_1t^2u_1 + u_1ts_2tu_1 + u_1ts_2^{-1}tu_1$. This proves (2). □

Lemma 4.4.

(1) *For all $i, j \in \{1,2\}$, $tu_itu_jt \subset \hat{A}^{(2)}$*
(2) $tu_2u_1tu_2t \subset \sum_{\alpha,\beta,\gamma} Rts_2^{\alpha}ts_1^{\beta}s_2^{\gamma}t$

Proof. We prove (1). If $i = 1$ or $j = 1$ this is clear by the commutation relations. One can thus assume $i = j = 2$, and consider $ts_2^a ts_2^b t$ with $a, b \in \{0,1,2\}$ since u_2 is R-spanned by $1, s_2$ and s_2^2. If $a = 0$ or $b = 0$ this is clear. If $a = b = 1$ then this is $(ts_2ts_2)t = s_2ts_2t^2 \in \hat{A}^{(2)}$; if $a = 1$ and $b = 2$, then this is $(ts_2ts_2)s_2t = s_2(ts_2ts_2)t = s_2^2ts_2t^2 \in \hat{A}^{(2)}$; the case $a = 2$ and $b = 1$ is similar. We thus only need to consider the case $a = b = 2$. Using $s_2^2 \in Rs_2^{-1} + Rs_2 + R$ we get from the preceding cases $ts_2^2ts_2^2t \in ts_2^{-1}ts_2^{-1}t + \hat{A}^{(2)}$; moreover $t \in R^{\times}t^{-1} + R$ hence $ts_2^{-1}ts_2^{-1}t \in R^{\times}t^{-1}s_2^{-1}t^{-1}s_2^{-1}t^{-1} + \hat{A}^{(2)}$. Now $(t^{-1}s_2^{-1}t^{-1}s_2^{-1})t^{-1} = s_2^{-1}t^{-1}s_2^{-1}t^{-2} \in \hat{A}^{(2)}$ and this concludes the proof of (1). (2) obviously follows from (1), as u_i is the R-linear span of $1, s_i$ and s_i^{-1}. □

Lemma 4.5.

(1) $ts_2s_1^{-1}ts_2^{-1}t \in A_3^{\times}ts_2^{-1}s_1ts_2^{-1}tA_3^{\times} + \hat{A}^{(2)}$
(2) $ts_2^{-1}s_1ts_2^{-1}t \in A_3^{\times}ts_2^{-1}s_1^{-1}ts_2tA_3^{\times} + \hat{A}^{(2)}$
(3) $ts_2s_1ts_2^{-1}t \in A_3^{\times}ts_2s_1^{-1}ts_2tA_3^{\times}$
(4) $ts_2s_1^{-1}ts_2t \in A_3^{\times}ts_2^{-1}s_1ts_2tA_3^{\times}$

Proof. We have

$$\begin{array}{rcl} s_1(ts_2s_1^{-1}ts_2^{-1}t) & = & t(s_1s_2s_1^{-1})ts_2^{-1}t \\ & = & ts_2^{-1}s_1s_2(ts_2^{-1}ts_2^{-1})s_2 \\ & \in & R^{\times}ts_2^{-1}s_1s_2(t^{-1}s_2^{-1}t^{-1}s_2^{-1})s_2 + \hat{A}^{(2)} \end{array}$$

and $ts_2^{-1}s_1s_2(t^{-1}s_2^{-1}t^{-1}s_2^{-1}) = ts_2^{-1}s_1s_2s_2^{-1}t^{-1}s_2^{-1}t^{-1} = ts_2^{-1}s_1t^{-1}s_2^{-1}t^{-1} \in R^{\times}ts_2^{-1}s_1ts_2^{-1}t + \hat{A}^{(2)}$, which proves (1). Now

$$\begin{array}{rcl} (ts_2^{-1}s_1ts_2^{-1}t)s_1^{-1} & = & ts_2^{-1}t(s_1s_2^{-1}s_1^{-1})t \\ & = & ts_2^{-1}ts_2^{-1}s_1^{-1}s_2t \\ & \in & R^{\times}t^{-1}s_2^{-1}t^{-1}s_2^{-1}s_1^{-1}s_2t + \hat{A}^{(2)} \end{array}$$

and $(t^{-1}s_2^{-1}t^{-1}s_2^{-1})s_1^{-1}s_2t = s_2^{-1}t^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2t = s_2^{-1}t^{-1}s_2^{-1}s_1^{-1}t^{-1}s_2t \in R^{\times}s_2^{-1}ts_2^{-1}s_1^{-1}ts_2t + \hat{A}^{(2)}$ and this proves (2).

We have

$$\begin{array}{lllll} & (ts_2s_1ts_2^{-1}t)s_1^{-1} & = & ts_2t(s_1s_2^{-1}s_1^{-1})t & = & ts_2ts_2^{-1}s_1^{-1}s_2t \\ = & s_2^{-1}(s_2ts_2t)s_2^{-1}s_1^{-1}s_2t & = & s_2^{-1}ts_2ts_2s_2^{-1}s_1^{-1}s_2t & = & s_2^{-1}ts_2s_1^{-1}ts_2t \end{array}$$

and this proves (3). Now $s_1ts_2s_1^{-1}ts_2t = t(s_1s_2s_1^{-1})ts_2t = ts_2^{-1}s_1(s_2ts_2t) = ts_2^{-1}s_1ts_2ts_2 = (ts_2^{-1}s_1ts_2t)s_2$ and this proves (4). □

Proposition 4.6. *(bimodule decomposition of $\hat{A}^{(3)}$)*

$$\hat{A}^{(3)} = \hat{A}^{(2)} + A_3ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2tA_3 + A_3ts_2s_1ts_2tA_3 + A_3ts_2s_1ts_2^{-1}tA_3$$

$$+A_3ts_2s_1^{-1}ts_2^{-1}tA_3 + A_3ts_2^{-1}s_1^{-1}ts_2^{-1}tA_3$$

Proof. Since $A_3 = u_1s_2s_1^{-1}s_2 + u_1u_2u_1 = s_2s_1^{-1}s_2u_1 + u_1u_2u_1$ we get

$$tA_3tA_3t \subset A_3ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2tA_3 + A_3tu_2tA_3tA_3 + A_3tA_3tu_2tA_3.$$

Now $tu_2tA_3t = Rt^2A_3t + Rts_2tA_3t + Rts_2^{-1}tA_3 \subset \hat{A}^{(2)} + Rts_2tA_3t + Rts_2^{-1}tA_3$. Using again $A_3 = s_2s_1^{-1}s_2u_1 + u_1u_2u_1$ we get $ts_2tA_3t \subset (ts_2ts_2)s_1^{-1}s_2u_1t + ts_2tu_1u_2u_1t \subset s_2ts_2ts_1^{-1}s_2tu_1 + ts_2tu_1u_2tu_1 \subset A_3tu_2u_1tu_2tA_3$; using $A_3 = s_2^{-1}s_1s_2^{-1}u_1 + u_1u_2u_1$ we get similarly $ts_2^{-1}tA_3t \subset (ts_2^{-1}ts_2^{-1})s_1s_2^{-1}u_1t + ts_2^{-1}tu_1u_2u_1t \subset s_2^{-1}ts_2^{-1}ts_1s_2^{-1}tu_1 + ts_2^{-1}tu_1u_2tu_1 \subset A_3tu_2u_1tu_2tA_3$. This yields $tu_2tA_3t \subset \hat{A}^{(2)} + A_3tu_2u_1tu_2tA_3$. In a similar way, we leave to the reader to check that $tA_3tu_2t \subset \hat{A}^{(2)} + A_3tu_2u_1tu_2tA_3$. This implies

$$tA_3tA_3t \subset \hat{A}^{(2)} + A_3tu_2u_1tu_2tA_3 + A_3ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2tA_3.$$

The conclusion then follows from lemmas 4.4 and 4.5. □

4.2. The bimodule $\hat{A}^{(2\frac{1}{2})}$.

Proposition 4.7.

1. $C \in ts_2s_1ts_2tA_3^\times$ *and* $C^{-1} \in A_3^\times ts_2^{-1}s_1^{-1}ts_2^{-1}t + \hat{A}^{(2)}$
2. $\hat{A}^{(3)} = \hat{A}^{(2\frac{1}{2})} + A_3ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2tA_3$
3. $\hat{A}^{(2\frac{1}{2})} = \hat{A}^{(2)} + A_3ts_2s_1ts_2t + A_3ts_2s_1ts_2^{-1}tA_3 + A_3ts_2s_1^{-1}ts_2^{-1}tA_3 + A_3ts_2^{-1}s_1^{-1}ts_2^{-1}t$

Proof. We have $C = ts_2s_1ts_2s_1ts_2s_1 = ts_2s_1ts_2ts_1s_2s_1 \in ts_2s_1ts_2tA_3^\times$. One gets similarly $C^{-1} \in A_3^\times t^{-1}s_2^{-1}s_1^{-1}t^{-1}s_2^{-1}t^{-1}$. Since $t^{-1} \in R^\times t + R$ this implies $C^{-1} \in A_3^\times ts_2^{-1}s_1^{-1}ts_2^{-1}t + \hat{A}^{(2)}$. This proves (1). (2) follows from proposition 4.6. Since C is central, (3) then follows from (1). □

We now compute the number of elements which are needed to generate $\hat{A}^{(2\frac{1}{2})}$ modulo $\hat{A}^{(2)}$ as a A_3-module. We need the following two lemmas.

Lemma 4.8.

(1) *For all* $\alpha \in \{0, 1, -1\}$, $(ts_2s_1)s_2^\alpha = s_1^\alpha(ts_2s_1)$

(2) $ts_2s_1ts_2^{\pm 1}tu_2 \subset \hat{A}^{(2)} + A_2ts_2s_1ts_2t + A_2ts_2s_1ts_2^{-1}t$

(3)

$$ts_2s_1ts_2^{\pm 1}tA_3 \subset \hat{A}^{(2)} + \sum_{a\in\{-1,0,1\}} \sum_{b,\varepsilon\in\{-1,1\}} A_2ts_2s_1ts_2^\varepsilon ts_1^bs_2^a + \sum_{\varepsilon\in\{-1,1\}} \left(A_2ts_2s_1ts_2^\varepsilon ts_1s_2^{-1}s_1 + A_2ts_2s_1ts_2^\varepsilon t\right)$$

(4)

$$ts_2s_1ts_2^{\pm 1}tA_3 \subset \hat{A}^{(2)} + A_3ts_2s_1ts_2t + \sum_{a\in\{-1,0,1\}} \sum_{b\in\{-1,1\}} A_2ts_2s_1ts_2^{-1}ts_1^bs_2^a$$

$$+ \left(A_3ts_2s_1ts_2^{-1}ts_1s_2^{-1}s_1 + A_2ts_2s_1ts_2^{-1}t\right)$$

(5) $\hat{A}^{(2)} + \sum_{\varepsilon\in\{-1,1\}} A_3ts_2s_1ts_2^{\pm 1}tA_3$ *is spanned as a A_3-module by $\hat{A}^{(2)}$ and 9 elements originating from the braid group.*

Proof. For (1), this is because $ts_2s_1s_2^\alpha = t(s_2s_1s_2^\alpha) = ts_1^\alpha s_2s_1 = s_1^\alpha ts_2s_1$. Since $ts_2^{\pm 1}tu_2 \subset \langle s_2, t\rangle$, proposition 4.2 implies

$$\begin{array}{rcl} ts_2s_1ts_2^{\pm 1}tu_2 & \subset & \hat{A}^{(2)} + \sum_{a\in\{-1,0,1\}} Rts_2s_1s_2^ats_2t + \sum_{a\in\{-1,0,1\}} Rts_2s_1s_2^ats_2^{-1}t \\ & & \hat{A}^{(2)} + \sum_{a\in\{-1,0,1\}} Rs_1^ats_2s_1ts_2t + \sum_{a\in\{-1,0,1\}} Rs_1^ats_2s_1ts_2^{-1}t \\ & & \hat{A}^{(2)} + A_2ts_2s_1ts_2t + A_2ts_2s_1ts_2^{-1}t \end{array}$$

that is (2). Then (3) is an immediate consequence of (2) and of the decomposition of A_3 as $\langle s_2\rangle$-module given by proposition 4.2 up to exchanging s_1 and s_2 (see also [26]). (4) is readily deduced because $ts_2s_1ts_2t$ commutes with A_3, and then (5) is clear. □

Lemma 4.9. *The image under ϕ of $\hat{A}^{(2)}+A_3ts_2^{-1}s_1^{-1}ts_2^{-1}t+A_3ts_2s_1^{-1}ts_2^{-1}tA_3$ is $\hat{A}^{(2)}+A_3ts_2s_1ts_2t+A_3ts_2s_1ts_2^{-1}tA_3$. Thus*

$$\hat{A}^{(2\frac{1}{2})} = \hat{A}^{(2)} + A_3ts_2s_1ts_2t + A_3ts_2s_1ts_2^{-1}tA_3 + A_3\phi(ts_2s_1ts_2t) + A_3\phi(ts_2s_1ts_2^{-1}t)A_3$$

Proof. This image is clearly $\hat{A}^{(2)} + A_3t^{-1}s_2s_1t^{-1}s_2^{-1}t^{-1} + A_3t^{-1}s_2^{-1}s_1t^{-1}s_2t^{-1}A_3$, that is $\hat{A}^{(2)} + A_3ts_2s_1ts_2^{-1}t + A_3ts_2^{-1}s_1ts_2tA_3$ by $t^{-1} \in R^\times t + R$, hence $\hat{A}^{(2)} + A_3ts_2s_1ts_2^{-1}t + A_3ts_2s_1^{-1}ts_2tA_3$ because $s_1^{-1}(ts_2^{-1}s_1ts_2t) = t(s_1^{-1}s_2^{-1}s_1)ts_2t) = ts_2s_1^{-1}s_2^{-1}(ts_2t) = (ts_2s_1^{-1}ts_2t)s_2^{-1}$. Now $ts_2s_1^{-1}ts_2t \in A_3^\times ts_2s_1ts_2^{-1}tA_3^\times$ by lemma 4.5 (3), and this concludes the proof of the lemma, the last equality being an obvious consequence.

□

These two lemmas imply the following proposition.

Proposition 4.10. *As a A_3-module, $\hat{A}^{(2\frac{1}{2})}$ is generated by $\hat{A}^{(2)}$ together with $2 \times 9 = 18$ elements originating from the braid group.*

Additional properties of $\hat{A}^{(2\frac{1}{2})}$ include the following two results.

Lemma 4.11. *Whatever the choices of signs $\pm$,*

1. $t^\pm u_2u_1u_2t^\pm u_1u_2t^\pm \subset \hat{A}^{(2\frac{1}{2})}$
2. $t^\pm u_2u_1t^\pm u_2u_1u_2t^\pm \subset \hat{A}^{(2\frac{1}{2})}$

Proof. (1). Since $t^{-1} \in Rt + R$ and $\hat{A}^{(2)} \subset \hat{A}^{(2\frac{1}{2})}$, it suffices to show $tu_2u_1u_2tu_1u_2t \subset \hat{A}^{(2\frac{1}{2})}$. Now $tu_2u_1u_2tu_1u_2t = tu_2u_1u_2u_1tu_2t$ and $u_2u_1u_2u_1 = A_3 = u_1u_2u_1u_2$, thus $tu_2u_1u_2u_1tu_2t = tu_1u_2u_1u_2tu_2t = u_1tu_2u_1u_2tu_2t$. Now $u_2tu_2t \subset \langle s_2, t\rangle$ hence, by proposition 4.2 (and applying the skew-automorphism induced by $s_2 \mapsto s_2^{-1}, t \mapsto t^{-1}$) we have $u_2tu_2t \subset u_2 + u_2tu_2 + ts_2tu_2 + ts_2^{-1}tu_2$ whence $tu_2u_1u_2tu_2t \subset \hat{A}^{(2)} + tu_2u_1ts_2tu_2 + tu_2u_1ts_2^{-1}tu_2 \subset \hat{A}^{(2\frac{1}{2})}$ by lemmas 4.4 (2) and lemma 4.5. This proves (1), and (2) follows from (1) under application of the skew-automorphism already mentionned.

□

Proposition 4.12.

1. $\hat{A}^{(2\frac{1}{2})} = \hat{A}^{(2)} + A_3\langle t\rangle u_2u_1\langle t\rangle u_2\langle t\rangle A_3 = \hat{A}^{(2)} + A_3\langle t\rangle u_2\langle t\rangle u_1u_2\langle t\rangle A_3$
2. $\hat{A}^{(2\frac{1}{2})}$ *is stable under ϕ and ψ.*
3. $\langle s_2, t\rangle u_1\langle s_2, t\rangle \subset \hat{A}^{(2\frac{1}{2})}$
4. $\langle t\rangle A_3\langle s_2, t\rangle \subset \hat{A}^{(2\frac{1}{2})}$
5. $\langle s_2, t\rangle A_3\langle t\rangle \subset \hat{A}^{(2\frac{1}{2})}$

Proof. (1) is an immediate consequence of the above, and (2) is a direct consequence of (1). Recall that (A) $\langle s_2, t\rangle = u_2 + u_2tu_2 + u_2ts_2t + u_2ts_2^{-1}t$ hence, applying $\phi \circ \psi$, we have (B) $\langle s_2, t\rangle = u_2 + u_2tu_2 + ts_2tu_2 + ts_2^{-1}tu_2$. In particular, $\langle s_2, t\rangle \subset \hat{A}^{(2)} \subset \hat{A}^{(2\frac{1}{2})}$, thus it is sufficient to show $\langle s_2, t\rangle s_1^\alpha\langle s_2, t\rangle \subset \hat{A}^{(2\frac{1}{2})}$ for $\alpha \in \{-1, 1\}$. Since $\hat{A}^{(2\frac{1}{2})}$ is a u_2-bimodule, because of (A) this amounts to proving (a) $tu_2s_1^\alpha\langle s_2, t\rangle \subset \hat{A}^{(2\frac{1}{2})}$ and (b) $ts_2^\varepsilon ts_1^\alpha\langle s_2, t\rangle \subset \hat{A}^{(2\frac{1}{2})}$ for all $\varepsilon \in \{-1, 1\}$. We start with (a). By (B),

$$tu_2s_1^\alpha\langle s_2, t\rangle \subset \hat{A}^{(2)} + tu_2s_1^\alpha ts_2tu_2 + tu_2s_1^\alpha ts_2^{-1}tu_2 \subset \hat{A}^{(2\frac{1}{2})}$$

by lemma 4.11, and this proves (a). We turn to (b). By (B),

$$ts_2^{\varepsilon}ts_1^{\alpha}\langle s_2,t\rangle \subset \hat{A}^{(2)} + ts_2^{\varepsilon}ts_1^{\alpha}u_2tu_2 + ts_2^{\varepsilon}ts_1^{\alpha}ts_2tu_2 + ts_2^{\varepsilon}ts_1^{\alpha}ts_2^{-1}tu_2.$$

Now $ts_2^{\varepsilon}ts_1^{\alpha}ts_2^{\pm 1}tu_2 = ts_2^{\varepsilon}s_1^{\alpha}t^2s_2^{\pm 1}tu_2 \subset \hat{A}^{(2\frac{1}{2})}$ and $ts_2^{\varepsilon}ts_1^{\alpha}u_2t \subset \hat{A}^{(2\frac{1}{2})}$, and this concludes the proof of (3). For proving (4), we use that $A_3 = u_1u_2u_1u_2$, hence $\langle t\rangle A_3\langle s_2,t\rangle \subset u_1\langle t\rangle u_2u_1u_2\langle s_2,t\rangle \subset \hat{A}^{(2\frac{1}{2})}$ because of (3). Now (5) is a consequence of (4) by applying ψ. □

4.3. **Computation of C^2 modulo $\hat{A}^{(2\frac{1}{2})}$.**

Lemma 4.13. $C^2 \in A_3^{\times}ts_2ts_1s_2^{-1}s_1ts_2tA_3^{\times} + \hat{A}^{(2\frac{1}{2})}$.

Proof. We actually prove $C^2 \in \hat{A}^{(2)} + A_3^{\times}ts_2ts_1s_2^{-1}s_1ts_2t(s_1s_2s_1) + A_3ts_2s_1^{-1}ts_2t(s_1^2s_2s_1) + A_3ts_2s_1ts_2t$. We have $C = (ts_2s_1)^3 \in A_3^{\times}ts_2s_1ts_2t$, hence $C^2 \in A_3^{\times}ts_2s_1ts_2tts_2s_1ts_2s_1ts_2s_1$ and

$$ts_2s_1ts_2t^2s_2s_1ts_2s_1ts_2s_1 \in R^{\times}ts_2s_1ts_2^2s_1ts_2s_1ts_2s_1 + Rts_2s_1ts_2ts_2s_1ts_2s_1ts_2s_1.$$

- We have

$$ts_2s_1ts_2^2s_1ts_2s_1ts_2s_1 \in R^{\times}ts_2s_1ts_2^{-1}s_1ts_2s_1ts_2s_1 + Rts_2s_1ts_2s_1ts_2s_1ts_2s_1 + Rts_2s_1ts_1ts_2s_1ts_2s_1.$$

We also have $ts_2s_1ts_2^{-1}s_1ts_2s_1ts_2s_1 = ts_2ts_1s_2^{-1}s_1ts_2ts_1s_2s_1$, and

$$\begin{array}{rcl} ts_2s_1ts_2s_1ts_2s_1ts_2s_1 & = & ts_2ts_1s_2s_1ts_2s_1ts_2s_1 \;=\; ts_2ts_2s_1s_2ts_2s_1ts_2s_1 \\ & = & s_2ts_2ts_1ts_2ts_2s_1s_2s_1 \;=\; s_2ts_2s_1t^2s_2ts_2s_1s_2s_1 \\ & \in & s_2ts_2s_1ts_2ts_2s_1s_2s_1 \;+\; \hat{A}^{(2)} \\ & \subset & A_3ts_2s_1ts_2tA_3 \;+\; \hat{A}^{(2)} \\ & \subset & A_3ts_2s_1ts_2t \;+\; \hat{A}^{(2)}. \end{array}$$

Finally, $ts_2s_1ts_1ts_2s_1ts_2s_1 = ts_2s_1^2t^2s_2s_1ts_2s_1 \in \hat{A}^{(2)} + Rts_2s_1^2ts_2s_1ts_2s_1$. Moreover,

$$ts_2s_1^2ts_2s_1ts_2s_1 \in Rts_2s_1^{-1}ts_2s_1ts_2s_1 + Rts_2s_1ts_2s_1ts_2s_1 + Rts_2ts_2s_1ts_2s_1.$$

But since $ts_2ts_2s_1ts_2s_1 = s_2ts_2ts_1ts_2s_1 = s_2ts_2s_1t^2s_2s_1 \in \hat{A}^{(2)}$ and $ts_2s_1ts_2s_1ts_2s_1 = ts_2s_1ts_2ts_1s_2s_1 \in ts_2s_1ts_2tA_3 \subset A_3ts_2s_1ts_2t$, we get

$$ts_2s_1ts_2^2s_1ts_2s_1ts_2s_1 \in ts_2ts_1s_2^{-1}s_1ts_2t(s_1s_2s_1) + A_3ts_2s_1ts_2t + A_3ts_2s_1^{-1}ts_2t(s_1s_2s_1) + \hat{A}^{(2)}$$

- We have

$$\begin{array}{rcl} ts_2s_1ts_2ts_2s_1ts_2s_1ts_2s_1 & = & ts_2s_1s_2ts_2ts_1ts_2s_1ts_2s_1 \;=\; ts_2s_1s_2ts_2t^2s_1s_2s_1ts_2s_1 \\ & = & ts_1s_2s_1ts_2t^2s_1s_2s_1ts_2s_1 \;=\; s_1ts_2s_1ts_2t^2s_1s_2s_1ts_2s_1 \end{array}$$

and $ts_2s_1ts_2t^2s_1s_2s_1ts_2s_1 \in Rts_2s_1ts_2ts_1s_2s_1ts_2s_1 + Rts_2s_1ts_2s_1s_2s_1ts_2s_1$. Moreover

$$\begin{array}{rcl} ts_2s_1ts_2s_1s_2s_1ts_2s_1 & = & ts_2s_1ts_1s_2s_1s_1ts_2s_1 \;=\; ts_2s_1^2ts_2ts_1^2s_2s_1 \\ & \in & Rts_2s_1^{-1}ts_2t(s_1^2s_2s_1) \;+\; Rts_2s_1ts_2ts_1^2s_2s_1 \;+\; ts_2ts_2ts_1^2s_2s_1. \end{array}$$

We have $ts_2s_1ts_2ts_1^2s_2s_1 \in ts_2s_1ts_2tA_3 \subset A_3ts_2s_1ts_2t + \hat{A}^{(2)}$, $ts_2ts_2ts_1^2s_2s_1 = s_2ts_2t^2s_1^2s_2s_1 \in \hat{A}^{(2)}$.

On the other hand, $ts_2s_1ts_2ts_1s_2s_1ts_2s_1 = ts_2s_1ts_2ts_2s_1s_2ts_2s_1 = ts_2s_1s_2ts_2ts_1s_2ts_2s_1 = ts_1s_2s_1ts_2ts_1s_2ts_2s_1 = s_1ts_2s_1ts_2ts_1s_2ts_2s_1 = s_1ts_2s_1ts_2s_1ts_2ts_2s_1 = s_1ts_2s_1ts_2s_1s_2ts_2ts_1 = s_1ts_2s_1ts_1s_2s_1ts_2ts_1 = s_1ts_2ts_1^2s_2s_1ts_2ts_1 \in A_3ts_2ts_1^2s_2s_1ts_2tA_3$. Moreover, $ts_2ts_1^2s_2s_1ts_2t \in Rts_2ts_1^{-1}s_2s_1ts_2t + Rts_2ts_1s_2s_1ts_2t + Rts_2ts_2s_1ts_2t$ and $ts_2ts_2s_1ts_2t = s_2ts_2ts_1ts_2t = s_2ts_2s_1t^2s_2t \in A_3ts_2s_1ts_2t + \hat{A}^{(2)}$, $ts_2ts_1s_2s_1ts_2t = ts_2ts_2s_1s_2ts_2t = s_2ts_2ts_1s_2ts_2t = s_2ts_2s_1t^2s_2ts_2 \in A_3ts_2s_1ts_2t + \hat{A}^{(2)}$, $ts_2ts_1^{-1}s_2s_1ts_2t = ts_2ts_2s_1s_2^{-1}ts_2t = s_2ts_2ts_1ts_2ts_2^{-1} = s_2ts_2s_1t^2s_2ts_2^{-1} \in A_3ts_2s_1ts_2tA_3 + \hat{A}^{(2)} \subset A_3ts_2s_1ts_2t + \hat{A}^{(2)}$. This concludes the proof. □

For subsequent use, we also need the following related computation.

Lemma 4.14.

(1) $ts_2s_1^{-1}ts_2ts_1C \in \hat{A}^{(2\frac{1}{2})}$
(2) $ts_2s_1^{-1}ts_2ts_1ts_2s_1ts_2t \in \hat{A}^{(2\frac{1}{2})}$

Proof. (1) is a clear consequence of (2), so we focus on (2). We have

$$\begin{array}{lllll} & ts_2s_1^{-1}ts_2ts_1ts_2s_1ts_2t & = & ts_2s_1^{-1}ts_2t^2s_1s_2s_1ts_2t & = & ts_2ts_1^{-1}s_2s_1t^2s_2s_1ts_2t \\ = & ts_2ts_2s_1s_2^{-1}t^2s_2s_1ts_2t & = & s_2ts_2ts_1s_2^{-1}t^2s_2s_1ts_2ts_2^{-1}s_2 & = & s_2ts_2ts_1s_2^{-1}t^2s_2s_1s_2^{-1}ts_2ts_2 \\ = & s_2ts_2ts_1s_2^{-1}t^2s_1^{-1}s_2s_1ts_2ts_2 & = & s_2ts_2ts_1s_2^{-1}s_1^{-1}t^2s_2s_1ts_2ts_2 & = & s_2ts_2ts_2^{-1}s_1^{-1}s_2t^2s_2s_1ts_2ts_2 \\ = & s_2s_2^{-1}ts_2ts_1^{-1}s_2t^2s_2s_1ts_2ts_2 & = & ts_2ts_1^{-1}s_2t^2s_2s_1ts_2ts_2 & & \\ \in & Rts_2ts_1^{-1}s_2ts_2s_1ts_2ts_2 & + & Rts_2ts_1^{-1}s_2s_2s_1ts_2ts_2. & & \end{array}$$

Now, on the one hand $ts_2ts_1^{-1}s_2s_2s_1ts_2t = ts_2ts_1^{-1}s_2^2s_1ts_2t = ts_2ts_2s_1^2s_2^{-1}ts_2t = s_2ts_2ts_1^2ts_2ts_2^{-1} = s_2ts_2s_1^2t^2s_2ts_2^{-1} \in \hat{A}^{(2\frac{1}{2})}$. On the other hand, $ts_2ts_1^{-1}s_2ts_2s_1ts_2t = ts_2s_1^{-1}ts_2ts_2s_1ts_2t = ts_2s_1^{-1}s_2ts_2ts_1ts_2t = ts_2s_1^{-1}s_2ts_2t^2s_1s_2t \in Rts_2s_1^{-1}s_2ts_2ts_1s_2t + Rts_2s_1^{-1}s_2ts_2s_1s_2t$. Then, $ts_2s_1^{-1}s_2ts_2s_1s_2t = ts_2s_1^{-1}s_2ts_1s_2s_1t = ts_2s_1^{-1}s_2ts_1s_2ts_1 \in \hat{A}^{(2\frac{1}{2})}$ by lemma 4.11, and $ts_2s_1^{-1}s_2ts_2ts_1s_2t = ts_2s_1^{-1}ts_2ts_2s_1s_2t = ts_2s_1^{-1}ts_2ts_1s_2s_1t = ts_2s_1^{-1}ts_2ts_1s_2ts_1 = ts_2ts_1^{-1}s_2s_1ts_2ts_1 = ts_2ts_2s_1s_2^{-1}ts_2ts_1 = s_2ts_2ts_1s_2^{-1}ts_2ts_1 = s_2ts_2ts_1ts_2ts_2^{-1}s_1 = s_2ts_2s_1t^2s_2ts_2^{-1}s_1 \in \hat{A}^{(2\frac{1}{2})}$. ∎

4.4. Properties of the bimodule $\hat{B}$.

Proposition 4.15.

1. *$\hat{B}$ is stable under ϕ and ψ.*
2. $\hat{B} = \hat{A}^{(2\frac{1}{2})} + A_3ts_2ts_1s_2^{-1}s_1ts_2t + A_3t^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2s_1^{-1}t^{-1}s_2^{-1}t^{-1}$
3. $\hat{B} = A_3\langle s_2, t\rangle A_3\langle s_2, t\rangle A_3$

Proof. Recall that $\hat{B} = \hat{A}^{(2\frac{1}{2})} + A_3C^2 + A_3C^{-2}$. Since $C \in A_3^{\times}ts_2s_1ts_2t$ and $C = (ts_2s_1)^3$, we have $C^{-1} = s_1^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2^{-1}t^{-1} = s_1^{-1}s_2^{-1}s_1^{-1}t^{-1}s_2^{-1}s_1^{-1}t^{-1}s_2^{-1}t^{-1} = s_1^{-1}s_2^{-1}s_1^{-1}\phi(ts_2s_1ts_2t) \in A_3^{\times}\phi(A_3^{\times}C)$ hence $C^{-1} \in A_3^{\times}\phi(C)$, and this implies $C^{-2} \in A_3^{\times}\phi(C^2)$. From this we deduce that $\hat{B}$ is ϕ-stable. Moreover, $\phi\circ\psi(C) \in ts_2ts_1s_2tA_3^{\times} = ts_2s_1ts_2tA_3^{\times} \in CA_3^{\times}$ hence $\hat{B}$ is $\phi\circ\psi$-stable, and thus also ψ-stable, that is (1). An immediate consequence of lemma 4.13 and of $C^{-2} \in A_3^{\times}\phi(C^2)$ is then that $\hat{B} = \hat{A}^{(2\frac{1}{2})} + A_3ts_2ts_1s_2^{-1}s_1ts_2tA_3 + A_3t^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2s_1^{-1}t^{-1}s_2^{-1}t^{-1}A_3 = \hat{A}^{(2\frac{1}{2})} + A_3ts_2ts_1s_2^{-1}s_1ts_2t + A_3t^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2s_1^{-1}t^{-1}s_2^{-1}t^{-1}$. Since $t^{-1} \in R^{\times}t + R$, from lemma 4.11 one gets $\hat{B} = \hat{A}^{(2\frac{1}{2})} + A_3ts_2ts_1s_2^{-1}s_1ts_2tA_3 + A_3ts_2^{-1}ts_1^{-1}s_2s_1^{-1}ts_2^{-1}tA_3 = \hat{A}^{(2\frac{1}{2})} + A_3ts_2ts_1s_2^{-1}s_1ts_2t + A_3ts_2^{-1}ts_1^{-1}s_2s_1^{-1}ts_2^{-1}t$ and (2). From this, and because of proposition 4.12, (3) is equivalent to $\langle s_2, t\rangle A_3\langle s_2, t\rangle \subset \hat{B}$, and we prove this now.

We have $\langle s_2, t\rangle \subset u_2 + u_2tu_2 + \sum_{\varepsilon\in\{-1,1\}} u_2ts_2^{\varepsilon}t$ hence $\langle s_2, t\rangle A_3\langle s_2, t\rangle \subset \hat{A}^{(2)} + u_2tA_3\langle s_2, t\rangle + \sum_{\varepsilon\in\{-1,1\}} u_2ts_2^{\varepsilon}tA_3\langle s_2, t\rangle$. Since $\langle s_2, t\rangle \subset u_2 + u_2tu_2 + \sum_{\varepsilon\in\{-1,1\}} ts_2^{\varepsilon}tu_2$, $u_2tA_3\langle s_2, t\rangle \subset \hat{A}^{(2)} + \sum_{\varepsilon\in\{-1,1\}} u_2tA_3ts_2^{\varepsilon}tu_2$ and $u_2tA_3ts_2^{\varepsilon}tu_2 \subset u_2tu_1u_2u_1u_2ts_2^{\varepsilon}tu_1 \subset A_3tu_2u_1u_2ts_2^{\varepsilon}tu_1 \subset \hat{A}^{(2\frac{1}{2})} \subset \hat{B}$. Moreover, $ts_2^{\varepsilon}tA_3\langle s_2, t\rangle \subset \hat{A}^{(2)} + ts_2^{\varepsilon}tA_3tu_2 + \sum_{\eta\in\{-1,1\}} ts_2^{\varepsilon}tA_3ts_2^{\eta}tu_2$. We have $ts_2^{\varepsilon}tA_3t \in \hat{A}^{(2\frac{1}{2})}$ by proposition 4.12 ; $ts_2^{\varepsilon}tA_3ts_2^{\eta}t \subset ts_2^{\varepsilon}t^{\varepsilon}A_3ts_2^{\eta}t + \hat{A}^{(2\frac{1}{2})} \subset t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}A_3ts_2^{\eta}t + \hat{A}^{(2\frac{1}{2})}$ by proposition 4.12, applying $t \in Rt^{\varepsilon} + R$ two times. Similarly, one gets $t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}A_3ts_2^{\eta}t \in t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}A_3t^{\eta}s_2^{\eta}t^{\eta} + \hat{A}^{(2\frac{1}{2})}$. Whatever the choice of $\alpha \in \{-1, 1\}$, one has $A_3 = u_2s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha} + u_2u_1u_2$ and

$$\begin{array}{rcl} t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}A_3t^{\eta}s_2^{\eta}t^{\eta} & \subset & t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}u_2s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}t^{\eta}s_2^{\eta}t^{\eta} + t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}u_2u_1u_2t^{\eta}s_2^{\eta}t^{\eta} \\ & \subset & u_2t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}t^{\eta}s_2^{\eta}t^{\eta} + u_2t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}u_1t^{\eta}s_2^{\eta}t^{\eta}u_2 \\ & \subset & u_2t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}t^{\eta}s_2^{\eta}t^{\eta} + \hat{A}^{(2\frac{1}{2})} \end{array}$$

by proposition 4.12, so we need to prove $t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}t^{\eta}s_2^{\eta}t^{\eta} \in \hat{B}$ for a suitable choice of $\alpha \in \{-1, 1\}$.

If $\eta = -\varepsilon$ we take $\alpha = \varepsilon$. Then $t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}t^{\eta}s_2^{\eta}t^{\eta} = t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}s_1^{\varepsilon}s_2^{-\varepsilon}s_1^{\varepsilon}t^{-\varepsilon}s_2^{-\varepsilon}t^{\varepsilon}$ and, up to applying ϕ, we can assume $\varepsilon = 1$. Then $t^{\varepsilon}s_2^{\varepsilon}t^{\varepsilon}s_1^{\varepsilon}s_2^{-\varepsilon}s_1^{\varepsilon}t^{-\varepsilon}s_2^{-\varepsilon}t^{\varepsilon} = ts_2ts_1s_2^{-1}s_1t^{-1}s_2^{-1}t = ts_2s_1ts_2^{-1}t^{-1}s_1s_2^{-1}t$. From $ts_2ts_2 = ts_2ts_2$ one gets $ts_2^{-1}t^{-1} = s_2^{-1}t^{-1}s_2^{-1}ts_2$, hence

$$\begin{array}{lllllll} & ts_2s_1ts_2^{-1}t^{-1}s_1s_2^{-1}t & = & t(s_2s_1s_2^{-1})t^{-1}s_2^{-1}ts_2s_1s_2^{-1}t & = & ts_1^{-1}s_2s_1t^{-1}s_2^{-1}ts_2s_1s_2^{-1}t \\ = & s_1^{-1}ts_2s_1t^{-1}s_2^{-1}t(s_2s_1s_2^{-1})t & = & s_1^{-1}ts_2s_1t^{-1}s_2^{-1}ts_1^{-1}s_2s_1t & = & s_1^{-1}ts_2t^{-1}(s_1s_2^{-1}s_1^{-1})ts_2ts_1 \\ = & s_1^{-1}ts_2t^{-1}s_2^{-1}s_1^{-1}s_2ts_2ts_1 & \in & A_3\langle s_2, t\rangle u_1\langle s_2, t\rangle A_3 & \subset & \hat{A}^{(2\frac{1}{2})} \end{array}$$

by proposition 4.12.

Otherwise, we have either $\varepsilon = \eta = 1$, in which case we take $\alpha = 1$ and get $t^\varepsilon s_2^\varepsilon t^\varepsilon s_1^\alpha s_2^{-\alpha} s_1^\alpha t^\eta s_2^\eta t^\eta = ts_2ts_1s_2^{-1}s_1ts_2t \in \hat{B}$, or we have $\varepsilon = \eta = -1$, in which case we take $\alpha = -1$ and get $\phi(ts_2ts_1s_2^{-1}s_1ts_2t) \in \hat{B}$. This concludes the proof. □

4.5. **Computation of C^3 modulo $\hat{B}$.** We first need to prove a few preliminary lemmas.

Lemma 4.16.

(1) $ts_2s_1^{-1}s_2ts_2t^2s_1^{-1}s_2t \in R^\times ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2t + \hat{B}$
(2) $ts_2s_1^{-1}s_2ts_2ts_1^{-1}s_2t \in \hat{B}$
(3) $ts_2ts_1^{-1}s_2s_1^3ts_2s_1^{-1}ts_2t \in R^\times ts_2ts_1^{-1}s_2ts_2s_1^{-1}ts_2t + \hat{B}$
(4) $ts_2ts_1^{-1}s_2s_1^{-1}ts_2s_1^{-1}ts_2t \in -c^{-1}bts_2ts_1^{-1}s_2ts_2s_1^{-1}ts_2t + \hat{B}$
(5) $ts_2ts_1^{-1}s_2s_1ts_2s_1^{-1}ts_2t \in \hat{A}^{(2\frac{1}{2})} \subset \hat{B}$

Proof. Clearly (1) is a consequence of (2), since $t^2 \in R^\times + Rt$, so we only need to prove (2). Now $ts_2s_1^{-1}(s_2ts_2t)s_1^{-1}s_2t = ts_2s_1^{-1}ts_2t(s_2s_1^{-1}s_2)t$, and $s_2s_1^{-1}s_2 \in s_2^{-1}s_1s_2^{-1}u_1 + u_1u_2u_1$, hence $ts_2s_1^{-1}ts_2t(s_2s_1^{-1}s_2)t \in ts_2s_1^{-1}ts_2ts_2^{-1}s_1s_2^{-1}tu_1 + ts_2s_1^{-1}ts_2tu_1u_2tu_1$. We have $ts_2s_1^{-1}(ts_2t)s_2^{-1}s_1s_2^{-1}t = t(s_2s_1^{-1}s_2^{-1})ts_2ts_1s_2^{-1}t = ts_1^{-1}s_2^{-1}s_1ts_2ts_1s_2^{-1}t = s_1^{-1}ts_2^{-1}t(s_1s_2s_1)ts_2^{-1}t = s_1^{-1}ts_2^{-1}ts_2s_1s_2ts_2^{-1}t \in \hat{A}^{(2\frac{1}{2})}$ by proposition 4.12, and $ts_2s_1^{-1}ts_2tu_1u_2t = ts_2ts_1^{-1}s_2u_1tu_2t \subset ts_2tA_3tu_2t \subset \hat{B}$ by proposition 4.15.

Since $s_1^3 = as_1^2 + bs_1 + c$, we have $s_1^2 = as_1 + b + cs_1^{-1}$ hence $s_1^3 = a(as_1 + b + cs_1^{-1}) + bs_1 + c = (a^2+b)s_1 + acs_1^{-1} + (ab+c)$. Thus, since $c = (ab+c) + ac(-c^{-1}b) \in R^\times$, (4) and (5) imply (3).

We first prove (5). We have $ts_2ts_1^{-1}s_2s_1ts_2s_1^{-1}ts_2t = ts_2t(s_1^{-1}s_2s_1)ts_2s_1^{-1}ts_2t = (ts_2ts_2)s_1s_2^{-1}ts_2s_1^{-1}ts_2t = s_2ts_2ts_1s_2^{-1}(ts_2t)s_1^{-1}s_2t = s_2ts_2ts_1(ts_2t)s_2^{-1}s_1^{-1}s_2t = s_2ts_2ts_1ts_2ts_2^{-1}s_1^{-1}s_2t = s_2ts_2s_1t^2s_2t(s_2^{-1}s_1^{-1}s_2)t = s_2ts_2s_1t^2s_2ts_1s_2^{-1}s_1^{-1}t = s_2ts_2t^2s_1s_2s_1ts_2^{-1}ts_1^{-1} = s_2ts_2t^2s_2s_1s_2ts_2^{-1}ts_1^{-1} \in \hat{A}^{(2\frac{1}{2})} \subset \hat{B}$ by proposition 4.12.

We prove (4). From the study of A_3, we have that $s_2^{-1}(s_1^{-1}s_2s_1^{-1}) = (s_2^{-1}s_1^{-1}s_2)s_1^{-1} = (s_1s_2^{-1}s_1^{-1})s_1^{-1} = s_1s_2^{-1}s_1^{-2}$. Moreover, $s_1^{-2} = c^{-1}s_1 - c^{-1}a - c^{-1}bs_1^{-1}$ hence $s_2^{-1}(s_1^{-1}s_2s_1^{-1}) = c^{-1}s_1s_2^{-1}s_1 - c^{-1}as_1s_2^{-1} - c^{-1}bs_1s_2^{-1}s_1^{-1}$. It follows that $ts_2ts_1^{-1}s_2s_1^{-1}ts_2s_1^{-1}ts_2t$ is equal to $s_2s_2^{-1}(ts_2t)s_1^{-1}s_2s_1^{-1}ts_2s_1^{-1}ts_2t$ and thus to

$$\begin{array}{rl} & s_2ts_2t(s_2^{-1}s_1^{-1}s_2s_1^{-1})ts_2s_1^{-1}ts_2t \\ = & s_2ts_2t(c^{-1}s_1s_2^{-1}s_1 - c^{-1}as_1s_2^{-1} - c^{-1}bs_1s_2^{-1}s_1^{-1})ts_2s_1^{-1}ts_2t \\ = & c^{-1}s_2ts_2ts_1s_2^{-1}s_1ts_2s_1^{-1}ts_2t - c^{-1}as_2ts_2ts_1s_2^{-1}ts_2s_1^{-1}ts_2t - c^{-1}bs_2ts_2ts_1s_2^{-1}s_1^{-1}ts_2s_1^{-1}ts_2t \end{array}$$

We deal separately with each of these three terms. We prove that the first one belongs to $\hat{B}$. We have

$$\begin{array}{rcllcl} ts_2ts_1s_2^{-1}s_1ts_2s_1^{-1}ts_2t & = & ts_2ts_1s_2^{-1}t(s_1s_2s_1^{-1})ts_2t & = & ts_2ts_1s_2^{-1}ts_2^{-1}s_1(s_2ts_2t) \\ & = & ts_2ts_1s_2^{-1}ts_2^{-1}s_1ts_2ts_2 & = & ts_2s_1(t)s_2^{-1}ts_2^{-1}s_1ts_2ts_2 \\ & \in & Rts_2s_1t^{-1}s_2^{-1}ts_2^{-1}s_1ts_2ts_2 & + & Rts_2s_1s_2^{-1}ts_2^{-1}s_1ts_2ts_2 \end{array}$$

Now $t(s_2s_1s_2^{-1})ts_2^{-1}s_1ts_2t = ts_1^{-1}s_2s_1ts_2^{-1}s_1ts_2t = s_1^{-1}ts_2ts_1s_2^{-1}s_1ts_2t \in A_3\langle s_2, t\rangle A_3\langle s_2, t\rangle \subset \hat{B}$, and similarly $ts_2s_1t^{-1}s_2^{-1}(t)s_2^{-1}s_1ts_2ts_2 \in Rts_2s_1t^{-1}s_2^{-1}t^{-1}s_2^{-1}s_1ts_2ts_2 + Rts_2s_1t^{-1}s_2^{-1}s_2^{-1}s_1ts_2ts_2$ with $ts_2s_1t^{-1}s_2^{-1}s_2^{-1}s_1ts_2t = ts_2t^{-1}s_1s_2^{-2}s_1ts_2t \in \hat{B}$. Finally,

$$\begin{array}{rcllcl} ts_2s_1(t^{-1}s_2^{-1}t^{-1}s_2^{-1})s_1ts_2t & = & ts_2s_1s_2^{-1}t^{-1}s_2^{-1}t^{-1}s_1ts_2t & = & t(s_2s_1s_2^{-1})t^{-1}s_2^{-1}s_1s_2t \\ & = & ts_1^{-1}s_2s_1t^{-1}s_2^{-1}s_1s_2t & = & s_1^{-1}ts_2s_1t^{-1}s_2^{-1}s_1s_2t \in \hat{A}^{(2\frac{1}{2})}. \end{array}$$

We now turn to the second one. We have $ts_2ts_1s_2^{-1}ts_2s_1^{-1}ts_2t = ts_2ts_1s_2^{-1}(ts_2t)s_1^{-1}s_2t = ts_2ts_1ts_2ts_2^{-1}s_1^{-1}s_2t = ts_2ts_1ts_2t(s_2^{-1}s_1^{-1}s_2)t = ts_2ts_1ts_2ts_1s_2^{-1}s_1^{-1}t = ts_2ts_1ts_2ts_1s_2^{-1}ts_1^{-1} \in \hat{A}^{(2\frac{1}{2})}$. We finally turn to the third one. We have $ts_2ts_1s_2^{-1}s_1^{-1}ts_2s_1^{-1}ts_2t = ts_2t(s_1s_2^{-1}s_1^{-1})ts_2s_1^{-1}ts_2t = (ts_2t)s_2^{-1}s_1^{-1}s_2ts_2s_1^{-1}ts_2t = s_2^{-1}ts_2ts_1^{-1}s_2ts_2s_1^{-1}ts_2t = s_2^{-1}ts_2ts_1^{-1}s_2ts_2s_1^{-1}ts_2t$. Altogether this proves $ts_2ts_1^{-1}s_2s_1^{-1}ts_2s_1^{-1}ts_2t \in \hat{B} - c^{-1}bs_2s_2^{-1}ts_2ts_1^{-1}s_2ts_2s_1^{-1}ts_2t = \hat{B} - c^{-1}bts_2ts_1^{-1}s_2ts_2s_1^{-1}ts_2t$, hence (4).

□

Lemma 4.17.

(1) $ts_2ts_1s_2^{-1}s_1ts_2^2s_1ts_2t \in R^\times ts_2ts_1s_2^{-1}s_1ts_2^{-1}s_1ts_2t + \hat{B}$
(2) $ts_2ts_1s_2^{-1}s_1ts_2s_1ts_2t \in \hat{A}^{(2\frac{1}{2})} \subset \hat{B}$
(3) $ts_2ts_1s_2^{-1}s_1ts_1ts_2t \in \hat{B}$

Proof. Since $s_2^2 \in R^\times s_2^{-1} + Rs_2 + R$, (1) is a consequence of (2) and (3). We first prove (2). We have

$$\begin{array}{rlll} & ts_2ts_1s_2^{-1}s_1ts_2s_1ts_2t & = ts_2ts_1s_2^{-1}t(s_1s_2s_1)ts_2t & = ts_2ts_1s_2^{-1}ts_2s_1(s_2ts_2t) \\ = & ts_2ts_1s_2^{-1}ts_2s_1ts_2ts_2 & = ts_2ts_1s_2^{-1}ts_2s_1(ts_2t)s_2^{-1}s_2^2 & = ts_2ts_1s_2^{-1}t(s_2s_1s_2^{-1})ts_2ts_2^2 \\ = & ts_2ts_1s_2^{-1}ts_1^{-1}s_2s_1ts_2ts_2^2 & = ts_2t(s_1s_2^{-1}s_1^{-1})ts_2s_1ts_2ts_2^2 & = (ts_2t)s_2^{-1}s_1^{-1}s_2ts_2s_1ts_2ts_2^2 \\ = & s_2^{-1}ts_2ts_1^{-1}s_2ts_2s_1ts_2ts_2^2 & = s_2^{-1}ts_2ts_1^{-1}(s_2ts_2t)s_1s_2ts_2^2 & = s_2^{-1}ts_2ts_1^{-1}ts_2ts_2s_1s_2ts_2^2 \\ = & s_2^{-1}ts_2s_1^{-1}t^2s_2t(s_2s_1s_2)ts_2^2 & = s_2^{-1}ts_2s_1^{-1}t^2s_2ts_1s_2s_1ts_2^2 & = s_2^{-1}ts_2s_1^{-1}t^2s_2ts_1s_2ts_1s_2^2 \end{array}$$

and, since $t^2 \in Rt+R$, $ts_2s_1^{-1}t^2s_2ts_1s_2t \in Rts_2s_1^{-1}ts_2ts_1s_2t+Rts_2s_1^{-1}s_2ts_1s_2t$; we have $ts_2s_1^{-1}s_2ts_1s_2t \in \hat{A}^{(2\frac{1}{2})}$ by lemma 4.11, and $ts_2s_1^{-1}ts_2ts_1s_2t = ts_2t(s_1^{-1}s_2s_1)ts_2t = ts_2ts_2s_1s_2^{-1}ts_2t \in \hat{A}^{(2\frac{1}{2})}$ by proposition 4.12. This proves (2). Then (3) follows from $ts_2ts_1s_2^{-1}s_1ts_1ts_2t = ts_2ts_1s_2^{-1}s_1^2t^2s_2t \in \langle s_2,t\rangle A_3\langle s_2,t\rangle \subset \hat{B}$. ☐

Lemma 4.18.

(1) $ts_2ts_1s_2^{-1}s_1ts_2t^2s_2s_1ts_2t \in R^\times ts_2ts_1s_2^{-1}s_1ts_2^2s_1ts_2t + \hat{B}$
(2) $ts_2ts_1s_2^{-1}s_1ts_2ts_2s_1ts_2t \in \hat{B}$.
(3) $ts_2s_1^2ts_2ts_2s_1t \in \hat{A}^{(2\frac{1}{2})}$
(4) $ts_2s_1^2ts_2ts_2s_1s_2^{-1}ts_2t \in \hat{B}$.

Proof. Since $t^2 = dt+e \in Rt+R^\times$, (1) is an immediate consequence of (2). We prove (2). We have $ts_2ts_1s_2^{-1}s_1ts_2ts_2s_1ts_2t = ts_2ts_1s_2^{-1}s_1(ts_2ts_2)s_1ts_2t = ts_2ts_1s_2^{-1}s_1s_2ts_2ts_1ts_2t = ts_2ts_1(s_2^{-1}s_1s_2)ts_2s_1t^2s_2t = ts_2ts_1s_1s_2s_1^{-1}ts_2s_1t^2s_2t = ts_2s_1^2ts_2t(s_1^{-1}s_2s_1)t^2s_2t = ts_2s_1^2ts_2ts_2s_1s_2^{-1}t^2s_2t$. Using $t^2 \in Rt+R$, we see that (2) is a consequence of (3) and (4). We have $ts_2s_1^2ts_2ts_2s_1t = ts_2s_1^2ts_2ts_2ts_1 \in \hat{A}^{(2\frac{1}{2})}$ by proposition 4.12 and this proves (3). We turn to (4). We have $ts_2s_1^2(ts_2ts_2)s_1s_2^{-1}ts_2t = ts_2s_1^2s_2ts_2ts_1s_2^{-1}(ts_2t) = ts_2s_1^2s_2ts_2ts_1ts_2ts_2^{-1} = ts_2s_1^2s_2ts_2s_1t^2s_2ts_2^{-1}$. Using $t^2 \in Rt+R$, we only need to prove $ts_2s_1^2s_2ts_2s_1ts_2t \in \hat{B}$, since $ts_2s_1^2s_2ts_2s_1s_2t = ts_2s_1^2s_2t(s_2s_1s_2)t = ts_2s_1^2s_2ts_1s_2s_1t = ts_2s_1^2s_2s_1ts_2ts_1 \in \langle s_2,t\rangle A_3\langle s_2,t\rangle A_3 \subset \hat{B}$. But $ts_2s_1^2s_2ts_2s_1ts_2t = ts_2s_1^2(s_2ts_2t)s_1s_2t = ts_2s_1^2ts_2ts_2s_1s_2t = ts_2ts_1^2s_2t(s_2s_1s_2)t = ts_2ts_1^2s_2ts_1s_2s_1t = ts_2ts_1^2s_2s_1ts_2ts_1 \in \langle s_2,t\rangle A_3\langle s_2,t\rangle A_3 \subset \hat{B}$, and this concludes the proof of (4), and thus of the lemma. ☐

Proposition 4.19. $C^3 \in A_3^\times ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2t + \hat{B}$

Proof. Since C^3 is central and $\hat{B}$ is a A_3-bimodule, it is sufficient to prove $C^3 \in A_3^\times ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2tA_3^\times + \hat{B}$. We have already proved $(s_1s_2s_1)^{-1}C^3 \in A_3^\times ts_2ts_1s_2^{-1}s_1ts_2t^2s_2s_1ts_2t(s_1s_2s_1)+A_3ts_2s_1^{-1}ts_2ts_1C+A_3C^2 + \hat{A}^{(2)} \subset A_3^\times ts_2ts_1s_2^{-1}s_1ts_2t^2s_2s_1ts_2tA_3^\times + \hat{B}$ by lemma 4.14, so we only need to prove $ts_2ts_1s_2^{-1}s_1ts_2t^2s_2s_1ts_2t \in A_3^\times ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2tA_3^\times + \hat{B}$. Now

$$\begin{array}{lll} & ts_2ts_1s_2^{-1}s_1ts_2t^2s_2s_1ts_2t & \\ \in & ts_2ts_1s_2^{-1}s_1ts_2^2s_1ts_2t + \hat{B} & \text{(by lemma 4.18)} \\ \subset & s_2^{-1}(s_2ts_2t)s_1s_2^{-1}s_1ts_2^{-1}s_1ts_2t + \hat{B} & \text{(by lemma 4.17)} \\ = & s_2^{-1}ts_2t(s_2s_1s_2^{-1})s_1ts_2^{-1}s_1ts_2t + \hat{B} & \\ = & s_2^{-1}ts_2ts_1^{-1}s_2s_1s_1ts_2^{-1}s_1ts_2ts_2s_2^{-1} + \hat{B} & \\ = & s_2^{-1}ts_2ts_1^{-1}s_2s_1s_1t(s_2^{-1}s_1s_2)ts_2ts_2^{-1} + \hat{B} & \\ = & s_2^{-1}ts_2ts_1^{-1}s_2s_1s_1t(s_1s_2s_1^{-1}ts_2ts_2^{-1} + \hat{B} & \\ = & s_2^{-1}ts_2ts_1^{-1}s_2s_1^3ts_2s_1^{-1}ts_2ts_2^{-1} + \hat{B} & \\ \subset & s_2^{-1}ts_2ts_1^{-1}(s_2ts_2t)s_1^{-1}s_2ts_2^{-1} + \hat{B} & \text{(by lemma 4.16 (3))} \\ = & s_2^{-1}ts_2ts_1^{-1}ts_2ts_2s_1^{-1}s_2ts_2^{-1} + \hat{B} & \\ = & s_2^{-1}ts_2s_1^{-1}t^2s_2ts_2s_1^{-1}s_2ts_2^{-1} + \hat{B} & \\ = & s_2^{-1}ts_2s_1^{-1}s_2ts_2t^2s_1^{-1}s_2ts_2^{-1} + \hat{B} & \\ \subset & s_2^{-1}ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2ts_2^{-1} + \hat{B} & \text{(by lemma 4.16 (1))} \end{array}$$

and this concludes the proof of the proposition. □

4.6. **Conclusion of the proof.**

Proposition 4.20.

(1) $\hat{A}^{(3)}t \subset \hat{A}^{(2\frac{1}{2})}t + \hat{A}^{(3)}$
(2) $\hat{A}^{(2\frac{1}{2})}t \subset \hat{A}^{(3)} + \hat{B}$
(3) $\hat{B}t \subset \hat{A}^{(3)} + \hat{B}$
(4) $\hat{A}^{(4)} = \hat{A}^{(3)} + \hat{B}$
(5) $\hat{A}^{(4)} = \hat{A}^{(5)} = \hat{A} = \hat{A}^{(3)} + A_3C^2 + A_3C^{-2}$
(6) $\hat{A}^{(4)} = \hat{A}^{(2\frac{1}{2})} + A_3C^2 + A_3C^{-2} + A_3C^3$

Proof. In order to simplify notations, we let $X = ts_2s_1^{-1}s_2ts_2s_1^{-1}s_2t$, $Y_+ = ts_2ts_1s_2^{-1}s_1ts_2t$, $Y_- = ts_2^{-1}ts_1^{-1}s_2s_1^{-1}ts_2^{-1}t$. By proposition 4.19, we have $C^3 \in A_3^\times X + \hat{B}$, hence $X \in A_3^\times C^3 + \hat{B} = A_3^\times C^3 + A_3C^2 + A_3C^{-2} + \hat{A}^{(2\frac{1}{2})}$. From this we deduce that, for all $m \in A_3$, $mX - Xm \in \hat{A}^{(2\frac{1}{2})}$, hence $A_3XA_3 + \hat{A}^{(2\frac{1}{2})} = A_3X + \hat{A}^{(2\frac{1}{2})}$, and $\hat{A}^{(3)} = \hat{A}^{(2\frac{1}{2})} + A_3XA_3 = \hat{A}^{(2\frac{1}{2})} + A_3X$. It follows that $\hat{A}^{(3)}t \subset \hat{A}^{(2\frac{1}{2})}t + A_3Xt$, and clearly $Xt \in \hat{A}^{(3)}$, whence $\hat{A}^{(3)}t \subset \hat{A}^{(2\frac{1}{2})}t + \hat{A}^{(3)}$ and (1). On the other hand,

$$\hat{A}^{(2\frac{1}{2})} = \hat{A}^{(2)} + A_3ts_2s_1ts_2tA_3 + A_3ts_2s_1ts_2^{-1}tA_3 + A_3ts_2s_1^{-1}ts_2^{-1}tA_3 + A_3ts_2^{-1}s_1^{-1}ts_2^{-1}tA_3$$

hence

$$\begin{array}{lcl} \hat{A}^{(2\frac{1}{2})}t & \subset & \hat{A}^{(2)}t + A_3ts_2s_1ts_2tA_3t + A_3ts_2s_1ts_2^{-1}tA_3t + A_3ts_2s_1^{-1}ts_2^{-1}tA_3t + A_3ts_2^{-1}s_1^{-1}ts_2^{-1}tA_3t \\ \hat{A}^{(2\frac{1}{2})}t & \subset & \hat{A}^{(3)} + A_3ts_2s_1ts_2tA_3t + A_3ts_2s_1ts_2^{-1}tA_3t + A_3ts_2s_1^{-1}ts_2^{-1}tA_3t + A_3ts_2^{-1}s_1^{-1}ts_2^{-1}tA_3t \end{array}$$

We have $A_3ts_2s_1ts_2tA_3t = A_3CA_3t = A_3Ct = A_3ts_2s_1ts_2t^2 \subset \hat{A}^{(3)}$, and similarly $ts_2^{-1}s_1^{-1}ts_2^{-1}t \in A_3^\times C^{-1} + \hat{A}^{(2)}$ implies $A_3ts_2^{-1}s_1^{-1}ts_2^{-1}tA_3t \subset A_3ts_2^{-1}s_1^{-1}ts_2^{-1}t^2 + \hat{A}^{(2)}t \subset \hat{A}^{(3)}$.

Moreover, by lemma 4.8,

$$ts_2s_1ts_2^{-1}tA_3t \subset \hat{A}^{(2)}t + \sum_{a\in\{-1,0,1\}} \sum_{b,\varepsilon\in\{-1,1\}} A_2ts_2s_1ts_2^\varepsilon ts_1^b s_2^a t + \sum_{\varepsilon\in\{-1,1\}} \left(A_2ts_2s_1ts_2^\varepsilon ts_1s_2^{-1}s_1t + A_2ts_2s_1ts_2^\varepsilon t^2\right)$$

hence

$$\begin{array}{lcl} ts_2s_1ts_2^{-1}tA_3t & \subset & \hat{A}^{(3)} + \sum_{a\in\{-1,0,1\}} \sum_{b,\varepsilon\in\{-1,1\}} A_2ts_2s_1ts_2^\varepsilon ts_1^b s_2^a t + \sum_{\varepsilon\in\{-1,1\}} \left(A_2ts_2s_1ts_2^\varepsilon ts_1s_2^{-1}s_1t\right) \\ & \subset & \hat{A}^{(3)} + \sum_{a\in\{-1,0,1\}} \sum_{b,\varepsilon\in\{-1,1\}} A_2ts_2ts_1s_2^\varepsilon s_1^b ts_2^a t + \sum_{\varepsilon\in\{-1,1\}} \left(A_2ts_2ts_1s_2^\varepsilon s_1ts_2^{-1}ts_1\right) \end{array}$$

and, by proposition 4.15, $ts_2s_1ts_2^{-1}tA_3t \subset \hat{A}^{(3)} + \hat{B} = \hat{A}^{(3)} + \hat{A}^{(2\frac{1}{2})} + A_3C^2 + A_3C^{-2} = \hat{A}^{(3)} + A_3C^2 + A_3C^{-2}$. This implies $\phi(ts_2s_1^{-1}ts_2^{-1}tA_3t) \subset \phi(ts_2s_1^{-1}ts_2^{-1}t)A_3t^{-1} \subset \hat{A}^{(3)} + A_3C^2 + A_3C^{-2}$ by lemma 4.9 and the above, whence (4).

We have $\hat{B} \subset \hat{A}^{(2\frac{1}{2})} + A_3ts_2ts_1s_2^{-1}s_1ts_2t + A_3t^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2s_1^{-1}t^{-1}s_2^{-1}t^{-1}$ by proposition 4.15, and we already know $\hat{A}^{(2\frac{1}{2})}t \subset \hat{A}^{(3)} + \hat{B}$. Since $t^{-1}s_2^{-1}t^{-1}s_1^{-1}s_2s_1^{-1}t^{-1}s_2^{-1}t^{-1}.t \in \hat{A}^{(3)}$ and $ts_2ts_1s_2^{-1}s_1ts_2t^2 \in Rts_2ts_1s_2^{-1}s_1ts_2t + \hat{A}^{(3)} \subset \hat{A}^{(3)} + \hat{B}$ because of $t^2 \in Rt + R$, this proves (3). From (3) and (4) we get $\hat{A}^{(4)}t \subset \hat{A}^{(3)}t + \hat{B}t \subset \hat{A}^{(4)} + \hat{B}t \subset \hat{A}^{(4)} + \hat{A}^{(3)} + \hat{B} = \hat{A}^{(4)}$. This implies $\hat{A}^{(5)} \subset \hat{A}^{(4)}$ hence $\hat{A}^{(4)} = \hat{A}^{(5)}$ and (5). As already noticed we have $\hat{A}^{(3)} = \hat{A}^{(2\frac{1}{2})} + A_3XA_3$, and $X \in A_3^\times C^3 + \hat{B}$ with C^3 central implies $\hat{A}^{(4)} = \hat{A}^{(3)} + \hat{B} = \hat{A}^{(2\frac{1}{2})} + \hat{B} + A_3C^3 = \hat{A}^{(2\frac{1}{2})} + A_3C^2 + A_3C^{-2} + A_3C^3$ and (6). □

Corollary 4.21. *As a A_3-module, $\hat{A}$ is generated by 54 elements.*

Proof. By the above property, $\hat{A}$ is generated by $\hat{A}^{(2\frac{1}{2})}$ and 3 elements. By proposition 4.10, $\hat{A}^{(2\frac{1}{2})}$ is generated by $\hat{A}^{(2)}$ and $2 \times 9 = 18$ elements. By proposition 4.3, $\hat{A}^{(2)} = \hat{A}^{(1)} + A_3ts_2tA_3 + A_3ts_2^{-1}tA_3 + A_3ts_2s_1^{-1}s_2tA_3$. Since ts_2t commutes with u_2, and because A_3 is a (free) u_2-module of rank (at most) 8, we know that $A_3ts_2tA_3$ is generated as a A_3-module by 8 éléments. Since $t \in R^\times t^{-1} + R$, we have $A_3ts_2^{-1}tA_3 + \hat{A}^{(1)} = A_3t^{-1}s_2^{-1}t^{-1}A_3 + \hat{A}^{(1)}$ and, because $t^{-1}s_2^{-1}t^{-1}$ commutes with u_2, this A_3-module is generated by $\hat{A}^{(1)}$ together with 8 éléments. Finally, because $A_3 = u_1s_2s_1^{-1}s_2 + u_1u_2u_1$, we have $(ts_2s_1^{-1}s_2t)s_1 = ts_2(s_1^{-1}s_2s_1)t = ts_2^2s_1s_2^{-1}t \in tu_1s_2s_1^{-1}s_2t +$

$tu_1u_2u_1t \subset u_1ts_2s_1^{-1}s_2t + u_1tu_2tu_1 \subset u_1ts_2s_1^{-1}s_2t + A_3ts_2tA_3 + A_3ts_2^{-1}tA_3 + \hat{A}^{(1)}$. Since A_3 is also a (free) u_1-module of rank 8, we deduced from this that $\hat{A}^{(2)}$ is generated by $\hat{A}^{(1)} + A_3ts_2tA_3 + A_3ts_2^{-1}tA_3$ together with 8 elements, and it follows that $\hat{A}^{(2)}$ is generated by $\hat{A}^{(1)}$ together with $2 \times 8 + 8 = 24$ elements. Finally, $\hat{A}^{(1)} = A_3 + A_3tA_3$ is A_3-generated par $1 + 8 = 9$ elements, since t commutes with u_1 and A_3 is generated by 8 elements as a u_1-module. This proves that $\hat{A}$ is generated as a A_3-module by $21 + 24 + 9 = 54$ elements. □

Since A_3 is a R-module of rank 24, this has for immediate consequence the following, which proves theorem 4.1.

Corollary 4.22. *As a R-module, $\hat{A}$ is spanned by* 1296 *elements.*

Remark 4.23. *The R-basis provided by this corollary is actually made out of elements of B, and contains* 1*.*

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