ON TERNARY QUOTIENTS OF CUBIC HECKE ALGEBRAS

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Abstract. We prove that the quotient of the group algebra of the braid group introduced by L. Funar in [F1] collapses in characteristic distinct from 2. In characteristic 2 we define several quotients of it, which are connected to the classical Hecke and Birman-Wenzl-Murakami quotients, but which admit in addition a symmetry of order 3. We also establish conditions on the possible Markov traces factorizing through it.

1. Introduction

Let B_n be the braid group on n strings ($n \geq 2$), that is the group defined by $n-1$ generators $s_1, \dots, s_{n-1}$ submitted to the relations $s_i s_j = s_j s_i$ whenever $i - j \geq 2$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for any $i = 1, \dots n-2$ (see e.g. [Bi] or [KM] for basic results on these groups).

This paper grew out as an attempt to understand the mysterious 'cubic Hecke algebras' defined by L. Funar and used in [F1] and [BF]. In [F1], an algebra $K_n(\gamma)$ for $\gamma \in k$ is defined over a commutative ring k as the quotient of the group algebra kB_n of the braid group B_n on n strands, by the relations $s_i^3 = \gamma$, and $s_{i+1} s_i^2 s_{i+1} + s_i s_{i+1}^2 s_i + s_i^2 s_{i+1} s_i + s_i s_{i+1} s_i^2 + s_i^2 s_{i+1}^2 + s_{i+1}^2 s_i^2 + \gamma s_i + \gamma s_{i+1} = 0$. Notice that the relations are equivalent to $s_1^3 = \gamma$, $s_2 s_1^2 s_2 + s_1 s_2^2 s_1 + s_1^2 s_2 s_1 + s_1 s_2 s_1^2 + s_1^2 s_2^2 + s_2^2 s_1^2 + \gamma s_1 + \gamma s_2 = 0$. The striking property of this algebra is that the latter relation involves only s_1, s_2 and that, as proved in [F1], it is a finitely generated k-module (hence finite dimensional over k if k is a field). Although many finite-dimensional cubic quotients of the (group algebra of the) braid groups have been defined, to our knowledge it is the only one which is not a quotient of the classical Birman-Wenzl-Murakami algebra and which can be defined from relations in kB_3. Notice that, whenever γ admits an invertible third root $\alpha \in k$ with $\alpha^3 = \gamma$, we have $K_n(\gamma) \simeq K_n(1)$ under $s_i \mapsto \alpha^{-1} s_i$ – and in particular always $K_n(-1) \simeq K_n(1)$. Moreover, $K_n(1)$ is a quotient of the group algebra $k\Gamma_n$, for $\Gamma_n = B_n / < s_1^3 >$. This group Γ_n is a semidirect product $\Gamma_n^0 \rtimes C_3$, with C_k denoting the cyclic group of order k, and the defining ideal of $K_n(1)$ has the remarkable property to be generated by a C_3-invariant ideal in $\mathbb{Z}\Gamma_3^0$ – thus deserving the name ternary used in the title.

By a theorem of Coxeter, Γ_n is finite if and only if $n \leq 5$. Moreover, in this case it is a finite complex reflection group, and, as was conjectured by Broué, Malle and Rouquier, $k\Gamma_n$ for $n \leq 5$ admits a flat deformation similar to the presentation of the ordinary Hecke algebra as a deformation of $k\mathfrak{S}_n$. This has been proved in [BM], Satz 4.7 for $n = 3, 4$, and recently in [M] for $n = 5$. Partly stimulated by this conjecture, the authors of [BF] constructed a deformation of $K_n(\gamma)$ (still finitely generated).

The main motivation in [F1] and [BF] is to construct link invariants. In [F1] it is claimed that $K_n(-1)$ admits a Markov trace with values in $\mathbb{Z}/6\mathbb{Z}$. A more general statement is claimed in [BF], that the constructed deformation provides a link invariant with values in

Date: February 4, 2012.

some extended ring. Around 2004-2005, S. Orevkov pointed out a gap in a part of [BF] devoted to the proof of the invariance of the trace under Markov moves, which originates in [F1]. In 2008, the second author of the present paper noticed that, when k is a field of characteristic 0, the 'tower of algebras' $K_n(1)$ collapsed, more precisely that $K_n(1) = 0$ for $n \geq 5$ (see theorem 4.8 below). However, when $k = \mathbb{Z}$, this tower does not collapse. This can be seen from the fact that the natural group morphisms $\Gamma_n \twoheadrightarrow C_3$ induce morphisms $\mathbb{Z}\Gamma_n \twoheadrightarrow \mathbb{Z}C_3 \twoheadrightarrow (\mathbb{Z}/8\mathbb{Z})C_3$ which factorize through $K_n(1)$.

1.1. **Statement of the main results.** Letting $K_n = K_n(1)$ we prove (see corollary 4.3 and theorems 4.8 and 4.9)

Theorem. *When $k = \mathbb{Z}$,*

(i) *K_n is a finite $\mathbb{Z}$-module for $n \geq 5$.*
(ii) *The exponent (as an abelian group) of K_n has the form $2^r 3^s$ for some r, s (depending on n) when $n \geq 5$.*
(iii) *The exponent of K_n is a power of 2 (*not *depending on n) when $n \geq 7$.*

When k is a field, in order to get a stably nontrivial structure, we thus need to assume that k has characteristic 2.

Theorem. *Assume k is a field of characteristic 2. For all n, there exists a quotient $\mathcal{H}_n$ of K_n, which has dimension $3(n!-1)$ and which embeds inside a product of three Hecke algebras. This algebra $\mathcal{H}_n$ is the quotient of $k\Gamma_n$ by the relation $s_1 s_2^{-1} + s_2 s_1^{-1} + s_1^{-1} s_2 + s_2^{-1} s_1 = 0$.*

We call this algebra the ternary Hecke algebra, as it can be defined as the quotient of $k\Gamma_n$ by the intersection of the three ideals whose corresponding quotients define the three possible Hecke algebras at third roots of 1.

Taking $k = \mathbb{Z}$, we let K_∞ denote the direct limit of the K_n under the natural morphisms $K_n \to K_{n+1}$, and we similarly define $\mathcal{H}_\infty$. Using the second definition above, $\mathcal{H}_\infty$ can be defined over $\mathbb{Z}/4\mathbb{Z}$.

We recall that a Markov trace on K_∞ is a $\mathbb{Z}$-module morphism $t : K_\infty \to M$, where M is some $\mathbb{Z}[u,v]$-module, which satisfies $t(xy) = t(yx)$ for all $x, y \in K_\infty$, $t(xs_n) = u\,t(x)$ and $t(xs_n^{-1}) = vt(x)$ for all x in (the image of) K_n. It can be shown that such a Markov trace, if it exists, is uniquely determined by the value $t(1)$, and takes values in $\mathbb{Z}[u,v]t(1) \subset M$.

Theorem.

(i) *If $t : K_\infty \to \mathbb{Z}[u,v]t(1)$ is a Markov trace, then $16t(1) = 0$, $4uv.t(1) = 4t(1)$, $4u^3.t(1) = 4v^3.t(1) = -4t(1)$ and $(3u^3 + 3v^3 - 3uv + 1)t(1) = 0$.*
(ii) *If $4t(1) = 0$, then t factors through $\mathcal{H}_\infty$ (defined over $\mathbb{Z}/4\mathbb{Z}$)*
(iii) *There exists a Markov trace $\bar{t} : \mathcal{H}_\infty \to (\mathbb{Z}/4\mathbb{Z})[u,v]$ with $\bar{t}(1) = \bar{1} \in \mathbb{Z}/4\mathbb{Z}$, which originates from the Markov traces on ordinary Hecke algebras.*

Modulo 4, the most general link invariant that can be defined this way is thus given by the following operation : take the Homfly polynomial, consider the three possible specialisations 'at third roots of 1', and reduce these three values modulo 4.

Finally, we investigate another quotient of K_n, that we denote $\mathcal{BMW}_n$ and which is obtained from the usual Birman-Wenzl-Murakami algebras by a similar 'ternary' operation. Computer calculations seems to indicate that this quotient is asymptotically very close to K_n. However, the study of this quotient is more delicate, and we get only partial results on it. This nethertheless shows that, over a field of characteristic 2, K_n is actually *larger* than all the quotients of $k\Gamma_n$ by relations on 3 strands that have been defined so far.

1.2. **Open problems.** The work leaves for now the following questions open :

(i) Over $\mathbb{Z}/4\mathbb{Z}$, and even over $\mathbb{Z}$, does $\mathcal{H}_n$ coincide with the quotient of the group algebra of Γ_n by the ideal generated by $s_1 s_2^{-1} - s_1^{-1} s_2 + s_2 s_1^{-1} - s_2^{-1} s_1$?

(ii) Which are the Markov traces on $K_n(1)$ with $4t(1) \neq 0$? Are there non-obvious ones ? (Notice that the natural projection $\Gamma_n \twoheadrightarrow C_3 =< s >$ obviously induces a Markov trace $t : K_n \twoheadrightarrow (\mathbb{Z}/8)C_3$ with $u = s$ and $v = s^2$.)

(iii) What is the minimal r ($r \geq 3$) such that $2^r K_\infty = 0$? Note that $2^5 K_\infty = 0$ by proposition 4.17.

(iv) We lack a nice description of the intersection of the defining ideals of the 'two Temperley-Lieb algebras', at third roots of 1 and in characteristic 2. This would help understanding $\mathcal{BMW}_n$ (see Definition 6.4).

(v) Do we have $\mathcal{BMW}_\infty = K_\infty(1)$, over a field of characteristic 2 ?

(vi) Are there 'nice generators' for the defining ideal of $\mathcal{BMW}_n$?

(vii) We did not study here the deformation of K_n proposed in [BF], although we hope our work now provides a firmer ground for it. See [M] for the characteristic 0 case.

(viii) Does $\mathcal{H}_n$ admit a 'nice' deformation, and a related Markov trace ?

(ix) Is there a nice description of the algebra $K_4(1)$ in characteristic 3 ?

(x) What are K_5, K_6 as modules over the ring $\mathbb{Z}_3$ of 3-adic integers ?

(xi) Are the natural morphisms $\Gamma_n \to \Gamma_m$ injective for $6 \leq n \leq m$?

1.3. **Notations.** Let G be a group. We denote by $\mathrm{Z}(G)$, resp. (G, G), the center, resp. derived subgroup of G, and we denote by G^{ab} the quotient $G/(G, G)$. If H is a group on which G acts by group automorphisms, we denote by $H \rtimes G$ the associated semi-direct product.

If A is a ring and G acts on A by ring automorphisms, we denote by $A \rtimes G$ the semi-direct product ring, that is the free A-module $\oplus_{g\in G} Ag$ endowed with multiplication $(ag).(a'g') = a(g.a')gg'$ for $a, a' \in A$, $g, g' \in G$.

If $n \geq 1$ is an integer, one denotes by C_n the cyclic group with n elements.

For k a field we let $\overline{k}$ denote an algebraic closure of k.

If G is a finite group, we denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G, that is trace characters of simple $\mathbb{C}G$-modules.

If A is a ring and $n \geq 1$ is an integer, one denotes by $\mathrm{Mat}_n(A)$ the ring of $n \times n$ matrices with coefficients in A. We will also use the more general notation $Mat_I(A)$ for I an arbitrary finite set. One denotes by Id_n the identity matrix. One denotes matrix transposition by $M \mapsto {}^tM$.

Let q be a power of a prime. We denote by $\mathbb{F}_q$ the field with q elements. We denote by $\mathrm{GL}_n(q) = \mathrm{GL}_n(\mathbb{F}_q)$, resp. $\mathrm{SL}_n(q) = \mathrm{SL}_n(\mathbb{F}_q)$ the general and special linear groups in $\mathrm{Mat}_n(\mathbb{F}_q)$. One denotes by $\mathrm{Sp}_{2n}(q) = \mathrm{Sp}_{2n}(\mathbb{F}_q)$ the multiplicative group of matrices $M \in \mathrm{Mat}_{2n}(\mathbb{F}_q)$ satisfying

$$ {}^tM \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}. $$

Let us denote by $a \mapsto \overline{a} = a^q$ the field automorphism of $\mathbb{F}_{q^2}$ order 2, which extends as a ring automorphism of $\mathrm{Mat}_n(\mathbb{F}_q)$ denoted in the same fashion. One denotes by $\mathrm{GU}_n(q)$ the subgroup of matrices $M \in \mathrm{GL}_n(q^2)$ such that

$$ {}^t\overline{M}M = \mathrm{Id}_n. $$

Denote $\mathrm{SU}_n(q) = \mathrm{GU}_n(q) \cap \mathrm{SL}_n(q^2)$. When $m \leq n$ we always consider $\mathrm{GU}_m(q)$ as the subgroup of $\mathrm{GU}_n(q)$ fixing the last $n-m$ elements of the canonical basis of $\mathbb{F}_{q^2}^n$.

1.4. **Acknowledgements.** The second author benefited from the ANR grant ANR-09-JCJC-0102-01, corresponding to the ANR project 'RepRed'.

2. Groups

2.1. **The groups Γ_n.** Let Γ_n be the quotient of B_n obtained by adding the extra relations $s_i^3 = 1$ for any $i = 1, \dots n-1$.

The following is due to Coxeter [Co] (see also [As2]).

Theorem 2.1. *Γ_2, Γ_3, Γ_4, Γ_5 are finite (complex) reflection groups, respectively denoted by $G(3,1,1) \simeq C_3$, $G_4 \simeq Q \rtimes C_3$ where Q is the quaternion of order 8 and C_3 acts by any automorphism of order 3, $G_{25} \simeq \mathrm{GU}_3(2)$, $G_{32} \simeq C_3 \times \mathrm{Sp}_4(3)$ in the Shephard-Todd classification. Their orders are respectively 3, 24, 648 and $155,520 = 2^7.3^5.5$. For $n \geq 6$, Γ_n is infinite.*

The following is due to Assion [As1].

Theorem 2.2. (i) *Every non-trivial normal subgroup of Γ_5 contains either $\left((s_1s_2)^3.(s_3s_4)^3\right)^3$ or $s_3.s_1.s_1^{(s_2s_3)^3}.s_1^{(s_2s_3)^3(s_3s_4)^3}$.*

(ii) *Let $U(m)$ be the quotient of Γ_{m+1} obtained by imposing the extra relation $\left((s_1s_2)^3.(s_3s_4)^3\right)^3 = 1$. Then it is isomorphic with $\mathrm{GU}_m(2)$ except when $m = 2$ mod 3 in which case*

$$U(m) = Y_{m-1} \rtimes \mathrm{GU}_{m-1}(2)$$

where $Y_{m-1} = \{(x,V) \mid x \in \mathbb{F}_4, V \in \mathbb{F}_4^{m-1} x + \overline{x} + {}^t\overline{V}.V = 0\}$ is endowed with the multiplication $(x,V).(x',V') = (x + x' +^t \overline{V}.V', V + V')$ and the action of GU_{m-1} is by $(x,V)^A = (x, A^{-1}V)$.

(iii) *For $n \geq 5$, the quotient of Γ_n by the relation $s_3.s_1.s_1^{(s_2s_3)^3}.s_1^{(s_2s_3)^3(s_3s_4)^3} = 1$ is a finite group, isomorphic to $\mathrm{Sp}_{n-1}(\mathbb{F}_3)$ if n is odd, and to the stabilizer of one vector in $\mathrm{Sp}_{n+1}(\mathbb{F}_3)$ if n is even.*

Remark 2.3. *In* [As1]*, the group $U(m)$ for $m = 2 \mod 3$ is defined in the projective unitary group $\mathrm{PGU}_{m+1}(2)$ as the centralizer of $\mathrm{Id}_{m+1} + E_{m+1}$ with E_{m+1} the matrix $E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in dimensions $m, m+1$ (zeros elsewhere). This clearly amounts to the subgroup of $\mathrm{GU}_{3k}(2)$ of elements fixing the sum $e_{3k-1} + e_{3k}$ of the two last elements of an orthonormal basis of $\mathbb{F}_4^{3k}$.*

For $n \leq m$, the classical embeddings $B_n \hookrightarrow B_m$ induce morphisms $\varphi_{n,m} : \Gamma_n \to \Gamma_m$. The length function $B_n \twoheadrightarrow \mathbb{Z}$ defined by $s_i \mapsto 1$ induces morphisms $l_n : \Gamma_n \twoheadrightarrow C_3$ such that $l_m \circ \varphi_{n,m} = l_n$. In particular, the finite index subgroup $\Gamma_n^0 = \mathrm{Ker}\, l_n$ of Γ_n is mapped to Γ_{n+1}^0 under $\varphi_{n,n+1}$.

Recall from [Bi, KM] that $\mathrm{Z}(B_n)$ is infinite cyclic, and generated for $n \geq 3$ by

$$z_n = (s_1 s_2 \dots s_{n-1})^n.$$

We gather here a few additional results on these groups. For explicit computations in the finite groups Γ_n for $n \leq 5$, we used the development version of the `CHEVIE` package for

GAP3 : in this package, the finite complex reflection groups G_4, G_{25}, G_{32} are represented as permutation groups on a set of 'complex roots', which makes some computations easy to do. This development version can be found at `http://www.math.jussieu.fr/~jmichel/chevie/index.html`.

Theorem 2.4. (i) *The image of* z_5 *in* Γ_5 *has order 6 and generates* $Z(\Gamma_5)$. *Under the isomorphism* $\Gamma_5 \simeq C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$, C_3 *is generated by* z_5^3, *while* $z_5^2 \in Z(\mathrm{Sp}_4(\mathbb{F}_3))$.

(ii) *Under* $B_5 \twoheadrightarrow \Gamma_5$, z_5^2 *is identified with* $s_3.s_1.s_1^{(s_2s_3)^3}.s_1^{(s_2s_3)^3(s_3s_4)^3}$ *and* z_5^3 *with* $\left((s_1s_2)^3.(s_3s_4)^3\right)^3$.

(iii) *The natural morphisms* $\Gamma_n \to \Gamma_m$ *are injective for* $n \leq 5$.

(iv) *The morphism* $\Gamma_5 \to \Gamma_6$ *admits a retraction, i.e. there exists a morphism* $p : \Gamma_6 \twoheadrightarrow \Gamma_5$ *with* $p \circ \varphi_{5,6} = \mathrm{Id}_{\Gamma_5}$. *In particular,* $\Gamma_6 = \Gamma_5 \ltimes \mathrm{Ker}\, p$. *It is given by* $p(s_5) = z_4^2 z_5^2$.

(v) *For every* n, Γ_n *is a semidirect product* $\Gamma_n^0 \rtimes C_3$, *and* Γ_n^0 *is the commutator subgroup of* Γ_n.

(vi) *For* $n \geq 2$, Γ_{n+1} *is normally generated by* $\varphi_{n,n+1}(\Gamma_n)$ *; For* $n \geq 3$, Γ_{n+1}^0 *is normally generated by* $\varphi_{n,n+1}(\Gamma_n^0)$.

Proof. Parts (i) and (ii) are easily checked by direct computations in $\Gamma_5 = G_{32}$ using CHEVIE (and in addition part (i) consists in well-known properties of the group G_{32}, also denoted $3 \times 2.S_4(3)$ in Atlas notation, see [Atlas] p. 26). For part (iii), the case $m \leq 5$ follows from the identification of $\Gamma_2, \Gamma_3, \Gamma_4$ with parabolic subgroups of G_{32} (see e.g. [BMR]). We thus can assume $n = 5$. Let $K = \mathrm{Ker}\, \varphi_{5,m}$. We have $K \subset \mathrm{Ker}\, l_5$ since $l_m \circ \varphi_{5,m} = l_5$. Since $l_5(z_5) = 5 \times (5-1) \mod 3$ we get $\mathrm{Ker}\, l_5 = \mathrm{Sp}_4(\mathbb{F}_3)$ and $K \triangleleft \mathrm{Sp}_4(\mathbb{F}_3)$. Since $\mathrm{Sp}_4(\mathbb{F}_3)$ is quasisimple we have $K = \{e\}$ or $K = Z(\mathrm{Sp}_4(\mathbb{F}_3)) =< z_5^3 >$ or $K = \mathrm{Sp}_4(\mathbb{F}_3)$. The third case is excluded because Γ_m is nontrivial and generated by conjugates of $\varphi_{2,m}(s_1)$, the case $K = Z(\mathrm{Sp}_4(\mathbb{F}_3))$ would imply the finiteness of $\Gamma_m \simeq \mathrm{Sp}_{m-1}(\mathbb{F}_3)$ by Assion's theorem, contradicting Coxeter's theorem. This proves (iii). Proving (iv) amounts to saying that $z_4^2 z_5^2 \in \Gamma_5$ has order 3, commutes with the s_i for $i \leq 3$, that is with Γ_4, which is clear, and that $s_4(z_4^2 z_5^2)s_4 = (z_4^2 z_5^2)s_4(z_4^2 z_5^2)$, which is easily checked using CHEVIE; this proves (iv). The first statement of part (v) is trivial, as the subgroup $< s_1 >$ generated by s_1 provides a complement to Γ_n^0 in Γ_n ; then, clearly $(\Gamma_n, \Gamma_n) \subset \mathrm{Ker}\, l_n = \Gamma_n^0$, as C_3 is abelian. In order to prove that $\Gamma_n^0 \subset (\Gamma_n, \Gamma_n)$, we consider the abelianization morphism $\pi_{\mathrm{ab}} : \Gamma_n \to \Gamma_n^{\mathrm{ab}}$. From the braid relations we have $\pi(s_i) = \pi(s_{i+1})$ for all i, hence $\pi(g) = \pi(s_1^{l_n(g)})$ for all $g \in \Gamma_n$; this proves $\pi(\Gamma_n^0) = \{1\}$ hence (v). Rewriting the braid relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ as $s_{i+1} = (s_i s_{i+1}) s_i (s_i s_{i+1})^{-1}$ we get that $\varphi_{n,n+1}(\Gamma_n)$ normally generates Γ_{n+1}. Now recall that, when G is a group generated by elements $a_1, \dots, a_r$, H a subgroup of G and $S \subset G$ a set of representatives of G/H with set-theoretic section $G/H \to S$ denoted $x \mapsto \bar{x}$, then H is generated by the $ya_i\overline{ya_i}^{-1}$ for $i \in [1, r]$ and $y \in S$ (see e.g. [MKS]). It follows that Γ_n^0 is generated by the $s_i s_1^{-1}$, $s_1 s_i s_1^{-2} = s_1 s_i s_1$, $s_1^2 s_i = s_1^{-1} s_i$, by taking $S = \{1, s_1, s_1^2\}$ as a set of representatives of $\Gamma_n/\Gamma_n^0 \simeq C_3$. Using $s_{i+1} = (s_i s_{i+1}) s_i (s_i s_{i+1})^{-1}$ we get that, for $i \geq 3$, $s_{i+1} s_1^{-1} = (s_i s_{i+1}) s_i s_1^{-1} (s_i s_{i+1})^{-1}$, $s_1 s_{i+1} s_1 = (s_i s_{i+1}) s_1 s_i s_1 (s_i s_{i+1})^{-1}$ and $s_1^{-1} s_{i+1} = (s_i s_{i+1}) s_1^{-1} s_i (s_i s_{i+1})^{-1}$. Thus, for $n \geq 4$, the generators of Γ_{n+1} involving s_n are conjugates of elements in $\varphi_{n,n+1}(\Gamma_n)$, and this proves (vi) for $n \geq 4$. The case $n = 3$ is easily checked by hand. ∎

Remark 2.5. *Part (iii) of Assion's theorem has been generalized by Wajnryb in* [Wa] *; the question of whether the natural morphisms* $\Gamma_n \to \Gamma_m$ *are injective for* $n \geq 6$ *seems to be open ; part (vi) is clearly false for* $n = 2$, *as* $\Gamma_2^0 = \{1\}$.

2.2. **Additional preliminaries on the groups $\Gamma_n, n \leq 5$.** The group $\Gamma_3 \simeq G_4$ is a semi-direct product $Q \rtimes C_3$ where Q is the quaternion group of order 8, C_3 is the cyclic group of order 3, and the semi-direct product is associated to any automorphism of Q of order 3. Writing classically $Q = <\mathbf{i}, \mathbf{j}>$ with $\mathbf{i}^2 = \mathbf{j}^2 = z$ the central element of order 2, $\mathbf{k} = \mathbf{ij}$ and $C = <s>$ with s acting on Q by the permutation $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, an isomorphism is obtained by $s_1 \mapsto s$ and $s_2 \mapsto \mathbf{i}^3 s$ (so that $s_1 s_2^2 \mapsto \mathbf{i}$).

Using the above morphisms we identify Γ_3 and therefore Q to subgroups of Γ_5. In the sequel we will need to use the Atlas character tables on elements of Q. For this we need to identify a few conjugacy classes in $\Gamma_5 = C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$. In Atlas notations, $\mathrm{Sp}_4(F_3) = 2.U_4(2)$ contains 2 classes of order 2. One of the two being central (hence corresponding to z_5^3), the value of the other one on any Brauer character in characteristic not 2 lies in the column labelled 2a of [AtMod]. Among the three classes of order 4 in $\mathrm{Sp}_4(\mathbb{F}_3)$, two are deduced one from the other by multiplication by z_5^3. It is easily checked that, if $x \in \Gamma_3 \subset \Gamma_5$ has order 4, then it is not conjugated to xz_5^3. It follows that the column of the ordinary or Brauer character table corresponding to x is the one labelled 4a in [AtMod]. We can thus read on the tables the values taken by elements of Q on ordinary and Brauer characters in characteristic prime to 2.

The group $\Gamma_5 \simeq C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$ and therefore $\mathrm{Sp}_4(\mathbb{F}_3)$ contains another useful quaternion subgroup Q_0, characterized up to Γ_5-conjugacy by $\mathrm{Z}(Q_0) =< z_5^3 >$. For later computations, an explicit description of this subgroup in terms of the generators will turn out useful. A 2-Sylow subgroup of Γ_5 is generated by the elements $a_1 = s_2^{-1} s_3 s_1 s_2^{-1} s_3 s_1 s_2^{-1} s_1^{-1}, a_2 = s_3^{-1} s_2 s_3^{-1} s_1 s_2 s_3 s_1, a_3 = s_4^{-1} s_3 s_4^{-1} s_3, a_4 = s_4 s_3^{-1} s_4 s_2 s_3 s_1 s_2^{-1} s_1 s_3 s_1$. Two generators of such a Q_0 are then given by $\mathbf{i}_0 = a_4^{-1} a_2 a_3 a_2$, $\mathbf{j}_0 = a_4^2 a_1$.

2.3. **The groups Y_m.** For $1 \leq r \leq m-1$ we let e_r denote the r-th vector of the canonical basis of $\mathbb{F}_4^{m-1}$, and we let $\pi : Y_m \to \mathbb{F}_4^{m-1}$ denote the canonical projection $(x, V) \mapsto V$. We choose $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$, and let $i_r = (e_r, \alpha)$, $j_r = (\alpha e_r, \alpha)$. Then i_r, j_r have order 4 and generate a quaternion subgroup Q_r of Y_m. It is easily checked that Y_m is a central product of the Q_r, namely the quotient of $Q_1 \times \cdots \times Q_{m-1}$ by the identification of the centers of $Q_1, \ldots, Q_{m-1}$. If z denotes the generator of $\mathrm{Z}(Y_m)$, the elements of Y_m can be uniquely written in the form $i_{r_1} i_{r_2} \ldots i_{r_u} j_{s_1} \ldots j_{s_v} z^{\epsilon}$ with $\epsilon \in \{0,1\}$ and $r_1, \ldots r_u$, $s_1, \ldots, s_v$ distinct indices.

In particular, the group Y_m is an extra-special group of type $2^{1+2(m-1)}$ (see [Go] § 5.5). In characteristic distinct from 2, such a group admits $m-1$ linear characters and a 2^{m-1}-dimensional irreducible representation, afforded by the tensor product of the 2-dimensional irreducible representations of the Q_r.

We need to recall some basic facts on the representations of the quaternion group. When k is a field of characteristic $p \neq 2$, the 1-dimensional representations are clearly defined over k. When k contains a primitive fourth root of unity ω, then the 2-dimensional representation can be defined over k, through $\mathbf{i} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{j} \mapsto \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$. It is also defined over $k = \mathbb{F}_3$, through $\mathbf{i} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{j} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It follows that, under these conditions on k, the 2^{m-1}-dimensional representation of Y_m can be explicitly defined over k.

3. Reminder on projective representations

Let G be a group and k a field. An action of G as algebra automorphisms of $\mathrm{Mat}_n(k)$ yields a projective representation $\bar{\rho} : G \to \mathrm{PGL}_n(k)$ by the Skolem-Noether theorem, hence a 2-cocycle $c : G \times G \to k^\times$ defined by $c(g_1, g_2) = \tilde{\rho}(g_1 g_2)\tilde{\rho}(g_2)^{-1}\tilde{\rho}(g_1)^{-1}$ where $\tilde{\rho} : G \to \mathrm{GL}_n(k)$ is a set-theoretic lifting of $\bar{\rho}$. It is always possible to choose $\tilde{\rho}(e) = \mathrm{Id}_n$, which we always assume from now on. Then the cocycle satisfies $c(e, g) = 1$ for all $g \in G$; we say that such a cocycle is *normalized*. The corresponding class $[c] \in H^2(G, k^\times)$ is trivial if and only if we can lift $\bar{\rho}$ to a linear representation $\rho : G \to \mathrm{GL}_n(k)$. In that case, if $c = \mathrm{d}\alpha$ for some $\alpha : G \to k^\times$, i.e. $c(g_1, g_2) = \alpha(g_1 g_2)\alpha(g_2)^{-1}\alpha(g_1)^{-1}$, then $\rho(g) = \alpha(g)^{-1}\tilde{\rho}(g)$ provides such a lifting. Under our assumption, such an α satisfies $\alpha(e) = 1$.

We recall the short exact sequences in low-dimensional group cohomology, provided by

(i) the universal coefficients exact sequence :

$$0 \to \mathrm{Ext}(H_1 G, k^\times) \to H^2(G, k^\times) \to \mathrm{Hom}(H_2 G, k^\times) \to 0.$$

(ii) the Künneth exact sequence :

$$0 \to \mathrm{Tor}(H_0 G, H_1 K) \oplus \mathrm{Tor}(H_1 G, H_0 K) \to H_2(G \times K) \to H_2 K \oplus (H_1 G \otimes H_1 K) \oplus H_2 G \to 0$$

We recall that, when G is finite, then $H_2 G$ is the so-called Schur multiplier of G.

Lemma 3.1. (i) *$H^2(\Gamma_3, k^\times) \simeq \mathrm{Ext}(C_3, k^\times)$ hence $H^2(\Gamma_3, k^\times) = 0$ when char.$k = 3$.*

(ii) *We have a short exact sequence $0 \to \mathrm{Ext}(C_3, k^\times) \to H^2(\mathrm{GU}(4,2), k^\times) \to \mathrm{Hom}(C_2, k^\times) \to 0$. If k is a finite field of characteristic 3, then $H^2(\mathrm{GU}(4,2), k) = C_2$.*

(iii) *Let $C_0 \simeq C_2 \times C_2$ denote the image of $Q_0 \subset \mathrm{Sp}_4(\mathbb{F}_3)$ inside $\mathrm{SU}(4,2) \simeq \mathrm{PSU}(4,2) \simeq \mathrm{PSp}_4(\mathbb{F}_3)$, and assume k is a finite field of characteristic 3. Then the restriction morphism $H^2(\mathrm{GU}(4,2), k^\times) \to H^2(C_0, k^\times)$ is injective.*

Proof. It is known that $H_2\Gamma_3 = 0$ (see [K] table 8.3), whence (i). We have $\mathrm{GU}(4,2) = C_3 \times \mathrm{SU}(4,2)$, and it is known that $H_2\,\mathrm{SU}(4,2) \simeq C_2$ (see [K] table 8.5) hence $H_2\,\mathrm{GU}(4,2) = C_2$ by Künneth, since $\mathrm{SU}(4,2)$ is perfect and $H_2 C_3 = 0$. Then the short exact sequence is the universal coefficients exact sequence. When k has characteristic 3, $\mathrm{Hom}(C_2, k^\times) \simeq C_2$ since $-1 \neq 1$ in k, and $k^\times$ is 3-divisible hence $\mathrm{Ext}(C_3, k^\times) = 0$, which proves (ii). The group $\Gamma_5 = C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$ provides a nonsplit central extension of $\mathrm{GU}(4,2)$, hence the nontrivial element of $H^2(\mathrm{GU}(4,2), C_2) \simeq H^2(\mathrm{GU}(4,2), k^\times)$. For $g_1, g_2 \in \mathrm{GU}(4,2)$ and arbitrary preimages $\tilde{g}_1, \tilde{g}_2, \widetilde{g_1 g_2}$ in $\Gamma_5 = C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$, it can be defined by $c(g_1, g_2) = 1$ if $\widetilde{g_1 g_2} = \tilde{g}_1\tilde{g}_2$ and $c(g_1, g_2) = -1$ otherwise. Restricting it to C_0 yields the cocycle associated to the extension $1 \to \mathrm{Z}(Q_0) \to Q_0 \to C_0 \to 1$ which is not split, hence (iii). ∎

Lemma 3.2. (i) *Let x, y be generators of C_2^2 and let $c : (C_2^2)^2 \to \mathbb{F}_3^\times$ be a normalized 2-cocycle. Its class $[c]$ is trivial in $H^2(C_2^2, \mathbb{F}_3^\times)$ if and only if $c(x, y) = c(y, x)$ and $c(x, x) = c(y, y) = 1$.*

(ii) *Let g be a generator of C_3 and let $c : C_3^2 \to \mathbb{F}_4^\times$ be a normalized 2-cocycle. Its class $[c]$ is trivial in $H^2(C_3, \mathbb{F}_4^\times)$ if and only if $c(g, g)c(g, g^{-1}) = 1$.*

Proof. The group $H^2(C_2^2, \mathbb{F}_3^\times) \simeq H^2(C_2^2, C_2)$ is an extension of $\mathrm{Hom}(H_2 C_2, \mathbb{F}_3^\times) = \mathrm{Hom}(C_2, \mathbb{F}_3^\times)$ by $\mathrm{Ext}((C_2)^2, \mathbb{F}_3^\times) = \mathrm{Ext}((C_2)^2, C_2) \simeq (C_2)^2$. We check that the normalized cocycles associated to the nonsplit extensions of C_2 by $C_2 \times C_2$ satisfy $c(x, y) = -c(y, x)$ when the extension is not abelian, and $|\{c(x, x), c(y, y)\}| = 2$ when it is. Conversely, all coboundaries satisfy $c(x, y) = c(y, x)$ and $c(x, x) = c(y, y)$, which proves (i). The proof of (ii) is similar and left to the reader. ∎

We will use the following in several instances.

Proposition 3.3. *Let G be a finite group, k a commutative ring and A a k-algebra. Let $f\colon G \to A^\times$ be a group morphism. This induces an action of G on A by conjugacy. Then the associated semi-direct product $A \rtimes G$ (defined by multiplication $ag.a'g' = af(g)a'f(g)^{-1}gg'$) is isomorphic with the (commutative) tensor product $A \otimes kG$.*

Proof. The map is $a\otimes g \mapsto af(g^{-1}).g$ since $(af(g^{-1}).g).bf(h^{-1}).h = af(g^{-1})f(g)bf(g^{-1})f(h^{-1}).gh = abf((gh)^{-1}).gh$ which is the image of $ab \otimes gh$. A reverse map is clearly afforded by $a.g \mapsto af(g) \otimes g$. ∎

The following essentially consists in making explicit a Morita equivalence summing up Mackey-Wigner's method of "little groups" (see [S] § 8.2 and [CE] ex. 18.6).

Proposition 3.4. *Let G a finite group (left) acting transitively on a set X. Let k be a commutative ring, and let A be the k-algebra $G \ltimes k^X$ where $k^X = \oplus_{x\in X} k\epsilon_x$ is endowed with the product law ($\epsilon_x\epsilon_{x'} = \delta_{x,x'}\epsilon_x$) and the action of G is induced by the one on X. Then any choice of $x_0 \in X$ with stabilizer $G_0 \subseteq G$ and any choice of a "section" $s\colon X \to G$ such that $s(x).x_0 = x$ for all $x \in X$, define a unique isomorphism*

$$A \longrightarrow \mathrm{Mat}_X(kG_0)$$

sending each $\epsilon_x \in k^X$ ($x \in X$) to $\theta(\epsilon_x) := E_{x,x}$, and each $g \in G$ to

$$\theta(g) := \sum_{x\in X} s(gx)^{-1}g.s(x)E_{gx,x}$$

(where $E_{x,y} \in \mathrm{Mat}_X(k)$ is the elementary matrix corresponding to $x, y \in X$).

Proof. Note that indeed $s(gx)^{-1}g.s(x) \in G_0$ since $s(gx).x_0 = gx = g.s(x).x_0$.

We assume $k = \mathbb{Z}$. The general case is deduced by tensor product $- \otimes_{\mathbb{Z}} k$.

Note that we are below actually checking explicitly that, denoting $i = \epsilon_{x_0}$, one has $A \simeq \mathrm{End}_{iAi}(Ai)^{\mathrm{opp}}$ where Ai is a A-bimodule-iAi isomorphic with $(iAi)^X$ as right iAi-module, with moreover $iAi = kG_0$ and $AiA = A$.

To check that the proposed formulae define a morphism between our algebras and in view of the law on A, it suffices to check that $\theta(g)\theta(g') = \theta(gg')$, $\theta(g)\theta(\epsilon_x) = \theta(\epsilon_{gx})\theta(g)$ and $\theta(\epsilon_x)\theta(\epsilon_{x'}) = \delta_{x,x'}\theta(\epsilon_x)$ for each $g, g' \in G$ and $x, x' \in X$.

We have $\theta(g)\theta(g') = \sum_{x,x'\in X} s(gx)^{-1}g.s(x).s(g'x')^{-1}g'.s(x')E_{gx,x}E_{g'x',x'}$. The product $E_{gx,x}E_{g'x',x'}$ is $E_{gx,x'}$ whenever $x = g'x'$, and is zero otherwise. When $x = g'x'$, we also have $s(x).s(g'x')^{-1} = 1$, so that $\theta(g)\theta(g') = \sum_{x'\in X} s(gg'x')^{-1}gg'.s(x')E_{gg'x',x'} = \theta(gg')$.

Samely, $\theta(g)\theta(\epsilon_x) = \sum_{x'\in X} s(gx')^{-1}g.s(x')E_{gx',x'}E_{x,x} = s(gx)^{-1}g.s(x)E_{gx,x}$, while $\theta(\epsilon_{gx})\theta(g) = \sum_{x'\in X} s(gx')^{-1}g.s(x')E_{gx,gx}E_{gx',x'} = s(gx)^{-1}g.s(x)E_{gx,x}$ since $gx = gx'$ if and only if $x = x'$.

The morphism is now clearly surjective by the equation above (with $x = x_0$) since any elementary matrix is then reached up to an element of G_0, and the elements of $G_0\epsilon_{x_0} \subseteq A$ surject on $\mathbb{Z}G_0.E_{x_0,x_0}$.

Isomorphism follows by noting that we have a surjective morphism between free $\mathbb{Z}$-modules of equal rank. Since it has to be split, it is an isomorphism. ∎

4. Algebras

We define and study a quotient of the group algebra of the groups Γ_n.

Definition 4.1. *We define* $\mathbf{q}$ *to be the sum of elements in* Q*, and* $\mathbf{c} = \mathbf{q}s_1$ *(or equivalently* $s_1 s_2 \mathbf{c} = \mathbf{q}$*), that is*

$$\begin{array}{rcl} \mathbf{q} & = & 1 + s_1 s_2^2 + s_2 s_1^2 + s_1^2 s_2 + s_2^2 s_1 + s_1 s_2 s_1 + s_1^2 s_2^2 s_1^2 + s_1 s_2^2 s_1 s_2^2 \in \mathbb{Z}\Gamma_3 \\ \mathbf{c} & = & s_2 s_1^2 s_2 + s_1 s_2^2 s_1 + s_1^2 s_2 s_1 + s_1 s_2 s_1^2 + s_1^2 s_2^2 + s_2^2 s_1^2 + s_1 + s_2 \end{array}$$

and $I_n = \mathbb{Z}\Gamma_n.\mathbf{q}.\mathbb{Z}\Gamma_n = (\mathbf{q}) = (\mathbf{c})$ *be the two-sided ideal it generates in* $\mathbb{Z}\Gamma_n$ *for any* $n \geq 3$*. Let* $K_n = \mathbb{Z}\Gamma_n / I_n$*.*

Note that K_n is the algebra $K_n(1)$ of the introduction.

If R denotes a (unital) commutative ring, we let RK_n denote the quotient of $R\Gamma_n$ by $RI_n = R\Gamma_n.q.R\Gamma_n \subset R\Gamma_n$. We have $RK_n \simeq K_n \otimes_{\mathbb{Z}} R$.

4.1. **First results.** As proved by L. Funar, for every n this algebra is a finitely generated $\mathbb{Z}$-module. For the convenience of the reader, we provide another (shorter) proof of the following result of [F1].

Proposition 4.2. *(Funar) Let* $\overline{A}_n$ *denote the image of the natural morphism* $K_n \to K_{n+1}$*. One has* $K_{n+1} = \overline{A}_n + \overline{A}_n s_n \overline{A}_n + \overline{A}_n s_n^2 \overline{A}_n$*.*

Proof. The case $n = 2$ is trivial, so we can proceed by induction. Let $C_n = \overline{A}_n + \overline{A}_n s_n \overline{A}_n + \overline{A}_n s_n^2 \overline{A}_n$. We have $1 \in C_n$ so we only need to prove that C_n is a left ideal. Since $\overline{A}_n C_n \subset C_n$ this amounts to saying $s_n C_n \subset C_n$, that is $s_n \overline{A}_n s_n^{\varepsilon} \overline{A}_n \subset C_n$ for $\varepsilon \in \{0, 1, 2\}$. By the induction assumption $\overline{A}_n s_n^{\varepsilon} \overline{A}_n = \overline{A}_{n-1} s_n^{\varepsilon} \overline{A}_n + \overline{A}_{n-1} s_{n-1} \overline{A}_{n-1} s_n^{\varepsilon} \overline{A}_n + \overline{A}_{n-1} s_{n-1}^{\varepsilon} \overline{A}_{n-1} s_n^{\varepsilon} \overline{A}_n$. Since s_n commutes with $\overline{A}_{n-1}$ we get $\overline{A}_n s_n^{\varepsilon} \overline{A}_n = s_n^{\varepsilon} \overline{A}_n + \overline{A}_{n-1} s_{n-1} s_n^{\varepsilon} \overline{A}_n + \overline{A}_{n-1} s_{n-1}^2 s_n^{\varepsilon} \overline{A}_n$. Now $s_n s_n^{\varepsilon} \overline{A}_n = s_n^{\varepsilon+1} \overline{A}_n \subset C_n$, $s_n \overline{A}_{n-1} s_{n-1} s_n^{\varepsilon} \overline{A}_n = \overline{A}_{n-1} s_n s_{n-1} s_n^{\varepsilon} \overline{A}_n$ and $s_n \overline{A}_{n-1} s_{n-1}^2 s_n^{\varepsilon} \overline{A}_n = \overline{A}_{n-1} s_n s_{n-1}^2 s_n^{\varepsilon} \overline{A}_n$. It is thus sufficient to show that $s_n s_{n-1} s_n^{\varepsilon} \in C_n$ and $s_n s_{n-1}^2 s_n^{\varepsilon} \in C_n$ for $\varepsilon \in \{0, 1, 2\}$. The case $\varepsilon = 0$ is obvious, $s_n s_{n-1} s_n = s_{n-1} s_n s_{n-1} \in C_n$, $s_n s_{n-1} s_n^2 = s_{n-1}^2 s_n s_{n-1} \in C_n$, $s_n s_{n-1}^2 s_n^2 = s_{n-1}^2 s_n^2 s_{n-1} \in C_n$, and there only remains to show that $s_n s_{n-1}^2 s_n \in C_n$. But $c \equiv 0$ in $\overline{A}_{n+1}$ implies, under conjugation by Γ_{n+1}, that $s_n s_{n-1}^2 s_n + s_{n-1} s_n^2 s_{n-1} + s_{n-1}^2 s_n s_{n-1} + s_{n-1} s_n s_{n-1}^2 + s_{n-1}^2 s_n^2 + s_n^2 s_{n-1}^2 + s_{n-1} + s_n = 0$ in $\overline{A}_{n+1}$, hence $s_n s_{n-1}^2 s_n \in C_n$, and this concludes the proof. ∎

Corollary 4.3. *For all* n*,* K_n *is a finitely generated* $\mathbb{Z}$*-module.*

The following lemma will be useful.

Lemma 4.4. *Let* p *be a prime,* H *a finite group,* S *is a simple* $\overline{\mathbb{F}_p}H$*-module, and* ϕ *is its Brauer character. Let* Q *is a* p'*-subgroup, that is a subgroup whose order is not divisible by* p*, then* $q := \sum_{t \in Q} t$ *annihilates* S *if and only if* $\phi(q) = 0$*. The same holds in characteristic* 0 *for arbitrary* H *and* Q *and* ϕ *the ordinary character of a simple* $\overline{\mathbb{Q}}H$*-module.*

Proof. We may replace H by Q itself and assume ϕ is the Brauer character of an arbitrary finite dimensional $\overline{\mathbb{F}}_p Q$-module S. Since Q is a p'-group, this module lifts to an $\mathcal{O}Q$-module $\widehat{S}$ where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$. Then ϕ is the ordinary character of $\widehat{S}$. Since q is an idempotent up to an invertible scalar of $\mathcal{O}$, we have $\phi(q) = 0$ if and only if $q\widehat{S} = 0$, and this is equivalent to $qS = 0$. The characteristic zero case is included in the above reasoning. ∎

The structure of K_3 and K_4 as $\mathbb{Z}$-modules can be obtained by computer means, as Γ_3 and Γ_4 are small enough : K_n is the quotient of $\mathbb{Z}\Gamma_n \simeq \mathbb{Z}^{|\Gamma_n|}$ by the submodule spanned by the elements $g_1\mathbf{q}g_2$ for $g_1, g_2 \in \Gamma_n$. Using the algorithms implemented in `GAP4` for computing the Smith normal form, we get the following.

Theorem 4.5. *As $\mathbb{Z}$-modules, $K_3 \simeq \mathbb{Z}^{21}$ and*

$$K_4 \simeq \mathbb{Z}^{183} \oplus (\mathbb{Z}/2\mathbb{Z})^{54} \oplus (\mathbb{Z}/3\mathbb{Z})^{48} \oplus (\mathbb{Z}/9\mathbb{Z})^{18}$$

The size of Γ_5 is too large for the same kind of computations to settle the case of K_5. However, we manage to get the following

Proposition 4.6. *The algebra $\mathbb{F}_2 K_5$ has dimension $3 \times 863 = 2589$.*

Proof. For computing this dimension we cannot rely on usual high-level mathematical software, and needed instead to write our own code. The computation is done as follows. Since $\mathbf{q} \in \mathbb{F}_2\Gamma_5^0$, we can content ourselves with computing the subspace spanned by the $g_1\mathbf{q}g_2$ for $g_1, g_2 \in \Gamma_5^0 = \mathrm{Sp}_4(\mathbb{F}_3)$. We can assume $g_1 \in \Gamma_5^0/N_{\Gamma_5^0}(Q_8)$ and $g_2 \in Q_8\backslash\Gamma_5^0$. Taking representatives in Γ_5^0 of these cosets, this leaves 90 possibilities for g_1 and 6480 for g_2. Encoding each entry on one bit, each vector in $\mathbb{F}_2\Gamma_5^0$ occupies 6480 bytes, and a basis of $\mathbb{F}_2\Gamma_5^0$ occupies about 330 MBytes. The encoding of elements of Γ_5^0 as matrices in $\mathrm{Sp}_4(\mathbb{F}_3)$ is more economic than encoding them as permutations, and the time-consuming procedures such as finding $90\times 6480\times 8$ times the position of an element in the list of the 51840 elements of Γ_5^0 can be optimized by using a numerical key and ordering these elements. Each time a new sequence of 8 elements is computed and converted into a new line vector, a Gauss elimination is performed (using `xor` operations on 4-bytes words) with respect to the precedingly obtained free family. We wrote a `C` program based on these ideas and computed the dimension of this submodule (this lasts a few hours on todays PCs). One gets 50977, hence $\dim \mathbb{F}_2K_5 = 3\times(51840-50977) = 3\times 863$. ∎

Theorem 4.7. *If k is an algebraically closed field with $k = 2k = 3k$ (i.e. its characteristic is $\neq 2, 3$), then $kK_3 \simeq \mathrm{Mat}_2(k) \times \mathrm{Mat}_2(k) \times \mathrm{Mat}_2(k) \times \mathrm{Mat}_3(k)$, $kK_4 \simeq \mathrm{Mat}_2(k)^3 \times \mathrm{Mat}_3(k) \times \mathrm{Mat}_9(k)^2$*

Proof. For the case $n = 3$, we first just assume $2k = k$ and k contains a primitive fourth root of unity ω. Then one has $kQ = k \times k \times k \times k \times \mathrm{Mat}_2(k)$ by the only k-algebra map such that

$$\mathbf{i} \mapsto (1, -1, 1, -1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) , \quad \mathbf{j} \mapsto (1, 1, -1, -1, \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}).$$

In kQ, $e_Q = \mathbf{q}/8$ is a central idempotent acting by 1 on the first coordinate above and by 0 on the others. So $kQ/kQe_Q \simeq k^3 \times \mathrm{Mat}_2(k)$ by the same map as above deleting the first coordinate, and $k\Gamma_3/k\Gamma_3e_Q$ is a semi-direct product $[k^3 \times \mathrm{Mat}_2(k)] \rtimes C_3$ where the generator of C_3 permutes cyclically the first three coordinates and acts on the summand $\mathrm{Mat}_2(k)$ according to $\mathbf{i} \mapsto \mathbf{j} \mapsto \mathbf{ij} \mapsto \mathbf{i}$, that is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\omega \\ -\omega & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This last action is by conjugacy by $\begin{pmatrix} -1 & \omega \\ 1 & \omega \end{pmatrix}$, so Proposition 3.3 implies that the corresponding semi-direct product is isomorphic with $\mathrm{Mat}_2(k) \otimes kC_3 = \mathrm{Mat}_2(kC_3)$.

Note that when moreover $3k = k$ and k contains a third root of unity, then $kC_3 \simeq k^3$ and $\mathrm{Mat}_2(kC_3) \simeq \mathrm{Mat}_2(k)^3$.

The other semi-direct product $k^3 \rtimes C_3$ is isomorphic with $\mathrm{Mat}_3(k)$ by identifying k^3 with diagonal matrices and sending the generator of C_3 to the permutation matrix of the appropriate cycle of order 3.

This gives the claim about kK_3.

We notice that the primes dividing the orders of Γ_3 and Γ_4 are $2, 3$. It follows that $k\Gamma_4$ is semisimple and that kK_4 is a direct sum of $B_\chi = \mathrm{Mat}_{\chi(1)}(k)$ among all irreducible Brauer characters χ corresponding to modules S with $\mathbf{q}S \neq 0$, that is $\chi(\mathbf{q}) \neq 0$ by Lemma 4.4. Equivalently, $\chi(\mathbf{q}) \neq 0$ means that the restriction of S to Γ_3 does not contain any 1-dimensional component. The ordinary character and induction tables of $\Gamma_3 = G_4$ and $\Gamma_4 = G_{25}$ are easily accessible using `CHEVIE`, so this readily provides the set of such characters and the conclusion. ∎

4.2. **Characteristic distinct from 2 and 3.**

Theorem 4.8. *If k is a field with $k = 2k = 3k$ (i.e. its characteristic is $\neq 2, 3$), then $kK_n = 0$ for $n \geq 5$.*

Proof. In order to prove $kK_n = 0$ for $n \geq 5$, it is sufficient to show that $kK_5 = 0$, as kK_n is generated by conjugates of the image of the natural morphism $K_5 \to K_n$. Since kK_5 is a quotient of $k\Gamma_5$ it is finite dimensional, so we can assume $k = \overline{k}$, as $\overline{k}K_5 = kK_5 \otimes_k \overline{k}$. The ordinary character table and elements of the complex reflection group $G_{32} = \Gamma_5$ are easy to deal with using `CHEVIE`. We get that no irreducible character of Γ_5 vanishes on $\mathbf{q}$, hence proving that kK_5 has no simple module by Lemma 4.4, hence $kK_5 = 0$, provided that the characteristic of k is not $2, 3$ or 5. For $p = 5$ we use that $\Gamma_5 = C_3 \times \mathrm{Sp}_4(\mathbb{F}_3)$, with $Q \subset \mathrm{Sp}_4(\mathbb{F}_3) \subset \Gamma_5$, hence $kK_5 = kC_3 \otimes (k\,\mathrm{Sp}_4(\mathbb{F}_3)/(\mathbf{q}))$. We check that no 5-modular Brauer character of $\mathrm{Sp}_4(\mathbb{F}_3)$ vanishes on $\mathbf{q}$ by using the table of Brauer characters provided by [AtMod], and the conclusion follows again from Lemma 4.4. ∎

4.3. **Characteristic 3.** This section is devoted to the proof of the following.

Theorem 4.9. *If k is a field of characteristic 3, then*

(i) $kK_3 \simeq \mathrm{Mat}_3(k) \times \mathrm{Mat}_2(kC_3)$.
(ii) $kK_5 \simeq kK_6 \simeq \mathrm{Mat}_{25}(kC_3)$,
(iii) $kK_n = 0$ *for* $n \geq 7$.

4.3.1. The case $n = 3$ has been treated at the start of the proof of Theorem 7 provided that k contains a 4-th root of 1. In the case $char.k = 3$ we remove that assumption. The irreducible representations of Q are defined over k. This is clear for the 1-dimensional ones, and the 2-dimensional one is given by $\mathbf{i} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{j} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The rest of the argument remains valid, provided that the cocycle given by the projective representation $C_3 \to \mathrm{Aut}(\mathrm{Mat}_2(k)) = \mathrm{PGL}_2(k)$ is zero in $H^2(C_3, k^\times)$. Since $H^2(C_3, k^\times) \simeq \mathrm{Ext}(C_3, k^\times) = 0$ when $char.k = 3$ this concludes the proof.

4.3.2. Case $n = 5$. Let us look at $\Gamma_5^0 = \mathrm{Sp}_4(3)$ whose group algebra contains q since Γ_5^0 contains all 2-elements of Γ_5. We have $K_5 = kC_3 \otimes A'_5$ where $A'_5 = k\Gamma_5^0/I'_5$ and I'_5 is the two-sided ideal of $k\Gamma_5^0$ generated by $\mathbf{q}$. In order to show that $kK_5 \simeq \mathrm{Mat}_{25}(kC_3)$, it suffices to check that $k\Gamma_5^0/I'_5 \simeq \mathrm{Mat}_{25}(k)$. We first assume that k is algebraically closed.

A first step is to check that all simple $k\Gamma_5^0$-modules except one are annihilated by I'_5. Using the table of Brauer characters of $\mathrm{Sp}_4(3) = 2.S_4(3)$, it is easy to check that only the simple $k\Gamma_5^0$-module M of dimension 25 is such that its Brauer character (with values in k) τ_M satisfies $\tau_M(\mathbf{q}) = 0$. So we have $\mathbf{q}M = 0$, by Lemma 4.4, $A'_5 \neq 0$, and the only simple $k\Gamma_5^0$-module which gives rise to a A'_5-module is this module M of dimension 25. Moreover, this module has no self-extension as $k\Gamma_5^0$-module by [B1] 12.2 (vi). So this unique simple A'_5-module has no selfextension, so is projective, hence $A'_5 \simeq \mathrm{Mat}_{25}(k)$ as claimed.

In case $k \neq \overline{k}$, we get from the above that $\overline{k}A'_5 \simeq \mathrm{Mat}_{25}(\overline{k})$. We prove that the 25-dimensional irreducible representation of $\mathrm{Sp}_4(\mathbb{F}_3)$ is defined over $\mathbb{F}_3$, which provides a non-trivial surjective morphism $kA'_5 \to \mathrm{Mat}_{25}(k)$, hence an isomorphism (e.g. by equality of dimensions). The proof goes as follows. We let $k = \mathbb{F}_3$. The 4-dimensional reflection representation of G_{32} is defined over $\mathbb{Z}[j]$, where $j = \exp(2i\pi/3)$, hence defines, after tensorisation by a suitable linear character, a 4-dimensional irreducible representation ρ_0 of $\mathrm{Sp}_4(\mathbb{F}_3)$ over $\mathbb{Z}[j]$. We let $\bar{\rho}_0 : \mathrm{Sp}_4(\mathbb{F}_3) \to \mathrm{GL}_4(\mathbb{F}_3)$ denote its reduction modulo the ideal $(3, j+1)$ (which is isomorphic to the standard representation of $\mathrm{Sp}_4(\mathbb{F}_3)$ over $\mathbb{F}_3$). We use the character table and the decomposition matrix of $\mathrm{Sp}_4(\mathbb{F}_3)$, as provided by [B1] (or by the package `CTblLib` of `GAP4`) to show the following :

- $S^2\rho_0$, $\Lambda^2\rho_0$, $\Lambda^2(S^2\rho_0)$ are absolutely irreducible, as well as $S^2\bar{\rho}_0$.
- The composition factors over $\overline{\mathbb{F}_3}$ of the 45-dimensional representation $\Lambda^2(S^2\bar{\rho}_0)$ are $S^2\bar{\rho}_0$ (twice) and the 25-dimensional irreducible (once).

Since $S^2(\Lambda^2\bar{\rho}_0)$ and $\Lambda^2\bar{\rho}$ are defined over $\mathbb{F}_3$, the same thus holds for our 25-dimensional representation.

Let us extract from the above the following proposition for future reference.

Proposition 4.10. *Let k be a field of characteristic 3. Under the isomorphism $\Gamma_5 \simeq C_3 \times \mathrm{Sp}_4(3)$, one has $\mathbf{q} \in k\,\mathrm{Sp}_4(3)$ and the only simple $k\,\mathrm{Sp}_4(3)$ annihilated by $\mathbf{q}$ is the only simple $k\,\mathrm{Sp}_4(3)/Z(\mathrm{Sp}_4(3)) = k\,\mathrm{SU}_4(2)$-module of dimension 25.*

4.3.3. Case $n = 6$. From the above, note that $z_5^3 - 1 \in I'_5$, since the isomorphism $kA'_5/I'_5 \to \mathrm{Mat}_{25}(k)$ is given by the 25-dimensional simple representation of $\Gamma'_5 = 2.S_4(3)$, which factorizes through $S_4(3)$ (see [AtMod]) hence has the center $< z_5^3 >$ of Γ'_5 in its kernel.

Therefore, by Theorem 2.2 (and Theorem 2.4 (ii)),

kK_6 is a quotient of the group algebra of $U(5) = Y_4 \rtimes \mathrm{GU}_4(2)$, the $\mathrm{GU}_4(2)$ term corresponding to $\Gamma_5/\mathrm{Z}(\Gamma_5^0)$. Note that $\mathbf{q}$ is a sum of elements of that group. Let us show that the simple kK_6-modules are all annihilated by $\mathbf{q}$ except the one which corresponds with the 25-dimensional Brauer character of $\mathrm{SU}_4(2)$. By Proposition 4.10, we are looking for the simple $kU(5)$-modules whose restriction to $\mathrm{SU}_4(2)$ annihilates $\mathbf{q}$, hence has all its composition factors isomorphic to the 25-dimensional representation singled out above.

From the description of $U(5)$ recalled in Theorem 2.2 (ii), we have $kU(5) = kY_4.\mathrm{GU}_4(2)$ where Y_4 is clearly an extra-special group of type 2^{1+8} (notation of [Atlas]). We have $\mathrm{Irr}(Y_4) = \mathrm{Irr}(kY_4) = \mathrm{Irr}(Y_4^{\mathrm{ab}}) \cup \{\chi_0\}$ where χ_0 is the irreducible character of degree 16 (see [Go] §5.5 on the characters of the extra-special groups).

If $\lambda \in \mathrm{Irr}(Y_4^{\mathrm{ab}})$, let e_λ be the sum of idempotents of kY_4 associated with elements of the orbit $U.\lambda \subset \mathrm{Irr}(Y_4^{\mathrm{ab}})$. Let us abbreviate $U = \mathrm{GU}_4(2)$ and let U_λ denote the stabilizer of λ in U by conjugacy.

By Proposition 3.4, $kY_4U.e_\lambda \simeq \mathrm{Mat}_{(U:U_\lambda)}(kU_\lambda)$, so the simple $kY_4U.e_\lambda$-modules are of dimensions $(U : U_\lambda)$ times the dimension of some simple kU_λ-module.

If $\lambda = 1$, then $U_\lambda = U$, so we find a block isomorphic to kU and the quotient by the ideal generated by $\mathbf{q}$ is $\mathrm{Mat}_{25}(k)$.

To study other stabilizers, note that $Y_4^{\mathrm{ab}} \simeq \mathrm{Irr}(Y_4^{\mathrm{ab}})$ by the hermitian form. This is U-equivariant, so we may consider those subgroups U_λ as stabilizers of non trivial elements V in the natural representation space $\mathbb{F}_4^4$.

If ${}^t\overline{V}V \neq 0$, then $\mathbb{F}_4^4 = \mathbb{F}_4.V \oplus V^\perp$ and U_λ then identifies with the unitary group on $V^\perp$, isomorphic with $\mathrm{GU}_3(2)$. By computing its Brauer character table (e.g. using GAP4), we get that its simple modules over k have dimensions 1,2,3, so we get dimensions $1,2,3 \times (\mathrm{GU}_4(2) : \mathrm{GU}_3(2))$ which is never a multiple of 25.

If ${}^t\overline{V}V = 0$, then V can be taken as the sum of last two vectors of an orthonormal basis, so that the computation of its stabilizer is similar to the one of Remark 2.3. Then U_λ identifies with a semi-direct product $Y_2 \rtimes \mathrm{GU}_2(2)$. By the discussion used above for $Y_4 \rtimes \mathrm{GU}_4(2)$, one can sort out the dimensions of the simple $k[Y_2 \rtimes \mathrm{GU}_2(2)]$-modules as follows. We first have $\mathrm{GU}_2(2) \simeq (C_3 \times C_3) \rtimes C_2$ with trivial Schur multiplier. So the simple projective representations of this group and the simple representations of its subgroups are of degree 1, hence the simple $k[Y_2 \rtimes \mathrm{GU}_2(2)]$-modules are of dimensions dividing 18. They are prime to 5, so that once multiplied with $(\mathrm{GU}_4(2) : \mathrm{GU}_2(2)) = 1440$ they give dimensions not a multiple of 25.

Let now e_0 be the idempotent of kY_4 corresponding to the only non linear character of the extra-special group Y_4. It is central in kY_4U, $e_0.kY_4 \simeq \mathrm{Mat}_{16}(k)$ by semi-simplicity and we have an action of U on the latter.

The following shows that $e_0kK_6 = 0$, thus establishing our claim.

Proposition 4.11. (i) *The above action of* $\mathrm{GU}_4(2)$ *on* $\mathrm{Mat}_{16}(k)$ *is induced by a morphism* $\mathrm{GU}_4(2) \to \mathrm{GL}_{16}(k)$ *and conjugacy.*
(ii) One has $e_0 \in kY_4U.\mathbf{q}.kY_4U$.

Proof. This action defines a projective representation $\mathrm{GU}(4,2) \to \mathrm{PGL}_{16}(k)$, and we need to show that it is linearizable, meaning that the induced element of $H^2(\mathrm{GU}(4,2), k^\times)$ is zero. By Lemma 3.1 it is sufficient to compute its image in $H^2(C_0, k^\times)$ where $C_0 \simeq (C_2)^2$ is the image of the quaternion group $Q_0 \subset \Gamma_5$ in $\mathrm{GU}(4,2) \simeq C_3 \times \mathrm{PSp}_4(\mathbb{F}_3)$. We compute it explicitly as follows. Since k has characteristic 3, we can assume that $k = \mathbb{F}_3$ and that the 16-dimensional representation ψ of Y_4 is defined over $\mathbb{F}_3$ by the matrix models given in Section 2.3. For x, y two generators of $C_0 \subset \mathrm{GU}(4,2)$, their actions on Y_4 define twisted representations $\psi_x = \psi \circ \mathrm{Ad}\, x$, $\psi_y = \psi \circ \mathrm{Ad}\, y$ of Y_4, which provides intertwinners $P_x, P_y \in \mathrm{GL}_{16}(k)$ and a normalized cocycle. We check that they satisfy $P_xP_y = P_yP_x$ and $P_x^2 = P_y^2 = \mathrm{Id}_{16}$. From Lemma 3.2 it follows that this cocycle is a coboundary, which concludes (i).

We let $U' = \mathrm{SU}(4,2) \subset \mathrm{GU}(4,2) = U$. From the above and Proposition 3.3 we get that $e_0kY_4 \rtimes U' \simeq \mathrm{Mat}_{16}(k) \rtimes U'$ is isomorphic to $\mathrm{Mat}_{16}(k) \otimes kU'$. If $\rho : Q \to \mathrm{GL}_{16}(k)$ denotes the restriction to $Q \subset \mathrm{GU}(4,2)$ of the representation defined above, $\mathbf{q}$ is mapped to $M = \sum_{g \in Q} \rho(g) \otimes g \in \mathrm{Mat}_{16}(k) \otimes kU'$ under this isomorphism. Then the ideal $e_0kY_4U'\mathbf{q}Y_4U'$ of e_0kY_4U' is mapped to the ideal generated by M inside $\mathrm{Mat}_{16}(k) \otimes kU' \simeq \mathrm{Mat}_{16}(kU')$. Every ideal of $\mathrm{Mat}_{16}(kU')$ being isomorphic to $\mathrm{Mat}_{16}(I)$ for some ideal I of kU', we get that this ideal is $\mathrm{Mat}_{16}(I)$ for I generated by the entries (m_{ij}) of the matrix M. In order to compute it we need to explicitly lift the representation $\bar{\rho} : Q \to \mathrm{PGL}_{16}(k)$ afforded by the intertwinners to a linear representation ρ. It is clearly sufficient to lift the generators $\mathbf{i}, \mathbf{j}$ of Q. Although any lifting will do, as $k^\times = \{-1, 1\}$ hence the set of the $\rho \otimes \chi$ for χ a linear character of Q covers all the possible liftings of the generators, we find that the four possible liftings are not equivalent as representations of Q, hence only one is the restriction of the linear representation of U' providing the isomorphism. Nevertheless, computing the entries of M in the four cases, we find that $\mathbf{ij}(z-1)$ belongs to all four possible vector subspaces of kQ spanned by the entries of M, where $z=\mathbf{i}^2=\mathbf{j}^2$. It follows that $z-1$ belongs to I. Since $U'=\mathrm{SU}(4,2)$ is simple, the conjugates of $z\in Q\subset U'$ generate U' hence I contains the augmentation ideal of kU'. As a consequence the quotient of $e_0kY_4\rtimes U'$ by $\mathbf{q}$ is either zero or isomorphic to $\mathrm{Mat}_{16}(k)$. Since the image of $kU'\subset e_0kY_4\rtimes U'$ factorize through $\mathrm{Mat}_{25}(k)$ it has to be 0. Since it generates $(e_0kY_4\rtimes U')/(\mathbf{q})$ we get $e_0kY_4U'=e_0kY_4U'\mathbf{q}Y_4U'$. ∎

4.3.4. Case $n\geq 7$. It suffices to show that $\mathbf{q}$ generates $k\Gamma_7$ as a two-sided ideal, to get the same in any $k\Gamma_n$ for any $n\geq 7$. By the argument at the start of 4.3.3 above, z_5^3-1 belong to the ideal generated by $\mathbf{q}$ in $k\Gamma_5$, hence to the ideal generated by $(\mathbf{q})$ in $k\Gamma_7$, and Theorem 2.2 (ii) then implies that kK_7 is a quotient of $k\,\mathrm{GU}_6(2)$ by the two-sided ideal generated by $\mathbf{q}\in k\,\mathrm{SU}_4(2)$.

Assume that $kK_7\neq 0$ and let S be a simple kK_7-module. We see it as a simple $k\,\mathrm{GU}_6(2)$-module such that $\mathbf{q}S=0$. Since the restriction of S to $\mathrm{SU}_4(2)$ is a module annihilated by $\mathbf{q}$, all its composition factors are isomorphic to the same 25-dimensional simple $k\,\mathrm{SU}_4(2)$-module. Its Brauer character ϕ_S then satisfies $\mathrm{Res}^{\mathrm{GU}_6(2)}_{\mathrm{SU}_4(2)}\phi_S=m.\phi_{25}$ where $m\geq 1$ is an integer and ϕ_{25} is a Brauer 3-modular character of degree 25.

In the table of Brauer characters of $\mathrm{GU}_6(2)$ (denoted by $3.U_6(2).3$ in the notations of [AtMod]), it should then appear as a character of degree $25m$ and with values in $m\mathcal{O}$ for the classes of elements of $\mathrm{SU}_4(2)\subset\mathrm{GU}_6(2)$ ($\mathcal{O}$ denotes the ring of integers of the 3-adic ring of a splitting field of $\mathrm{GU}_6(2)$).

Since the publication of [AtMod], this table has been computed and made available in `GAP4` (package `CTblLib 1.1.3`), so we can check that only two characters match the condition on degree, and it is for $m=111$ and 154. But the condition on values is satisfied in neither case. (see table 1).

4.4. **Even characteristic.** Here we choose another equivalent description of K_n and introduce a new element $\mathbf{b}\in\mathbb{Z}\Gamma_3$ that will prove important to our study of characteristic 2.

Definition 4.12. *Let*

$$\mathbf{b}=s_1s_2^{-1}+s_2^{-1}s_1+s_1^{-1}s_2+s_2s_1^{-1}$$

Note that $\mathbf{b}+s_1s_2^{-1}\mathbf{b}=\mathbf{q}$, and in particular $(\mathbf{q})\subset(\mathbf{b})$.

In characteristic 2, we will not get a complete description of kK_n. This section is devoted to the description of kK_n for $n\in\{3,4\}$, and to preliminary results on the ideal generated by b. We will prove the following, letting $z\mapsto\bar{z}$ denote the element $z\mapsto z^2$ of $\mathrm{Gal}(\overline{\mathbb{F}_2}/\mathbb{F}_2)$.

Theorem 4.13. *If k is a field of characteristic 2, then*

(i) $kK_3\simeq k\Gamma_3/J(k\Gamma_3)^4\simeq(kQ/J(kQ)^4)\rtimes C_3$.
(ii) *When $k\supset\mathbb{F}_4$, $kK_4\simeq kK_3\oplus\mathrm{Mat}_3(k\Gamma_3/I_q)\oplus\mathrm{Mat}_3(k\Gamma_3/\overline{I_q})$ with $I_q=M_qC_3\subset k\Gamma_3$, M_q a 4-dimensional ideal of kQ with $J(kQ)^3\subset M_q\subset J(kQ)^2$, $M_q+\overline{M_q}=J(kQ)^2$.*

For $n=3$ this is a consequence of the following.

Proposition 4.14. *Keep k of characteristic 2. Then $J(k\Gamma_3)^4=(\mathbf{q})\subset(\mathbf{b})=J(k\Gamma_3)^3$.*

```
gap> Display(CharacterTableFromLibrary("3.U6(2).3") mod 3);
3.U6(2).3mod3

      2     15    15   14  12  11  11  9  8  8  1  .  6  5   1   .   .
      3      8     6    4   4   5   3  2  3  2  3  2  2  1   2   2   2
      5      1     1    .   .   .   .  .  .  .  1  .  .  .   1   .   .
      7      1     .    .   .   .   .  .  .  .  .  1  .  .   .   .   .
     11      1     .    .   .   .   .  .  .  .  .  .  .  .   .   1   1

            1a    2a   2b  2c  4a  4b 4c 4d 4e 5a 7a 8a 8b 10a 11a 11b
     2P     1a    1a   1a  1a  2a  2a 2b 2b 2b 5a 7a 4b 4c  5a 11b 11a
     3P     1a    2a   2b  2c  4a  4b 4c 4d 4e 5a 7a 8a 8b 10a 11a 11b
     5P     1a    2a   2b  2c  4a  4b 4c 4d 4e 1a 7a 8a 8b  2a 11a 11b
     7P     1a    2a   2b  2c  4a  4b 4c 4d 4e 5a 1a 8a 8b 10a 11b 11a
    11P     1a    2a   2b  2c  4a  4b 4c 4d 4e 5a 7a 8a 8b 10a  1a  1a

X.1          1     1    1   1   1   1  1  1  1  1  1  1  1   1   1   1
X.2         21   -11    5  -3   5  -3  1 -3  1  1  .  1 -1  -1  -1  -1
X.3        210    50    2  -6  18  10 -2  2 -2  .  .  2  .   .   1   1
X.4        229    69   21  13   5  -3  1  5  1 -1 -2 -3 -1  -1  -2  -2
X.5        364   -84   12  -4  -4  12  4 -4 -4 -1  .  .  .   1   1   1
X.6        560   -80  -16  16  16 -16  .  .  .  .  .  .  .   .  -1  -1
X.7       2775   375   39  15 -57  15 -5 -9  3  .  3  3  1   .   3   3
X.8       1365  -235   53 -27 -11   5 -3 -3  5  .  . -3  1   .   1   1
X.9       1540   260    4 -28 -12 -12  4  4 -4  .  .  .  .   .   .   .
X.10     10395 -1125   27  27 -69  27 -5  3  3  .  .  3 -1   .   .   .
X.11      3850   170  -38   2  26  34 -2 -6  6  .  . -2  .   .   .   .
X.12     18711 -1161  -57  63  -9  15  3 -1 -5  1  . -1  1  -1   .   .
X.13     18711  1431   87  -9  -9  -9 -9 -1 -1  1  . -1 -1   1   .   .
X.14     25515  -405 -117  27  27 -21  3  3  3  .  . -1 -1   .   A  /A
X.15     25515  -405 -117  27  27 -21  3  3  3  .  . -1 -1   .  /A   A
X.16     40095  1215  -81  -9 -81  -9  3 -9  3  . -1  3  1   .   .   .

A = -E(11)-E(11)^3-E(11)^4-E(11)^5-E(11)^9
  = (1-ER(-11))/2 = -b11
```

TABLE 1. Brauer character table for $3.U_6(3).3$, after GAP4

Proof. As before, we let $z = (s_1 s_2)^3$, $\mathbf{i} = s_2 s_1^{-1} z^{-1}$, $s_2^2 s_1 = \mathbf{k} = \mathbf{ij} \in Q$. We have $\mathbf{b} = [s_1, s_2^2] + [s_1^2, s_2] = (\mathbf{i} + \mathbf{ij})(1 + z)$, hence $\sigma_{\mathbf{j}} = \sum_{x \in <\mathbf{j}>} x = \mathbf{ib} \in (\mathbf{b})$ and similarly $\sigma_{\mathbf{i}} = (\mathbf{ib})^{s_1^2}, \sigma_{\mathbf{ij}} = (\mathbf{ib})^{s_1} \in (\mathbf{b})$. Let $K = k\sigma_{\mathbf{i}} \oplus k\sigma_{\mathbf{j}} \oplus k\sigma_{\mathbf{ij}} \subset kQ$. It is easily checked to be a 2-sided ideal, stable under s_1-conjugation. Since Q is a 2-group, the Jacobson radical $J(kQ)$ is the 7-dimensional augmentation ideal, and in particular $1 + \mathbf{i} \in J(kQ)$. By Jennings theorem (see [B2] thm. 3.14.6) one easily gets that $\sum_{r \geq 0} t^r \dim_k J(kQ)^r / J(kQ)^{r+1} = 1 + 2t + 2t^2 + 2t^3 + t^4$ hence $J(kQ)^5 = 0$, $\dim J(kQ)^4 = 1$, $\dim J(kQ)^3 = 3$ and $\dim J(kQ)^2 = 5$. In particular $J(kQ)^4$ coincides with the simple submodule $k\mathbf{q}$. We have $\sigma_x = (1 + x)^3$ for $x \in \{\mathbf{i}, \mathbf{j}, \mathbf{ij}\}$, so $K \subset J(kQ)^3$ hence $K = J(kQ)^3$ by equality of dimensions. The ideal $J(kQ)$ of kQ being stable under s_1-conjugation, we get that $J(kQ)C_3 = C_3 J(kQ)$ is an ideal of $k\Gamma_3 = kQ \ltimes C_3$ with $(J(kQ)C_3)^5 = 0$ hence $J(kQ)C_3 \subset J(k\Gamma_3)$. We have $\dim J(kQ)C_3 = 21$ and

$\dim J(k\Gamma_3) = 24-3 = 21$ because $k\Gamma_3$ admits 3 simple 1-dimensional modules (when $k \supset \mathbb{F}_4$), hence $J(kQ)C_3 = J(k\Gamma_3)$.

From $J(kQ)C_3 = C_3 J(kQ)$ and $C_3C_3 = C_3$ we get that $J(k\Gamma_3)^n = J(kQ)^n C_3$. It follows that the ideal $(\mathbf{b})$ in $k\Gamma_3$ is $KC_3 = J(kQ)^3C_3 = J(k\Gamma_3)^3$, and the one generated by $\mathbf{q}$ is $k\mathbf{q}.C_3 = J(kQ)^4C_3 = J(k\Gamma_3)^4$. ∎

We now consider the case $n = 4$. We let K denote the kernel of the natural morphism $\Gamma_4 \twoheadrightarrow \Gamma_3$. It is the extra-special group 3^{1+2} with exponent 3. Generators are given by $a = s_1s_3^{-1}$,$u = (s_1s_2^2s_1)^{-1}(s_3s_2^2s_3)$, $\zeta = (s_1s_2s_3)^4 \in Z(\Gamma_4)$, and we have $(a,u) = aua^{-1}u^{-1} = \zeta$. The action of Γ_3 on K is given by $s_1us_1^{-1} = au$, $s_2as_2^{-1} = u^{-1}\zeta a$, $(s_1,a) = (s_2,u) = (s_1,\zeta) = (s_2,\zeta) = 1$.

We assume $k \supset \mathbb{F}_4$. The irreducible representations of K are defined over k. Choosing $j \in \mathbb{F}_4 \setminus \mathbb{F}_2$, an irreducible 3-dimensional representation $R : K \to \mathrm{GL}_3(\mathbb{F}_4)$ is given by

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ j & j^2 & 0 \\ 1 & 1 & j \end{pmatrix} u \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \zeta \mapsto j^2$$

and another one is afforded by its Galois conjugate $\bar{R}$. Then $kK = k^9 \oplus \mathrm{Mat}_3(k) \oplus \mathrm{Mat}_3(k)$, and $k\Gamma_4 = (k^9 \rtimes \Gamma_3) \oplus (\mathrm{Mat}_3(k) \rtimes_R \Gamma_3) \oplus (\mathrm{Mat}_3(k) \rtimes_{\bar{R}} \Gamma_3)$. We will prove that $\mathrm{Mat}_3(k) \rtimes \Gamma_3 \simeq \mathrm{Mat}_3(k\Gamma_3)$ and describe an explicit isomorphism. For $g \in \Gamma_3$ we denote $R^g : x \mapsto R(gxg^{-1})$. From $R^g \simeq R$ for every $g \in \Gamma_3$ we get a projective representation $\rho : \Gamma_3 \to \mathrm{PGL}_3(k)$; by explicit computations we check that this ρ can be lifted to a linear representation $\tilde{\rho} : \Gamma_3 \to \mathrm{GL}_3(k)$ given by

$$s_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ j & j^2 & 0 \\ j & j & 1 \end{pmatrix} \quad s_2 \mapsto \begin{pmatrix} j^2 & 0 & j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, an explicit isomorphism $\mathrm{Mat}_3(k) \rtimes \Gamma_3 \to \mathrm{Mat}_3(k) \otimes k\Gamma_3$ is given by $1 \otimes g \mapsto \tilde{\rho}(g) \otimes g$. The ideal generated by $\mathbf{q} \in k\Gamma_3$ in $\mathrm{Mat}_3(k) \rtimes \Gamma_3$ then corresponds to $\mathrm{Mat}_3(I_q)$ with I_q the ideal generated in $k\Gamma_3$ by the entries of $\sum_{x\in Q} \rho(x)$. By computer we find that I_q has dimension 12, and is generated by $s_1^{-1}s_2s_1 + j^2s_1s_2^{-1}s_1 + j^2s_2s_1^{-1}s_2 + js_2^{-1}s_1^{-1} + j^2s_1^{-1}s_2^{-1}$. We also check that $I_q = M_qC_3$ with M_q the 4-dimensional ideal of kQ generated by $1 + js_1s_2s_1 + j^2(s_1s_2)^3 + js_1s_2^{-1} + js_2^{-1}s_1$, that $J(kQ)^3 \subset M_q \subset J(kQ)^2$.

Similarly, we get that the ideal generated by $\mathbf{b}$ in $\mathrm{Mat}_3(k) \rtimes \Gamma_3$ corresponds to $\mathrm{Mat}_3(I_b)$ with I_b an ideal of dimension 21 that contains $s_1^{-1}s_2 + 1$. since $(k\Gamma_3)/(s_1^{-1}s_2 + 1) = k(\Gamma_3/s_1^{-1}s_2) \simeq kC_3$ we get $I_b = (s_1^{-1}s_2 + 1)$ and $\mathrm{Mat}_3(k\Gamma_3)/\mathrm{Mat}_3(I_b) \simeq \mathrm{Mat}_3(kC_3)$.

Lemma 4.15. *The images of the elements* $r_1 = s_2s_3^2 + s_1^2s_2 + s_1s_2^2 + s_3s_1^2 + s_2^2s_3 + s_1s_3^2$ *and* $r_2 = s_2^2s_3 + s_1 + s_2 + s_2s_3s_1^2 + s_2^2s_3s_1 + s_1^2s_3^2$ *of* $k\Gamma_4$ *inside* $\mathrm{Mat}_3(k) \rtimes \Gamma_3$ *lie inside the image of* $(\mathbf{b})$.

Proof. We first write r_1, r_2 inside $kK \rtimes \Gamma_3$. We get $r_1 = u^{-1}\zeta as_2s_1^2 + s_1^2s_2 + s_1s_2^2 + a^{-1} + au^{-1}as_2^2s_1 + a$ and $r_2 = \zeta^{-1}uas_2^2s_1^2 + s_1 + s_2 + auas_2 + au^{-1}as_2^2s_1^2 + as_1$. We need to prove that they map to 0 through the composite of the morphisms $kK \rtimes \Gamma_3 \to \mathrm{Mat}_3(k) \rtimes \Gamma_3 \to \mathrm{Mat}_3(k) \otimes k\Gamma_3 \twoheadrightarrow \mathrm{Mat}_3(kC_3)$, that is that $R(u^{-1}\zeta a)\tilde{\rho}(s_2s_1^2) + \tilde{\rho}(s_1^2s_2) + \tilde{\rho}(s_1s_2^2) + R(a^{-1}) + R(au^{-1}a)\tilde{\rho}(s_2^2s_1) + R(a) = 0$ and $R(\zeta^{-1}ua)\tilde{\rho}(s_2^2s_1^2) + \tilde{\rho}(s_1) + \tilde{\rho}(s_2) + R(aua)\tilde{\rho}(s_2) + R(au^{-1}a)\tilde{\rho}(s_2^2s_1^2) + R(a)\tilde{\rho}(s_1) = 0$. This follows from a straightforward computation. ∎

The case of the other 3-dimensional representations is similar and can moreover be deduced from the first one by Galois action : letting $x \mapsto \bar{x}$ denote the nontrivial element of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, $(\mathbf{q})$ corresponds to the ideal $\overline{I_q} = \overline{M_q}C_3$, and we check $M_q + \overline{M_q} = J(kQ)^2$.

We now turn to the 1-dimensional representations $\rho_{\alpha,\beta} : K \to \mathbb{F}_4^\times$ defined by $a \mapsto j^\alpha, u \mapsto j^\beta, \zeta \mapsto 1$ for $\alpha, \beta \in \{0,1,2\}$. We have $\rho_{\alpha,\beta}^{s_1}(a) = j^\alpha$, $\rho_{\alpha,\beta}^{s_1}(u) = j^{\alpha+\beta}$, $\rho_{\alpha,\beta}^{s_2}(a) = j^{\alpha-\beta}$, $\rho_{\alpha,\beta}^{s_2}(u) = j^\beta$. Identifying the possible (α, β) with $\mathbb{F}_3^2$, the Γ_3-action on the classes of representations thus corresponds to the identification of Γ_3 with $\mathrm{SL}_2(\mathbb{F}_3)$ given by

$$s_1 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad s_2 \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

It follows that there are two orbits, of cardinalities 1 and 8. We have $k^9 \rtimes \Gamma_3 = k\Gamma_3 \oplus k^{\Gamma_3/C} \rtimes \Gamma_3$ with C the stabilizer of a nonzero vector in $\mathbb{F}_3^2$.

We apply Proposition 3.4 with $C = < s_1 >$ and Q making a representative system of Γ_3/C. Then, under the isomorphism $k^{\Gamma_3/C} \rtimes \Gamma_3 \simeq \mathrm{Mat}_Q(kC_3)$, $g \in Q$ is mapped to $\sum_{u \in Q} E_{gu,u}$. In particular, $\mathbf{q} \in kQ$ is mapped to

$$\sum_{v \in Q} \sum_{u \in Q} E_{vu,u} = \sum_{u,v \in Q} E_{u,v}.$$

The ideal of $k^{\Gamma_3/C} \rtimes \Gamma_3$ generated by $\mathbf{q}$ is then mapped to $\mathrm{Mat}_8(I)$ for I the ideal of kC_3 generated by 1, hence is the full block $k^{\Gamma_3/C} \rtimes \Gamma_3$. Since $\mathbf{q} \in (\mathbf{b})$, the same holds for the ideal generated by $\mathbf{b}$.

Proposition 4.16. *The elements $r_1, r_2 \in k\Gamma_4$ of Lemma 4.15 belong to the 2-sided ideal generated by* $\mathbf{b}$.

Proof. We showed that $k\Gamma_4/(\mathbf{b})$ is isomorphic to $(k\Gamma_3/(\mathbf{b})) \oplus \mathrm{Mat}_3(kC_3) \oplus \mathrm{Mat}_3(kC_3)$. The images of r_1, r_2 in both $\mathrm{Mat}_3(kC_3)$ is 0 by Lemma 4.15, and it is readily checked that $r_1 \mapsto \mathbf{b}$ and $r_2 \mapsto 0$ through $k\Gamma_4 \twoheadrightarrow k\Gamma_3$. The conclusion follows. ∎

4.5. **A finer description of K_5 as a $\mathbb{Z}$-module.** Similar algorithms as the ones used in the proof of proposition 4.6 enabled us, using several months of CPU time, to determine the structure of $(\mathbb{Z}/32\mathbb{Z})K_5$ as a $(\mathbb{Z}/32\mathbb{Z})$-module. Combined with our study of odd characteristic, this implies the following.

Proposition 4.17. *As a $\mathbb{Z}$-module, $K_5 \simeq (K_5^0)^3$ with*

$$K_5^0 \simeq (\mathbb{Z}/2\mathbb{Z})^{744} \times (\mathbb{Z}/4\mathbb{Z})^{38} \times (\mathbb{Z}/8\mathbb{Z})^{80} \times (\mathbb{Z}/16\mathbb{Z}) \times G$$

where G is an abelian 3-group with $\dim_{\mathbb{F}_3} G \otimes \mathbb{F}_3 = 25^2 = 625$.

5. A ternary Hecke algebra in characteristic 2

We assume that k is a field of characteristic 2 with $k \supset \mathbb{F}_4 = \{0, 1, j, j^2\}$. Recall that $\mathbf{b}$ and $\mathbf{q}$ are defined in Definitions 4.12 and 4.1.

Definition 5.1. *For $\alpha, \beta \in k$, and $n \geq 3$, we define the following.*
Let $J_n(\alpha, \beta) = k\Gamma_n.(s_1 - \alpha)(s_1 - \beta).k\Gamma_n$
Let $H_n(\alpha, \beta) = k\Gamma_n / J_n(\alpha, \beta)$.
Let $J_n = J_n(1, j) \cap J_n(1, j^2) \cap J_n(j, j^2)$.

The aim of this section is essentially to prove the following. In particular, we see that kK_n never collapses and actually has dimension $\geq 3(n!-1)$. Recall that $\mathbf{q} \in k\Gamma_n.\mathbf{b}.k\Gamma_n$ (see Proposition 4.14)

Theorem 5.2. *Let $n \geq 3$. Then $J_n = k\Gamma_n.\mathbf{b}.k\Gamma_n$ as a 2-sided ideal of $k\Gamma_n$ and $k\Gamma_n/J_n$ has dimension $3(n!-1)$.*

Notice that $J_n(\alpha,\beta)$ contains $(s_i-\alpha)(s_i-\beta)$ for arbitrary $1 \leq i < n$.

Lemma 5.3. *Assume $n \geq 3$. We have $I_n \subset J_n(\alpha,\beta)$ whenever $\alpha^3 = \beta^3 = 1$ and $\alpha \neq \beta$.*

Proof. We need to show that $\mathbf{c} \equiv 0$ modulo $J_n(\alpha,\beta)$. From $s_1^2 \equiv (\alpha+\beta)s_1 - \alpha\beta$ we get $s_2s_1^2s_2 \equiv (\alpha+\beta)s_2s_1s_2 - \alpha\beta s_2^2 \equiv (\alpha+\beta)s_2s_1s_2 - \alpha\beta(\alpha+\beta)s_2 + (\alpha\beta)^2$ and symmetrically $s_1s_2^2s_1 \equiv (\alpha+\beta)s_1s_2s_1 - \alpha\beta(\alpha+\beta)s_1 + (\alpha\beta)^2$, thus $s_1s_2^2s_1 + s_2s_1^2s_2 = 2(\alpha+\beta)s_1s_2s_1 - \alpha\beta(\alpha+\beta)(s_1+s_2) + 2\alpha^2\beta^2$. From the same equation we get $s_1^2s_2s_1 \equiv (\alpha+\beta)s_1s_2s_1 - \alpha\beta s_2s_1$ and $s_1s_2s_1^2 \equiv (\alpha+\beta)s_1s_2s_1 - \alpha\beta s_1s_2$ hence $s_1^2s_2s_1 + s_1s_2s_1^2 \equiv 2\alpha\beta s_1s_2s_1 - \alpha\beta(s_1s_2+s_2s_1)$. Finally $s_1^2s_2^2 \equiv ((\alpha+\beta)s_1 - \alpha\beta)((\alpha+\beta)s_2 - \alpha\beta) \equiv (\alpha+\beta)^2s_1s_2 - \alpha\beta(\alpha+\beta)(s_1+s_2) + (\alpha\beta)^2$ and symetrically $s_2^2s_1^2 \equiv (\alpha+\beta)^2s_2s_1 - \alpha\beta(\alpha+\beta)(s_1+s_2) + (\alpha\beta)^2$. Altogether this yields

$$\mathbf{c} \equiv 4(\alpha+\beta)s_1s_2s_1 + \big((\alpha+\beta)^2 - \alpha\beta\big)(s_2s_1+s_1s_2) + (1-3\alpha\beta(\alpha+\beta))(s_1+s_2) + 4\alpha^2\beta^2.$$

Since $(\alpha+\beta)^2 - \alpha\beta = \alpha^2+\alpha\beta+\beta^2 = 0$ and $\alpha\beta(\alpha+\beta) = (\beta/\alpha)+(\alpha/\beta) = -1$ we get $\mathbf{c} \equiv 4(\alpha+\beta)s_1s_2s_1 + 4(s_1+s_2) + 4\alpha^2\beta^2$. Since $4=0$ this concludes the proof. ∎

Recall $J_n = J_n(1,j) \cap J_n(1,j^2) \cap J_n(j,j^2)$. From the above lemma, $J_n \supset I_n$ and obviously kK_n surjects onto $k\Gamma_n/J_n$, while $k\Gamma_n/J_n$ embeds into $H_n(1,j) \times H_n(1,j^2) \times H_n(j,j^2)$.

In order to deal with quotients of an intersection of three ideals we will need, here and later on, the following two lemmas.

Lemma 5.4. *Let A be a (possibly non-commutative) unital ring, I_1, I_2, I_3 three 2-sided ideals, such that $A = I_1+I_2+I_3$. Then $I_1 + I_2 \cap I_3 = (I_1+I_2) \cap (I_1+I_3)$.*

Proof. Denote $I = (I_1+I_2)\cap(I_1+I_3)$. Then the inclusion $I_1 + (I_2 \cap I_3) \subseteq I$ is trivial. On the other hand, we have $I = I(I_1+I_2+I_3) = I(I_1+I_2) + I.I_3 \subseteq (I_1+I_3).(I_1+I_2) + (I_1+I_2).I_3 \subseteq I_1 + I_3I_2 + I_2I_3 \subseteq I_1 + (I_2 \cap I_3)$. ∎

Lemma 5.5. *Let A be an abelian group, I, J, K subgroups of A with $I+J+K = A$. We define morphisms*

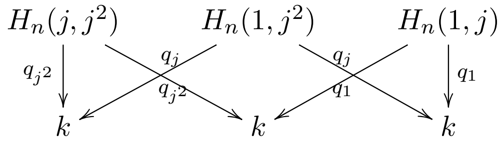
$$A/I\cap J\cap K \xrightarrow{d_1} A/I \times A/J \times A/K \xrightarrow{d_2} A/(J+K) \times A/(I+K) \times A/(I+J)$$

where d_2 is induced by $(a,b,c) \mapsto (b-c, a-c, a-b)$ and d_1 is the natural (injective) map. Then $d_2 \circ d_1 = 0$, d_2 is surjective and $\operatorname{Ker} d_2/\operatorname{Im} d_1 \simeq (K+I)\cap(K+J)/K+I\cap J$.

Proof. $d_2 \circ d_1 = 0$ is clear. $\operatorname{Im} d_2$ contains $A/(I+J) = 0 \times 0 \times A/(I+J)$, as $A/(I+J) = (I+J+K)/(I+J)$ is clearly $d_2(K/I \times 0 \times 0)$, where K/I denotes the image of K in A/I, hence d_2 is surjective by symmetry. An element of $\operatorname{Ker} d_2$ is the class of a triple $(a,b,c) \in A^3$ with $a-b = i+j$, $b-c = j'+k$, $a-c = i'+k'$ for some $i,i' \in I$, $j,j' \in J$, $k,k' \in K$, hence of a $(a-i, b+j, c) = (a', a', c)$ with $a' = a-i = b+j$. One has $a'-c = b-c+j = a-i-c \in (K+I)\cap(K+J)$. Conversely, the class of any (a,a,c) with $a-c \in (K+I)\cap(K+J)$ belongs to $\operatorname{Ker} d_2$.

On the other hand, such a triple (a,a,c) originates from A iff there exists $i \in I$, $j \in J$, $k \in K$ such that $a+i = a+j = c+k$, which means $c \in K + I\cap J$. This proves $\operatorname{Ker} d_2/\operatorname{Im} d_1 \simeq (K+I)\cap(K+J)/(K+I\cap J)$ under $(a,a,c) \mapsto a-c$. ∎

When $\alpha, \beta \in \mu_3(k)$ with $\alpha \neq \beta$, we let $q_\alpha : H_n(\alpha, \beta) \to k$ denote the natural morphism sending each s_i to α.



Definition 5.6. *We let $\mathcal{H}_n$ denote the subalgebra of $H_n(j, j^2) \oplus H_n(1, j^2) \oplus H_n(1, j)$ made of the triples (x_1, x_j, x_{j^2}) such that $q_\alpha(x_{\alpha'}) = q_\alpha(x_{\alpha''})$ whenever $\{\alpha, \alpha', \alpha''\} = \mu_3(k)$.*

We also denote $U_n = k\Gamma_n/(\mathbf{b})$ (see Definition 4.12).

Recall that $J_n = J_n(j, j^2) \cap J_n(1, j^2) \cap J_n(1, j)$.

Proposition 5.7. *The image of the natural embedding $k\Gamma_n/J_n \hookrightarrow H_n(j, j^2) \oplus H_n(1, j^2) \oplus H_n(1, j)$ is $\mathcal{H}_n$. We have $\dim \mathcal{H}_n = 3(n! - 1)$ and $kK_n \twoheadrightarrow \mathcal{H}_n$.*

Proof. We want to apply lemmas 5.4 and 5.5 to the ideals $J_n(\alpha, \beta)$ for α, β distinct third roots of 1. If $\mu_3(k) = \{\alpha, \beta, \gamma\}$, we first prove $J_n(\alpha, \beta) + J_n(\alpha, \gamma) = k\Gamma_n(s_1 - \alpha)k\Gamma_n$: under φ, we can assume for this $\alpha = 1$, $\beta = j$, $\gamma = j^2$ and we have $(s_1 - 1)(s_1 - j) - (s_1 - 1)(s_1 - j^2) = (1 + 2j)(s_1 - 1)$ with $1 + 2j$ invertible. This implies at once that the sum of the three $J_n(\alpha, \beta)$ have for sum $k\Gamma_n$. We also get that the map d_2 of 5.5 defines $\mathcal{H}_n$ as its kernel. We thus get the first statement of our proposition by applying Lemma 5.5 in a case where Lemma 5.4 ensures that the sequence of maps is exact. This exactness also implies the claim on dimensions since each Hecke algebra has dimension $n!$. ∎

Remark 5.8. *Lemma 5.3 and Proposition 5.7 hold true with k replaced by $(\mathbb{Z}/4\mathbb{Z})[j] = (\mathbb{Z}/4\mathbb{Z})[x]/(x^2 + x + 1)$, with the same proofs (except, of course, for the statement on the dimension).*

Remark 5.9. *We have natural morphisms $\mathcal{H}_n \twoheadrightarrow H_n(\alpha, \beta)$, hence every simple $H_n(\alpha, \beta)$-module M provides a simple module for $\mathcal{H}_n$, and for $k\Gamma_n$. Since char.$k = 2$ and $s_1 \in \Gamma_n$ has order 3, the action of s_1 is semisimple, and so we can assume that the induced morphism $k\Gamma_n \to \mathrm{End}_k(M)$ either factorizes through $K_n(\alpha)$ (up to exchanging α and β) or does not factorize through any of the $J_n(u, v)$ for $\{u, v\} \neq \{\alpha, \beta\}$. It follows that a collection of non-isomorphic simple modules for $\mathcal{H}_n$ is afforded by the simple $H_n(\alpha, \beta)$-modules of dimension at least 2 and the three 1-dimensional modules defined by $s_1 \mapsto \alpha$ for a given $\alpha \in \mu_3(k)$. By the same argument, the same holds for indecomposable modules as well.*

Proposition 5.10. *For $n \geq 3$, J_n is generated as a 2-sided ideal by* $\mathbf{b}$ *(see Definition 4.12).*

It is easily checked that, for $n \geq 3$, $\mathbf{b} \in J_n$. Indeed, we have $\mathbf{b} \in J_n(\alpha, \beta)$ because the image of $\mathbf{b} = s_1 s_2^2 + s_2^2 s_1 + s_1^2 s_2 + s_2 s_1^2$ in $H_n(\alpha, \beta) = k\Gamma_n/J_n(\alpha, \beta)$ is $2(\alpha + \beta)(s_1 s_2 + s_2 s_1) + 2\alpha\beta(s_1 + s_2) \equiv 0$, using $s_i^2 \equiv (\alpha + \beta)s_i + \alpha\beta$ and char. $k = 2$.

Recall $U_n = k\Gamma_n/(\mathbf{b})$. In view of Proposition 5.7, in order to prove $J_n = (\mathbf{b})$, it is enough to check that $\dim U_n \leq \dim k\Gamma_n/J_n = 3(n! - 1)$. We need a lemma, where we abuse notations by letting U_k denote the image of U_k in U_{n+1}.

Lemma 5.11. *For $n \geq 2$, one has*

(i) $U_{n+1} = U_n + U_n s_n U_n + U_n s_n^2 U_n$
(ii) $U_{n+1} = U_n + U_n s_n U_n + U_n s_n^2$
(iii) *If $k < n$, $r, t \in \{0, 1, 2\}$ we have $s_k^r s_1^t s_n^2 \equiv s_1^{r+t} s_n^2$ modulo $U_n + U_n s_n$.*

(iv) $U_{n+1} = U_n + U_n s_n U_n + U_2 s_n^2$

Proof. Item (i) is a consequence of $\mathbf{q} \in (\mathbf{b})$ by Proposition 4.2.

We now prove (ii). One has $s_2^2 s_1 = s_1 s_2^2 + s_1^2 s_2 + s_2 s_1^2 + \mathbf{b}$. By (1), U_{n+1} is spanned by U_n, $U_n s_n U_n$ and the $w_1 s_n^2 w_2$ for w_1, w_2 positive words in the s_i for $i \leq n-1$. We let $l(w_2)$ denote the length of w_2 with respect to these generators. If, as a word, $w_2 = s_r w_2'$ with $r \leq n-2$, then $w_1 s_n^2 w_2 = w_1 s_n^2 s_r w_2' = w_1 s_r s_n^2 w_2'$ with $l(w_2') < l(w_2)$. If $w_2 = 1$ is the empty word, then $w_1 s_n^2 w_2 \in U_n s_n^2$. Otherwise, we have $w_2 = s_{n-1} w_2'$. By conjugating $\mathbf{b}$, we get $s_n^2 s_{n-1} \equiv s_{n-1} s_n^2 + s_{n-1}^2 s_n + s_n s_{n-1}^2 \mod (\mathbf{b})$ hence

$$w_1 s_n^2 s_{n-1} w_2' \equiv w_1 s_{n-1} s_n^2 w_2' + w_1 s_{n-1}^2 s_n w_2' + w_1 s_n s_{n-1}^2 w_2' \mod (\mathbf{b})$$

On the other hand, $l(w_2') < l(w_2)$ and $w_1 s_{n-1}^2 s_n w_2' + w_1 s_n s_{n-1}^2 w_2' \in U_n s_n U_n$, so we can conclude by induction on the length of w_2.

We first note that (iii) is trivial for $n = 2$, so we assume $n \geq 3$. It is also trivial for $r = 0$, so we can assume $r \in \{1, 2\}$. We first deal with the case $t = 0$. We let $V_n = U_n + U_n s_n$ and we use that, in $k\Gamma_4/(\mathbf{b})$, $s_2 s_3^2 = (s_1^2 s_2 + s_1 s_2^2) + (s_1^2 s_3 + s_2^2 s_3) + s_1 s_3^2$ and $s_2^2 s_3^2 = (s_1 + s_2) + (s_2 s_1^2 s_3 + s_2^2 s_1 s_3) + s_1^2 s_3^2$ (see Proposition 4.16). Here and in the following, all congruences are modulo additive subgroups. By conjugation we thus get $s_{n-1} s_n^2 \equiv (s_{n-2}^2 s_{n-1} + s_{n-2} s_{n-1}^2) + (s_n s_{n-2}^2 + s_{n-1}^2 s_n) + s_{n-2} s_n^2 \mod (\mathbf{b})$ whenever $n \geq 3$, and in particular $s_{n-1} s_n^2 \equiv s_{n-2} s_n^2$ modulo V_n and also $s_{n-1}^2 s_n^2 \equiv s_{n-2}^2 s_n^2$ modulo V_n. We need to prove that $s_k^r s_n^2 \equiv s_1^r s_n^2 \mod V_n$ for all $k < n$ and $r \in \{1, 2\}$, or, equivalently, that $s_k^r s_n^2 \equiv s_{k+1}^r s_n^2 \mod V_n$ for all $k < n-1$ and $r \in \{1, 2\}$. We prove this by decreasing induction, the case $k = n-2$ being already known. Let now $k < n-2$. Notice that V_n is both a left U_n-module and a U_{n-1}-bimodule. Modulo V_n, we have by the induction hypothesis and the commutation relations that $s_{k+1}^a s_k^b s_n^2 \equiv s_{k+1}^a s_n^2 s_k^b \equiv s_{k+2}^a s_n^2 s_k^b \equiv s_k^b s_{k+2}^a s_n^2 \equiv s_k^b s_{k+1}^a s_n^2$ for all $a, b \in \{1, 2\}$. On the other hand, $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ hence $s_k s_{k+1} s_k s_n^2 \equiv s_k^2 s_{k+1} s_n^2 \equiv s_{k+1} s_k^2 s_n^2$ is equal modulo V_n to $s_{k+1} s_k s_{k+1} s_n^2 \equiv s_{k+1}^2 s_k s_n^2$. Multiplying on the left by $s_k^{-1} s_{k+1}^{-1}$ we thus get $s_k s_n^2 \equiv s_k^{-1} s_{k+1} s_k s_n^2 \equiv s_k^{-1} s_k s_{k+1} s_n^2 \equiv s_{k+1} s_n^2$ hence the conclusion for $t = 0$. For arbitrary t, we then have $s_k^r s_1^t s_n^2 \equiv s_k^r s_k^t s_n^2 \equiv s_k^{r+t} s_n^2 \equiv s_1^{r+t} s_n^2$.

Notice that (ii) and (iv) are the same statement for $n = 2$, so we can again assume $n \geq 3$. Since $U_n + U_n s_n \subset U_n + U_n s_n U_n$, (iii) implies $U_n s_n^2 \subset U_2 s_n^2 + U_n + U_n s_n U_n$ hence (iv) follows from (ii). ∎

For $0 \leq k \leq n$, we let $s_{n,k} = s_n s_{n-1} \dots s_{n-k+1}$ with the convention that $s_{n,0} = 1$ and $s_{n,1} = s_n$. We let $U_n^k = U_n s_{n,k}$ (hence $U_n^0 = U_n$). Similarly, we let $x_{n,k} = s_n s_{n-1} \dots s_{n-k+2} s_{n-k+1}^2$ for $1 \leq k \leq n$, with the convention $x_{n,1} = s_n^2$.

Lemma 5.12. (i) *If $r \leq n-1$, $1 \leq k \leq n$ and $c \in \{0, 1, 2\}$, then $s_r s_1^c x_{n,k} \in s_1^{c+1} x_{n,k} + U_n^0 + \dots + U_n^k$.*

(ii) *For $w \in \Gamma_n$, $w x_{n,k} \in s_1^{l(w)} x_{n,k} + U_n^0 + \dots + U_n^n$, where $l : \Gamma_n \twoheadrightarrow \mathbb{Z}/3\mathbb{Z}$ is $s_i \mapsto 1$.*

Proof. We first deal with (i). Notice that the statement is trivial for $n \leq 2$, so we can assume $n \geq 3$ and in particular $s_1 s_n = s_n s_1$. We prove the statement by induction on k, for all n. The case $k = 1$ being known by the previous lemma, we can assume $k \geq 2$. Let $r \leq n-1$. We first consider the case $r \leq n-2$. Then $s_r s_1^c x_{n,k} = s_r s_1^c s_n s_{n-1} \dots s_{n-k+1}^2 = s_n s_r s_1^c s_{n-1} \dots s_{n-k+1}^2 = s_n s_r s_1^c x_{n-1,k-1}$. By the induction hypothesis we have $s_r s_1^c x_{n-1,k-1} \equiv s_1^{c+1} x_{n-1,k-1}$ modulo

$U_{n-1}^0 + \cdots + U_{n-1}^{k-1}$ hence $s_n s_r s_1^c x_{n-1,k-1} \equiv s_n s_1^{c+1} x_{n-1,k-1}$ modulo $s_n U_{n-1}^0 + \cdots + s_n U_{n-1}^{k-1}$. Noticing that $s_n U_{n-1}^j = s_n U_{n-1} s_{n-1,j} = U_{n-1} s_n s_{n-1,j} = U_{n-1} s_{n,j+1} \subset U_n s_{n,j+1}$ we get that $s_n s_r s_1^c x_{n-1,k-1} \equiv s_n s_1^{c+1} x_{n-1,k-1} \equiv s_1^{c+1} s_n x_{n-1,k-1} \equiv s_1^{c+1} x_{n,k}$ modulo $U_n + U_n^1 + \cdots + U_n^k$.

We now consider the case $r = n-1$. For clarity, we let $b = n-k+1$. Then, using $s_{b+1} s_b^2 = s_{b+1}^2 s_b + s_b s_{b+1}^2 + s_b^2 s_{b+1}$, we get that $s_{n-1} s_1^c x_{n,k} = A + B + C$ with

$$\left\{ \begin{array}{lcl} A & = & s_{n-1} s_1^c s_n s_{n-1} \dots s_{b+2} s_{b+1}^2 s_b \\ B & = & s_{n-1} s_1^c s_n s_{n-1} \dots s_{b+2} s_b s_{b+1}^2 \\ C & = & s_{n-1} s_1^c s_n s_{n-1} \dots s_{b+2} s_b^2 s_{b+1} \end{array} \right.$$

First note that $C = s_{n-1} s_1^c s_b^2 s_n s_{n-1} \dots s_{b+2} s_{b+1} \in U_n s_{n,k-1} = U_n^{k-1}$. By the induction hypothesis, $A = (s_{n-1} s_1^c s_n s_{n-1} \dots s_{b+2} s_{b+1}^2) s_b$ is congruent to $(s_1^{c+1} s_n s_{n-1} \dots s_{b+2} s_{b+1}^2) s_b = s_1^{c+1} x_{n,k-1} s_b$ modulo $(U_n + U_n^1 + \cdots + U_n^{k-1}) s_b \subset U_n + U_n^1 + \cdots + U_n^{k-2} + U_n^k$. Now $s_1^{c+1} x_{n,k-1} s_b = s_1^{c+1} s_{n,k-2} s_{b+1}^2 s_b$ and using again $s_{b+1}^2 s_b = s_{b+1} s_b^2 + s_b s_{b+1}^2 + s_b^2 s_{b+1}$ we get that

$$s_1^{c+1} x_{n,k-1} s_b = s_1^{c+1} x_{n,k} + s_1^{c+1} s_n \dots s_{b+2} s_b s_{b+1}^2 + s_1^{c+1} s_n \dots s_{b+2} s_b^2 s_{b+1}.$$

We have $s_1^{c+1} s_n \dots s_{b+2} s_b^2 s_{b+1} = s_1^{c+1} s_b^2 s_{n,k-1} \in U_n^{k-1}$. Moreover, $s_1^{c+1} s_n \dots s_{b+2} s_b s_{b+1}^2 = s_1^{c+1} s_b x_{n,k-1}$, and by the induction hypothesis, we have $s_b x_{n,k-1} \in s_1 x_{n,k-1} + U_n + \cdots + U_n^{k-1}$. Hence $A \in s_1^{c+1} x_{n,k} + s_1^{c+2} x_{n,k-1} + U_n + \cdots + U_n^k$.

We now consider B. We have $s_b s_{b+1}^2 \in s_1 s_{b+1}^2 + U_{b+1} + U_{b+1} s_{b+1}$ by Lemma 5.11. Moreover $s_{n-1} s_1^c s_n \dots s_{b+2} U_{b+1} = s_{n-1} U_{b+1} s_n \dots s_{b+2} \subset U_n s_n \dots s_{b+2}$ and similarly $s_{n-1} s_1^c s_n \dots s_{b+2} U_{b+1} s_{b+1} \subset U_n s_n \dots s_{b+1}$, hence $B \in s_{n-1} s_1^c \dots s_{b+2} s_1 s_{b+1}^2 + U_n^{k-2} + U_n^{k-1}$ i.e. $B \in s_{n-1} s_1^{c+1} x_{n,k-1} + U_n^{k-2} + U_n^{k-1} \subset s_1^{c+2} x_{n,k-1} + U_n + U_n^1 + \cdots + U_n^{k-1}$ by the induction hypothesis. Altogether this yields $A + B + C \in s_1^{c+1} x_{n,k} + U_n + \cdots + U_n^k$ and the conclusion for (i).

Part (ii) is an immediate consequence of (i), as we have $s_r x_{n,k} \equiv s_1 x_{n,k}$ and $s_r^2 x_{n,k} = s_r s_r x_{n,k} \equiv s_r s_1 x_{n,k} \equiv s_1^2 x_{n,k}$ modulo $U_n^0 + U_n^1 + \cdots + U_n^n$ whenever $r < n$, and the s_r for $r < n$ generate Γ_n. ∎

Proposition 5.13. *Let $n \geq 2$. Then* $\dim U_n = 3(n! - 1)$ *and*

$$U_{n+1} = U_n \oplus U_n^1 \oplus \cdots \oplus U_n^n \oplus U_2 x_{n,1} \oplus \cdots \oplus U_2 x_{n,n}$$

Proof. We first prove that $U_{n+1} = U_n + U_n^1 + \cdots + U_n^n + U_2 x_{n,1} + \cdots + U_2 x_{n,n}$ by induction on n. Assuming this to be true for n, we have $U_{n+2} = U_{n+1} + U_{n+1} s_{n+1} U_{n+1} + U_2 x_{n+1,1}$ by Lemma 5.11, and

$$U_{n+1} s_{n+1} U_{n+1} \subset U_{n+1} s_{n+1} (U_n + \cdots + U_n^n) + U_{n+1} s_{n+1} U_2 x_{n,1} + \cdots + U_{n+1} s_{n+1} U_2 x_{n,n}.$$

But, for $k = 1, \dots, n$, $U_{n+1} s_{n+1} U_n^k = U_{n+1} s_{n+1} U_n s_{n,k} = U_{n+1} U_n s_{n+1} s_{n,k} = U_{n+1}^{k+1}$ and $U_{n+1} s_{n+1} U_2 x_{n,k} \subset U_{n+1} s_{n+1} x_{n,k} = U_{n+1} x_{n+1,k+1}$. Therefore $U_{n+2} \subset \sum_{k=1}^n U_{n+1} x_{n+1,k+1} + \sum_{k=1}^n U_2 x_{n+1,k}$. On the other hand, $U_{n+1} x_{n+1,k+1} \subset U_2 x_{n+1,k+1} + U_{n+1} + \cdots + U_{n+1}^{n+1}$ by Lemma 5.12. It follows that $U_{n+2} \subset U_{n+1} + U_{n+1}^1 + \cdots + U_{n+1}^{n+1} + U_2 x_{n+1,1} + \cdots + U_2 x_{n+1,n+1}$ and we conclude by induction.

We then prove that $\dim U_n \leq 3(n! - 1)$, again by induction on n. Since $U_{n+1} = U_n + U_n^1 + \cdots + U_n^n + U_2 x_{n,1} + \cdots + U_2 x_{n,n}$, we get $\dim U_{n+1} \leq (n+1) \dim U_n + 3n \leq 3(n+1)! - 3(n+1) + 3n = 3((n+1)! - 1)$.

Finally, since U_n maps onto $\mathcal{H}_n$ we know $\dim U_n \geq 3(n! - 1)$ hence $\dim U_n = 3(n! - 1)$. It follows that all inequalities above are equalities and the sum is direct, which concludes the proof. ∎

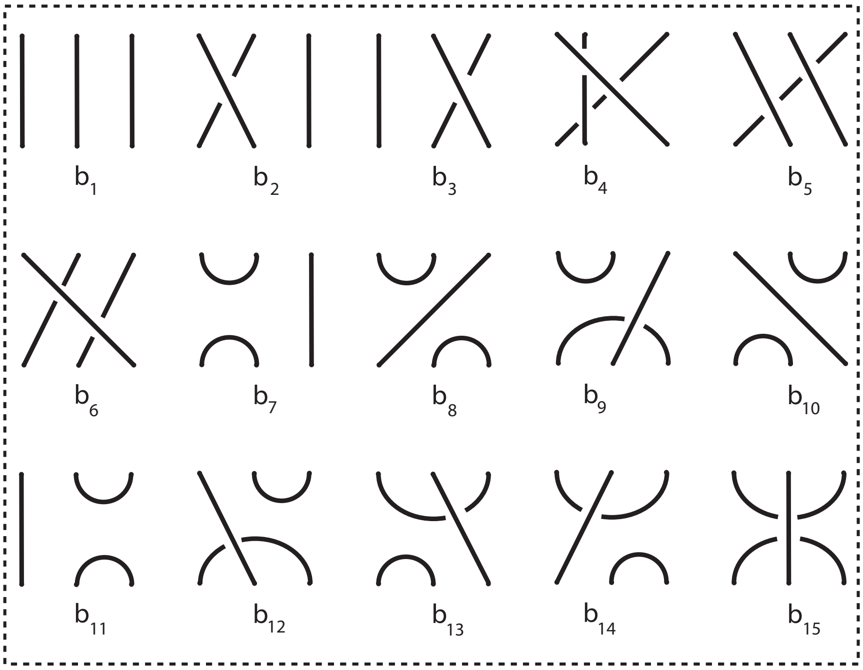


Figure 1. Basis for BW_3

1	$\mapsto$	b_1	$s_1 s_2^{-1}$	$\mapsto$	$b_2 + b_6 + b_8 + b_{11} + b_{14}$
$s_1^{-1} s_2$	$\mapsto$	$b_3 + b_6 + b_7 + b_8 + b_9$	$s_2 s_1^{-1}$	$\mapsto$	$b_3 + b_5 + b_{13}$
$s_2^{-1} s_1$	$\mapsto$	$b_2 + b_5 + b_{12}$	$s_1 s_2 s_1$	$\mapsto$	b_4

$s_1^{-1} s_2^{-1} s_1^{-1}$	$\mapsto$	$b_2 + b_3 + b_4 + b_5 + b_6 + b_9 + b_{10} + b_{14} + b_{15}$
$s_2^{-1} s_1 s_2^{-1} s_1$	$\mapsto$	$b_1 + b_2 + b_3 + b_5 + b_6 + b_7 + b_{10} + b_{11} + b_{12} + b_{13} + b_{15}$

Table 2. The map $\mathbb{F}_4 Q_8 \twoheadrightarrow BW_3$

$b_1 s$	$=$	b_2	$b_2 s$	$=$	$b_1 + b_2 + b_7$	$b_3 s$	$=$	b_5
$b_4 s$	$=$	$b_4 + b_6 + b_{10}$	$b_5 s$	$=$	$b_3 + b_5 + b_{13}$	$b_6 s$	$=$	b_4
$b_7 s$	$=$	b_7	$b_8 s$	$=$	b_9	$b_9 s$	$=$	$b_7 + b_8 + b_9$
$b_{10} s$	$=$	b_{10}	$b_{11} s$	$=$	b_{12}	$b_{12} s$	$=$	$b_{10} + b_{11} + b_{12}$
$b_{13} s$	$=$	b_{13}	$b_{14} s$	$=$	b_{15}	$b_{15} s$	$=$	$b_{13} + b_{14} + b_{15}$

Table 3. Multiplication by s in BW_3

6. A ternary Birman-Wenzl algebra

6.1. **Birman-Wenzl algebras.** If k is a ring and $x, \lambda, q \in k^\times$, $\delta \in k$ with $\delta = q - q^{-1}$, and $x\delta = \delta - \lambda + \lambda^{-1}$ the Birman-Wenzl algebra BW_n is defined by generators $s_1^{\pm}, \dots, s_{n-1}^{\pm}$, $e_1, \dots, e_{n-1}$ and relations

(i) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

(ii) $s_i s_j = s_j s_i$ for $|j-i| \geq 2$
(iii) $e_i s_{i-1}^{\pm 1} e_i = \lambda^{\mp 1} e_i$
(iv) $s_i - s_i^{-1} = \delta(1-e_i)$
(v) $s_i e_j = e_j s_i$ for $|j-i| \geq 2$
(vi) $e_i e_j = e_j e_i$ for $|j-i| \geq 2$
(vii) $s_i e_i = e_i s_i = \lambda e_i$
(viii) $s_i s_j e_i = e_j e_i = e_j s_i s_j$ for $|j-i| = 1$
(ix) $e_i^2 = x e_i$
(x) $e_i e_{i\pm 1} e_i = e_i$.

It is a free k-module of dimension $(2n-1).(2n-3).\dots.3.1$, isomorphic to Kaufmann's tangle algebra (see [MW]). In case δ is invertible, the e_i can be expressed in terms of the s_i. This algebra can then be described as the quotient of the group algebra kB_n with relations (3), (7) ,(8), (9), (10) where e_i is defined as $1-\delta^{-1}(s_i - s_i^{-1})$. Relation (7) is then equivalent to (7') : $(s_i - \lambda)(s_i + q^{-1})(s_i - q) = 0$, and a straightforward calculation shows that it implies (9). Now notice that the pair (s_i, s_{i+1}) is conjugated in B_n to the pair (s_{i+1}, s_i), hence (3) can be rewritten as $e_i s_j^{\pm 1} e_i = \lambda^{\mp 1} e_i$ whenever $|j-i| = 1$. Then (10) is easily seen to be a consequence of (3) and (9), hence of (3) and (7). The relation (8) can be shown to be implied by (3) and (7') (see [We] §3). Finally, note that conjugation in the braid groups shows that (3) is equivalent to (3') : $e_1 s_2^{\pm 1} e_1 = \lambda^{\mp 1} e_1$.

A natural quotient of BW_n is obtained by adding the relation $e_i = 0$, or equivalently $s_i - s_i^{-1} = \delta$. This quotient is naturally isomorphic to the Hecke algebra $kB_n/(s_i - q)(s_i + q^{-1})$.

We now specialize to the specific instance we are interested in, by taking $k = \mathbb{F}_4$, $q = j \in \mathbb{F}_4 \setminus \mathbb{F}_2$ hence $\delta = 1$, and $x = 1$. Then relation (7') is $s_i^3 = 1$, which means that BW_n is the quotient of $k\Gamma_n$ by the relations $(3')$, which we split as the two relations $(3'_\pm) : e_1 s_2^{\pm} e_1 = e_1$. It can be checked (e.g. by computer) that the ideals generated by $(3'_+)$ and $(3'_-)$ have dimension 8 in $k\Gamma_3$, while their sum has dimension 9, as is known by $\dim BW_3 = 15$. Note that the relations $(3'_\pm)$ can also be rewritten in our case $e_1(s_2^{\pm} + 1)e_1 = 0$.

Definition 6.1. *Let $r_w^{\pm} = e_1(s_2^{\pm} + 1)e_1 \in k\Gamma_3$, that is, writing $\Gamma_3 = Q_8 \rtimes C_3$,*

$$\left\{ \begin{array}{lcl} r_w^+ & = & (1 + \mathbf{i}z + \mathbf{j}z + \mathbf{k}z)(1 + s + s^2) \\ r_w^- & = & (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})(1 + s + s^2) \end{array} \right.$$

We notice for future use that the three 1-dimensional representations of $k\Gamma_n$ factor through BW_n. Indeed, the two non-trivial ones factor through the Hecke algebra $H_n(j, j^2)$, which is a quotient of BW_n, while BW_n admits the representation $s_i \mapsto 1$, $e_i \mapsto 1$, which induces the trivial representation of Γ_n.

6.2. **Another quotient of K_n in characteristic 2.** We use the representation of BW_n in terms of tangles, taking for convention that the product xy of the tangles x and y is obtained by putting y *below* x. Following [MW], a basis for BW_n is given by a basis of the algebra of Brauer diagrams and an arbitrary choice of over and under crossings. The basis chosen for BW_3 is pictured in figure 1, with $s_1 = s = b_2$; the morphism $kQ_8 \to BW_3$ and the multiplication on the right by $s = b_2$ are tabulated in tables 2 and 3, respectively.

Let $\varphi \in \mathrm{Aut}(kB_n)$ be defined by $\varphi(s_i) = js_i$. It induces an automorphism of $k\Gamma_n$ of order 3, and $\varphi^3 = \mathrm{Id}$. Let $\mathcal{B}_1^n$ be the kernel of $k\Gamma_n \twoheadrightarrow BW_n$, namely the ideal $(3'_+) + (3'_-)$. We have $\varphi(\mathbf{q}) = \mathbf{q}$, and we let $\mathcal{B}_j^n = \varphi(\mathcal{B}_1^n)$, $\mathcal{B}_{j^2}^n = \varphi^2(\mathcal{B}_1^n)$,

Proposition 6.2. (i) *The natural morphism $k\Gamma_n \twoheadrightarrow BW_n$ factors through K_n*

(ii) *When $n = 3$, its kernel is contained in $J(k\Gamma_3)$*
(iii) *We have $\mathbf{q} \in \mathcal{B}^n = \mathcal{B}_1^n \cap \mathcal{B}_j^n \cap \mathcal{B}_{j^2}^n$*
(iv) $(\mathbf{q}) = \mathcal{B}^3$
(v) $1 + z_3 \in \mathcal{B}_+ = \mathcal{B}_1 + \mathcal{B}_j + \mathcal{B}_{j^2}$
(vi) $k\Gamma_n/\mathcal{B}_+ = kC_3$ *for* $n \geq 5$.
(vii) *For $n = 3$, $\mathcal{B}_+ = J(kQ_8)^2 C_3 = J(k\Gamma_3)^2$.*

Proof. Part (i) means that q is mapped to 0, which can be checked on table 2. Part (ii) is because $k\Gamma_3$ has only 3 simple representations, all coming from BW_n, hence all annihilated by the kernel. Part (iii) follows from $\varphi(\mathbf{q}) = \mathbf{q}$. For part (iv), notice that $\mathcal{B}^3$ is stabilized by φ, hence $\mathcal{B}^3 = I_1 \oplus I_j \oplus I_j^2$ with $I_\alpha = \mathrm{Ker}(\varphi - \alpha)$, as *char.k* $\neq 3$. On the other hand, $k\Gamma_3 = \bigoplus_{i=0}^2 (kQ_8)s^i$ and $\varphi(s^i) = j^i s^i$. Since $\mathcal{B}^3$ is an ideal, $I_j = I_1 s, I_{j^2} = I_1 s^2$, hence $\mathcal{B}_3 = I_1 \rtimes C_3$ with I_1 an ideal of kQ_8. This ideal is the kernel of the natural map $kQ_8 \to BW_3$, and it is easy to determine from table 2. We find $I_1 = k\mathbf{q}$, hence (iv). In Q_8, we have $1 + z = (1 + z\mathbf{i} + z\mathbf{j} + z\mathbf{k}) + z(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$, hence $(1 + z)S \in \mathcal{B}_0$ with $S = 1 + s + s^2$. Since $S, \varphi(S), \varphi^2(S)$ span kC_3, we have $1+z \in \mathcal{B}_+$, which proves (v). We have $\Gamma_5 = \Gamma_5^0 \rtimes C_3$, with Γ_5^0 the normal subgroup of Γ_5 generated by z_3. Since $\mathcal{B}_+$ is invariant under φ, we have $\mathcal{B}_+ = I \rtimes C_3$ with $I \subset k\Gamma_5^0$. Now $1+z_3 \in \mathcal{B}_+$ hence $1+z_3 \in I$, and $k\Gamma_5^0/I = k(\Gamma_5^0/ \ll z_3 \gg) = k$. The ideal I is then the augmentation ideal of $k\Gamma_5^0$, $k\Gamma_5/\mathcal{B}_+ = kC_3$. This implies $k\Gamma_n/\mathcal{B}_+ = kC_3$ for all $n \geq 5$, hence (vi). For $n = 3$, we have similarly $\mathcal{B}_+ = IC_3$ for some ideal I of kQ_8. We have $(1+\mathbf{i})(1+\mathbf{j}) = 1+\mathbf{i}+\mathbf{j}+\mathbf{k}$, and $1+z\mathbf{i}+z\mathbf{j}+z\mathbf{k} = z(1+\mathbf{i})(1+\mathbf{j})+(1+\mathbf{i})^2$ hence $I \subset J(kQ_8)^2$. We know that $\dim J(kQ_8)^2 = 5$ and we compute that $\dim \mathcal{B}_+ = 15$, which proves (vii). ∎

Remark 6.3. *For $n = 4$, $\mathcal{B}_+$ has dimension 639.*

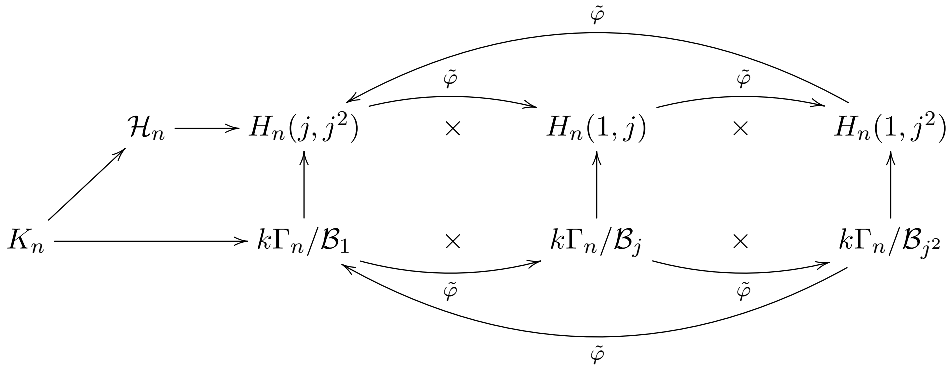
The proposition above enables us to define the following quotient of K_n.

Definition 6.4. *We define the algebra $\mathcal{BMW}_n$ as $k\Gamma_n/\mathcal{B} = k\Gamma_n/(\mathcal{B}_1 \cap \mathcal{B}_j \cap \mathcal{B}_{j^2})$. It is a quotient of K_n.*

6.3. **A natural embedding.** Let $(T_w, w \in \mathfrak{S}_n)$ denote the standard basis of the Hecke algebra under consideration (see [Hu]) and $\ell : \mathfrak{S}_n \to \mathbb{Z}_{\geq 0}$ the Coxeter length. For $\alpha \in \mu_3(k)$, we let $E_n(\alpha) = \sum_{w \in \mathfrak{S}_n} \alpha^{\ell(w)} T_w$. In particular $E_3(\alpha) = \alpha^3 s_1 s_2 s_1 + \alpha^2 s_1 s_2 + \alpha^2 s_2 s_1 + \alpha s_1 + \alpha s_2 + 1 = s_1 s_2 s_1 + \alpha^2 s_1 s_2 + \alpha^2 s_2 s_1 + \alpha s_1 + \alpha s_2 + 1$. We recall from ([GL], §4.3) that the Temperley-Lieb algebra $TL_n(1, j)$ is $H_n(1, j)/E_3(j^2) = H_n(1, j)/E_3(j^{-1})$ (notice that a slight renormalization of the Artin generators is needed from the original formulations there). It has dimension the n-th Catalan number $C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$.

Let $\{\alpha, \beta, \gamma\} = \mu_3$. We introduce the involutive automorphism τ_γ of $k\Gamma_n$ defined by $s_i \mapsto \gamma^2 s_i^{-1}$. It maps $(s_i+\alpha)(s_i+\beta)$ to $s\alpha\beta(s+\beta)(s+\alpha)$, hence induces an involutive automorphism $\tau_{\alpha,\beta}$ of $H_n(\alpha, \beta)$. The automorphism φ induces isomorphisms $\tilde{\varphi} : H_n(\alpha, \beta) \to H_n(j^2\alpha, j^2\beta)$ making the natural diagram commute.

$$\begin{array}{ccc} k\Gamma_n & \xrightarrow{\varphi} & k\Gamma_n \\ \downarrow & & \downarrow \\ H_n(\alpha, \beta) & \xrightarrow[\tilde{\varphi}]{} & H_n(j^2\alpha, j^2\beta) \end{array}$$



Let $ITL_n^j(1,j)$ denote the ideal of $H_n(1,j)$ generated by $E_3(j^2) = E_3(j^{-1})$. Then $ITL_n^1(1,j) = \tau_{1,j} ITL_n^+(1,j)$ is the ideal generated by $\tau_{1,j}E_3(j^{-1}) = E_3(1)$. A straightforward computation shows more generally that $\tau_\gamma E_3(\alpha^{-1}) = E_3(\beta^{-1})$. We define more generally

Definition 6.5. *For $\{\alpha,\beta,\gamma\} = \mu_3(k)$, we define $ITL_n^\alpha(\alpha,\beta) = ITL_n^\alpha(\beta,\alpha)$ as the 2-sided ideal of $H_n(\alpha,\beta)$ generated by $E_3(\alpha^{-1})$.*

With this definition, we have $\tau_{\alpha,\beta}(ITL_n^\alpha(\alpha,\beta)) = ITL_n^\beta(\beta,\alpha)$. Moreover, we have $\varphi(E_3(x)) = E_3(jx)$, hence $\tilde{\varphi}$ maps $ITL_n^\alpha(\alpha,\beta) \subset H_n(\alpha,\beta)$ to the ideal of $H_n(j^2\alpha, j^2\beta)$ generated by $E_3(j\alpha^{-1}) = E_3((j^2\alpha)^{-1})$, that is $ITL_n^{j^2\alpha}(j^2\alpha, j^2\beta)$.

Lemma 6.6. *In $H_n(\alpha,\beta)$, let $M_n(\alpha)$ and $M_n(\beta)$ denote the kernels of the natural morphisms $H_n(\alpha,\beta) \to k$ defined by $s_i \mapsto \alpha$ and $s_i \mapsto \beta$, respectively. We have*

(i) *$ITL_n^\alpha(\alpha,\beta) \subset M_n(\alpha) \cap M_n(\beta)$*
(ii) *$ITL_n^\alpha(\alpha,\beta) + ITL_n^\beta(\alpha,\beta) = M_n(\alpha) \cap M_n(\beta)$ for $n \geq 5$.*
(iii) *For all n, $M_n(\alpha) \cap M_n(\beta)$ is generated by $s_1s_2 + s_2s_1$.*

Proof. Part (i) comes from the fact that $E_3(\alpha^{-1})$ is mapped to 0 under both $s_i \mapsto \alpha$ and $s_i \mapsto \beta$, as is easily checked. We now deal with part (ii). For $n = 5$ We check by computer that $s_1s_2s_4 + 1 \in I = ITL_n^\alpha(\alpha,\beta) + ITL_n^\beta(\alpha,\beta)$ when $n = 5$, hence for $n \geq 5$. It follows that $H_n(\alpha,\beta)/I$ is a quotient of $k\Gamma_n/N$, where N is the normal subgroup of Γ_n generated by $w = s_1s_2s_4$. Note that $w \in \Gamma_n^0 = \mathrm{Ker}(\Gamma_n \twoheadrightarrow C_3)$. In particular, for $n = 5$, w belongs to $\mathrm{Sp}_4(\mathbb{F}_3)$, and one easily check that $N = \mathrm{Sp}_4(\mathbb{F}_3) = \Gamma_n^0$ in this case, by quasi-simplicity of $\mathrm{Sp}_4(\mathbb{F}_3)$. By Theorem 2.4 (vi) it follows that $N = \Gamma_n^0$ for all $n \geq 5$. Thus $H_n(\alpha,\beta)/I$ is a quotient of $kC_3 = k[s]/(s^3-1)$ of dimension at least 2, and even of $k[s]/(s+\alpha)(s+\beta)$, which has dimension 2. It follows that $H_n(\alpha,\beta)/I$ has dimension 2 hence $I = M_n(\alpha)\cap M_n(\beta)$ hence (ii). In order to prove (iii), we first note that $x = s_1s_2 + s_2s_1$ is mapped to 0 under the maps $s_i \mapsto \alpha$ and $s_i \mapsto \beta$, hence $(x) \subset K = M_n(\alpha) \cap M_n(\beta)$. It is then sufficient to show that $H_n(\alpha,\beta)/(x)$ has dimension 2. From the presentation of $H_n(\alpha,\beta)$ one gets that adding $s_1s_2 = s_2s_1$ implies $s_i = s_j$ for all i,j hence $H_n(\alpha,\beta)/(x) = k[s]/(s+\alpha)(s+\beta)$ has dimension 2. This proves (iii).

∎

Remark 6.7. *In the characteristic 0 (semisimple) case with generic parameters, the sum of the two copies of the Temperley-Lieb ideals is the whole Hecke algebra for $n \geq 5$, as the corresponding quotient has irreducible representations labelled by the Young diagrams with at most 2 rows and 2 columns, and there are clearly no such diagram of size more than 4.*

Lemma 6.8. *In $H_n(1,j)$, we have $r_w^+ \equiv j^2E_3(j^2), r_w^- \equiv jE_3(j^2), \varphi(r_w^+) \equiv 0, \varphi(r_w^-) \equiv 0, \varphi^2(r_w^+) \equiv jE_3(1)$ and $\varphi^2(r_w^-) \equiv j^2E_3(1)$.*

Proof. Straightforward computation from the equations $s_i^2 + j^2 s_i + j = 0$ and $s_i^{-1} = j^2 s_i + j$. ∎

Lemma 6.9. *Let $n \geq 3$ and $\pi : k\Gamma_n \twoheadrightarrow \mathcal{H}_n \hookrightarrow H_n(j,j^2) \times H_n(1,j) \times H_n(1,j^2)$. Then $\pi(\mathcal{B}_1) \subset 0 \times ITL_n^j \times ITL_n^{j^2}$, $\pi(\mathcal{B}_j) \subset ITL_n^j \times 0 \times ITL_n^1$, $\pi(\mathcal{B}_{j^2}) \subset ITL_n^{j^2} \times ITL_n^1 \times 0$*

Proof. Let $p_\gamma : \mathcal{H}_n \to H_n(\alpha,\beta)$, for $\{\alpha,\beta,\gamma\} = \mu_3$. The induced map from K_n, also denoted by p_γ, factors through $k\Gamma_n/\mathcal{B}_{\gamma^{-1}}$. We have $p_1(\mathcal{B}_1) = 0$, $p_{j^2}(\mathcal{B}_1) = ITL_n^j(1,j)$ by the lemma. We have $p_\gamma \circ \varphi = \tilde{\varphi} \circ p_{\gamma j}$ hence $p_\gamma \circ \varphi^2 = \tilde{\varphi}^2 \circ p_{\gamma j^{-2}}$ and, using the lemma and the commutative diagrams above, $p_j(\mathcal{B}_1) = \tilde{\varphi}(p_{j^2}(\varphi^{-1}(\mathcal{B}_1))) = \tilde{\varphi}(p_{j^2}(\mathcal{B}_{j^2})) = \tilde{\varphi}(ITL_n^1(1,j)) = ITL_n^{j^2}(1,j^2)$. ∎

Proposition 6.10. *Recall $\mathbf{b} = s_1 s_2^{-1} + s_2 s_1^{-1} + s_1^{-1} s_2 + s_2^{-1} s_1$.*

(i) *$\mathbf{b} \in (\mathcal{B}_1 + \mathcal{B}_j) \cap (\mathcal{B}_1 + \mathcal{B}_{j^2}) \cap (\mathcal{B}_j + \mathcal{B}_{j^2})$ for $n \geq 4$*
(ii) *In $H_4(1,j)$, one has $ITL_4^1 \cap ITL_4^j = \{0\}$*
(iii) *For $n \geq 4$, the inclusions of Lemma 6.9 are equalities.*
(iv) *For $n \geq 5$, $\dim k\Gamma_n/(\mathcal{B}_1 + \mathcal{B}_j) = 2\dim TL_n - 1$.*
(v) *$\mathcal{B}_1 + \mathcal{B}_j \cap \mathcal{B}_{j^2} = (\mathcal{B}_1 + \mathcal{B}_j) \cap (\mathcal{B}_1 + \mathcal{B}_{j^2})$ for $n = 4$.*
(vi) *For $n \geq 4$, $\mathbf{b} \in \mathcal{B}_1 + \mathcal{B}_j \cap \mathcal{B}_{j^2}$.*

Proof. For proving (i) one needs to check that $\mathbf{b} \in \mathcal{B}_1 + \mathcal{B}_j$, as $\varphi(\mathbf{b}) = \mathbf{b}$, and one needs to do it only for $n = 4$, which follows from a computer check. (ii) also follows from a computer check. As a consequence of $(\mathbf{b}) \subset \mathcal{B}_1 + \mathcal{B}_j$, $k\Gamma_n/\mathcal{B}_1 + \mathcal{B}_j$ is a quotient of $k\Gamma_n/(\mathbf{b})$, that is of the ternary Hecke algebra. Letting again $\pi : k\Gamma_n \twoheadrightarrow \mathcal{H}_n \subset H_n(1,j) \times H_n(1,j^2) \times H_n(j,j^2)$ denote the natural map, we have $\pi(\mathcal{B}_1 + \mathcal{B}_j) = \pi(\mathcal{B}_1) + \pi(\mathcal{B}_j) \subset ITL_n^j \times ITL_n^j \times (ITL_n^1 + ITL_n^{j^2})$. When $n = 4$ a computer check shows that the two sides of this inclusion have the same dimension (which is 40). Since $ITL_4^1 \cap ITL_4^{j^2} = \{0\}$ by (ii), this implies that the inclusions of Lemma 6.9 are equalities for $n = 4$, say for $\mathcal{B}_1$. This means that $\pi(\mathcal{B}_1)$ contains, for $n = 4$ hence for all $n \geq 4$, the elements $(0, E_3(j^{-1}), 0)$ and $(0, 0, E_3(j^{-2}))$; it follows that, for $n \geq 4$, $\pi(\mathcal{B}_1)$ contains $0 \times ITL_n^j \times ITL_n^{j^2}$, hence is equal to it. Since π commutes with φ this implies (iii) also for all the $\mathcal{B}_\gamma$. For (iv), let $\pi : k\Gamma_n \twoheadrightarrow k\Gamma_n/(\mathbf{b}) = \mathcal{H}_n$ denote the natural projection. Since $\mathbf{b} \in \mathcal{B}_1 + \mathcal{B}_j$, the dimension of $k\Gamma_n/(\mathcal{B}_1 + \mathcal{B}_j)$ is

$$\dim \mathcal{H}_n/\pi(\mathcal{B}_1 + \mathcal{B}_j) = -3 + \dim \frac{H_n(j,j^2) \times H_n(1,j) \times H_n(1,j^2)}{ITL_n^j \times ITL_n^j \times (ITL_n^1 + ITL_n^{j^2})} = -3 + 2\dim TL_n + 2$$

for $n \geq 5$ by Lemma 6.6, which proves (iv). (v) is proved by a direct computer check, and (vi) is a trivial consequence of (v) and (i). ∎

Remark 6.11. *1) $\mathbf{b} \notin \mathcal{B}_1 + \mathcal{B}_j$ for $n = 3$. 2) Using a computer one can prove that the natural map $K_3 \to K_4$ is injective.*

It follows from the proposition that $\pi(\mathcal{B}_1 + \mathcal{B}_j) = ITL_n^j \times ITL_n^j \times (ITL_n^1 + ITL_n^{j^2})$ for all $n \geq 4$. By Lemma 6.6, for $n \geq 5$ this is $ITL_n^j \times ITL_n^j \times M_n(1) \cap M_n(j^2)$, and likewise $\pi(\mathcal{B}_1 + \mathcal{B}_{j^2}) = ITL_n^{j^2} \times (ITL_n^1 + ITL_n^j) \times ITL_n^{j^2}$. Letting $\cap ITL$ denote $ITL_n^\alpha(\alpha,\beta) \cap ITL_n^\beta(\alpha,\beta)$, we have $\pi\left((\mathcal{B}_1 + \mathcal{B}_{j^2}) \cap (\mathcal{B}_1 + \mathcal{B}_j)\right) = \pi(\mathcal{B}_1 + \mathcal{B}_{j^2}) \cap \pi(\mathcal{B}_1 + \mathcal{B}_j) = \cap ITL_n \times ITL_n^j \times ITL_n^{j^2}$, because

$(\mathcal{B}_1+\mathcal{B}_{j^2})\cap(\mathcal{B}_1+\mathcal{B}_j)$ contains $\mathrm{Ker}\,\pi$ for $n\geq 4$ by (1). Also, $\pi(\mathcal{B}_j)\cap\pi(\mathcal{B}_{j^2}) = \cap ITL_n\times 0\times 0$, hence $\pi(\mathcal{B}_1)+\pi(\mathcal{B}_j)\cap\pi(\mathcal{B}_{j^2}) = \pi\left((\mathcal{B}_1+\mathcal{B}_{j^2})\cap(\mathcal{B}_1+\mathcal{B}_j)\right)$. This implies $\pi(\mathcal{B}_1+\mathcal{B}_j\cap\mathcal{B}_{j^2})\subset \cap ITL_n\times ITL_n^j\times ITL_n^{j^2}$. We check by computer that the dimensions on both sides are equal for $n=4$. This proves that $\pi(\mathcal{B}_1+\mathcal{B}_j\cap\mathcal{B}_{j^2})$ contains $(0,E_3(j^{-1}),0)$,$(0,0,E_3(j^{-2})$.

In order to have the property that $\mathcal{B}_1+\mathcal{B}_j\cap\mathcal{B}_{j^2} = (\mathcal{B}_1+\mathcal{B}_{j^2})\cap(\mathcal{B}_1+\mathcal{B}_j)$ it would be sufficient to control $\cap ITL_n$ in the sense that, if $\pi(\mathcal{B}_j\cap\mathcal{B}_{j^2}) = \cap ITL_n$ for some n, and ITL_m for $m\geq n$ is generated by elements in $\cap ITL_n$, this would prove $\pi(\mathcal{B}_j\cap\mathcal{B}_{j^2}) = \cap ITL_m$ for all $m\geq n$. This at first seems not be such an obstacle as, in the semisimple case, $\cap ITL$ is generated by $ab\in\cap ITL_5$ (or ba) with $a=E_3(1)$ and $b=1+j^2s_3+j^2s_4+js_3s_4+js_4s_3+s_3s_4s_3$ a conjugate of $E_3(j^2)$, and ab is clearly in the image of $\mathcal{B}_j\cap\mathcal{B}_{j^2}$. However, by computer calculation, we get that the situation is much more complicated in our case, as shown by the next lemma, which gathers the result of computer calculations.

Lemma 6.12. *Inside $H_n(1,j)$, we have*

(i) $\dim\cap ITL_5 = 38$.
(ii) *For $n=5$, $(ab)=(ba)$ and $\dim(ab)=36$.*
(iii) $\cap ITL_5 = (ab)\oplus kE_5(1)\oplus kE_5(j^2)$
(iv) $\dim\cap ITL_6 = 458$
(v) *For $n=6$, $\dim(ab)=454$, the ideal generated by $\cap ITL_5\subset\cap ITL_6$ has dimension 456 and contains $E_6(1),E_6(j^2)$.*
(vi) $\dim\cap ITL_7 = 4184$
(vii) *For $n=7$, $\dim(ab)=4180$, and $\cap ITL_7$ is generated by $\cap ITL_6\subset\cap ITL_7$.*

The fact that $\cap ITL_7$ is generated by $\cap ITL_6$ is checked as follows : we find randomly a (complicated) element in $\cap ITL_6$ which it generates as an ideal, and check that this element also generates $\cap ITL_7$. Note that the following always holds true.

Lemma 6.13. *For all $n\geq 3$ and $\alpha,\beta\in\mu_3(k)$, we have $E_n(\alpha^{-1})\in H_n(\alpha,\beta)E_3(\alpha^{-1})$.*

Proof. Let $h=\sum_{w\in D}\alpha^{-\ell(w)}T_w$ with $\ell:\mathfrak{S}_n\to\mathbb{Z}$ the Coxeter length and D the representative system of $\mathfrak{S}_n/\mathfrak{S}_3$ consisting of $\mathfrak{S}_3$-reduced elements on the right so that any element $\sigma\in\mathfrak{S}_n$ writes uniquely $\sigma=w\sigma'$ with $w\in D,\sigma'\in\mathfrak{S}_3$ and $\ell(\sigma)=\ell(w)+\ell(\sigma')$ (see [Hu] §1.10). Then clearly $E_n(\alpha^{-1}) = hE_3(\alpha^{-1})$. ∎

Lemma 6.14. *Let $n\geq 5$. Then*

$$\dim\mathcal{BMW}_n = 3(\dim BW_n - 2\dim TL_n + 2) - \dim\frac{(\mathcal{B}_1+\mathcal{B}_j)\cap(\mathcal{B}_1+\mathcal{B}_{j^2})}{\mathcal{B}_1+\mathcal{B}_j\cap\mathcal{B}_{j^2}}$$

Proof. Recall from Proposition 6.10 that $\dim k\Gamma_n/(\mathcal{B}_1+\mathcal{B}_j) = 2\dim TL_n - 1$. We apply Lemma 5.5 with $A=\mathcal{B}_+$, $I=\mathcal{B}_1$, $J=\mathcal{B}_j$, $K=\mathcal{B}_{j^2}$. We get $\dim A/(I\cap J\cap K) = \dim\mathrm{Im}\,d_1 = \dim\mathrm{Ker}\,d_2 - \dim(K+I)\cap(K+J)/(K+I\cap J)$, and, since d_2 is onto, $\dim\mathrm{Ker}\,d_2 = 3\dim\mathcal{B}_+/\mathcal{B}_1 - \dim\mathcal{B}_+/(\mathcal{B}_1+\mathcal{B}_j) = 3\dim k\Gamma_n/\mathcal{B}_1 - 3\dim k\Gamma_n/(\mathcal{B}_1+\mathcal{B}_j)$.

Since $\dim k\Gamma_n/\mathcal{B}_+ = 3$, we get $\dim\mathcal{BMW}_n = \dim k\Gamma_n/I\cap J\cap K = 3+\dim\mathcal{B}_+/(I\cap J\cap K) = 3+3\dim BMW_n - 3(2\dim TL_n - 1) - \dim\frac{(\mathcal{B}_1+\mathcal{B}_j)\cap(\mathcal{B}_1+\mathcal{B}_{j^2})}{\mathcal{B}_1+\mathcal{B}_j\cap\mathcal{B}_{j^2}}$ whence the conclusion. ∎

Remark 6.15. *In particular, since $\dim TL_5 = 42$ and $\dim BMW_5 = 945$ one gets $\dim\mathcal{BMW}_5 = 3\times 863 - \dim\frac{(\mathcal{B}_1+\mathcal{B}_j)\cap(\mathcal{B}_1+\mathcal{B}_{j^2})}{\mathcal{B}_1+\mathcal{B}_j\cap\mathcal{B}_{j^2}}$, to be compared with $\dim\mathbb{F}_2K_5 = 3\times 863$ (see proposition 4.6).*

7. Markov traces

7.1. **Definitions and conditions for $n = 3$.** In this section we deal with Markov traces. We let $K_n = K_n(1)$, and denote K_∞ the direct limit of the K_n under the natural morphisms $K_n \to K_{n+1}$. Letting $A = \mathbb{Z}[u,v]$, we denote AB_∞, $A\Gamma_\infty$ the direct limits of the group algebras AB_n, $A\Gamma_n$, respectively.

Definition 7.1. *A Markov trace is a pair (t, R), where R is a $\mathbb{Z}[u,v]$-module and $t \in \mathrm{Hom}_A(AB_\infty, R)$ satisfying*

- $t(xy) = t(yx)$ *for all* $x, y \in AB_\infty$
- $t(xs_n) = ut(x)$ *for all* $x, y \in AB_{n-1}$
- $t(xs_n^{-1}) = vt(x)$ *for all* $x, y \in AB_{n-1}$

A Markov trace is said to factorize through a quotient H of the A-algebra AB_∞ if it lies in the image of $\mathrm{Hom}_A(H, R) \to \mathrm{Hom}_A(AB_\infty, R)$.

We now assume that t is a Markov trace that factors through K_∞. This means that it factors through $A\Gamma_\infty$, and that $t(g_1\mathbf{q}g_2) = 0$ for all $g_1, g_2 \in \Gamma_\infty$, or equivalently that $t(\mathbf{q}g) = 0$ for all $g \in \Gamma_\infty$, and finally these conditions for $g \in \Gamma_3$ reduce to $t(\mathbf{q}) = t(\mathbf{q}s_1) = t(\mathbf{q}s_1^2) = 0$. A direct computation shows that these equations imply the following.

Lemma 7.2. *If t is a Markov trace that factors through K_∞, then* $4(u^2+v)t(1) = 4(v^2+u)t(1) = 0$ *and* $t(z_3) = -(1+6uv)t(1)$

Notice that a Markov trace factorizing through K_∞ takes values in $At(1) \subset R$, and that, as a consequence of Proposition 4.2, it is uniquely determined by the value of $t(1) \in R$.

It should be noted that $\{z_3\}$ is the only conjugacy class in Γ_3 that does not meet any gs_2^ε for $g \in \Gamma_2$ and $\varepsilon \in \{0,1,2\}$. Let $A\Gamma_\infty$ denote the direct limit of the $A\Gamma_n$. Of course a Markov trace on K_∞ induces a Markov trace on $A\Gamma_\infty$. A Markov trace on $A\Gamma_\infty$ then induces elements $\tau_n \in \mathrm{Hom}_A(A\Gamma_n, R)$ for all n (recall from Theorem 2.4 that $A\Gamma_\infty$ contains the $A\Gamma_n$ for $n \leq 5$). The condition $\tau_n(xy) = \tau_n(yx)$ means that τ_n is actually a function on the conjugacy classes of Γ_n. For instance, a consequence of the special property of $\{z_3\}$ mentionned above is that any such τ_3 is defined uniquely by the values $\tau_3(1)$ and $\tau_3(z_3)$. In the following section we looked at the conjugacy classes of Γ_4 and Γ_5, and checked whether one could define functions τ_4, τ_5 such that τ_4, τ_5 vanish on the ideal generated by $\mathbf{q}$.

7.2. **Conditions for $n = 4$.** In order to shorten computations with words in the s_i's, we will use when convenient the notation $ijk...$ for $s_is_js_k...$, with $\text{-}i$ meaning s_i^{-1} (for instance $s_1s_2^{-1}s_3 = 1\text{-}23$).

Lemma 7.3. *If t is a Markov trace that factors through K_∞, then* $(3u^3+3v^3-5uv-1)t(1) = 0$.

Proof. We consider $x = s_2s_1^{-1}s_3s_2^{-1}$ and $y = s_2^{-1}s_1s_3s_2^2s_3s_2s_1$ in Γ_4. In K_∞ we have $-s_3s_2^2s_3 \equiv s_2s_3^{-1}s_2 + s_2^{-1}s_3s_2 + s_2s_3s_2^{-1} + s_2^{-1}s_3^{-1} + s_3^{-1}s_2^{-1} + s_2 + s_3$. Then $t(y) = -t(s_2^{-1}s_1s_2s_3^{-1}s_2s_2s_1) - t(s_2^{-1}s_1s_2^{-1}s_3s_2s_2s_1) - t(s_2^{-1}s_1s_2s_3s_2^{-1}s_2s_1) - t(s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1) - t(s_2^{-1}s_1s_3^{-1}s_2^{-1}s_2s_1) - t(s_2^{-1}s_1s_2s_2s_1) -$

$t(s_2^{-1}s_1s_3s_2s_1)$. We have

$$\begin{array}{rclclcl}
t(s_2^{-1}s_1s_2s_3^{-1}s_2s_2s_1) & = & t(s_2^{-1}s_1s_2^{-1}s_1s_2s_3^{-1}) & = & v\,t(s_2^{-1}s_1s_2^{-1}s_1s_2) & = & v\,t(s_1s_2s_2^{-1}s_1s_2^{-1}) \\
 & = & v\,t(s_1^{-1}s_2^{-1}) & = & v^3t(1) & & \\
t(s_2^{-1}s_1s_2^{-1}s_3s_2s_2s_1) & = & t(s_2^{-1}s_1s_2^{-1}s_1s_2^{-1}s_3) & = & u\,t(s_2^{-1}s_1s_2^{-1}s_1s_2^{-1}) & = & u\,t(s_1s_2^{-1}s_2^{-1}s_1s_2^{-1}) \\
 & = & u\,t(s_1s_2s_1s_2^{-1}) & = & u\,t(s_2s_1s_2s_2^{-1}) & = & u\,t(s_1s_2) \\
 & = & u^3t(1) & & & & \\
t(s_2^{-1}s_1s_2s_3s_2^{-1}s_2s_1) & = & t(s_1s_2^{-1}s_1s_2s_3) & = & u\,t(s_1s_2^{-1}s_1s_2) & = & u\,t(s_1s_2s_1s_2^{-1}) \\
 & = & u\,t(s_2s_1s_2s_2^{-1}) & = & u\,t(s_1s_2) & = & u^3t(1) \\
t(s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1) & = & t(s_2s_1s_2^{-1}s_1s_2^{-1}s_3^{-1}) & = & v\,t(s_2s_1s_2^{-1}s_1s_2^{-1}) & = & v\,t(s_1s_2^{-1}s_2s_1s_2^{-1}) \\
 & = & v\,t(s_1^{-1}s_2^{-1}) & = & v^3t(1) & & \\
t(s_2^{-1}s_1s_3^{-1}s_2^{-1}s_2s_1) & = & t(s_1s_2^{-1}s_1s_3^{-1}) & = & v\,t(s_1s_2^{-1}s_1) & = & v\,t(s_1^{-1}s_2^{-1}) \\
 & = & v^3t(1) & & & & \\
t(s_2^{-1}s_1s_2s_2s_1) & = & t(s_2^2s_1s_2^2s_1) & = & t(z_3) & = & (-1-6uv)t(1) \\
t(s_2^{-1}s_1s_3s_2s_1) & = & t(s_2s_1s_2^{-1}s_1s_3) & = & u\,t(s_2s_1s_2^{-1}s_1) & = & u\,t(s_1s_2s_1s_2^{-1}) \\
 & = & u\,t(s_2s_1s_2s_2^{-1}) & = & u\,t(s_1s_2) & = & u^3t(1)
\end{array}$$

hence $t(y) = (-3u^3-3v^3+1+6uv)t(1)$. One has $t(x) = t(s_1^{-1}s_3) = uv\,t(1)$. It is easily checked that x and y belong to $G = \mathrm{Ker}(\Gamma_4 \twoheadrightarrow \Gamma_3)$, which is an extra-special group 3^{1+2} which contains $z_4 = (s_1s_2s_3)^4$, hence $(G,G) = Z(G) = Z(\Gamma_4) =< z_4 >$. We prove that $y = xz_4$. From the braid relations we get $(s_1s_2s_3)^3 = 123123123 = 121121321 = s_1s_2s_1^2s_2s_1s_3s_2s_1$, hence $y = xz_4$ means that $s_2^2s_1s_3s_2^2 = s_2s_1^2s_3s_2^2s_1s_2s_3s_1s_2s_1^2s_2s_1$; this comes from the equalities $211322123121121 = 211322121321121 = 211322212321121 = 21312321121 = 211132321121 = 223221121 = 223221212 = 223222122 = 223122 = 221322$. Clearly $x \notin Z(\Gamma_4) = Z(G)$. For an extra-special group, the conjugacy classes not lying in $Z(G)$ are determined by their images in $G/(G,G) = G/Z(G)$, hence x, y are conjugated in G hence in Γ_4. This proves $t(x) = t(y)$ hence $(3u^3+3v^3-5uv-1)t(1) = 0$ in R. ∎

Lemma 7.4. *If t is a Markov trace that factors through K_∞, then $16t(1) = 0$, $4uv\,t(1) = 4t(1)$, $4u^3t(1) = 4v^3t(1) = -4t(1)$.*

Proof. We recall (32-3) = (-232) and (3-23) = −(2-32)−(-232)−(23-2)−(-2-3)−(-3-2)−(2)−(3) and note that $t(z_3) = (-1-6uv)t(1)$, $t(12121) = t(11211) = t(11112) = t(12) = u^2t(1)$. We will compute $t(a)$ and $t(b)$ with $a =$ (2-312-3121) and $b =$ (-3231-2312). It can be checked by hand that, in Γ_4, we have $ac = cb$ with $c =$ (2-13-2), hence $t(a) = t(b)$.

We first compute $t(a) = t$(2-312-3121). We have t(2-312-3121) $= t$(21-32-3121) $= t$(21332-3121) $= t$(213-232121) $= -t$(212-322121)$-t$(21-2322121)$-t$(2123-22121)$-t$(21-2-32121)$-t$(21-3-22121)$-t$(2122121) $- t$(2132121)

- t(212-322121) $= t$(22121212-3) $= v\,t$(22121212) $= v\,t$(22212121) $= v\,t$(12121) $= u^2v\,t(1)$
- t(21-2322121) $= t$(2212121-23) $= u\,t$(2212121-2) $= u\,t$(-22212121) $= u\,t$(212121) $= u\,t$(121212) $= u\,t(z_3)$
- t(2123-22121) $= t$(2123121) $= t$(1212123) $= u\,t$(121212) $= u\,t(z_3)$
- t(21-2-32121) $= t$(212121-2-3) $= v\,t$(212121-2) $= v\,t$(-2212121) $= v\,t$(12121) $= u^2v\,t(1)$
- t(21-3-22121) $= t$(21-3121) $= t$(12121-3) $= v\,t$(12121) $= u^2v\,t(1)$
- t(2122121) $= t$(2122212) $= t$(2112) $= t$(1122) $= v^2t(1)$
- t(2132121) $= t$(2121213) $= u\,t$(212121) $= u\,t$(121212) $= u\,t(z_3)$

hence $t(2\text{–}312\text{–}3121) = (-3u^2v - 3u\,t(z_3) - v^2)t(1) = (-3u^2v + 3u(1+6uv) - v^2)t(1) = (3u + 15u^2v - v^2)t(1)$.

We now compute $t(b) = t(\text{–}3231\text{–}2312)$. We have $t(\text{–}3231\text{–}2312) = t(\text{–}3213\text{–}2312) = -t(\text{–}3212\text{–}3212) - t(\text{–}321\text{–}23212) - t(\text{–}32123\text{–}212) - t(\text{–}321\text{–}2\text{–}312) - t(\text{–}321\text{–}3\text{–}212) - t(\text{–}321212) - t(\text{–}321312)$ and

- $t(\text{–}3212\text{–}3212) = t(\text{–}3121\text{–}3212) = t(1\text{–}32\text{–}31212) = t(1332\text{–}31212) = t(13\text{–}2321212)$
- $t(\text{–}321\text{–}23212) = t(\text{–}3\text{–}1213212) = t(\text{–}1\text{–}3231212) = t(\text{–}133231212) = t(\text{–}132321212) = t(\text{–}123221212) = t(221212\text{–}123) = u\,t(221212\text{–}12) = u\,t(\text{–}12221212) = u\,t(212) = u\,t(122) = u^2v\,t(1)$
- $t(\text{–}32123\text{–}212) = t(\text{–}31213\text{–}212) = t(1\text{–}3231\text{–}212) = t(133231\text{–}212) = t(132321\text{–}212) = t(123221\text{–}212) = t(221\text{–}212123) = ut(221\text{–}21212) = ut(2221\text{–}2121) = ut(1\text{–}2121) = ut(1\text{–}2212) = ut(112) = u^2v\,t(1)$
- $t(\text{–}321\text{–}2\text{–}312) = t(\text{–}3\text{–}121\text{–}312) = t(\text{–}1\text{–}32\text{–}3112) = t(\text{–}1332\text{–}3112) = t(\text{–}13\text{–}232112)$
- $t(\text{–}321\text{–}3\text{–}212) = t(\text{–}32\text{–}31\text{–}212) = t(332\text{–}31\text{–}212) = t(3\text{–}2321\text{–}212)$
- $t(\text{–}321212) = t(21212\text{–}3) = v\,t(21212) = v\,t(22122) = v\,t(22221) = v\,t(21) = v\,t(12) = u^2vt(1)$
- $t(\text{–}321312) = t(\text{–}323112) = t(3323112) = t(3232112) = t(2322112) = t(2211223) = ut(221122) = ut(112222) = ut(112) = u^2v\,t(1)$

We have $t(21212) = t(22122) = t(12222) = t(12) = u^2t(1)$, $t(221212\text{–}12) = t(221212112) = t(112221212) = t(111212) = t(212) = t(122) = uv\,t(1)$, $t(13\text{–}2321212) = t(213\text{–}232121) = (3u + 15u^2v - v^2)t(1)$ as we already computed, hence $t(\text{–}13\text{–}232112) = t(113\text{–}232112) = t(13\text{–}2321121) = t(13\text{–}2321212) = (3u + 15u^2v - v^2)t(1)$, $t(3\text{–}2321\text{–}212) = t(3\text{–}23212212) = t(3\text{–}23212121) = t(13\text{–}2321212) = (3u + 15u^2v - v^2)t(1)$. We thus get $t(\text{–}3231\text{–}2312) = (-3(3u + 15u^2v - v^2) - 4u^2v)t(1) = (-9u + 3v^2 - 49u^2v)t(1)$. We have that (–3231–2312) is conjugated to (2–312–3121) hence $t(b) = (3u + 15u^2v - v^2)t(1) = (-9u + 3v^2 - 49u^2v)t(1)$. Therefore $t(a) = t(b)$ means $(64u^2v + 12u - 4v^2)t(1) = 0$. Since $4v^2t(1) = -4u\,t(1)$ and $4u^2v\,t(1) = (4u^2)v\,t(1) = -4v^2t(1) = 4u\,t(1)$, this means $(64u + 12u + 4u)t(1) = 0$, i.e. $80u\,t(1) = 0$. Since $80 = 16 \times 5$ and we know $2^rt(1) = 0$ for some r, there exists $g, h \in \mathbb{Z}$ with $2^rg + 5h = 1$ hence $80hut(1) = 16ut(1) = 0$. From $4vt(1) = -4u^2t(1)$ we then get $16v\,t(1) = 0$. By Lemma 7.3 we have $(3u^3 + 3v^3 - 5uv - 1)t(1) = 0$, whence $16u\,t(1) = 16vt(1) = 0$ implies $16t(1) = 0$. Moreover, $0 = 4 \times (3u^3 + 3v^3 - 5uv - 1)t(1) = (12u^3 + 12v^3 - 20uv - 4)t(1) = (-12uv - 12uv - 20uv - 4)t(1) = (-44uv - 4)t(1)$ because $4u^3t(1) = -4uv\,t(1) = 4v^3t(1)$. Since $-44t(1) = 4t(1)$. This proves $4uv\,t(1) = 4t(1)$, and $4u^3t(1) = 4v^3t(1) = -4t(1)$.

∎

Remark 7.5. *Over $A = \mathbb{Z}[u,v]/(16, 4(u^2+v), 4(v^2+u), 3u^3+3v^3-5uv-1)$, one can define a 'Markov trace' for $n = 4$ extending a given τ_3 originating from $MT(K_\infty, R)$, namely a linear map $\tau_4 : A\Gamma_4 \to A$ with $\tau_4(xy) = \tau_4(yx)$ and, when $x \in A\Gamma_3$, $\tau_4(xs_3) = u\tau_3(x)$, $\tau_4(xs_3^{-1}) = v\tau_3(x)$. This can be checked as follows : for each one of the 24 conjugacy classes of Γ_4, one takes an element in it and find a word in s_1, s_2, s_3 representing it ; we then get a value for the Markov trace by the implicit algorithm used to prove Proposition 4.2. This class function naturally extends to a trace $\tau_4 : A\Gamma_4 \to A$, and we check that, for each $g_0 \in \Gamma_3$, we have $\tau_4(g_0s_3) = u\tau_3(g_0)$, $\tau_4(g_0s_3^{-1}) = v\tau_3(g_0)$. Finally, we check that this τ_4 factorizes through K_4, that is that $\tau_4(g_1\mathbf{q}g_2) = 0$ for each $g_1, g_2 \in \Gamma_4$ and, as before, $\mathbf{q}$ is the sum of the elements of $Q_8 \subset \Gamma_3$. Since g_1 can be taken in $\Gamma_4/N_{\Gamma_4}(Q_8)$ and g_2 can be taken in $Q_8\backslash\Gamma_4$, there is only 729 conditions $\tau_4(g_1\mathbf{q}g_2) = 0$ to check. Since τ_4 is already a class function this number of equations reduces drastically to 18, so we can check that τ_4 indeed factors through K_4.*

When $n = 5$, we check similarly that there is a linear map $\tau_5 : A\Gamma_5 \to A$ with $\tau_5(xy) = \tau_5(yx)$ and, when $x \in A\Gamma_4$, $\tau_5(xs_4) = u\tau_4(x)$, $\tau_5(xs_4^{-1}) = v\tau_4(x)$: the computations in `GAP` *take only a lot more time, and we use the software* `Macaulay 2` *in order to automatize equality checking inside A. The conditions for t to factorize through K_5 amount to 243 equalities in A, which we check to be true using* `Macaulay 2`*.*

The two lemmas above can be combined to show the following.

Lemma 7.6. *If t is a Markov trace that factors through K_∞, then $(u+v+1)(u+jv+j^2)(u+j^2v+j)t(1) = (u^3+v^3-3uv+1)t(1) = 0$.*

Proof. $(u+v+1)(u+jv+j^2)(u+j^2v+j) = u^3+v^3-3uv+1$ holds true in $\mathbb{Z}[j]$, and $(u^3+v^3-3uv+1)t(1) = 0$ because $u^3+v^3-3uv+1 = (4u^3+4)+(4v^3+4)-2\times(4uv-4)-(3u^3+3v^3-5uv-1)-16$. ∎

7.3. **Markov traces modulo 4.** In this section we prove that Markov traces exist modulo 4. We let $R = (\mathbb{Z}/4\mathbb{Z})[j]$, that is $(\mathbb{Z}/4\mathbb{Z})[x]/(x^2+x+1)$, and consider the reduction $\bar{t} : K_\infty \to R[u,v]\bar{t}(1)$, with values in $R \otimes_{\mathbb{Z}/4\mathbb{Z}} (\mathbb{Z}[u,v]t(1)/4t(1))$. Here we let $\mu_3 = \{1, j, j^2\}$. Since $4\bar{t}(1) = 0$, we have $0 = (3u^3+3v^3-5uv-1)\bar{t}(1) = -(u^3+uv+v^3+1)\bar{t}(1) = -(u+v+1)(u+jv+j^2)(u+j^2v+j)\bar{t}(1)$ hence a natural map

$$R[u,v]/(u^3+v^3+uv+1) \to \tilde{M} = \prod_{\gamma\in\mu_3} R[u,v]/(v+\gamma u+\gamma^2) \simeq R[u]^3.$$

It can be checked (e.g. using `Macaulay 2`) that the intersection of the ideals $(v+\gamma u+\gamma^2)$ in $R[u,v]$ is equal to their product (u^3+v^3+uv+1), so the above map is injective. Now consider the Hecke algebras $H_n(\alpha,\beta)$ over $R = (\mathbb{Z}/4\mathbb{Z})[j]$, their direct limit $H_\infty(\alpha,\beta)$, and introduce their Markov trace $tr_\gamma : H_\infty(\alpha,\beta) \to R[u] \simeq R[u,v]/(v+\gamma u+\gamma^2)$ for $\{\alpha,\beta,\gamma\} = \mu_3$, such that $tr_\gamma(gs_n) = utr_\gamma(g)$ and $tr_\gamma(gs_n^{-1}) = (\gamma u+\gamma^{-1})tr_\gamma(g) = vtr_\gamma(g)$ for $g \in H_n(\alpha,\beta)$.

They extend to Markov traces $K_\infty \to R[u,v]/(v+\gamma u+\gamma^2)$. Then a convenient Markov trace $\bar{t} : K_\infty \to R[u,v]/(u^3+uv+v^3+1)$ can be defined by $\bar{t}(g) = (tr_\gamma(g))_{\gamma\in\mu_3}$; indeed, this defines at first a map to the cyclic $R[u,v]/(u^3+uv+v^3+1)$-module generated by $\bar{t}(1) \in \tilde{M}$, which is free of rank 1 as $R[u,v]/(u^3+uv+v^3+1) \hookrightarrow \tilde{M}$. In particular, $\bar{t}(g) = 0$ for g in the ideal $J_n(\alpha,\beta)$ of $R\Gamma_n$ defining $H_n(\alpha,\beta)$, for every α,β. It follows that $\bar{t}$ vanishes on J, hence factorizes through the direct limit $\mathcal{H}_\infty$ of the $\mathcal{H}_n = R\Gamma_n/J$. Finally the proof of Lemma 5.3 says that $\mathbf{c} \in J$ not only modulo 2 but modulo 4, hence $\bar{t}$ factorizes also through $K_n(1)$, so this $\bar{t}$ is indeed a Markov trace on K_∞.

Proposition 7.7. *Any Markov trace t on K_∞ with $4t(1) = 0$ factorizes through $\mathcal{H}_\infty$, and is induced by the Markov traces of the Hecke algebras $H_\infty(\alpha,\beta)$.*

Remark 7.8.
(i) *Over $(\mathbb{Z}/4\mathbb{Z})[j]$, and even over $\mathbb{Z}[j]$, denoting $\mathbf{b} = s_1s_2^{-1} - s_1^{-1}s_2 + s_2s_1^{-1} - s_2^{-1}s_1$, one still gets that $\mathbf{b}$ belongs to the intersection of the ideals $J_n(\alpha,\beta)$. Do we still have $\mathcal{H}_n = R\Gamma_n/(\mathbf{b})$, for $R = (\mathbb{Z}/4\mathbb{Z})[j]$ or even $R = \mathbb{Z}[j]$?*
(ii) *A natural question is whether the Birman-Wenzl algebra is still a quotient of $R\Gamma_n/(\mathbf{q})$ when $R = (\mathbb{Z}/4\mathbb{Z})[j]$ ($\lambda = 1$, $\delta = j - j^2 = 1+2j$). The answer is no, as a straightforward though tedious calculation shows that, over $\mathbb{Z}[j]$, $\mathbf{q}$ is mapped inside BW_3 to $(1-\delta+\delta^2-\delta^3)b_1 + (-2\delta+\delta^2-\delta^3)b_2 + (\delta^2-3\delta)b_3 + 2b_4 + (2-\delta-\delta^2)b_5 + (3-\delta)b_6 + (\delta^2-\delta^3)b_7 + (\delta-2\delta^2-\delta^3)b_8 + (2\delta+\delta^2+\delta^3)b_9 + (\delta-\delta^3)b_{10} + 2\delta^3 b_{11} + (\delta-\delta^2-2\delta^3)b_{12} + (\delta^2+\delta)b_{13} + (\delta-\delta^2)b_{14} + (\delta-\delta^2)b_{15}$, which is nonzero modulo 4.*

7.4. **Comparison with the claims of** [F1]**.** In order to make the comparison with [F1] easier, we switch our notations to the ones there. We first briefly review the setting used in [F1]. In [F1], elements $z, z' \in \mathbb{C}^{\times}$ are chosen, $A = A(z, z')$ is defined to be the subring of $\mathbb{C}$ generated by z, z', the $K_n(\gamma)$ are defined over A with $\gamma \in A$, and the direct limit $K_\infty = K_\infty(\gamma)$ of the $K_n = K_n(\gamma)$ is introduced. Let K_n^{ab} be quotient of the module K_n by the submodule $[K_n, K_n]$ spanned by the $xy - yx$ for $x, y \in K_n$, and K_∞^{ab} be the direct limit of the K_n^{ab}.

For R some fixed A-module, the following A-modules are defined :

$$\begin{array}{rcl} AF(K_\infty, R) & = & \{t \in \mathrm{Hom}_A(K_\infty, R) \mid t(xs_ny) = zt(xy), t(xs_n^{-1}y) = z't(xy), x, y \in K_n\} \\ MT(K_\infty, R) & = & \{t \in \mathrm{Hom}_A(K_\infty^{\mathrm{ab}}, R) \mid t(xs_ny) = zt(xy), t(xs_n^{-1}y) = z't(xy), x, y \in K_n^{\mathrm{ab}}\} \end{array}$$

Since, for $a, b \in K_{n+1}^{\mathrm{ab}}$, $ab = ba$, we have $t(xs_ny) - zt(xy) = t(yxs_n - zyx)$ and $t(xs_n^{-1}y) - z't(xy) = t(yxs_n^{-1} - z'yx)$. It follows that $MT(K_\infty, R) = \mathrm{Hom}_A(L(K_\infty), R)$ with $L(K_\infty)$ the quotient of K_∞^{ab} by the A-submodule spanned by the $xs_n - zx, xs_n^{-1} - z'x$ for $x \in K_n$.

Then is introduced an A-module M defined as the quotient of K_n by the A-submodule spanned by the $as_ib - zab, as_i^2b - tab$ for $a, b \in K_i$ and $i < n$ (by abusing notations, here K_i means the image of K_i in K_n), and with $t = \gamma z'$. Since K_{n+1} is the sum of the $K_n s_n^{\varepsilon} K_n$ for $\varepsilon \in \{0, 1, 2\}$ we have $AF(K_n, R) = \mathrm{Hom}_A(M, R)$. The author of [F1] incorrectly identifies this space with $R \otimes_A M$. More generally, most of the arguments in [F1] implicitely assume that the A-modules involved are free, which is incorrect in view of our results. In particular, for a nontrivial $t \in MT(K_\infty, R)$ to exist, it is claimed that z, z' have to be related by the relations $(z')^2 = -z, z^2 = -z'$, these coming from $t(\mathbf{q}s_1) = 0$, $t(\mathbf{q}s_1^2) = 0$ (in the notations of [F1], $\mathbf{q}s_1 = R_0$, $\mathbf{q}s_1^2 = R_1$ and $\mathbf{q} = R_2$). Actually, one finds that, if t is such a Markov trace, then $t(\mathbf{q}s_1^2) = 4(z^2 + z')t(1)$, $t(\mathbf{q}s_1) = 4((z')^2 + z)t(1)$ and $t(\mathbf{q}) = t(z_3) + 6zz't(1) + t(1)$, with $z_3 = (s_1s_2)^3$. Of course division by 4 is not licit in general.

8. Appendix : the 25-dimensional representation of $S_4(3)$

A crucial tool for investigating K_n in characteristic 3 has been the 25-dimensional irreducible representation of $S_4(3)$, denoted φ_5 in [AtMod] (see section 4.3.2). We proved and used that it is defined over $\mathbb{F}_3$, and we computed an explicit matrix model for it. We provide in figures 2 to 5 the images of the Artin generators in such a model, so that the reader have the possibility to check some of the computations of this paper. In order to save space, the following convention has been adopted for representing elements in $\mathbb{F}_3$: a dot $\cdot$ represents 0, a black square ■ represents -1 and an empty square □ represents 1.

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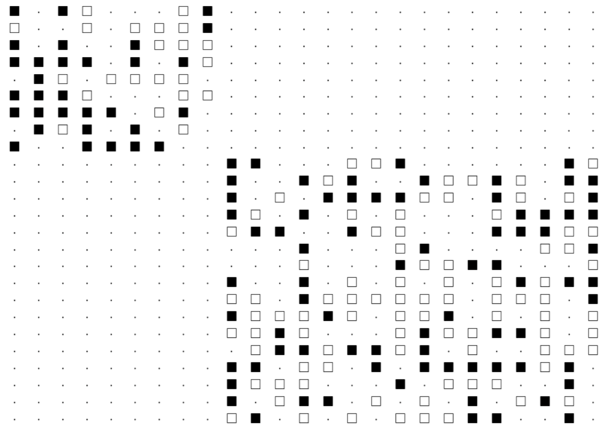

FIGURE 2. $\varphi_5(s_1)$

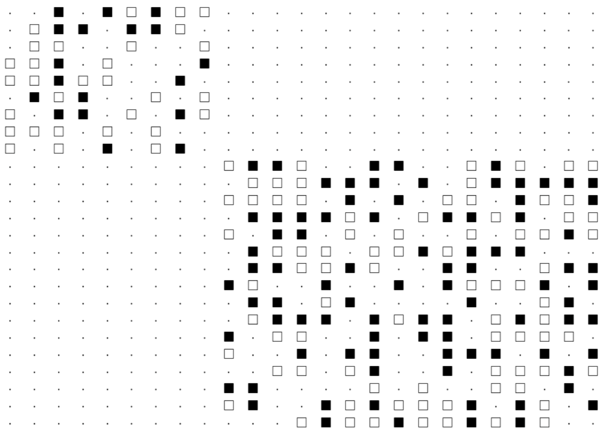

FIGURE 3. $\varphi_5(s_2)$

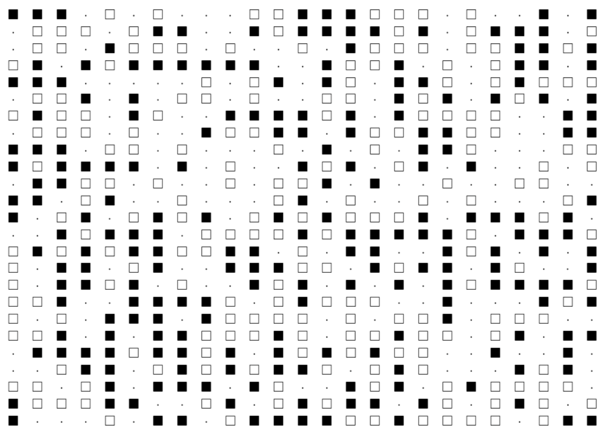

FIGURE 4. $\varphi_5(s_3)$

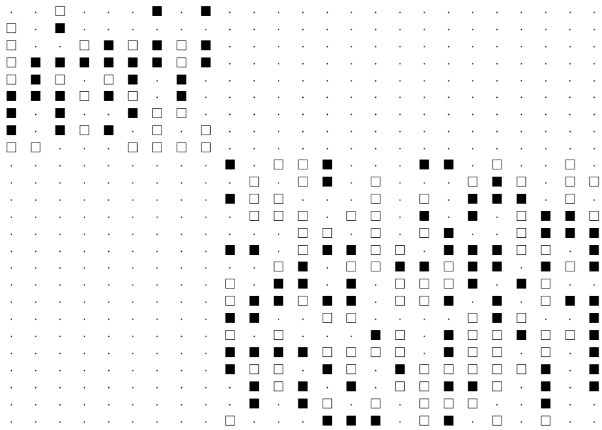

FIGURE 5. $\varphi_5(s_4)$

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