THE CUBIC HECKE ALGEBRA ON AT MOST 5 STRANDS

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To the memory of Johann Gustav Hermes, who worked 10 years on completing the construction of the 65537-gon and on producing the corresponding beautiful artwork of drawings and numbers, these days known as 'Der Koffer' in Göttingen's library.

Abstract. We prove that the quotient of the group algebra of the braid group on 5 strands by a generic cubic relation has finite rank. This was conjectured in 1998 by Broué, Malle and Rouquier and has for consequence that this algebra is a flat deformation of the group algebra of the complex reflection group G_{32}, of order 155,520.

1. Introduction

In 1957 H.S.M. Coxeter proved (see [7]) that the quotient of the braid group B_n on $n \geq 2$ strands by the relations $s_i^k = 1$, where $s_1, \dots, s_{n-1}$ denote the usual Artin generators, is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. This means that, besides the obvious case $k = 2$, which leads to the symmetric group, and the case $n = 2$, there is only a finite number of such groups. They all turn out to be irreducible complex reflection groups, namely finite subgroups of $\mathrm{GL}_n(\mathbf{C})$ generated by endomorphisms which fix an hyperplane (so-called pseudo-reflections), and which leave no proper subspace invariant. In the classical classification of such objects, due to Shephard and Todd, they are nicknamed as G_4, G_8, G_{16} for $n = 3$ and $k = 3, 4, 5$, G_{25}, G_{32} for $n = 4, 5$ and $k = 3$.

In 1998, M. Broué, G. Malle and R. Rouquier conjectured (see [4]) that the group algebra of complex reflection groups admit flat deformations similar to the Hecke algebra of a Weyl or Coxeter group. They actually introduced natural deformations of such group algebras, called them the (generic) Hecke algebra associated to such a group, and they conjectured that these were flat deformations, and in particular that they have finite rank. For the groups we are interested in, this conjecture actually amounts to saying that the quotients of the group algebra RB_n by the relations $s_i^k + a_{k-1}s_i^{k-1} + \dots + a_1 s_i + a_0 = 0$, where $R = \mathbf{Z}[a_{k-1}, \dots, a_1, a_0, a_0^{-1}]$, is a flat deformation of the group algebra RW, where $W = B_n/s_i^k$ (note that we actually use a slightly smaller ring than the one used in [4] and [3]). This conjecture was proved in [3] for all the five groups above but the largest case G_{32} (the proof for G_{25} is however only sketched there).

According to [4] (see the proof of theorem 4.24 there) only the following needs to be proved : that the algebra is spanned over R by $|W|$ elements. This is what we prove here.

Theorem 1.1. *The generic Hecke algebra associated to $W = G_{32}$ is spanned by $|W|$ elements, and is thus a free R-module of rank $|W|$ which becomes isomorphic to the group algebra of W after a suitable extension of scalars.*

More precisely, according to [10] corollary 7.2, a convenient extension of scalars would be $\mathbf{Q}(\zeta_3, (\zeta_3^{-r}u_r)^{\frac{1}{6}}, r = 0, 1, 2)$ where ζ_3 is a primitive 3rd root of 1 and $X^3 + a_2X^2 + a_1X + a_0 = (X - u_0)(X - u_1)(X - u_2)$ or, better, the algebraic extension of $\mathbf{Q}(\zeta_3)(u_0, u_1, u_2)$ generated by $\sqrt{u_0u_1}$ and $\sqrt[3]{u_0u_1u_2}$ (see [10] table 8.2 and proposition 5.1).

In the general setting of complex reflection groups, it is known that this conjecture is true

- for the general series (usually denoted $G(de, e, r)$) of complex reflection groups (by work of Ariki and Ariki-Koike),
- for most of the exceptional groups of rank 2 by [3] and [12], which are numbered G_4 to G_{22}, and by [8] for all exceptional groups of rank 2 over a larger ring than expected,

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- for the Coxeter groups.

The remaining cases are in rank 4 the groups G_{29} ([12] however proves it over the field of fraction by computer means), G_{31}, G_{32}, in rank 5 the group G_{33} and in rank 6 the group G_{34}. All but G_{32}, whose case we settled here, have all their pseudo-reflections of order 2.

In the case studied here, we actually prove more. Here and in the sequel we denote A_n the quotient of RB_n by the generic cubic relation $s_i^3 - as_i^2 - bs_i - c = 0$. The usual embedding $B_n \hookrightarrow B_{n+1}$ induces a natural morphism $A_n \to A_{n+1}$, hence a A_n-bimodule structure on A_{n+1}. For $n \leq 4$, we give a decomposition of A_{n+1} as A_n-bimodule. This immediately provides an explicit R-basis of A_n for $n \leq 5$, made of images of braids in B_n. Recall that the orders of G_4, G_{25} and G_{32} are 24, 648, 155520.

The following theorem is a recollection of the main results of this article : see in particular theorems 3.2, 4.1, 6.21 and 6.26 as well as corollary 5.12, and recall that the argument of [4] theorem 4.24 (which involves a transcendantal monodromy construction) shows that proving that the Hecke algebra of type W is R-generated by $|W|$ elements ensures that this Hecke algebra is free as a R-module, with basis the given $|W|$ elements. Moreover, notice that, if we have an inclusion of parabolic subgroups $W_0 \subset W$ with corresponding Hecke algebras $H_0 \subset H$, knowing the conjecture for H_0 and that H is generated by $|W/W_0|$ elements as a H_0-module proves (1) the conjecture for H and (2) that H is free as a H_0-module, with basis these elements. Indeed, letting $N = |W/W_0|$ the assumption provides a H_0-module morphism $H_0^N \to H$; composing with $(R^{|W_0|})^N \simeq H_0^N$ this yields a surjective morphism $R^{|W|} \to H$ which is an isomorphism by the argument of [4]. This proves that the original morphism $H_0^N \to H$ has no kernel either, and so is an isomorphism.

Theorem 1.2.

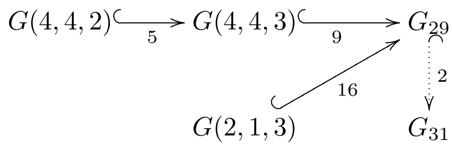
- *Let* $S_2 = \{1, s_1, s_1^{-1}\} \subset B_2$. *One has* $|S_2| = 3$ *and* S_2 *provides an* R*-basis of* A_2.
- *Let* $S_3 = S_2 \sqcup S_2 s_2^{\pm} S_2 \sqcup S_2 s_2^{-1} s_1 s_2^{-1} \subset B_3$. *One has* $|S_3| = 24$ *and* S_3 *provides a* R*-basis of* A_3.
- A_4 *is a free* A_3*-module of rank 27. A basis of this* A_3*-module is provided by elements of the braid group (including* 1*) which map to a system of representatives of* G_{25}/G_4.
- A_4 *is a free* R*-module of rank 648. A basis of this* R*-module is provided by elements of the braid group including* 1 *which map to all* G_{25}.
- A_4 *is a free* $A_2 \otimes_R A_2 \simeq \langle s_1, s_3 \rangle$*-module of rank* 72. *A basis of this* $\langle s_1, s_3 \rangle$*-module is provided by elements of the braid group including* 1 *which map to a system of representatives of* $G_{25}/(\mathbf{Z}/3\mathbf{Z})^2$.
- A_5 *is a free* A_4*-module of rank* 240. *A basis is provided by elements of the braid group including* 1 *which map to a system of representatives of* G_{32}/G_{25}.
- A_5 *is a free* R*-module of rank* 155,520. *A basis of this* R*-module is provided by elements of the braid group which include* 1 *and which map to all* G_{32}.

Corollary 1.3. *The natural map* $A_n \to A_{n+1}$ *is injective for* $2 \leq n \leq 4$.

We describe the plan of the proof. Our method is inductive. We find generators of A_{n+1} as a A_n-bimodule, and only then as a A_n-module. After some preliminaries in section 2 we do the case of A_3 in section 3. The structure of A_4 as a A_3-module is obtained in section 4. Before considering A_5, we provide in section 5 an alternative description of A_4, this time as a $\langle s_1, s_3 \rangle$-module. In addition to providing an alternative proof of the conjecture for A_4, this is used in the decomposition of A_5 as a A_4-module. This decomposition is obtained in section 6. We first obtain a decomposition of A_5 as a A_4-bimodule, and introduce a filtration of A_5 by simpler A_4-bimodules. The latest step of the filtration has original generators originating from the center of the braid group, and this turns out to be the crucial reason why this filtration terminates, thus proving that A_5 is a R-module of finite rank. For proving this crucial property one needs a lengthy calculation which is postponed in section 7. We conclude the section 6 and the proof of the main theorem by studying the structure as A_4-modules of the A_4-bimodules involved there.

1.1. **Perspectives.** It seems likely that our methods can be used to attack the conjecture for other complex reflection groups of higher rank. One indeed has the following standard inclusions of parabolic subgroups (except for the dotted line, which is not a parabolic inclusion). The number associated to the inclusion is the number of double classes. Note again that the groups of rank at least 3 for which the conjecture remains open have all their reflections of order 2.

$$G(3,3,2) \underset{4}{\hookrightarrow} G(3,3,3) \underset{4}{\hookrightarrow} G(3,3,4) \underset{6}{\hookrightarrow} G_{33} \underset{13}{\hookrightarrow} G_{34}$$



For instance, 8 of the 9 double classes of $W = G_{29} = \langle g_1, g_2, g_3, g_4 \rangle$ with respect to $W_0 = G(4,4,3) = \langle g_2, g_3, g_4 \rangle$ have for representatives $g_1^{\varepsilon} z$ for $z \in Z(W)$ and $\varepsilon \in \{0,1\}$. If we had a practical knowledge of the braid groups of type G_{29} and $G(4,4,3)$ of the same level than the one we have for the usual braid group, the methods used here would then probably yield a proof of the conjecture for G_{29} in the same way we managed to get one for G_{32}, as this kind of phenomenon (that the most complicated double classes are mainly represented by central elements) is crucial in our proof. Similarly, if $G_{34} = \langle s_1, \dots, s_6 \rangle$ with $G_{33} = \langle s_1, \dots, s_5 \rangle$, one can check that 12 of the 13 double classes have for representative a term of the form $z s_6^{\varepsilon}$ for $\varepsilon \in \{0,1\}$ and z a central element of G_{34}.

Another natural question is whether similar deformations exist for a higher number of strands. Indeed, although it is known that the groups $\Gamma_n = B_n/s_i^3$ are infinite for $n \geq 6$, it was proved in [1] (see also [5]) that $\Gamma_n^{(1)} = \Gamma_n/z_5^2$ and $\Gamma_n^{(2)} = \Gamma_n/z_5^3$ are finite for arbitrary $n \geq 5$, and are related to symplectic group over $\mathbf{F}_3$ and to unitary groups over $\mathbf{F}_2$, respectively. Here z_5 denotes the image of the generator $(s_1 s_2 s_3 s_4)^5$ of the braid group on 5 strands into $\Gamma_n, n \geq 5$, which has order 6 in Γ_5. It is thus tempting to look for deformations of the group algebras of $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$ for arbitrary n that would be quotients of the group algebra of the braid group by a generic cubic relations *and* other relations probably involving z_5.

1.2. **Applications.** We mention the following consequences. A first one concerns the study of linear representations of the (usual) braid groups. A consequence of the proof in [3] for the cases G_4, G_8 and G_{16} was a classification of the linear representations of the braid group B_3 in which the image of s_1 (and thus of all s_i) is killed by a polynomial of degree at most 5 : indeed, such a representation has to factorize through the corresponding Hecke algebra. This proves that such representations have a very rigid structure, a result rediscovered in [13]. A similar consequence of this new result is a classification of the linear representations of the braid group B_n for n at most 5 in which the image of s_1 is killed by a cubic polynomial.

A second one is about the cubic invariants of knots and links. The algebras connected to cubic invariants, including the Kauffman polynomial and the Links-Gould polynomial, are quotients of A_n. Our result gives the structure of A_5 ; in order to prove it, we actually establish its decomposition as a A_4-bimodule, which may be useful in order to understand the possible Markov traces factorizing through A_n.

Specifically, in [5], we used the representation theory of the group G_{32} to prove that an algebra $K_n(\gamma)$ introduced by L. Funar in [9] for studying knot invariants collapsed for large n over a field of characteristic distinct from 2, and in characteristic 0 for $n \geq 5$. An immediate consequence of the present result is that our argument in characteristic 0 applies verbatim to prove that the deformation $K_n(\alpha, \beta)$ introduced by P. Bellingeri and L. Funar in [2] also collapses for $n \geq 5$. We provide the details below.

Theorem 1.4. *The generic algebra $K_n(\alpha, \beta)$ introduced in* [2] *is zero for $n \geq 5$.*

Proof. We use the notations of [2]. Let $\mathbf{k}$ be an algebraically closed extension of $\mathbf{Q}(\alpha, \beta)$. The $\mathbf{Z}[\alpha, \beta]$-algebra $K_n(\alpha, \beta)$ is defined as the quotient of the group algebra $\mathbf{Z}[\alpha, \beta] B_n$ by the two-sided ideal generated by the elements $s_i^3 - \alpha s_i^2 - \beta s_i - 1$ and another element $q \in \mathbf{Z}[\alpha, \beta] B_3 \subset \mathbf{Z}[\alpha, \beta] B_n$.

We let $\varphi : \mathbf{Z}[a,b,c,c^{-1}] \to \mathbf{Z}[\alpha,\beta]$ be the specialization $a \mapsto \alpha$, $b \mapsto \beta$, $c \mapsto 1$, and let A_n^0 denote $A_n \otimes_\varphi \mathbf{Z}[\alpha,\beta]$. Obviously $K_n(\alpha,\beta)$ is a quotient of A_n^0, more precisely the quotient of A_n^0 by the two-sided ideal generated by (the canonical image of) q. Let $\mathbf{k}$ denote an algebraically closed extension of $\mathbf{Q}(\alpha,\beta)$. We have $A_3^0 \otimes_{\mathbf{Z}[\alpha,\beta]} \mathbf{k} \simeq \mathbf{k}G_4 \simeq \mathbf{k}^3 \oplus Mat_2(\mathbf{k})^3 \oplus Mat_3(\mathbf{k})$, and the ideal generated by q is by definition the factor $\mathbf{k}^3$ in this decomposition (see remark 1.3 in [2]). As a consequence, the $\mathbf{k}$-algebra $\mathbf{k}K_5(\alpha,\beta)$ is the quotient of the semisimple algebra $\mathbf{k}A_5^0 \simeq \mathbf{k}G_{32}$ by the following two-sided ideal : make the direct sum of all the direct factors $Mat_N(\mathbf{k})$ whose corresponding irreducible representations have at least one 1-dimensional component in their restriction to $\mathbf{k}A_3^0$. Now, to the expense of possibly enlarging $\mathbf{k}$, the isomorphisms between the algebras A_n^0 and the corresponding group algebras can be chosen in such a way that the following diagram commutes (e.g. by theorem 2.9 of [11] – see also remark 2.11 there).

$$\begin{array}{ccccc} \mathbf{k}A_3^0 & \longrightarrow & \mathbf{k}A_4^0 & \longrightarrow & \mathbf{k}A_5^0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{k}G_4 & \longrightarrow & \mathbf{k}G_{25} & \longrightarrow & \mathbf{k}G_{32} \end{array}$$

As in [5], the induction table between the (ordinary) characters of G_4 of G_{32} then shows that *all* direct factors $Mat_N(\mathbf{k})$ satisfy this property, and thus the two-sided ideal is all A_5^0. It follows that $K_5(\alpha,\beta) = 0$, whence $K_n(\alpha,\beta) = 0$ for $n \geq 5$, as $K_n(\alpha,\beta)$ is generated by conjugates of the image of $K_5(\alpha,\beta)$.

□

2. Preliminaries and notations

We let $R = \mathbf{Z}[a,b,c,c^{-1}]$ and let B_n denote the braid group on n strands, generated by the braids $s_1,\dots,s_{n-1}$ with relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|j-i| \geq 2$. The cubic Hecke algebra A_n for $n \geq 2$ is the quotient of the group algebra RB_n by the relations $s_i^3 = a s_i^2 + b s_i + c$. We identify s_i to their images in A_n. Notice that, since c is invertible in R, s_i is still invertible, and we have the equivalent relations $s_i^2 = a s_i + b + c s_i^{-1}$, etc.

The group algebra RB_n admits the automorphism $s_i \mapsto s_{n-i}$, which induces an automorphism of A_n, as a R-algebra. The automorphism $s_i \mapsto s_i^{-1}$ of B_n induces an automorphism Φ of A_n as a $\mathbf{Z}$-algebra, defined by $s_i \mapsto s_i^{-1}$, $a \mapsto -bc^{-1}$, $b \mapsto -ac^{-1}$, $c \mapsto c^{-1}$, and similarly the skew-automorphism Ψ of B_n defined by $s_i \mapsto s_i^{-1}$ induces a skew-automorphism of A_n as a $\mathbf{Z}$-algebra.

In the sequel we will denote u_i the R-subalgebra of A_n generated by s_i (or equivalently by s_i^{-1}).

The following equalities hold in the braid group, and thus also in A_n. We state them as a lemma because of their importance in the sequel. Notice that they transform an element of the form $s_{i+1}^{\pm} s_i^{\epsilon} s_{i+1}^{\mp}$ into an element of $u_i u_{i+1} u_i$.

Lemma 2.1. *For $\alpha \in \{-1,1\}$, we have $s_{i+1}^{\alpha} s_i^{\alpha} s_{i+1}^{-\alpha} = s_i^{-\alpha} s_{i+1}^{\alpha} s_i^{\alpha}$ and $s_{i+1}^{\alpha} s_i^{-\alpha} s_{i+1}^{-\alpha} = s_i^{-\alpha} s_{i+1}^{-\alpha} s_i^{\alpha}$, that is*

$$\begin{array}{rcl} s_{i+1} s_i s_{i+1}^{-1} & = & s_i^{-1} s_{i+1} s_i \\ s_{i+1} s_i^{-1} s_{i+1}^{-1} & = & s_i^{-1} s_{i+1}^{-1} s_i \\ s_{i+1}^{-1} s_i s_{i+1} & = & s_i s_{i+1} s_i^{-1} \\ s_{i+1}^{-1} s_i^{-1} s_{i+1} & = & s_i s_{i+1}^{-1} s_i^{-1} \end{array}$$

Lemma 2.2.

(1) $s_{i+1}^{\pm} s_i^{\epsilon} s_{i+1}^{\mp} \in u_i u_{i+1} u_i$
(2) $s_{i+1}^{\pm} s_i^{\pm} s_{i+1}^{\epsilon} \in u_i u_{i+1} u_i$
(3) $s_{i+1}^{\epsilon} s_i^{\pm} s_{i+1}^{\pm} \in u_i u_{i+1} u_i$

Proof. The first item is a direct consequence of lemma 2.1, and the latter two items are consequences of (1) and of the braid relations $s_i^{\pm} s_{i+1}^{\pm} s_i^{\pm} = s_i^{\pm} s_{i+1}^{\pm} s_i^{\pm}$. □

Lemma 2.3.

(1) *For all* $x \in u_i$, $(s_{i+1}^{-1} s_i s_{i+1}^{-1})x \in x(s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i$.
(2) *For all* $x \in u_i$, $(s_{i+1} s_i^{-1} s_{i+1})x \in x(s_{i+1} s_i^{-1} s_{i+1}) + u_i u_{i+1} u_i$.

Proof. (2) is a consequence of (1) up to applying an automorphism of A_n, so we restrict ourselves to proving (1). Since u_i is generated as a R-algebra by s_i^{-1}, we only need to prove $(s_{i+1}^{-1} s_i s_{i+1}^{-1}) s_i^{-1} \in s_i^{-1}(s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i$. We use $s_i = c s_i^{-2} + b s_i^{-1} + a$, $s_i^{-2} = c^{-1} s_i - ac^{-1} - bc^{-1} s_i^{-1}$ and the braid relations, and get

$$\begin{array}{rcl}
(s_{i+1}^{-1} s_i s_{i+1}^{-1}) s_i^{-1} & = & s_{i+1}^{-1} s_i s_{i+1}^{-1} s_i^{-1} \\
& = & s_{i+1}^{-1} (c s_i^{-2} + b s_i^{-1} + a) s_{i+1}^{-1} s_i^{-1} \\
& = & c s_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} s_i^{-1} + b s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1} s_i^{-1} + a s_{i+1}^{-1} s_{i+1}^{-1} s_i^{-1} \\
& = & c s_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} s_i^{-1} + b s_i^{-1} s_{i+1}^{-1} s_i^{-1} s_i^{-1} + a s_{i+1}^{-1} s_{i+1}^{-1} s_i^{-1} \\
& \in & c s_{i+1}^{-1} s_i^{-1} (s_i^{-1} s_{i+1}^{-1} s_i^{-1}) + u_i u_{i+1} u_i \\
& \in & c (s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}) s_i^{-1} s_{i+1}^{-1} + u_i u_{i+1} u_i \\
& \in & c s_i^{-1} s_{i+1}^{-1} s_i^{-2} s_{i+1}^{-1} + u_i u_{i+1} u_i \\
& \in & c s_i^{-1} s_{i+1}^{-1} (c^{-1} s_i - ac^{-1} - bc^{-1} s_i^{-1}) s_{i+1}^{-1} + u_i u_{i+1} u_i \\
& \in & s_i^{-1} s_{i+1}^{-1} (s_i - a - b s_i^{-1}) s_{i+1}^{-1} + u_i u_{i+1} u_i \\
& \in & s_i^{-1} s_{i+1}^{-1} s_i s_{i+1}^{-1} - a s_i^{-1} s_{i+1}^{-1} s_{i+1}^{-1} - b s_i^{-1} (s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1}) + u_i u_{i+1} u_i \\
& \in & s_i^{-1} s_{i+1}^{-1} s_i s_{i+1}^{-1} - a s_i^{-1} s_{i+1}^{-1} s_{i+1}^{-1} - b s_i^{-1} s_i^{-1} s_{i+1}^{-1} s_i^{-1} + u_i u_{i+1} u_i \\
& \in & s_i^{-1} (s_{i+1}^{-1} s_i s_{i+1}^{-1}) + u_i u_{i+1} u_i
\end{array}$$

□

Lemma 2.4. $s_{i+1}^{-1} s_i s_{i+1}^{-1} \in c^{-1}(s_{i+1} s_i^{-1} s_{i+1}) s_i + u_i u_{i+1} u_i$

Proof. We have $(s_{i+1} s_i^{-1} s_{i+1}) s_i = s_{i+1}(s_i^{-1} s_{i+1} s_i) = s_{i+1} s_{i+1} s_i s_{i+1}^{-1} = s_{i+1}^2 s_i s_{i+1}^{-1}$ by lemma 2.1. Since $s_{i+1}^2 = a s_{i+1} + b + c s_{i+1}^{-1}$ we get $(s_{i+1} s_i^{-1} s_{i+1}) s_i = (a s_{i+1} + b + c s_{i+1}^{-1}) s_i s_{i+1}^{-1} = a s_{i+1} s_i s_{i+1}^{-1} + b s_i s_{i+1}^{-1} + c s_{i+1}^{-1} s_i s_{i+1}^{-1} \in c s_{i+1}^{-1} s_i s_{i+1}^{-1} + u_i u_{i+1} u_i$ since $s_{i+1} s_i s_{i+1}^{-1} \in u_i u_{i+1} u_i$ by lemma 2.1. □

3. The algebra A_3

We identify A_2 with its image in A_3 under $s_i \mapsto s_i$, that is with the subalgebra of A_3 generated by s_1. Lemma 2.1 provides the following equalities

$$\begin{array}{rcl}
s_2 s_1 s_2^{-1} & = & s_1^{-1} s_2 s_1 \\
s_2 s_1^{-1} s_2^{-1} & = & s_1^{-1} s_2^{-1} s_1 \\
s_2^{-1} s_1 s_2 & = & s_1 s_2 s_1^{-1} \\
s_2^{-1} s_1^{-1} s_2 & = & s_1 s_2^{-1} s_1^{-1}
\end{array}$$

Lemma 3.1. $s_2^{-1} s_1 s_2^{-1} A_2 \subset A_2 s_2^{-1} s_1 s_2^{-1} + u_2 u_1 u_2$ *and* $s_2 s_1^{-1} s_2 A_2 \subset A_2 s_2 s_1^{-1} s_2 + u_2 u_1 u_2$

Proof. Straightforward consequences of lemma 2.3 □

Theorem 3.2.

(1) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2^{-1} s_1 s_2^{-1} A_2$
(2) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2 s_1^{-1} s_2 A_2$
(3) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2 s_1^{-1} s_2 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + s_2 s_1^{-1} s_2 A_2$
(4) $A_3 = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + A_2 s_2^{-1} s_1 s_2^{-1} = A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2 + s_2^{-1} s_1 s_2^{-1} A_2$

Proof. Up to applying Φ, (2) is a consequence of (1). Then (3) and (4) are consequences of (1) and (2) by the above lemma. We now prove (1), and let U denote its RHS. It is clearly a A_2-submodule of A_3 which contains 1, so we only need to prove $s_2 U \subset U$. Note that, clearly, $u_1 u_2 u_1 \subset U$. We first prove $u_2 u_1 u_2 \subset U$. Since we know $u_1 u_2 \subset U$, $u_2 u_1 \subset U$, this means that $w = s_2^{\alpha} s_1^{\beta} s_2^{\gamma} \in U$ for all $\alpha, \beta, \gamma \in \{-1, 1\}$. If α and β have opposite signs this element belongs to $u_1 u_2 u_1 \subset U$ by lemma 2.1, so we can assume $\alpha = \beta$. If $\alpha = \beta = \gamma$, then the braid relations imply $w \in u_2 u_1 u_2 \subset U$. Thus only remains $w \in \{s_2^{-1} s_1 s_2 - 1, s_2 s_1^{-1} s_2\}$. Clearly $s_2^{-1} s_1 s_2^{-1} \in U$, and $s_2 s_1^{-1} s_2 \in c(s_2^{-1} s_1 s_2^{-1}) s_1^{-1} + u_1 u_2 u_1 \subset u_1 U u_1 = U$ by lemma 2.4. We thus proved $u_2 u_1 u_2 \subset U$. We now prove $s_2 U \subset U$. Clearly $s_2(A_2 + A_2 s_2 A_2 + A_2 s_2^{-1} A_2) \subset u_2 u_1 u_2 u_1 \subset U u_1 \subset U$, so we need to prove $s_2u_1s_2^{-1}s_1s_2^{-1} \subset U$. But $s_2u_1s_2^{-1} \subset u_1u_2u_1$ by lemma 2.1 hence $s_2u_1s_2^{-1}s_1s_2^{-1} \subset u_1u_2u_1u_2 \subset u_1U \subset U$. This proves the claim.

□

Corollary 3.3. *We have $A_3 = u_1u_2u_1u_2 = u_2u_1u_2u_1$. Moreover,*

$$\begin{array}{rclcl} A_3 & = & u_1u_2u_1 + u_2u_1u_2 + Rs_1^{-1}s_2s_1^{-1}s_2 & = & u_1u_2u_1 + u_2u_1u_2 + Rs_2^{-1}s_1s_2^{-1}s_1 \\ & = & u_1u_2u_1 + u_2u_1u_2 + Rs_1s_2^{-1}s_1s_2 & = & u_1u_2u_1 + u_2u_1u_2 + Rs_2s_1^{-1}s_2s_1 \end{array}$$

Corollary 3.4. *Let $n \geq 3$. For all $1 \leq i,j \leq n-1$, we have in A_n the equality $u_iu_ju_iu_j = u_ju_iu_ju_i$.*

This theorem implies that A_3 is a free R-module of finite rank, consequently that $A_3 \subset A_3 \otimes_R K \simeq Mat_3(K) \oplus Mat_2(K)^3 \oplus K^3$ where K is a sufficiently large extension of the quotient field of R, and the isomorphism is explicitely given by the matrix models of the irreducible representations of A_3. From this it is simply a linear algebra matter to check equalities in A_3, or to express a given element in a given basis. We used this approach to get the following identities in A_3.

Lemma 3.5.

$$\begin{array}{rcl} s_2^{-1}s_1s_2^{-1}s_1s_2^{-1}s_1 & = & \frac{-(c+ab)a}{c^2}s_1 + \frac{a}{c}s_1s_2 + \frac{a}{c}s_1^{-1}s_2s_1\frac{-ab}{c}s_2^{-1}s_1s_2^{-1} + \frac{-ab}{c}s_1^{-1} + \frac{ab}{c^2}s_2s_1 \\ & & +s_1^{-1}s_2s_1^{-1} - \frac{b}{c}s_2^{-1}s_1s_2^{-1}s_1 - \frac{ab^2}{c^2}s_2^{-1}s_1 + \frac{b}{c}s_1^{-1}s_2 \\ & & -\frac{a}{c}s_1s_2^{-1}s_1 + \frac{b}{c}s_2s_1^{-1} - \frac{b^2}{c}s_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1}s_1^{-1} \end{array}$$

Lemma 3.6.

$$\begin{array}{rcl} s_1s_2^{-1}s_1s_2^{-1} & = & s_2^{-1}s_1s_2^{-1}s_1 + \frac{a}{c}s_1s_2 - \frac{a}{c}s_2s_1 - \frac{ab}{c}s_1s_2^{-1} + \frac{ab}{c}s_2^{-1}s_1 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1} \\ s_2s_1^{-1}s_2s_1^{-1} & = & s_2^{-1}s_1s_2^{-1}s_1 + a(s_1^{-1}s_2s_1^{-1} - s_2^{-1}s_1s_2^{-1}) - \frac{ab}{c}s_1s_2^{-1} + \frac{ab}{c}s_1^{-1}s_2 + \frac{b}{c}s_1s_2^{-1}s_1 \\ & & -\frac{b}{c}s_2s_1^{-1}s_2 \\ s_1^{-1}s_2s_1^{-1}s_2 & = & s_2^{-1}s_1s_2^{-1}s_1 + \frac{a}{c}s_1s_2 - as_2^{-1}s_1s_2^{-1} - \frac{a}{c}s_2s_1 + as_1^{-1}s_2s_1^{-1} \\ & & -\frac{ab}{c}s_1s_2^{-1} + \frac{b}{c}s_1s_2^{-1}s_1 + \frac{ab}{c}s_2s_1^{-1} - \frac{b}{c}s_2s_1^{-1}s_2 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1} \end{array}$$

As a consequence, we get

Lemma 3.7.

$$\begin{array}{rcl} s_1s_2^{-1}s_1s_2^{-1} - s_2^{-1}s_1s_2^{-1}s_1 & = & \frac{a}{c}s_1s_2 - \frac{a}{c}s_2s_1 + -\frac{ab}{c}s_1s_2^{-1} + \frac{ab}{c}s_2^{-1}s_1 + bs_2^{-1}s_1^{-1} - bs_1^{-1}s_2^{-1} \\ s_2s_1^{-1}s_2s_1^{-1} - s_1^{-1}s_2s_1^{-1}s_2 & = & \frac{ab}{c}s_1^{-1}s_2 - \frac{a}{c}s_1s_2 + \frac{a}{c}s_2s_1 - \frac{ab}{c}s_2s_1^{-1} - bs_2^{-1}s_1^{-1} + bs_1^{-1}s_2^{-1} \end{array}$$

4. The algebra A_4 as a A_3 (bi)module

We identify A_3 with its image in A_4, and denote $sh(A_3)$ the R-subalgebra of A_4 generated by s_2, s_3, s_4. It is the image of A_3 under the 'shift' morphism $s_i \mapsto s_{i+1}$. The goal of this section is to prove the following theorem.

Theorem 4.1.

(1) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}A_3 + A_3s_3s_2^{-1}s_1s_2^{-1}s_3A_3$
(2) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3$
(3) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}A_3 + s_3s_2^{-1}s_1s_2^{-1}s_3A_3$

We denote U the right-hand side of (1). We notice that $sh(A_3) \subset U$, because of theorem 3.2 (2). Also notice that $\Psi(U) = U$ and $\Phi(U) = U$ because of lemma 2.4.

Lemma 4.2. $u_3A_3u_3 \subset U$.

Proof. By theorem 3.2 we have $A_3 = u_1u_2u_1 + u_1s_2^{-1}s_1s_2^{-1}$ hence $u_3A_3u_3 \subset u_3u_1u_2u_1u_3 + u_3u_1s_2^{-1}s_1s_2^{-1}u_3$. But $u_3u_1u_2u_1u_3 = u_1u_3u_2u_3u_1 \subset u_1sh(A_3)u_1 \subset u_1Uu_1 \subset U$, and $u_3u_1s_2^{-1}s_1s_2^{-1}u_3 = u_1u_3s_2^{-1}s_1s_2^{-1}u_3$ so we need to prove $s_3^{\alpha}s_2^{-1}s_1s_2^{-1}s_3^{\beta} \in U$ for $\alpha, \beta \in \{-1, 1\}$. The case $(\alpha, \beta) = (1, 1)$ is clear by definition of U. When $(\alpha, \beta) = (-1, -1)$, we have $s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} \in c^{-1}s_3^{-1}s_2s_1^{-1}s_2s_1s_3^{-1} + s_3^{-1}u_1u_2u_1s_3^{-1}$ that is $s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} \in s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_1 + u_1s_3^{-1}u_2s_3^{-1}u_1 \subset U + u_1sh(A_3)u_1 \subset U$. When $(\alpha, \beta) = (1, -1)$ we get $s_3s_2^{-1}s_1s_2^{-1}s_3^{-1} = s_3s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2^{-1})s_2 = s_3s_2^{-1}s_1s_3^{-1}s_2^{-1}s_3^{-1}s_2 = (s_3s_2^{-1}s_3^{-1})s_1s_2^{-1}s_3^{-1}s_2 = s_2^{-1}s_3^{-1}(s_2s_1s_2^{-1})s_3^{-1}s_2 \in s_2^{-1}s_3^{-1}u_1u_2u_1s_3^{-1}s_2 \subset s_2^{-1}u_1s_3^{-1}u_2s_3^{-1}u_1s_2 \subset A_3sh(A_3)A_3 \subset U$. The case $(-1, 1)$ is similar.

□

Lemma 4.3. $u_3 A_3 u_3 A_3 u_3 \subset A_3 u_3 A_3 u_3 A_3$

Proof. For $x \in A_3$, we say that x has at most p factors if it belongs to as $u_{\sigma(1)} \dots u_{\sigma(p)}$ for some $\sigma : [1,p] \to \{1,2\}$. By theorem 3.2 the minimal number of factors for such an x is at most 4. We let $x, y \in A_3$, with minimal number of factors p and q, and prove that $u_3 x u_3 y u_3 \subset A_3 u_3 A_3 u_3 A_3$ by induction on (p,q) in lexicographic order. Note that, since $\Psi(U) = U$, we may assume $p \geq q$. Moreover, since $A_3 = u_1 u_2 u_1 u_2$ and $u_3(u_1 u_2 u_1 u_2) u_3 y u_3 = u_1 u_3 u_2 u_1 u_2 u_3 y u_3$, we can assume $p \leq 3$ (hence $q \leq 3$).

The case $q = 0$ is trivial. If $x \in u_1 u_{\sigma(2)} \dots u_{\sigma(p)}$, we have $u_3 x u_3 y u_3 \in u_3 u_1 u_{\sigma(2)} \dots u_{\sigma(p)} u_3 y u_3 = u_1 u_3 u_{\sigma(2)} \dots u_{\sigma(p)} u_3 y u_3$ and we are reduced to the case $(p-1,q)$. Similarly, if $y \in u_{\tau(1)} \dots u_{\tau(q-1)} u_1$, we are reduced to $(p, q-1)$. As a consequence, the only non-trivial case for $p \leq 1$ is $u_3 u_2 u_3 u_2 u_3 \subset sh(A_3) \subset A_3 u_3 A_3 u_3 A_3$ because of theorem 3.2.

We consider the case $(p,q) = (2,1)$. The only nontrivial case is $u_3 u_2 u_1 u_3 u_2 u_3$. We need to prove $s_3^{\alpha} u_2 u_1 s_3^{\beta} u_2 s_3^{\gamma} \subset A_3 u_3 A_3 u_3 A_3$ for all $\alpha, \beta, \gamma \in \{-1, 1\}$. Because $s_3^{\alpha} u_2 u_1 (s_3^{\beta} u_2 s_3^{\gamma}) = (s_3^{\alpha} u_2 s_3^{\beta}) u_1 u_2 s_3^{\gamma}$ this is clear by lemma 2.1 unless α, β and γ are all the same. We thus need to prove $s_3^{\alpha} s_2^{\beta} u_1 s_3^{\alpha} s_2^{\gamma} s_3^{\alpha} \subset A_3 u_3 A_3 u_3 A_3$ for all $\beta, \gamma \in \{-1, 1\}$. Since $s_3^{\alpha} s_2^{\alpha} s_3^{\alpha} = s_2^{\alpha} s_3^{\alpha} s_2^{\alpha}$ we can assume $\beta = -\alpha$ and $\gamma = -\alpha$ and consider $s_3^{\alpha} s_2^{-\alpha} u_1 s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha}$. By lemma 2.4 we have $s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha} \in s_3^{-\alpha} s_2^{\alpha} s_3^{-\alpha} u_2 + u_2 u_3 u_2$ hence $s_3^{\alpha} s_2^{-\alpha} u_1 s_3^{\alpha} s_2^{-\alpha} s_3^{\alpha} \subset s_3^{\alpha} s_2^{-\alpha} u_1 s_3^{-\alpha} s_2^{\alpha} s_3^{-\alpha} u_2 + s_3^{\alpha} s_2^{-\alpha} u_1 u_2 u_3 u_2 \subset s_3^{\alpha} s_2^{-\alpha} u_1 s_3^{-\alpha} s_2^{\alpha} s_3^{-\alpha} u_2 + u_3 A_3 u_3 A_3$ and we already noticed

$$s_3^{\alpha} s_2^{-\alpha} u_1 s_3^{-\alpha} s_2^{\alpha} s_3^{-\alpha} u_2 = (s_3^{\alpha} s_2^{-\alpha} s_3^{-\alpha}) u_1 s_2^{\alpha} s_3^{-\alpha} u_2 \subset u_2 u_3 u_2 u_1 u_2 u_3 u_2 \subset A_3 u_3 A_3 u_3 A_3.$$

All cases for $(p,q) = (2,2)$ can be easily reduced to smaller values by commutation relations. The only a priori irreducible case for $(p,q) = (3,1)$ is $u_3 u_2 u_1 u_2 u_3 u_2 u_3$. Since $u_2 u_3 u_2 u_3 \subset u_2 u_3 u_2 + u_3 u_2 u_3$ by theorem 3.2, we are reduced to case $(2,1)$.

For the case $(p,q) = (3,2)$, we can use a similar argument : the only nontrivial case is $u_3 u_2 u_1 u_2 u_3 u_1 u_2 u_3 = u_3 u_2 u_1 u_2 u_1 u_3 u_2 u_3$ and $u_2 u_1 u_2 u_1 \subset u_2 u_1 u_2 + u_1 u_2 u_1$, and we are reduced to smaller cases.

The only remaining case is thus $(p,q) = (3,3)$. Since $x \in A_3 = u_1 u_2 u_1 + u_1 s_2^{-1} s_1 s_2^{-1}$ and $y \in A_3 = u_1 u_2 u_1 + s_2^{-1} s_1 s_2^{-1} u_1$ we are reduced to considering $s_3^{\alpha} s_2^{-1} s_1 s_2^{-1} s_3^{\beta} s_2^{-1} s_1 s_2^{-1} s_3^{\gamma}$ for $\alpha, \beta, \gamma \in \{-1, 1\}$. Up to applying Φ if necessary, we can assume $\beta = -1$. Then $s_3^{\alpha} s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3^{\gamma} = s_3^{\alpha} s_2^{-1} s_1 s_3^{-1} s_2^{-1} s_3^{-1} s_1 s_2^{-1} s_3^{\gamma} = (s_3^{\alpha} s_2^{-1} s_3^{-1}) s_1 s_2^{-1} s_1 (s_3^{-1} s_2^{-1} s_3^{\gamma}) \subset u_2 u_3 u_2 A_3 u_2 u_3 u_2 \subset A_3 u_3 A_3 u_3 A_3$ by lemma 2.2, and this concludes the proof. ☐

We let $U_0 = A_3 u_3 A_3 + A_3 s_3 s_2^{-1} s_3 A_3 = A_3 u_3 A_3 + A_3 s_3^{-1} s_2 s_3^{-1} A_3 = A_3 sh(A_3) A_3 \subset U$.

Lemma 4.4.

(1) $s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} A_3 \subset A_3 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + U_0$
(2) $s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} A_2 \subset A_2 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + U_0$
(3) $s_3 s_2^{-1} s_1 s_2^{-1} s_3 A_3 \subset A_3 s_3 s_2^{-1} s_1 s_2^{-1} s_3 + U_0$
(4) $s_3 s_2^{-1} s_1 s_2^{-1} s_3 A_2 \subset A_2 s_3 s_2^{-1} s_1 s_2^{-1} s_3 + U_0$

Statements (3) and (4) are consequences of (1) and (2) by application of Φ, and (1) and (2) are immediate consequences of the more detailed lemma below.

Lemma 4.5.

(1) *For all* $x \in A_2$, $s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} x \in x s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} + U_0$.
(2) $(s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) s_2^{-1} \in s_1^{-1} s_2^{-1} s_1 (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + U_0$
(3) *For all* $x \in A_3$, $(s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) x \in x^{s_1} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + U_0$ *(where* $x^{s_1} = s_1^{-1} x s_1$*).*

Proof. We first prove (1). We have

$$\begin{aligned}
(s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) s_1^{-1} &= s_3^{-1} (s_2 s_1^{-1} s_2) s_1^{-1} s_3^{-1} \\
&\in s_3^{-1} s_1^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} + s_3^{-1} u_1 u_2 u_1 s_3^{-1} \\
&\subset s_3^{-1} s_1^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} + u_1 s_3^{-1} u_2 s_3^{-1} u_1 \\
&\subset s_1^{-1} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + A_3 sh(A_3) A_3 \\
&\subset s_1^{-1} (s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}) + U_0
\end{aligned}$$

by lemma 2.3. Since s_1^{-1} generates A_2 this proves (1).

We now prove (2). We have

$$\begin{array}{rcl} (s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} & = & s_3^{-1}(s_2s_1^{-1}s_2)s_3^{-1}s_2^{-1} \\ & \in & cs_3^{-1}(s_2^{-1}s_1s_2^{-1})s_1^{-1}s_3^{-1}s_2^{-1} + s_3^{-1}u_1u_2u_1s_3^{-1}s_2^{-1} \\ & \subset & cs_3^{-1}(s_2^{-1}s_1s_2^{-1})s_1^{-1}s_3^{-1}s_2^{-1} + U_0 \end{array}$$

by lemma 2.4. By lemma 2.3 it follows that

$$\begin{array}{rlcl} & (s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})s_2^{-1} & \in & cs_3^{-1}s_1^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_2^{-1} + U_0 \\ = & cs_1^{-1}s_3^{-1}s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2^{-1}) + U_0 & = & cs_1^{-1}s_3^{-1}s_2^{-1}s_1s_3^{-1}s_2^{-1}s_3^{-1} + U_0 \\ = & cs_1^{-1}(s_3^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_3^{-1} + U_0 & = & cs_1^{-1}s_2^{-1}s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1} + U_0 \\ \subset & cs_1^{-1}s_2^{-1}s_3^{-1}c^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} + U_0 & = & s_1^{-1}s_2^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} + U_0 \end{array}$$

again by lemma 2.4. Since

$$s_1^{-1}s_2^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_1s_3^{-1} \in s_1^{-1}s_2^{-1}s_3^{-1}s_1(s_2s_1^{-1}s_2)s_3^{-1} + U_0 \subset s_1^{-1}s_2^{-1}s_1s_3^{-1}(s_2s_1^{-1}s_2)s_3^{-1} + U_0$$

by lemma 2.3, this proves (2)

Since A_3 is generated by s_1^{-1} and s_2^{-1} and U_0 is a $A_3 - A_3$ submodule of A_4, we need to check (3) only for $x = s_1^{-1}$ and $x = s_2^{-1}$, and we just did. □

Proof of theorem 4.1.

Since $1 \in U$ and U is a A_3-submodule of A_4, in order to prove (1) one need to prove $s_3U \subset U$. Clearly $U \subset A_3u_3A_3u_3A_3$ hence $s_3U \subset u_3A_3u_3A_3u_3A_3 \subset A_3u_3A_3u_3A_3$ by lemma 4.3, and $A_3(u_3A_3u_3)A_3 \subset A_3UA_3 = U$ by lemma 4.2 which proves the claim. Then (2) and (3) are consequences of (1) by lemma 4.4. This concludes the proof of the theorem.

We now let $w^+ = s_3s_2^{-1}s_1s_2^{-1}s_3$, and $w^- = s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} \in A_4$. We recall that $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3 \subset A_4$ is a sub-bimodule, and let $U^+ = A_3w^+ + U_0$.

Let $w_0 = s_3s_2s_1^2s_2s_3$. It is classical that, already in the braid group B_4, w_0 commutes with s_1 and s_2. Thus clearly $A_3w_0A_3 = A_3w_0$ and $A_3w_0^{-1}A_3 = A_3w_0^{-1}$. The lemma below thus provides another explanation of lemma 4.4 above.

Lemma 4.6.

(1) $w_0 \in A_3^\times w^+ + U_0$, $w_0^{-1} \in A_3^\times w^- + U_0$
(2) $U^+ = A_3w_0 + U_0$
(3) $s_3A_3s_3^{-1} \subset U_0$, $s_3^{-1}A_3s_3 \subset U_0$
(4) $s_3A_3s_3 \subset U^+$

Proof. We have $w_0 = s_3(s_2s_1^2s_2)s_3 \in cs_3s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1s_2s_3 + Rs_3s_2^2s_3$. Clearly $s_3s_2^2s_3 \in U_0$ and $s_3(s_2s_1s_2)s_3 = s_3(s_1s_2s_1)s_3 = s_1s_3s_2s_3s_1 \in U_0$. Moreover, by lemmas 2.4 and 2.3, $s_3(s_2s_1^{-1}s_2)s_3 \in cs_3s_1^{-1}(s_2^{-1}s_1s_2^{-1})s_3 + s_3u_1u_2u_1s_3 \subset cs_1^{-1}w^+ + U_0$ and thus $w_0 \in A_3^\times w^+ + U_0$. As a consequence, $w_0^{-1} = s_3^{-1}s_2^{-1}s_1^{-2}s_2^{-1}s_3^{-1} = \Phi(w_0) \in \Phi(A_3^\times)\Phi(w^+) + \Phi(U_0) = A_3^\times w^- + U_0$, and this proves (1). By definition we have $U^+ = A_3w^+ + U_0 \subset A_3(A_3^\times w_0 + U_0) + U_0 \subset A_3w_0 + U_0$, and conversely $A_3w_0 + U_0 \subset A_3(A_3^\times w^+ + U_0) + U_0 \subset U^+$; this proves (2). (3) and (4) are given by the proof of lemma 4.2.

□

An immediate consequence is the following variation on theorem 4.1.

Theorem 4.7.

(1) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0A_3 + A_3w_0^{-1}A_3$
(2) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0 + A_3w_0^{-1}$
(3) $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + w_0A_3 + w_0^{-1}A_3$

From this one easily gets the following generating set of A_4 as A_3-module. Another generating set can be found in [3] §4B.

Proposition 4.8. *As a left A_3-module, A_4 is generated by the 27 elements*

$$\{1, s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}, s_3s_2^{-1}s_1s_2^{-1}s_3, s_3, s_3^{-1}, s_3^{\pm}s_2^{\pm}, s_3^{\pm}s_2^{\pm}s_1^{\pm},$$

$$s_3^{\pm}s_2^{-1}s_1s_2^{-1}, s_3s_2^{-1}s_3, s_3s_2^{-1}s_3s_1^{\pm}, s_3s_2^{-1}s_3s_1s_2^{-1}s_1, s_3s_2^{-1}s_3s_1^{\pm}s_2^{\pm}\}.$$

Proof. We denote S the set of 27 elements of the statement and L its span as a A_3-module. We have $A_4 = A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3$ by theorem 4.1, and clearly $A_3 + A_3s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + A_3s_3s_2^{-1}s_1s_2^{-1}s_3 \subset L$. Moreover, since $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + A_2s_2^{-1}s_1s_2^{-1}$ we have

$$A_3s_3^{\alpha}A_3 = A_3s_3^{\alpha} + \sum_{\substack{\varepsilon\in\{-1,0,1\}\\ \beta\in\{-1,1\}}} A_3s_3^{\alpha}s_2^{\beta}s_1^{\varepsilon} + A_3s_3^{\alpha}s_2^{-1}s_1s_2^{-1} \subset L$$

for any $\alpha \in \{-1,1\}$. It remains to prove $A_3s_3s_2^{-1}s_3A_3 \subset L$. Since $A_3 = A_2 + A_2s_2A_2 + A_2s_2^{-1}A_2 + s_2^{-1}s_1s_2^{-1}A_2$, we have $A_3s_3s_2^{-1}s_3A_3 = A_3s_3s_2^{-1}s_3A_2 + A_3s_3s_2^{-1}s_3A_2s_2^{-1}A_2 + A_3s_3s_2^{-1}s_3A_2s_2A_2 + A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2$. Clearly $A_3s_3s_2^{-1}s_3A_2$ is A_3-spanned by the $s_3s_2^{-1}s_3s_1^{\varepsilon}$ for $\varepsilon \in \{0,1,-1\}$ hence $A_3s_3s_2^{-1}s_3A_2 \subset L$. Now $s_3s_2^{-1}s_3s_2^{-1} \in s_2^{-1}s_3s_2^{-1}s_3 + u_2u_3 + u_3u_2$ by lemma 3.6, hence $A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2 \subset A_3s_3s_2^{-1}s_3s_1s_2^{-1}A_2 + A_3u_3u_2s_1s_2^{-1}A_2 + A_3u_3s_1s_2^{-1}A_2$, that is

$$A_3s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}A_2 \subset A_3s_3s_2^{-1}s_3s_1s_2^{-1}A_2 + A_3u_3A_3.$$

We thus only need to show $A_3s_3s_2^{-1}s_3A_2s_2^{\beta}A_2 \subset L$ for $\beta \in \{-1,1\}$. This module is A_3-spanned by the $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_1^{\gamma}$ for $\alpha,\gamma \in \{0,1,-1\}$. The elements belong to S when $\gamma = 0$, so we can assume $\gamma \in \{-1,1\}$. When $\alpha = 0$, in case $\beta = 1$ we have $s_3(s_2^{-1}s_3s_2)s_1^{\gamma} = s_3s_3s_2s_3^{-1}s_1^{\gamma} \in u_3s_2s_3^{-1}s_1^{\gamma}$. This latter module is spanned by the $s_3^{\varepsilon}s_2s_3^{-1}s_1^{\gamma}$ for $\varepsilon \in \{-1,0,1\}$. In case $\varepsilon = 0$ such an element belongs to $A_3u_3A_3 \subset L$; when $\varepsilon = 1$ we can use $(s_3s_2s_3^{-1})s_1^{\gamma} = s_2^{-1}s_3s_2s_1^{\gamma} \in A_3s_3s_2s_1^{\gamma} \in L$; when $\varepsilon = -1$ we have $s_3^{-1}s_2s_3^{-1}s_1^{\gamma} \in A_3s_3s_2^{-1}s_3s_1^{\gamma}$ by lemmas 2.4 and 2.3, and $s_3s_2^{-1}s_3s_1^{\gamma} \in S$. We can thus assume $\alpha \neq 0$.

We consider first the case $\gamma = -\alpha$. We have $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_1^{-\alpha} = s_3s_2^{-1}s_3s_2^{-\alpha}s_1^{\beta}s_2^{\alpha}$. Then, either $\alpha = 1$ and, by lemma 3.6,

$$(s_3s_2^{-1}s_3s_2^{-1})s_1^{\beta}s_2 \in s_2^{-1}s_3s_2^{-1}s_3s_1^{\beta}s_2 + u_2u_3s_1^{\beta}s_2 + u_3u_2s_1^{\beta}s_2 \subset L,$$

or $\alpha = -1$ and $s_3(s_2^{-1}s_3s_2)s_1^{\beta}s_2^{-1} = s_3s_3s_2s_3^{-1}s_1^{\beta}s_2^{-1} \in u_3s_2s_3^{-1}s_1^{\beta}s_2^{-1}$. This latter module is spanned by the $s_3^{\varepsilon}s_2s_3^{-1}s_1^{\beta}s_2^{-1}$ which clearly belong to L for $\varepsilon = 0$ and, because of $s_3s_2s_3^{-1} = s_2^{-1}s_3s_2$, for $\varepsilon = 1$; in case $\varepsilon = -1$ it is readily shown to belong to L by lemmas 2.4 and 2.3 applied to $s_3^{-1}s_2s_3^{-1}$.

We can now assume $\gamma = \alpha$. In case $\beta = \alpha = \gamma$, we have $s_3s_2^{-1}s_3(s_1^{\alpha}s_2^{\alpha}s_1^{\alpha}) = s_3s_2^{-1}s_3s_2^{\alpha}s_1^{\alpha}s_2^{\alpha}$ and, when $\alpha = 1$ we have $s_3(s_2^{-1}s_3s_2)s_1s_2 = s_3s_3s_2s_3^{-1}s_1s_2 \in u_3s_2s_3^{-1}s_1s_2 \subset L$ by similar arguments as for $u_3s_2s_3^{-1}s_1^{\beta}s_2^{-1}$; when $\alpha = -1$, we have, by lemma 3.6,

$$(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2^{-1} \in s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2^{-1} + u_3u_2s_1^{-1}s_2^{-1} + u_2u_3s_1^{-1}s_2^{-1} \subset L.$$

We thus only need to consider the $s_3s_2^{-1}s_3s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}$. By lemmas 2.4 and 2.3, we have $s_1^{-1}s_2s_1^{-1} \in s_2(s_1s_2^{-1}s_1) + u_2u_1u_2$, and $s_3s_2^{-1}s_3u_2u_1u_2$ belongs to the A_3-span of our list by our previous arguments. It follows that it only remains to consider $s_3s_2^{-1}s_3s_1s_2^{-1}\ s_1$, which belongs to our list, and $s_3(s_2^{-1}s_3s_2)s_1s_2^{-1}\ s_1 = s_3^2s_2s_3^{-1}s_1s_2^{-1}\ s_1$, which lies in the linear span of the $s_3^{\varepsilon}s_2s_3^{-1}s_1s_2^{-1}\ s_1$ for $\varepsilon \in \{-1,0,1\}$. Clearly this element belongs to L in case $\varepsilon = 0$, when $\varepsilon = 1$ it also belongs to L because of $(s_3s_2s_3^{-1})s_1s_2^{-1}\ s_1 = s_2^{-1}s_3s_2s_1s_2^{-1}\ s_1 \in A_3u_3A_3 \subset L$, and when $\varepsilon = -1$ lemmas 2.4 and 2.3 applied to $s_3^{-1}s_2s_3^{-1}$ show that

$$s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}\ s_1 \in A_3s_3s_2^{-1}s_3s_1s_2^{-1}\ s_1 + A_3u_3A_3 \subset L,$$

and this concludes the proof. □

For subsequent use we prove here the following lemma.

Lemma 4.9. $w_0^2 \in A_3^{\times}w_0^{-1} + U^+$.

Proof. We have $w_0^2 = s_3s_2(s_1^2)s_2s_3^2s_2s_1^2s_2s_3 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^2s_2s_3 + Rs_3s_2s_1s_2s_3^2s_2s_1^2s_2s_3 + Rs_3s_2^2s_3^2s_2s_1^2s_2s_3$, and $s_3(s_2s_1s_2)s_3^2s_2s_1^2s_2s_3 = s_3s_1s_2s_1s_3^2s_2s_1^2s_2s_3 = s_1(s_3s_2s_3)s_3s_1s_2s_1^2s_2s_3 = s_1s_2(s_3s_2s_3)s_1s_2s_1^2s_2s_3 = s_1s_2s_2s_3s_2s_1s_2s_1^2s_2s_3 \in U_0^+$ by lemma 4.6, while $s_3s_2^2(s_3^2)s_2s_1^2s_2s_3 \in Rs_3s_2^2s_3^{-1}s_2s_1^2s_2s_3 + Rs_3s_2^2s_3s_2s_1^2s_2s_3 + Rs_3s_2^3s_2s_1^2s_2s_3$, clearly $s_3s_2^3s_2s_1^2s_2s_3 \in U^+$ by lemma 4.6,

$$\begin{array}{rlrl} & s_3s_2^2s_3s_2s_1^2s_2s_3 & = & s_3s_2(s_2s_3s_2)s_1^2s_2s_3 \\ = & s_3s_2s_3s_2s_3s_1^2s_2s_3 & = & (s_3s_2s_3)s_2s_1^2(s_3s_2s_3) \\ = & s_2s_3s_2s_2s_1^2s_2s_3s_2 \in U^+ & & \end{array}$$

by lemma 4.6, and finally

$$(s_3s_2^2s_3^{-1})s_2s_1^2s_2s_3 = s_2^{-1}(s_3^2)s_2^2s_1^2s_2s_3 \in Rs_2^{-1}s_3^{-1}s_2^2s_1^2s_2s_3 + Rs_2^{-1}s_3s_2^2s_1^2s_2s_3 + Rs_2s_1^2s_2s_3 \subset U^+$$

by lemma 4.6.

Thus, $w_0^2 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^2s_2s_3 + U^+$. Now, $s_3s_2s_1^{-1}s_2s_3^2s_2(s_1^2)s_2s_3 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1^{-1}s_2s_3^2s_2s_1s_2s_3 + Rs_3s_2s_1^{-1}s_2s_3^2s_2^2s_3$. We have $s_3s_2s_1^{-1}s_2s_3^2(s_2s_1s_2)s_3 = s_3s_2s_1^{-1}s_2s_3^2s_1s_2s_1s_3 = s_3s_2(s_1^{-1}s_2s_1)s_3^2s_2s_3s_1 = s_3s_2^2s_1(s_2^{-1}s_3^2s_2)s_3s_1 = s_3s_2^2s_1s_3s_2^2s_3^{-1}s_3s_1 = s_3s_2^2s_1s_3s_2^2s_1 \in U^+$ by lemma 4.6. Now $s_3s_2s_1^{-1}s_2(s_3^2)s_2^2s_3 \in Rs_3s_2s_1^{-1}s_2s_3^{-1}s_2^2s_3 + Rs_3s_2s_1^{-1}s_2s_3s_2^2s_3 + Rs_3s_2s_1^{-1}s_2^3s_3$; we have $s_3s_2s_1^{-1}s_2^3s_3 \in U^+$ by lemma 4.6, $s_3s_2s_1^{-1}s_2s_3s_2^2s_3 = s_3s_2s_1^{-1}(s_2s_3s_2)s_2s_3 = s_3s_2s_1^{-1}s_3s_2s_3s_2s_3 = (s_3s_2s_3)s_1^{-1}s_2(s_3s_2s_3) = s_2s_3s_2s_1^{-1}s_2s_2s_3s_2 \in U^+$ by lemma 4.6, and $s_3s_2s_1^{-1}s_2(s_3^{-1}s_2^2s_3) = s_3s_2s_1^{-1}s_2s_2s_3^2s_2^{-1} \in s_3s_2s_1^{-1}s_2^2s_2(R + Rs_3 + Rs_3^{-1})s_2^{-1} \subset U^+$ by lemma 4.6.

Thus $w_0^2 \in R^\times s_3s_2s_1^{-1}s_2s_3^2s_2s_1^{-1}s_2s_3 + U^+$. Now,

$$s_3s_2s_1^{-1}s_2(s_3^2)s_2s_1^{-1}s_2s_3 \in R^\times s_3s_2s_1^{-1}s_2s_3^{-1}s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1^{-1}s_2s_3s_2s_1^{-1}s_2s_3 + Rs_3s_2s_1^{-1}s_2^2s_1^{-1}s_2s_3.$$

We have

$$\begin{array}{rclcl} s_3s_2s_1^{-1}(s_2s_3s_2)s_1^{-1}s_2s_3 & = & s_3s_2s_1^{-1}s_3s_2s_3s_1^{-1}s_2s_3 & = & (s_3s_2s_3)s_1^{-1}s_2s_1^{-1}(s_3s_2s_3) \\ & = & s_2s_3s_2s_1^{-1}s_2s_1^{-1}s_2s_3s_2 & \in & U^+ \end{array}$$

by lemma 4.6, $s_3s_2s_1^{-1}s_2^2s_1^{-1}s_2s_3 \in U^+$ by lemma 4.6, hence $w_0^2 \in R^\times s_3s_2s_1^{-1}s_2s_3^{-1}s_2s_1^{-1}s_2s_3 + U^+$. Using $s_2s_1^{-1}s_2 \in A_2^\times s_2^{-1}s_1s_2^{-1} + u_1u_2u_1$ (see lemmas 2.4 and 2.3), we get $s_3(s_2s_1^{-1}s_2)s_3^{-1}s_2s_1^{-1}s_2s_3 \in A_2^\times s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1^{-1}s_2s_3 + s_3u_1u_2u_1s_3^{-1}s_2s_1^{-1}s_2s_3$. Since

$$s_3u_1u_2u_1s_3^{-1}s_2s_1^{-1}s_2s_3 = u_1(s_3u_2s_3^{-1})u_1s_2s_1^{-1}s_2s_3 \subset u_1s_2^{-1}u_3s_2u_1s_2s_1^{-1}s_2s_3 \subset U^+$$

by lemma 4.6, we have $w_0^2 \in A_2^\times s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1^{-1}s_2s_3 + U^+$. Now

$$\begin{array}{rlclcl} & s_3s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2)s_1^{-1}s_2s_3 & = & s_3s_2^{-1}s_1s_3s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3 & = & s_3s_2^{-1}s_1s_3s_2^{-1}s_1^{-1}(s_3^{-1}s_2s_3) \\ = & s_3s_2^{-1}s_1s_3(s_2^{-1}s_1^{-1}s_2)s_3s_2^{-1} & = & s_3s_2^{-1}s_1s_3s_1s_2^{-1}s_1^{-1}s_3s_2^{-1} & = & s_3s_2^{-1}s_3s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} \end{array}$$

and, using $s_3s_2^{-1}s_3 \in u_2^\times s_3^{-1}s_2s_3^{-1} + u_2u_3u_2$, we get

$$s_3s_2^{-1}s_3s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} \in u_2^\times s_3^{-1}s_2s_3^{-1}s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} + u_2u_3u_2s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1}.$$

We have $u_2u_3u_2s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} \in U^+$ by lemma 4.6, and

$$s_3^{-1}s_2s_3^{-1}s_1^2s_2^{-1}s_1^{-1}s_3s_2^{-1} = s_3^{-1}s_2s_1^2(s_3^{-1}s_2^{-1}s_3)s_1^{-1}s_2^{-1} = s_3^{-1}s_2s_1^2s_2s_3^{-1}s_2^{-1}s_1^{-1}s_2^{-1}.$$

Thus $w_0^2 \in A_3^\times s_3^{-1}s_2s_1^2s_2s_3^{-1}(s_2^{-1}s_1^{-1}s_2^{-1}) + U^+$. Since $s_3^{-1}s_2(s_1^2)s_2s_3^{-1} \in R^\times s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} + Rs_3^{-1}s_2s_1s_2s_3^{-1} + Rs_3^{-1}s_2^2s_3^{-1}$ and clearly $s_3^{-1}s_2^2s_3^{-1} \in U_0$, $s_3^{-1}(s_2s_1s_2)s_3^{-1} = s_3^{-1}s_1s_2s_1s_3^{-1} = s_1s_3^{-1}s_2s_3^{-1}s_1 \in U_0$, we have $w_0^2 \in A_3^\times w^-(s_2^{-1}s_1^{-1}s_2^{-1}) + U^+$, hence $w_0^2 \in A_3^\times w_0^{-1}(s_2^{-1}s_1^{-1}s_2^{-1}) + U^+$ by lemma 4.6 (1). Since w_0 commutes with s_1 and s_2 this yields $w_0^2 \in A_3^\times w_0^{-1} + U^+$.

□

5. The algebra A_4 as a $\langle s_1, s_3\rangle$ (bi)module

Let $B = \langle s_1, s_3\rangle$ denote the subalgebra (with 1) of A_4 generated by s_1 and s_3. In order to describe A_5 as a A_4-module we will need the description of A_4 as a B-module, that we do in this section. Note that this will provide another proof of the conjecture of [4] for A_4.

First note that there are *three* automorphisms or skew-automorphisms of the pair (A_4, B) : in addition to the automorphism Φ and the skew-automorphism Ψ, there is the automorphism $\mathrm{Ad}\,\Delta : x \mapsto \Delta x \Delta^{-1}$, where Δ is the image in A_4 of Garside's Δ in the braid group on 4 strands, that is $\Delta = s_1s_2s_3s_1s_2s_1 = s_1(s_2s_3s_1s_2)s_1$; this automorphism exchanges s_1 and s_3 and fixes s_2.

We denote $A_4^{[0]} = B$, $A_4^{[n+1]} = A_4^{[n]}u_2B = A_4^{[n]} + A_4^{[n]}s_2B + A_4^{[n]}s_2^{-1}B$, and in particular $A_4^{[1]} = B + Bs_2B + Bs_2^{-1}B$.

We first prove several lemmas.

Lemma 5.1.

(1) *For $i,j \in \{1,3\}$ we have $u_2u_iu_2u_ju_2 \subset A_4^{[2]}$.*
(2) *For $i,j,k \in \{1,3\}$ we have $u_2u_iu_2u_ju_ku_2 \subset A_4^{[2]}$ and $u_2u_iu_ju_2u_ku_2 \subset A_4^{[2]}$*

Proof. We prove (1). If $i=j$, up to applying $\mathrm{Ad}\,\Delta$ we can assume $i=j=1$ and the statement is a consequence of the study of A_3, as $u_2u_1u_2u_1u_2 \subset A_3 \subset u_1u_2u_1u_2+u_1u_2u_1$. Thus we can assume $i \neq j$, and by using $\mathrm{Ad}\,\Delta$ and Ψ we only need to consider $X = s_2^{\alpha}s_1^{\beta}s_2^{\gamma}s_3^{\delta}s_2^{\varepsilon}$ with $\alpha,\dots,\varepsilon \in \{-1,1\}$. If $\alpha=-\gamma$ or $\gamma=-\varepsilon$, then we get $X \in A_4^{[2]}$ by using $s_2^{\alpha}s_1^{\beta}s_2^{-\alpha} = s_1^{-\alpha}s_2^{\beta}s_1^{\alpha}$ and $s_2^{\gamma}s_3^{\delta}s_2^{-\gamma} = s_3^{-\gamma}s_2^{\delta}s_3^{\gamma}$. Up to applying Φ we can thus assume $\alpha=\gamma=\varepsilon=1$, that is $X = s_2s_1^{\beta}s_2s_3^{\delta}s_2$. If $\beta=1$ or $\delta=1$ we get $X \in A_4^{[2]}$ by $s_2s_1s_2 = s_1s_2s_1$ and $s_2s_3s_2 = s_3s_2s_3$. One can thus assume $X = s_2s_1^{-1}s_2s_3^{-1}s_2$. By lemmas 2.4 and 2.3 we have $s_2s_1^{-1}s_2 \in u_1^{\times}s_2^{-1}s_1s_2^{-1}+u_1u_2u_1$ hence $X \in u_1^{\times}s_2^{-1}s_1s_2^{-1}s_3s_2+A_4^{[2]}$ and $s_2^{-1}s_1(s_2^{-1}s_3s_2) = s_2^{-1}s_1s_3s_2s_3^{-1} \in A_4^{[2]}$, and this concludes the proof of (1).

We prove (2). Up to applying Ψ we can confine ourselves to prove $u_2u_iu_2u_ju_ku_2 \subset A_4^{[2]}$. By (1) and $u_j^2 = u_j$, $u_k^2 = u_k$ we can assume $j \neq k$, that is $\{j,k\} = \{1,3\}$. Up to applying $\mathrm{Ad}\,\Delta$ we can assume $i=1$, hence we want to prove $u_2u_1u_2u_1u_3u_2 \subset A_4^{[2]}$. We have $u_2u_1u_2u_1 \subset A_3 = u_1u_2u_1u_2+u_1u_2u_1$ hence $u_2u_1u_2u_1u_3u_2 \subset u_1u_2u_1u_2u_3u_2+u_1u_2u_1u_3u_2 \subset A_4^{[2]}$ by (1). □

Lemma 5.2.

$$A_4^{[3]} \subset A_4^{[2]} + \sum_{\alpha,\beta\in\{-1,1\}} Bs_2^{\alpha}(s_1s_3^{-1})^{\beta}s_2^{\alpha}(s_1s_3^{-1})^{\beta}s_2^{\alpha}B + \sum_{\alpha,\beta\in\{-1,1\}} Bs_2^{\alpha}(s_1s_3)^{\beta}s_2^{-\beta}(s_1s_3)^{\beta}s_2^{\varepsilon}B$$

Proof. We only need to prove that all the terms of the form $s_2^{\alpha}s_1^{\beta_1}s_3^{\beta_3}s_2^{\gamma}s_1^{\delta_1}s_3^{\delta_3}s_2^{\varepsilon}$ belong to the RHS, as all the over natural linear generators of $A_4^{[3]}$ belong to $A_4^{[2]}$ by lemma 5.1. We remark that the RHS is stable under Φ, Ψ and $\mathrm{Ad}\,\Delta$.

We first assume $\beta_1 = -\delta_1$. Then

$$s_2^{\alpha}s_1^{\beta_1}s_3^{\beta_3}s_2^{\gamma}s_1^{\delta_1}s_3^{\delta_3}s_2^{\varepsilon} = s_2^{\alpha}s_3^{\beta_3}(s_1^{\beta_1}s_2^{\gamma}s_1^{-\beta_1})s_3^{\delta_3}s_2^{\varepsilon} = s_2^{\alpha}s_3^{\beta_3}s_2^{-\beta_1}s_1^{\gamma}s_2^{\beta_1}s_3^{\delta_3}s_2^{\varepsilon}.$$

If $\alpha = \beta_1$ or $\varepsilon = -\beta_1$, such a term belongs to $A_4^{[2]}$ by $s_2^{\alpha}s_3^{\beta_3}s_2^{-\alpha} = s_3^{-\alpha}s_2^{\beta_3}s_3^{\alpha}$ or $s_2^{-\varepsilon}s_3^{\delta}s_2^{\varepsilon} = s_3^{\varepsilon}s_2^{\delta}s_3^{-\varepsilon}$ and lemma 5.1. We thus only need to consider $X = s_2^{-\beta_1}s_3^{\beta_3}s_2^{-\beta_1}s_1^{\gamma}s_2^{\beta_1}s_3^{\delta_3}s_2^{\beta_1}$. Since $s_2^{-\beta_1}s_3^{-\beta_1}s_2^{-\beta_1} = s_3^{-\beta_1}s_2^{-\beta_1}s_3^{-\beta_1}$ and $s_2^{\beta_1}s_3^{\beta_1}s_2^{\beta_1} = s_3^{\beta_1}s_2^{\beta_1}s_3^{\beta_1}$, by lemma 5.1 we can assume $\beta_3 = \beta_1$ and $\delta = -\beta_1$, that is $X = s_2^{-\beta_1}s_3^{\beta_1}s_2^{-\beta_1}s_1^{\gamma}s_2^{\beta_1}s_3^{-\beta_1}s_2^{\beta_1}$. By applying Φ and Ψ we can assume $X = s_2s_3^{-1}s_2s_1s_2^{-1}s_3s_2^{-1}$. By lemma 2.4 we have $s_2^{-1}s_3s_2^{-1} \in s_2s_3^{-1}s_2u_3^{\times}+u_3u_2u_3$ hence $s_2s_3^{-1}s_2s_1(s_2^{-1}s_3s_2^{-1}) \in s_2s_3^{-1}s_2s_1s_2s_3^{-1}s_2u_3^{\times}+s_2s_3^{-1}s_2s_1u_3u_2u_3$. We have $s_2s_3^{-1}s_2s_1u_3u_2u_3 \subset A_4^{[2]}$ by lemma 5.1 and $s_2s_3^{-1}s_2s_1s_2s_3^{-1}s_2$ belongs to the RHS, which concludes this case.

The case $\beta_3 = -\delta_3$ is a consequence of the previous case by applying $\mathrm{Ad}\,\Delta$. We thus only need to consider $X = s_2^{\alpha}s_3^{\beta_3}s_1^{\beta_1}s_2^{\gamma}s_1^{\beta_1}s_3^{\beta_3}s_2^{\varepsilon}$. First assume $\gamma = \beta_1$. Then $X = s_2^{\alpha}s_3^{\beta_3}(s_1^{\gamma}s_2^{\gamma}s_1^{\gamma})s_3^{\beta_3}s_2^{\varepsilon} = s_2^{\alpha}s_3^{\beta_3}s_2^{\gamma}s_1^{\gamma}s_2^{\gamma}s_3^{\beta_3}s_2^{\varepsilon}$ belongs as before to $A_4^{[2]}$ by lemma 5.1 and elementary transformations, unless $\varepsilon = \gamma$, $\alpha = \gamma$, and then $\beta_3 = -\gamma$. In that case $X = s_2^{\alpha}s_3^{-\alpha}(s_2^{\alpha}s_1^{\alpha}s_2^{\alpha})s_3^{-\alpha}s_2^{\alpha} = s_2^{\alpha}s_3^{-\alpha}s_1^{\alpha}s_2^{\alpha}s_1^{\alpha}s_3^{-\alpha}s_2^{\alpha}$ belongs to the RHS. We can thus assume $\gamma \neq \beta_1$ and, applying $\mathrm{Ad}\,\Delta$, $\gamma \neq \beta_3$, hence we can assume $\beta_1 = \beta_3 = -\gamma$. Then $X = s_2^{\alpha}s_3^{\gamma}s_1^{\gamma}s_2^{-\gamma}s_1^{\gamma}s_3^{\gamma}s_2^{\varepsilon}$, which belongs to the RHS, and this concludes the proof.

□

Lemma 5.3. *Let $\alpha,\beta,\varepsilon \in \{-1,1\}$. Then $s_2^{\alpha}(s_1s_3)^{\beta}s_2^{-\beta}(s_1s_3)^{\beta}s_2^{\varepsilon}$ belongs to*

$$A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs_2^{\delta}(s_1s_3)^{\delta}s_2^{-\delta}(s_1s_3)^{\delta}s_2^{\delta}B + \sum_{\delta\in\{-1,1\}} Bs_2^{-\delta}(s_1s_3)^{\delta}s_2^{-\delta}(s_1s_3)^{\delta}s_2^{-\delta}B$$

Proof. First assume $\alpha = \beta$. Then $X = s_2^\beta s_1^\beta s_3^\beta s_2^{-\beta} s_3^\beta (s_1^\beta s_2^\varepsilon s_1^{-\beta}) s_1^\beta = s_2^\beta s_1^\beta (s_3^\beta s_2^{-\beta} s_3^\beta s_2^{-\beta}) s_1^\varepsilon s_2^\beta s_1^\beta \in s_2^\beta s_1^\beta s_2^{-\beta} s_3^\beta s_2^{-\beta} s_3^\beta s_1^\varepsilon s_2^\beta s_1^\beta + s_2^\beta s_1^\beta u_2 u_3 s_1^\varepsilon s_2^\beta s_1^\beta + s_2^\beta s_1^\beta u_3 u_2 s_1^\varepsilon s_2^\beta s_1^\beta$ by lemma 3.6. Now $s_2^\beta s_1^\beta u_2 u_3 s_1^\varepsilon s_2^\beta s_1^\beta \subset A_4^{[2]}$ and $s_2^\beta s_1^\beta u_3 u_2 s_1^\varepsilon s_2^\beta s_1^\beta \subset A_4^{[2]}$ by lemma 5.1. We thus only need to consider

$$X = (s_2^\beta s_1^\beta s_2^{-\beta}) s_3^\beta s_2^{-\beta} s_3^\beta s_1^\varepsilon s_2^\beta = s_1^{-\beta} s_2^\beta s_1^\beta s_3^\beta s_2^{-\beta} s_3^\beta s_1^\varepsilon s_2^\beta$$

hence, if $\varepsilon = -\beta$, we get

$$X = s_1^{-\beta} . s_2^\beta s_3^\beta (s_1^\beta s_2^{-\beta} s_1^{-\beta}) s_3^\beta s_2^\beta = s_1^{-\beta} . s_2^\beta s_3^\beta s_2^{-\beta} s_1^{-\beta} (s_2^\beta s_3^\beta s_2^\beta) = s_1^{-\beta} . s_2^\beta s_3^\beta s_2^{-\beta} s_1^{-\beta} s_3^\beta s_2^\beta s_3^\beta \in A_4^{[2]}$$

by lemma 5.1. We can thus assume $\varepsilon = \beta$, in which case X clearly belongs to the space we want.

This concludes the case $\alpha = \beta$, hence also the case $\varepsilon = \beta$ by application of Φ and Ψ. Thus we can assume $\alpha = -\beta = \varepsilon$, and the conclusion is clear in this case. □

Lemma 5.4. *For $\alpha, \beta \in \{-1, 1\}$, we have*

$$s_2^\alpha (s_1 s_3^{-1})^\beta s_2^\alpha (s_1 s_3^{-1})^\beta s_2^\alpha \in A_4^{[2]} + \sum_{\delta \in \{-1,1\}} B s_2^\delta (s_1 s_3)^\delta s_2^{-\delta} (s_1 s_3)^\delta s_2^\delta B$$

Proof. The RHS is stable under $\mathrm{Ad}\,\Delta$ and Φ, hence we can assume $\alpha = \beta = 1$, and thus we only need to consider $X = s_2 s_1 s_3^{-1} s_2 s_1 s_3^{-1} s_2 = s_2 s_1 (s_3^{-1} s_2 s_3^{-1}) s_1 s_2 \in s_2 s_1 u_2 s_3 s_2^{-1} s_3 s_1 s_2 + s_2 s_1 u_2 u_3 u_2 s_1 s_2$ by lemmas 2.4 and 2.3. We have

$$s_2 s_1 u_2 u_3 u_2 s_1 s_2 \subset \sum_{a \in \{0,1,-1\}} s_2 s_1 s_2^a u_3 u_2 s_1 s_2$$

and,

- if $a = 0$ we have $s_2 s_1 u_3 u_2 s_1 s_2 \subset A_4^{[2]}$ by lemma 5.1 ;
- if $a = 1$ we have $(s_2 s_1 s_2) u_3 u_2 s_1 s_2 = s_1 s_2 s_1 u_3 u_2 s_1 s_2 \subset A_4^{[2]}$ by lemma 5.1 ;
- if $a = -1$ we have $(s_2 s_1 s_2^{-1}) u_3 u_2 s_1 s_2 = s_1^{-1} s_2 s_1 u_3 u_2 s_1 s_2 \subset A_4^{[2]}$ by lemma 5.1

hence $X \in s_2 s_1 u_2 s_3 s_2^{-1} s_3 s_1 s_2 + A_4^{[2]}$. The module $s_2 s_1 u_2 s_3 s_2^{-1} s_3 s_1 s_2$ is R-spanned by the $Y(a) = s_2 s_1 s_2^a s_3 s_2^{-1} s_3 s_1 s_2$ for $a \in \{-1, 0, 1\}$. We have $Y(0) = s_2 s_1 s_3 s_2^{-1} s_3 s_1 s_2 \in RHS$, $Y(1) = (s_2 s_1 s_2) s_3 s_2^{-1} s_3 s_1 s_2 = s_1 s_2 s_1 s_3 s_2^{-1} s_3 s_1 s_2 \in RHS$ and

$$Y(-1) = (s_2 s_1 s_2^{-1}) s_3 s_2^{-1} s_3 s_1 s_2 = s_1^{-1} s_2 s_1 s_3 s_2^{-1} s_3 s_1 s_2 \in RHS,$$

and this concludes the proof. □

In the braid group on 4 strands , we have

$$\Delta = (s_1 s_2 s_3)(s_1 s_2) s_1 = (s_1 s_3)(s_2 s_1 s_3 s_2) = (s_2 s_1 s_3 s_2)(s_1 s_3)$$

hence the same equalities hold in A_4. In the remaining part of this section, we let $s = s_2$, $p = s_1 s_3 = s_3 s_1$, hence $\Delta = spsp = psps$. Note that $\Delta p = p\Delta$, $\Delta s = s\Delta$. It follows that $\Delta^2 = p.sps.\Delta = p.\Delta.sps = p(psps)sps = p^2.sps^2ps$, $\Delta^3 = p^2.sps^2ps.\Delta = p^2.\Delta.sps^2ps = p^3.sps^2ps^2p$, and $\Delta^4 = p^4.sps^2ps^2ps^2ps$.

We thus have $\Delta^2 = p^2.sps^2ps$ hence $p^{-2}\Delta^2 \in R^\times sps^{-1}ps + Rspsps + Rsp^2s$ and we known $sp^2s \in A_4^{[2]}$, $(spsp)s = psps^2 \in A_4^{[2]}$ by lemma 5.1. It follows that

$$(*) \quad \begin{array}{rcl} p^{-2}\Delta^2 & \in & R^\times sps^{-1}ps + Rpsps^2 + Rsp^2s \\ p^{-2}\Delta^2 & \in & R^\times sps^{-1}ps + A_4^{[2]} \end{array}$$

Applying Φ, we have $\Phi(\Delta) = \Phi(s_1 s_2 s_3 s_1 s_2 s_1) = s_1^{-1} s_2^{-1} s_3^{-1} s_1^{-1} s_2^{-1} s_1^{-1} = (s_1 s_2 s_1 s_3 s_2 s_1)^{-1} = \Delta^{-1}$, hence, since $\Phi(p) = p^{-1}$,

$$(*) \quad p^{-2}\Delta^2 \quad \in \quad R^\times s^{-1} p^{-1} s p^{-1} s^{-1} + A_4^{[2]}$$

Lemma 5.5.

(1) $s_2^{-1} p s_2^{-1} p s_2^{-1} . s_1^{-1} \in u_1^\times s_2 p^{-1} s_2 p^{-1} s_2 + A_4^{[2]}$
(2) $s_2^{-1} p s_2^{-1} p s_2^{-1} B \subset B s_2 p^{-1} s_2 p^{-1} s_2 + B s_2^{-1} p s_2^{-1} p s_2^{-1} + A_4^{[2]}$
(3) $s_2 p^{-1} s_2 p^{-1} s_2 B \subset B s_2 p^{-1} s_2 p^{-1} s_2 + B s_2^{-1} p s_2^{-1} p s_2^{-1} + A_4^{[2]}$

Proof. $X = s_2^{-1}ps_2^{-1}ps_2^{-1}.s_1^{-1} = s_2^{-1}ps_2^{-1}s_3(s_1s_2^{-1}.s_1^{-1}) = s_2^{-1}ps_2^{-1}s_3s_2^{-1}s_1^{-1}s_2 = s_2^{-1}s_1(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2 \in s_2^{-1}s_1s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2 + s_2^{-1}s_1u_2u_3s_1^{-1}s_2 + s_2^{-1}s_1u_3u_2s_1^{-1}s_2$ by lemma 3.6. We have $s_2^{-1}s_1u_2u_3s_1^{-1}s_2 \subset A_4^{[2]}$ and $s_2^{-1}s_1u_3u_2s_1^{-1}s_2 \subset A_4^{[2]}$ by lemma 5.1, hence

$$\begin{array}{rcl} X & \in & (s_2^{-1}s_1s_2^{-1})s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ & \subset & u_1^{\times}s_2s_1^{-1}(s_2s_3s_2^{-1}s_3s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ & \subset & u_1^{\times}s_2s_1^{-1}s_3^{-1}s_2s_3^2s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \\ & \subset & u_1^{\times}s_2s_1^{-1}s_3^{-1}s_2p^{-1}s_2 + u_1s_2s_1^{-1}s_3^{-1}s_2s_3s_1^{-1}s_2 + u_1s_2s_1^{-1}s_3^{-1}s_2s_1^{-1}s_2 + u_1u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 + A_4^{[2]} \end{array}$$

We know $s_2s_1^{-1}s_3^{-1}s_2s_1^{-1}s_2 \in A_4^{[2]}$ by lemma 5.1, $s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1^{-1}s_2 = s_2s_1^{-1}s_2s_3(s_2^{-1}s_1^{-1}s_2) == s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_1^{-1} \in A_4^{[2]}$ by lemma 5.1, and $u_2u_1s_3s_2^{-1}s_3s_1^{-1}s_2 = u_2s_3u_1s_2^{-1}s_1^{-1}s_3s_2$ is the sum of $u_2s_3s_2^{-1}s_1^{-1}s_3s_2 \subset A_4^{[2]}$ (by lemma 5.1) and of the $u_2s_3s_1^{\alpha}s_2^{-1}s_1^{-1}s_3s_2$ for $\alpha \in \{-1,1\}$. Since $u_2s_3(s_1^{\alpha}s_2^{-1}s_1^{-1})s_3s_2 = u_2s_3s_2^{-1}s_1^{-1}(s_2^{\alpha}s_3s_2) = u_2s_3s_2^{-1}s_1^{-1}s_3s_2s_3^{\alpha} \subset A_4^{[2]}$ by lemma 5.1, and this proves (1). To get (2) from (1), we use $s_2^{-1}ps_2^{-1}ps_2^{-1}.s_3^{-1} \in u_3^{\times}s_2p^{-1}s_2p^{-1}s_2^{-1} + A_4^{[2]}$, that we get from (1) by applying $\mathrm{Ad}\,\Delta$, and the fact that B is generated as a unital R-algebra by s_1^{-1} and s_3^{-1}. This proves (2), and then (3) follows from (2) by a direct application of Φ. ☐

From all this we deduce the following.

Lemma 5.6.

(1) $A_4^{[3]} = A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta} + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^{\delta}s^{-\delta}p^{\delta}s^{-\delta}$

(2) $A_4 = A_4^{[3]}$

Proof. As a consequence of lemmas 5.2 and 5.3, we get

$$A_4^{[3]} = A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta}B + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^{\delta}s^{-\delta}p^{\delta}s^{-\delta}B.$$

We know $s^{-\delta}p^{\delta}s^{-\delta}p^{\delta}s^{-\delta}B \subset A_4^{[2]} + \sum_{\varepsilon\in\{-1,1\}} Bs^{-\varepsilon}p^{\varepsilon}s^{-\varepsilon}p^{\varepsilon}s^{-\varepsilon}$ by lemma 5.5 hence

$$A_4^{[3]} = A_4^{[2]} + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^{\delta}s^{-\delta}p^{\delta}s^{-\delta} + \sum_{\delta\in\{-1,1\}} Bs^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta}B$$

and finally $s^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta} \in R^{\times}p^{-\delta}\Delta^{2\delta} + A_4^{[2]}$ by (*), hence $s^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta}B \subset p^{-\delta}\Delta^{2\delta}B + A_4^{[2]} = p^{-\delta}B\Delta^{2\delta} + A_4^{[2]} = B\Delta^{2\delta} + A_4^{[2]} \subset Bs^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta} + A_4^{[2]}$ and this concludes the proof of (1). Now $A_4^{[3]}$ is a R-submodule of A_4 which contains 1, which is stable under right-multiplication by s_1 and s_3 by definition. Moreover, in view of (1), we have

$$A_4^{[3]}s_2 \subset A_4^{[2]}s + \sum_{\delta\in\{-1,1\}} Bs^{\delta}p^{\delta}s^{-\delta}p^{\delta}s^{\delta}s + \sum_{\delta\in\{-1,1\}} Bs^{-\delta}p^{\delta}s^{-\delta}p^{\delta}s^{-\delta}s \subset A_4^{[3]}$$

hence $A_4^{[3]}$ is also stable under right multiplication by s_2, hence it is a right-ideal of A_4 containing 1, hence (2). ☐

We let $x_{\pm} = s^{\pm}p^{\pm}s^{\mp}p^{\pm}s^{\pm}$ and $y_{\pm} = s^{\pm}p^{\mp}s^{\pm}p^{\mp}s^{\pm}$.

Lemma 5.7.

(1) $sBsps \subset A_4^{[2]}$

(2) $sBs^{-1}ps \subset Rsps^{-1}psA_4^{[2]}$

Proof. The R-module $sBsps$ is spanned by $s^2ps \in A_4^{[2]}$, the $ss_isps \in A_4^{[2]}$ for $i \in \{1,3\}$ by lemma 5.1, $s(psps) = s(spsp) = s^2psp \in A_4^{[2]}$, $s_2s_1(s_3^{-1}s_2s_3)s_1s_2 = s_2s_1s_2s_3(s_2^{-1}s_1s_2) = s_2s_1s_2s_3s_1s_2s_1^{-1} \in A_4^{[2]}$ by lemma 5.1, the image of this latest one by $\mathrm{Ad}\,\Delta$, and by

$$s_2s_1^{-1}s_3^{-1}s_2s_3s_1s_2 = s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1s_2 = s_2s_1^{-1}s_2s_3(s_2^{-1}s_1s_2) = s_2s_1^{-1}s_2s_3s_1s_2s_1^{-1} \in A_4^{[2]}$$

by lemma 5.1, and this proves (1).

Now $sBs^{-1}ps$ is R-spanned by $sps^{-1}ps$ and

- the $ss^{-1}ps = ps \in A_4^{[2]}$
- the $ss_is^{-1}ps \in A_4^{[2]}$ for $i \in \{1,3\}$ by lemma 5.1
- $s_2s_1(s_3^{-1}s_2^{-1}s_3)s_1s_2 = s_2s_1s_2s_3^{-1}(s_2^{-1}s_1s_2) = s_2s_1s_2s_3^{-1}s_1s_2s_1^{-1} \in A_4^{[2]}$ for $i \in \{1,3\}$ by lemma 5.1
- $\Delta.s_2s_1s_3^{-1}s_2^{-1}s_3s_1s_2\Delta^{-1} \in A_4^{[2]}$
- $s_2s_1^{-1}(s_3^{-1}s_2^{-1}s_3)s_1s_2 = s_2s_1^{-1}s_2s_3^{-1}(s_2^{-1}s_1s_2) = s_2s_1^{-1}s_2s_3^{-1}s_1s_2s_1^{-1} \in A_4^{[2]}$ for $i \in \{1,3\}$ by lemma 5.1

and this proves (2). □

We want to express Δ^3 in terms of the $x_\pm$ and $y_\pm$. We recall that $\Delta^2 \in R^\times p^2sps^{-1}ps + Rp^3sps^2 + Rp^2sp^2s$ hence $\Delta^3 \in R^\times p^2sps^{-1}ps\Delta + Rp^3sps^2\Delta + Rp^2sp^2s\Delta$. We have

- $sps^{-1}ps\Delta \in Rsp\Delta + Rsps\Delta + Rsps^{-1}\Delta$ and
  (1) $sps^{-1}\Delta = sps^{-1}(spsp) = sp^2sp \in A_4^{[2]}$,
  (2) $sps\Delta = sps(spsp) = sps^2psp \in R^\times sps^{-1}psp + Rspspsp + Rsp^2sp$, and we have $(spsp)sp = psps^2p \in A_4^{[2]}$, $sp^2sp \in A_4^{[2]}$, hence $sps\Delta \in R^\times sps^{-1}psp + A_4^{[2]}$.
  (3) $sps^{-1}\Delta = sps^{-1}spsp = sp^2sp \in A_4^{[2]}$
  hence $sps^2\Delta \in R^\times sps^{-1}psp + A_4^{[2]}$.
- $sp^2s\Delta = sp^2s^2psp \in R^\times sp^2s^{-1}psp + Rsp^2spsp + Rsp^2psp$, and $sp^2(spsp) = sp^2psps = sp^3sps$, $sp^2psp \in A_4^{[2]}$.

It follows that

$$\Delta^3 \in R^\times p^2sp^2sp^2s + Rp^3sps^{-1}psp + Rp^2sp^2s^{-1}psp + Rp^2sp^3sps + A_4^{[2]}$$

From (*) we have $p^2sps^{-1}ps \in \Delta^2 + A_4^{[2]}$ hence $p^3sps^{-1}psp \in p\Delta^2p + A_4^{[2]} = p^2\Delta^2 + A_4^{[2]}$ hence $p^3sps^{-1}psp \in R^\times p^4.sps^{-1}ps + A_4^{[2]}$. By lemma 5.7, we have $sp^2s^{-1}ps \in Rx_+ + A_4^{[2]}$ hence $p^2sp^2s^{-1}psp \in Rp^2x_+p + A_4^{[2]} \subset Bx_+ + A_4^{[2]}$. Since $sp^3sps \in A_4^{[2]}$ this leads to

$$\Delta^3 \in R^\times p^2sp^2sp^2s + Bx_+ + A_4^{[2]}.$$

Since $s_i^2 = as_i + b + cs_i^{-1}$ we have $p^2 = s_1^2s_3^2 = (as_1 + b + cs_1^{-1})(as_3 + b + cs_3^{-1}) \in a^2p + c^2p^{-1} + W$ with W the R-span of $s_1s_3^{-1}, s_3s_1^{-1}, s_1, s_3, s_1^{-1}, s_3^{-1}, 1$. After easy applications of lemma 5.1 it follows that $sp^2sp^2s \in c^4sp^{-1}sp^{-1}s + RspsBs + RsBsps + A_4^{[2]}$. Since $spsBs + sBsps \subset A_4^{[2]}$ by lemma 5.7 we get

$$\Delta^3 \in c^4p^2y_+ + Bx_+ + A_4^{[2]}$$

and

$$\Delta^{-3} = \Phi(\Delta^3) \in c^{-4}p^{-2}y_- + Bx_+ + A_4^{[2]}$$

Now we have $\Delta^3s_1 = s_3\Delta^3 \in c^4s_3p^2y_+ + Bx_+ + A_4^{[2]}$ and $\Delta^3s_1 \in c^4p^2y_+s_1 + Bx_+B + A_4^{[2]}$, $\Delta^3s_1 \in c^4p^2u_1^\times y_- + Bx_+B + A_4^{[2]}$ by lemma 5.5 (1), $\Delta^3s_1 \in c^4p^2u_1^\times y_- + Bx_+ + A_4^{[2]}$ by using $p^{-2}\Delta^2 \in R^\times x_+ + A_4^{[2]}$. It follows that $c^4s_3p^2y_+ \in c^4p^2u_1^\times y_- + Bx_+ + A_4^{[2]}$ hence

$$\begin{cases} y_+ & \in & By_- + Bx_+ + A_4^{[2]} \\ y_- & \in & By_+ + Bx_+ + A_4^{[2]} \end{cases}$$

As a consequence we get the following.

Proposition 5.8.

$$A_4 = A_4^{[3]} = A_4^{[2]} + Bx_+ + Bx_- + By_+ = A_4^{[2]} + Bx_+ + Bx_- + By_-$$

For $x \in A_4^\times$, we let $[x]$ denote its class in $B^\times \backslash A_4^\times / B^\times$, and we write $x \sim y$ for $[x] = [y]$.

Lemma 5.9. *Let $E_2 = \{s_2^\alpha s_1^\beta s_3^\gamma s_2^\delta \mid \alpha, \beta, \gamma, \delta \in \{0, 1, -1\}\} \subset A_4^\times$. The image of $[E_2]$ of E_2 in $B^\times \backslash A_4^\times / B^\times$ has cardinality at most 13, and is equal to $\mathcal{S}_2$, with*

$$\begin{aligned} \mathcal{S}_2 = & \{[1], [s_2], [s_2^{-1}], [s_2s_1^{-1}s_2], [s_2s_3^{-1}s_2], [s_2s_1s_3s_2], [s_2s_1^{-1}s_3s_2], [s_2s_1^{-1}s_3^{-1}s_2], [s_2s_1s_3^{-1}s_2^{-1}], \\ & [s_2s_1^{-1}s_3s_2^{-1}], [s_2s_1s_3^{-1}s_2^{-1}], [s_2^{-1}s_1s_3s_2^{-1}], [s_2^{-1}s_1^{-1}s_3^{-1}s_2^{-1}]\} \end{aligned}$$

Proof. Clearly $\mathcal{S}_2 \subset [E_2]$, hence we only need to prove $[E_2] \subset \mathcal{S}_2$. In view of $s_2^{\alpha} s_i^{\alpha} s_2^{\alpha} = s_i^{\alpha} s_2^{\alpha} s_i^{\alpha}$, $s_2^{-1} s_i^{\alpha} s_2 = s_i s_2^{\alpha} s_i^{-1}$, $s_2 s_i^{\alpha} s_2^{-1} = s_i^{-1} s_2^{\alpha} s_i$ for $\alpha \in \{-1,1\}$ and $i \in \{1,3\}$, we have $[s_2^{\alpha} s_i^{\beta} s_2^{\gamma}] \in \mathcal{S}_2$ for all α, β, γ. Among the $s_2 s_1^{\alpha} s_3^{\beta} s_2$ for $\alpha, \beta \in \{-1,1\}$, we have $[s_2 s_1^{\alpha} s_3^{\beta} s_2] \in \mathcal{S}_2$ because $s_2 s_1^{\alpha} s_3^{\beta} s_2 \sim s_2 s_1^{-1} s_3 s_2$: indeed, we have the identity $s_2 s_1^{-1} s_3 s_2 = s_1^{-1} s_3 (s_2 s_1 s_3^{-1} s_2) s_1^{-1} s_3$ in the braid group on 4 strands (because $s_1^{-1} s_3 s_2 s_1 s_3^{-1} s_2 s_1^{-1} s_3 = s_1^{-1} (s_3 s_2 s_3^{-1}) s_1 s_2 s_1^{-1} s_3 = s_1^{-1} s_2^{-1} s_3 (s_2 s_1 s_2) s_1^{-1} s_3 = s_1^{-1} s_2^{-1} s_3 s_1 s_2 s_1 s_1^{-1} s_3 = s_1^{-1} s_2^{-1} s_1 (s_3 s_2 s_3) = (s_1^{-1} s_2^{-1} s_1) s_2 s_3 s_2 = s_2 s_1^{-1} s_2^{-1} s_2 s_3 s_2 = s_2 s_1^{-1} s_3 s_2$). Among the $s_2 s_1^{\alpha} s_3^{\beta} s_2^{-1}$ for $\alpha, \beta \in \{-1,1\}$, we have $[s_2 s_1^{\alpha} s_3^{\beta} s_2^{-1}] \in \mathcal{S}_2$ because $s_2 s_1 s_3 s_2^{-1} \sim s_2 s_1 s_3^{-1} s_2$: indeed, we have $s_1 (s_2 s_1 s_3 s_2^{-1}) s_3^{-1} = (s_1 s_2 s_1) s_3 s_2^{-1} s_3^{-1} = s_2 s_1 (s_2 s_3 s_2^{-1}) s_3^{-1} = s_2 s_1 s_3^{-1} s_2 s_3 s_3^{-1} = s_2 s_1 s_3^{-1} s_2$.

Again for $\alpha, \beta \in \{-1,1\}$, we have $[s_2^{-1} s_1^{\alpha} s_3^{\beta} s_2] \in \mathcal{S}_2$ because of the following identities

(1) $s_2^{-1} s_1^{-1} s_3 s_2 \sim s_2 s_1 s_3^{-1} s_2^{-1}$
(2) $s_2^{-1} s_1 s_3 s_2 \sim s_2 s_1 s_3^{-1} s_2$
(3) $s_2^{-1} s_1 s_3^{-1} s_2 \sim s_2 s_1^{-1} s_3 s_2^{-1}$
(4) $s_2^{-1} s_1^{-1} s_3^{-1} s_2 \sim s_2 s_1^{-1} s_3^{-1} s_2^{-1}$

We prove these identities now. We have $s_3 (s_2 s_3^{-1} s_1 s_2^{-1}) s_1^{-1} = (s_3 s_2 s_3^{-1}) s_1 s_2^{-1} s_1^{-1} = s_2^{-1} s_3 (s_2 s_1 s_2^{-1}) s_1^{-1} = s_2^{-1} s_3 s_1^{-1} s_2 s_1 s_1^{-1} = s_2^{-1} s_3 s_1^{-1} s_2$ hence $s_2 s_3^{-1} s_1 s_2^{-1} \sim s_2^{-1} s_3 s_1^{-1} s_2$ that is (1). By applying $\mathrm{Ad}\, \Delta$ this implies $s_2 s_1^{-1} s_3 s_2^{-1} \sim s_2^{-1} s_1 s_3^{-1} s_2$ that is (3). We have $s_3^{-1} (s_2^{-1} s_1 s_3 s_2) s_1 = (s_3^{-1} s_2^{-1} s_3) s_1 s_2 s_1 = s_2 s_3^{-1} (s_2^{-1} s_1 s_2) s_1 = s_2 s_3^{-1} s_1 s_2 s_1^{-1} s_1 = s_2 s_3^{-1} s_1 s_2$ hence $s_2^{-1} s_1 s_3 s_2 \sim s_2 s_3^{-1} s_1 s_2$ that is (2).

We have $s_1 (s_2^{-1} s_1^{-1} s_3 s_2^{-1}) s_3^{-1} = (s_1 s_2^{-1} s_1^{-1}) s_3 s_2^{-1} s_3^{-1} = s_2^{-1} s_1^{-1} (s_2 s_3 s_2^{-1}) s_3^{-1} = s_2^{-1} s_1^{-1} s_3^{-1} s_2 s_3 s_3^{-1} = s_2^{-1} s_1^{-1} s_3^{-1} s_2$ hence $s_2^{-1} s_1^{-1} s_3 s_2^{-1} \sim s_2^{-1} s_1^{-1} s_3^{-1} s_2$. Moreover, we have $s_1 (s_2 s_1^{-1} s_3^{-1} s_2^{-1}) s_3^{-1} = s_1 s_2 s_1^{-1} (s_3^{-1} s_2^{-1} s_3^{-1}) = s_1 (s_2 s_1^{-1} s_2^{-1}) s_3^{-1} s_2^{-1} = s_1 s_1^{-1} s_2^{-1} s_1 s_3^{-1} s_2^{-1} = s_2^{-1} s_1 s_3^{-1} s_2^{-1}$ hence $s_2 s_1^{-1} s_3^{-1} s_2^{-1} \sim s_2^{-1} s_1 s_3^{-1} s_2^{-1}$. Applying Δ we get $s_2 s_1^{-1} s_3^{-1} s_2^{-1} \sim s_2^{-1} s_3 s_1^{-1} s_2^{-1}$, hence $s_2 s_1^{-1} s_3^{-1} s_2^{-1} \sim s_2^{-1} s_3 s_1^{-1} s_2^{-1} \sim s_2^{-1} s_1^{-1} s_3^{-1} s_2$ hence (4).

Now, for $\alpha, \beta \in \{-1,1\}$, we have $[s_2^{-1} s_1^{\alpha} s_3^{\beta} s_2^{-1}] \in \mathcal{S}_2$ because $s_2^{-1} s_1 s_3^{-1} s_2^{-1} \sim s_2 s_1^{-1} s_3^{-1} s_2^{-1}$ and $s_2^{-1} s_1^{-1} s_3 s_2^{-1} \sim s_2 s_1^{-1} s_3^{-1} s_2^{-1}$ as we proved above, and this concludes the proof. □

From this we get

$$\begin{aligned} A_4 &= \textstyle\sum_{\sigma \in [E_2]} B \sigma B + B x_+ + B x_- + B y_- \\ &= \textstyle\sum_{\sigma \in \mathcal{S}_2} B \sigma B + B x_+ + B x_- + B y_- \end{aligned}$$

We write $\mathcal{S}_2 = \mathcal{S}_2^1 \cup \mathcal{S}_2^{\Delta} \cup \mathcal{S}_2^{\alpha} \cup \mathcal{S}_2^{\beta} \cup \mathcal{S}_2^0$ with

$$\begin{aligned} \mathcal{S}_2^1 &= \{[1], [s_2], [s_2^{-1}]\} \\ \mathcal{S}_2^{\Delta} &= \{[s_2 s_1 s_3 s_2], [s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}]\} \\ \mathcal{S}_2^{\alpha} &= \{[s_2 s_1^{-1} s_2], [s_2 s_3^{-1} s_2]\} \\ \mathcal{S}_2^{\beta} &= \{[s_2 s_1 s_3^{-1} s_2^{-1}], [s_2 s_1^{-1} s_3 s_2^{-1}]\} \\ \mathcal{S}_2^0 &= \{[s_2 s_1^{-1} s_3 s_2], [s_2 s_1^{-1} s_3^{-1} s_2], [s_2 s_1^{-1} s_3^{-1} s_2^{-1}], [s_2^{-1} s_1 s_3 s_2^{-1}]\} \end{aligned}$$

Recall that $B = u_1 u_3 = u_3 u_1$, with u_i the unital subalgebra generated by s_i. We prove the following.

Lemma 5.10.

(1) $s_2 s_1 s_3 s_2 B \subset B s_2 s_1 s_3 s_2$, $s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1} B \subset B s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}$
(2) $s_2 s_1^{-1} s_2 B \subset B s_2 s_1^{-1} s_2 u_3 + A_4^{[1]}$, $s_2 s_3^{-1} s_2 B \subset B s_2 s_3^{-1} s_2 u_1 + A_4^{[1]}$
(3) $s_2 s_1 s_3^{-1} s_2^{-1} B \subset B s_2 s_1 s_3^{-1} s_2^{-1} u_1$, $s_2 s_1^{-1} s_3 s_2^{-1} B \subset B s_2 s_1^{-1} s_3 s_2^{-1} u_3$

Proof. We have $\Delta = s_1 s_3 (s_2 s_1 s_3 s_2) = (s_2 s_1 s_3 s_2) s_1 s_3$ and $\Delta B = B \Delta$ hence $s_2 s_1 s_3 s_2 B = (s_1 s_3)^{-1} \Delta B = B (s_1 s_3)^{-1} \Delta = B (s_2 s_1 s_3 s_2)$. Applying Φ (or considering Δ^{-1}) we get $s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1} B = B s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}$ hence (1).

By lemma 3.6 we have $(s_2 s_1^{-1} s_2) s_1^{-1} \in s_1^{-1} (s_2 s_1^{-1} s_2) + A_4^{[1]}$ hence $(s_2 s_1^{-1} s_2) u_1 \subset u_1 (s_2 s_1^{-1} s_2) + A_4^{[1]}$ and $(s_2 s_1^{-1} s_2) u_1 \subset B (s_2 s_1^{-1} s_2) + A_4^{[1]}$. Since $B = u_1 u_3$ this yields $(s_2 s_1^{-1} s_2) B = (s_2 s_1^{-1} s_2) u_1 u_3 \subset B (s_2 s_1^{-1} s_2) u_3 + A_4^{[1]}$. Using $\mathrm{Ad}\, \Delta$ this implies $(s_2 s_3^{-1} s_2) B \subset B (s_2 s_3^{-1} s_2) u_1 + A_4^{[1]}$, hence (2). Finally, $(s_2 s_1 s_3^{-1} s_2^{-1}) s_3 = s_2 s_1 (s_3^{-1} s_2^{-1} s_3) = (s_2 s_1 s_2) s_3^{-1} s_2^{-1} = s_1 (s_2 s_1 s_3^{-1} s_2^{-1})$ hence $(s_2 s_1 s_3^{-1} s_2^{-1}) u_3 \subset$

$B(s_2s_1s_3^{-1}s_2^{-1})$ whence, using $B = u_3u_1$, $(s_2s_1s_3^{-1}s_2^{-1})B \subset B(s_2s_1s_3^{-1}s_2^{-1})u_1$ and, applying $\mathrm{Ad}\,\Delta$, $(s_2s_1^{-1}s_3s_2^{-1})B \subset B(s_2s_1^{-1}s_3s_2^{-1})u_3$, which proves (3). □

Proposition 5.11.

(1) $A_4^{[1]} = B + Bs_2B + Bs_2^{-1}B$ *is equal to*

$$B + \sum_{a,b\in\{0,1,-1\}} Bs_2s_1^as_3^b + \sum_{a,b\in\{0,1,-1\}} Bs_2^{-1}s_1^as_3^b$$

(2) $A_4^{[2]} = Bu_2A_4^{[1]} = A_4^{[1]}u_2B$ *is equal to*

$$A_4^{[1]} + \sum_{x\in\mathcal{S}_2^{\Delta}} Bx + \sum_{a\in\{0,1,-1\}} Bs_2s_1^{-1}s_2s_3^a + \sum_{a\in\{0,1,-1\}} Bs_2s_3^{-1}s_2s_1^a + \sum_{a\in\{0,1,-1\}} Bs_2s_1s_3^{-1}s_2^{-1}s_1^a$$

$$+ \sum_{a\in\{0,1,-1\}} Bs_2s_1^{-1}s_3s_2^{-1}s_3^a + \sum_{\substack{x\in\mathcal{S}_2^0\\ a,b\in\{0,1,-1\}}} Bxs_1^as_3^b$$

(3) $A_4 = A_4^{[3]} = A_4^{[2]} + Bx_+ + Bx_- + By_-$

Proof. (1) is clear, (3) has been proved before, and (2) is an immediate consequence of $A_4^{[2]} = A_4^{[1]} + \sum_{x\in\mathcal{S}_2} BxB$ and of lemma 5.10. □

Corollary 5.12. *As a B-module, A_4 is generated by 72 elements, which are images of elements of the braid group on 4 strands.*

Proof. By proposition 5.11, $A_4^{[1]}$ is generated by $1+9+9=19$ elements, $A_4^{[2]}$ by $A_4^{[1]}$ and $|\mathcal{S}_2^{\Delta}| + 4\times 3 + 9\times|\mathcal{S}_2^0| \leq 2+12+9\times 4 = 50$ elements, and $A_4^{[3]}$ by $A_4^{[2]}$ and 3 elements. Thus $A_4 = A_4^{[3]}$ is generated by 72 elements, all originating from the braid group. □

6. The algebra A_5

Recall $w^+ = s_3s_2^{-1}s_1s_2^{-1}s_3$, $w^- = s_3^{-1}s_2s_1^{-1}s_2s_3^{-1} \in A_4$. Our first goal in this section is to prove the following theorem.

Theorem 6.1.

$$\begin{aligned} A_5 \;=\; & A_4 + A_4s_4A_4 + A_4s_4^{-1}A_4 + A_4s_4s_3^{-1}s_4A_4 + A_4s_4^{-1}s_3s_2^{-1}s_3s_4^{-1}A_4 + A_4s_4s_3^{-1}s_2s_3^{-1}s_4A_4 \\ & +A_4s_4^{-1}w^+s_4^{-1}A_4 + A_4s_4w^-s_4A_4 + A_4s_4^{-1}w^-s_4^{-1}A_4 + A_4s_4w^+s_4A_4 + A_4s_4w^-s_4w^-s_4A_4 \\ & +A_4s_4w^+s_4^{-1}w^+s_4A_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4 \end{aligned}$$

We denote again U the right-hand side. We let $A_5^{(0)} = A_4$ and $A_5^{(n+1)} = A_5^{(n)}u_4A_4$. This defines an increasing sequence of A_4 sub-bimodules of A_5. An immediate consequence of theorem 4.1 is $sh(A_4) \subset U$. Also, we have $u_4 \subset U$ hence $A_5^{(1)} = A_4u_4A_4 \subset U$.

Lemma 6.2. $u_4A_4u_4 \subset U$*, hence* $A_5^{(2)} = A_4u_4A_4u_4A_4 \subset U$*.*

Proof. According to theorem 4.1, we have $A_4 = A_3 + A_3s_3A_3 + A_3s_3^{-1}A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w^- + A_3w^+$, hence $u_4A_4u_4 \subset A_3u_4 + A_4u_4u_3u_4A_4 + A_4u_4s_3s_2^{-1}s_3u_4A_3 + A_3u_4w^-u_4 + A_3u_4w^+u_4$. We have $A_3u_4 + A_4u_4u_3u_4A_4 + A_4u_4s_3s_2^{-1}s_3u_4A_3 \subset A_4sh(A_4)A_4 \subset A_4UA_4 \subset U$. Moreover, since by definition $s_4^{\alpha}w^{\beta}s_4^{\alpha} \in U$ for all $\alpha,\beta \in \{-1,1\}$, we have $s_4A_4s_4 \subset U$, $s_4^{-1}A_4s_4^{-1} \subset U$, and we only need to prove $s_4w^{\pm}s_4^{-1} \in U$ and $s_4^{-1}w^{\pm}s_4 \in U$. We have $w^{\pm} \in s_3^{\alpha}A_3s_3^{\alpha}$ for some $\alpha \in \{-1,1\}$, hence $s_4^{\alpha}w^{\pm}s_4^{-\alpha} \in s_4^{\alpha}s_3^{\alpha}A_3s_3^{\alpha}s_4^{-\alpha} = s_3^{-\alpha}(s_3^{\alpha}s_4^{\alpha}s_3^{\alpha})A_3s_3^{\alpha}s_4^{-\alpha} = s_3^{-\alpha}s_4^{\alpha}s_3^{\alpha}s_4^{\alpha}A_3s_3^{\alpha}s_4^{-\alpha} \subset A_4s_4^{\alpha}s_3^{\alpha}A_3(s_4^{\alpha}s_3^{\alpha}s_4^{-\alpha}) \subset A_4s_4^{\alpha}s_3^{\alpha}A_3s_3^{-\alpha}s_4^{\alpha}s_3^{\alpha}$ by lemma 2.1. Now

$$A_4s_4^{\alpha}s_3^{\alpha}A_3s_3^{-\alpha}s_4^{\alpha}s_3^{\alpha} \subset A_4(s_4^{\alpha}A_4s_4^{\alpha})A_4 \subset A_4UA_4 \subset U,$$

as we already proved. □

6.1. The A_4-bimodule $A_5^{(3)}/A_5^{(2)}$: first reduction.

Proposition 6.3. *If $p \le 5$, $q \le 5$ and $(p,q) \neq (5,5)$, then for all $x \in u_4 u_{i_1} \dots u_{i_p} u_4 u_{j_1} \dots u_{j_q} u_4$ we have $x \in A_4 u_4 A_4 u_4 A_4$, for all choices of $i_1, \dots, i_p, j_1, \dots, j_q \in \{1,2,3\}$, unless $(p,q) \in \{(5,4),(4,5)\}$ and $x \in s_4 u_3 u_2 u_1 u_3 u_2 s_4 u_1 u_3 u_2 u_3 s_4 \cup s_4^{-1} u_3 u_2 u_1 u_3 u_2 s_4^{-1} u_1 u_3 u_2 u_3 s_4^{-1}$.*

Proof. Note that $sh(A_4) \subset A_4 u_4 A_4 u_4 A_4$. By application of Ψ we may assume $p \ge q \ge 1$. We prove the statement by induction on (p,q), using lexicographic ordering. By commutation relations we can assume $i_1 \notin \{1,2\}$ hence $i_1 = 3$, and similarly $j_q = 3$. In case $(p,q) = (1,1)$ we have then $u_4 u_3 u_4 u_3 u_4 \subset sh(A_4) \subset A_4 u_4 A_4 u_4 A_4$. More generally, in the cases $(1,1)$, $(2,1)$, $(2,2)$, $(3,1)$, using only commutation relations we check that the corresponding algebras are necessarily included in $A_4 sh(A_4) A_4 \subset A_4 u_4 A_4 u_4 A_4$.

If $(p,q) = (3,2)$, the only case which is not clearly included in $A_4 sh(A_4) A_4$ is $u_4 u_3 u_2 u_1 u_4 u_2 u_3 u_4 = u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4$, and we have $u_4 u_3 u_4 \subset u_3 s_4 s_3^{-1} s_4 + u_3 u_4 u_3$ by theorem 3.2 hence $u_4 u_3 u_4 u_2 u_1 u_2 u_3 u_4 \subset u_3 s_4 s_3^{-1} s_4 u_2 u_1 u_2 u_3 u_4 + u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_4 \subset u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4 u_3 u_4 + A_4 u_4 A_4 u_4 A_4$. Again $s_4 u_3 u_4 \subset s_4^{-1} s_3 s_4^{-1} u_3 + u_3 u_4 u_3$ by theorem 3.2 hence $u_3 s_4 s_3^{-1} u_2 u_1 u_2 (s_4 u_3 u_4) \subset u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4^{-1} s_3 s_4^{-1} + A_4 u_4 A_4 u_4 A_4$ and $u_3 s_4 s_3^{-1} u_2 u_1 u_2 s_4^{-1} s_3 s_4^{-1} = u_3 (s_4 s_3^{-1} s_4^{-1}) u_2 u_1 u_2 s_3 s_4^{-1} = u_3 s_3^{-1} s_4^{-1} s_3 u_2 u_1 u_2 s_3 s_4^{-1} \subset A_4 u_4 A_4 u_4 A_4$.

If $(p,q) = (3,3)$, the corresponding algebra is either included in $A_4 sh(A_4) A_4 \subset U$, or can be reduced using commutation relations to the case $(3,2)$, or we are dealing with the remaining case $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4$ (or its image under the natural anti-isomorphism $u_4 u_3 u_2 u_3 u_4 u_1 u_2 u_3 u_4$). We want to prove $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 u_4 \subset A_4 u_4 A_4 u_4 A_4$. Up to using the natural isomorphism induced by $s_i \mapsto s_i^{-1}$, we only need to prove $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 s_4 \subset A_4 u_4 A_4 u_4 A_4$. We have $u_4 u_3 u_2 u_1 u_4 u_3 u_2 u_3 s_4 = u_4 u_3 u_4 u_2 u_1 u_3 u_2 u_3 s_4$ and we know from theorem 3.2 that $u_4 u_3 u_4 \subset A_4 s_4^{-1} s_3 s_4^{-1} + u_3 u_4 u_3$, hence $u_4 u_3 u_4 u_2 u_1 u_3 u_2 u_3 s_4 \subset A_4 s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 + A_4 u_4 A_4 u_4 A_4$. Now, using $u_3 u_2 u_3 \subset s_3 s_2^{-1} s_3 u_2 + u_2 u_3 u_3$ we have

$$s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 \subset s_4^{-1} s_3 s_4^{-1} u_2 u_1 s_3 s_2^{-1} s_3 u_2 s_4 + s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_2 u_3 u_2 s_4.$$

But $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_2 u_3 u_2 s_4 = s_4^{-1} s_3 u_2 u_1 s_4^{-1} u_2 u_3 s_4 u_2 \subset A_4 u_4 A_4 u_4 A_4$ by the induction assumption, hence $s_4^{-1} s_3 s_4^{-1} u_2 u_1 u_3 u_2 u_3 s_4 \subset s_4^{-1} s_3 s_4^{-1} u_2 u_1 s_3 s_2^{-1} s_3 s_4 u_2 + A_4 u_4 A_4 u_4 A_4$. Now we need to prove $s_4^{-1} s_3 s_4^{-1} s_2^{\alpha} s_1^{\beta} s_3 s_2^{-1} s_3 s_4 \in A_4 u_4 A_4 u_4 A_4$ for $\alpha, \beta \in \{-1, 1\}$. If $\alpha = 1$, this holds true because

$$\begin{array}{rcl} s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} s_3 s_4 & = & s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} (s_3 s_4 s_3) s_3^{-1} \\ & = & s_4^{-1} s_3 s_4^{-1} s_2 u_1 s_3 s_2^{-1} s_4 s_3 s_4 s_3^{-1} \\ & = & s_4^{-1} s_3 s_2 u_1 (s_4^{-1} s_3 s_4) s_2^{-1} s_3 s_4 s_3^{-1} \\ & = & s_4^{-1} s_3 s_2 u_1 s_3 s_4 s_3^{-1} s_2^{-1} s_3 s_4 s_3^{-1} \\ & = & s_4^{-1} (s_3 s_2 s_3) u_1 s_4 (s_3^{-1} s_2^{-1} s_3) s_4 s_3^{-1} \\ & = & s_4^{-1} s_2 s_3 s_2 u_1 s_4 s_2 s_3^{-1} s_2^{-1} s_4 s_3^{-1} \\ & = & s_2 s_4^{-1} s_3 s_2 u_1 s_4 s_2 s_3^{-1} s_4 s_2^{-1} s_3^{-1} \\ & \subset & A_4 u_4 A_4 u_4 A_4 \end{array}$$

by the induction assuption. We thus assume $\alpha = -1$. If $\beta = 1$, then

$$\begin{array}{rcll} s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 (s_3 s_2^{-1} s_3) s_4 & \subset & s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_4 u_2 & + A_4 u_4 A_4 u_4 A_4 \text{ (lemmas 2.4 + 2.3)} \\ & \subset & s_1 s_1^{-1} s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_3^{-1} s_2 s_3^{-1} s_4 u_2 & + A_4 u_4 A_4 u_4 A_4 \\ & \subset & s_1 s_4^{-1} s_3 s_4^{-1} (s_1^{-1} s_2^{-1} s_1) s_3^{-1} s_2 s_3^{-1} s_4 u_2 & + A_4 u_4 A_4 u_4 A_4 \\ & \subset & s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_2 s_3^{-1} s_4 u_2 & + A_4 u_4 A_4 u_4 A_4 \\ & \subset & s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} sh(A_3) s_4 u_2 & + A_4 u_4 A_4 u_4 A_4 \\ & \subset & s_1 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 u_2 s_4 u_2 & + A_4 u_4 A_4 u_4 A_4 \end{array}$$

by theorem 3.2 and the induction assumtion, and we already proved $s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 u_2 s_4 = s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_3 s_4 u_2 \subset A_4 u_4 A_4 u_4 A_4$. The remaining case is then $(\alpha, \beta) = (-1,-1)$, for which we have

$$\begin{array}{rcll}
s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_4 & \subset & s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1^{-1}s_3^{-1}s_2s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4 \text{ (lemma 2.4)}\\
& \subset & s_3s_3^{-1}(s_4^{-1}s_3s_4^{-1})s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4\\
& \subset & s_3(s_4^{-1}s_3s_4^{-1})s_3^{-1}s_2^{-1}s_3^{-1}s_1^{-1}s_2s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4 \text{ (lemma 2.3)}\\
& \subset & A_4s_4^{-1}s_3s_4^{-1}(s_3^{-1}s_2^{-1}s_3^{-1})s_1^{-1}s_2s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_3^{-1}(s_2^{-1}s_1^{-1}s_2)s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_3^{-1}s_1s_2^{-1}s_1^{-1}s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1(s_3^{-1}s_2^{-1}s_3^{-1})s_4s_1^{-1}A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2^{-1}s_4s_1^{-1}A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4s_4^{-1}s_3s_4^{-1}s_2^{-1}s_1s_2^{-1}s_3^{-1}s_4s_2^{-1}A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4s_4^{-1}s_3s_2^{-1}s_1s_4^{-1}s_2^{-1}s_3^{-1}s_4A_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_4u_4A_4u_4A_4 & \text{(induction assumption)}
\end{array}$$

and this concludes the case $(p,q) = (3,3)$.

All cases $(4,q)$ for $q = 1,2,4$ can be easily reduced to smaller cases by using commutation relations and relations $u_iu_ju_iu_j = u_ju_iu_ju_i$. Most cases for $(4,3)$ can also be reduced this way, except for one remaining case $u_4u_3u_2u_3u_1u_4u_3u_2u_3u_4$. Using Φ, we only need to prove $u_4u_3u_2u_3u_1s_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$. Using the induction assumption and theorem 3.2 on $sh(A_3)$, we get

$$\begin{array}{rcll}
u_4(u_3u_2u_3)u_1s_4(u_3u_2u_3)u_4 & \subset & u_4u_2s_3s_2^{-1}s_3u_1s_4s_3s_2^{-1}s_3u_2u_4 & +A_4u_4A_4u_4A_4\\
& \subset & A_3u_4s_3s_2^{-1}u_1(s_3s_4s_3)s_2^{-1}s_3u_4A_3 & +A_4u_4A_4u_4A_4\\
& \subset & A_3u_4s_3s_2^{-1}u_1s_4s_3s_4s_2^{-1}s_3u_4A_3 & +A_4u_4A_4u_4A_4\\
& \subset & A_3(u_4s_3s_4)s_2^{-1}u_1s_3s_2^{-1}(s_4s_3u_4)A_3 & +A_4u_4A_4u_4A_4\\
& \subset & A_3u_3u_4u_3s_2^{-1}u_1s_3s_2^{-1}u_3u_4u_3A_3 & +A_4u_4A_4u_4A_4
\end{array}$$

by lemma 2.1, which proves the claim.

We now deal with the cases $(5,q)$ with $1 \leq q < 5$. We can assume that $u_{i_1}\dots u_{i_p} = u_3u_2u_1u_2u_3$ or $u_{i_1}\dots u_{i_p} = u_3u_2u_1u_3u_2$, because otherwise we can reduce to smaller cases by using commutation relations and the relation $u_au_bu_au_b = u_bu_au_bu_a$. From this remark one easily checks that the cases $(5,1)$ are readily reduced to smaller cases, and also the cases $(5,2)$ except for the case $u_4u_3u_2u_1u_2u_3u_4u_2u_3u_4 = u_4u_3u_2u_1u_2u_3u_2u_4u_3u_4$ that we tackle now : we have $u_3u_2u_1u_2u_3u_2 \subset A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ by theorem 4.1, hence

$$\begin{array}{rcl}
u_4u_3u_2u_1u_2u_3u_2u_4u_3u_4 & \subset & u_4A_3u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4u_3u_4\\
& \subset & A_3u_4u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4\\
& \subset & A_3u_4u_3u_4u_2u_1u_2u_1u_3u_4 + A_3u_4u_3u_2u_3u_4u_2u_1u_2u_1u_3u_4\\
& & +A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4\\
& \subset & A_3u_4u_3u_4u_2u_1u_2u_3u_4A_2 + A_3u_4u_3u_2u_3u_4u_2u_1u_2u_3u_4A_2\\
& & +A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4
\end{array}$$

using $A_3 = u_2u_1u_2u_1$, and we are thus reduced to smaller cases.

When $(p,q) = (5,3)$, the only nontrivial case (up to commutation and $u_au_bu_au_b = u_bu_au_bu_a$ relations) is $u_4u_3u_2u_1u_2u_3u_4u_3u_2u_3u_4$. We have $u_2u_3u_4u_3u_2u_3u_4 \subset sh(A_4) \subset A_4u_4A_4 + sh(A_3)u_4u_3u_4A_4 + u_4u_3u_2u_3u_4A_4$ by theorem 4.1, hence

$$u_4u_3u_2u_1u_2u_3u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4 + u_4u_3u_2u_1sh(A_3)u_4u_3u_4A_4 + u_4u_3u_2u_1u_4u_3u_2u_3u_4A_4$$

and we have $u_4u_3u_2u_1u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$ by the induction assumption, and, since $sh(A_3) = u_2u_3u_2u_3$ by theorem 3.2,

$$\begin{array}{rcccl}
u_4u_3u_2u_1sh(A_3)u_4u_3u_4 & \subset & u_4u_3u_2u_1u_2u_3u_2(u_3u_4u_3u_4) & = & u_4u_3u_2u_1u_2u_3u_2u_4u_3u_4u_3\\
& & & = & u_4u_3u_2u_1u_2u_3u_4u_2u_3u_4u_3
\end{array}$$

and we are reduced to case $(5,2)$.

When $(p,q) = (5,4)$, the only nontrivial cases are

$$u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4 \text{ and } u_4u_3u_2u_1u_3u_2u_4u_3u_1u_2u_3u_4.$$

In the first case, $u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4 = u_4u_3u_2u_1u_2u_3u_2u_1u_2u_4u_3u_4 \subset u_4A_4u_4u_3u_4$. By theorem 4.1, we have $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ hence

$$\begin{array}{rcl} u_4A_4u_4u_3u_4 & \subset & u_4A_3u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4u_3u_4 \\ & \subset & A_3u_4u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_3u_4A_3u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4 \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by the induction assumption and $A_3 = u_2u_1u_2u_1$.

In the second case, we need to consider the sets $s_4^{\alpha}u_3u_2u_1u_3u_2s_4^{\beta}u_3u_1u_2u_3s_4^{\gamma}$ with $\alpha, \beta, \gamma \in \{-1, 1\}$, and we can assume that two of them have distinct signs, otherwise we are in the exceptional case of the statement. Up to using Φ and Ψ, we can assume $\gamma = 1$ and $\beta = -1$. We are thus considering expressions of the type $u_4u_3u_2u_1u_3u_2s_4^{-1}u_3u_1u_2u_3s_4 = u_4u_3u_2u_3u_1u_2u_1s_4^{-1}u_3u_2u_3s_4$. Notice that

$$\begin{array}{rcl} & u_4u_3u_2u_3(u_2u_1u_2)s_4^{-1}u_3u_2u_3s_4 & = & u_4(u_3u_2u_3u_2)u_1u_2s_4^{-1}u_3u_2u_3s_4 \\ = & u_4u_2u_3u_2u_3u_1u_2s_4^{-1}u_3u_2u_3s_4 & = & u_2u_4u_3u_2u_3u_1u_2s_4^{-1}u_3u_2u_3s_4 \end{array}$$

hence reduces to smaller cases. As a consequence, among the natural spanning set of $u_1u_2u_1$, only the $s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}$ do not reduce to smaller cases, and so we may restrict ourselves to these. Moreover, using $u_3u_2u_3 \subset u_2s_3^{-1}s_2s_3^{-1} + u_2u_3u_2$ and $u_3u_2u_3 \subset s_3s_2^{-1}s_3u_2 + u_2u_3u_2$ we are reduced to expressions of the form $u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4^{-1}s_3s_2^{-1}s_3s_4$. We then have

$$\begin{array}{rcl} u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4^{-1}s_3s_2^{-1}s_3s_4 & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4^{-1}s_3s_2^{-1}(s_3s_4s_3)s_3^{-1} \\ & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4^{-1}s_3s_2^{-1}s_4s_3s_4s_3^{-1} \\ & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}(s_4^{-1}s_3s_4)s_2^{-1}s_3s_4s_3^{-1} \\ & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4s_3^{-1}s_2^{-1}s_3s_4s_3^{-1} \\ & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4(s_3^{-1}s_2^{-1}s_3)s_4s_3^{-1} \\ & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4s_2s_3^{-1}s_2^{-1}s_4s_3^{-1} \\ & = & u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4s_2s_3^{-1}s_4s_2^{-1}s_3^{-1} \end{array}$$

so we now need to prove that $u_4s_3^{-1}s_2s_3^{-1}s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_3s_4s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$. When $\alpha = 1$ we get

$$\begin{array}{rcl} u_4s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}s_1s_3s_4s_2s_3^{-1}s_4 & = & u_4s_3^{-1}s_2s_1(s_3^{-1}s_2^{-1}s_3)s_1s_4s_2s_3^{-1}s_4 \\ & = & u_4s_3^{-1}(s_2s_1s_2)s_3^{-1}s_2^{-1}s_1s_4s_2s_3^{-1}s_4 \\ & = & u_4s_3^{-1}s_1s_2s_1s_3^{-1}s_2^{-1}s_1s_4s_2s_3^{-1}s_4 \\ & = & s_1u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}s_1s_4s_2s_3^{-1}s_4 \\ & = & s_1u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}s_4s_1s_2s_3^{-1}s_4 \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by the induction assumption. When $\alpha = -1$ we get

$$\begin{array}{rcl} u_4s_3^{-1}s_2s_3^{-1}s_1^{-1}s_2s_1^{-1}s_3s_4s_2s_3^{-1}s_4 & = & u_4s_3^{-1}s_2s_1^{-1}(s_3^{-1}s_2s_3)s_1^{-1}s_4s_2s_3^{-1}s_4 \\ & = & u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_2^{-1}s_1^{-1}s_4s_2s_3^{-1}s_4 \\ & = & u_4s_3^{-1}s_2s_1^{-1}s_2s_3(s_2^{-1}s_1^{-1}s_2)s_4s_3^{-1}s_4 \\ & = & u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_1^{-1}s_4s_3^{-1}s_4 \\ & = & u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_1s_2^{-1}s_4s_3^{-1}s_4s_1^{-1} \\ & = & u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_1s_2^{-1}s_3^{-1}s_4s_1^{-1} \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by the induction assumption.

This concludes the case $(5, 4)$ and the proof of the proposition. □

Lemma 6.4.

$$u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4 \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)u_2 + A_4u_4A_4u_4A_4$$

Proof. By proposition 6.3 it is enough to prove

$$s_4u_3u_2u_3u_1u_2u_1s_4u_3u_2u_3s_4 \subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4u_2 + A_4u_4A_4u_4A_4$$

and, as noted in the proof of proposition 6.3, we can restrict to the forms $s_4u_3u_2u_3s_1^{\alpha}s_2^{-\alpha}s_1^{\alpha}s_4u_3u_2u_3s_4$. Moreover, since $s_1s_2^{-1}s_1 = s_2^{-1}s_1^{-1}\ s_2s_1^2 \in s_2^{-1}s_1^{-1}\ s_2u_1$, and $s_1^{-1}s_2u_1 \subset Rs_1^{-1}s_2s_1^{-1} + u_2u_1u_2$, we get

$$\begin{array}{rcl} s_4u_3u_2u_3s_1s_2^{-1}s_1s_4u_3u_2u_3s_4 & \subset & s_4u_3u_2u_3s_2^{-1}s_1^{-1}\ s_2u_1s_4u_3u_2u_3s_4 \\ & \subset & s_4(u_3u_2u_3u_2)s_1^{-1}\ s_2u_1s_4u_3u_2u_3s_4 \\ & \subset & s_4(u_2u_3u_2u_3)s_1^{-1}\ s_2u_1s_4u_3u_2u_3s_4 \\ & \subset & u_2s_4u_3u_2u_3s_1^{-1}\ s_2u_1s_4u_3u_2u_3s_4 \\ & \subset & u_2s_4u_3u_2u_3s_1^{-1}\ s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4(u_3u_2u_3u_2)\ u_1u_2s_4u_3u_2u_3s_4 \\ & \subset & u_2s_4u_3u_2u_3s_1^{-1}\ s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4u_2u_3u_2u_3\ u_1u_2s_4u_3u_2u_3s_4 \\ & \subset & u_2s_4u_3u_2u_3s_1^{-1}\ s_2s_1^{-1}s_4u_3u_2u_3s_4 + u_2s_4u_3u_2u_3\ u_1u_2s_4u_3u_2u_3s_4 \\ & \subset & u_2s_4u_3u_2u_3s_1^{-1}\ s_2s_1^{-1}s_4u_3u_2u_3s_4 + A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3. We can thus restrict to $s_4u_3u_2u_3s_1^{-1}\ s_2s_1^{-1}s_4u_3u_2u_3s_4$. Moreover, using that $u_3u_2u_3 \subset u_2s_3s_2^{-1}s_3 + u_2u_3u_2$ and $u_3u_2u_3 \subset s_3^{-1}s_2s_3^{-1}u_2 + u_2u_3u_2$ leads to $s_4u_3u_2u_3s_1^{-1}\ s_2s_1^{-1}s_4u_3u_2u_3s_4 \subset u_2s_4s_3s_2^{-1}s_3s_1^{-1}\ s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4u_2 + A_4u_4A_4u_4A_4$ by proposition 6.3. Now

$$\begin{array}{rcl} s_4s_3s_2^{-1}s_3s_1^{-1}\ s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 & = & s_1^{-1}s_1s_4s_3s_2^{-1}s_3s_1^{-1}\ s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_4s_3(s_1s_2^{-1}s_1^{-1})s_3\ s_2s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}(s_2s_3\ s_2)s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3s_2\ s_3s_1^{-1}s_4s_3^{-1}s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3s_2\ s_1^{-1}(s_3s_4s_3^{-1})s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3s_2\ s_1^{-1}s_4^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}(s_3s_4s_3)s_2^{-1}s_1^{-1}s_3s_2\ s_1^{-1}s_4^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_4s_3s_4s_2^{-1}s_1^{-1}s_3s_2\ s_1^{-1}s_4^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_4s_3s_2^{-1}s_1^{-1}(s_4s_3s_4^{-1})s_2\ s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_4s_3s_2^{-1}s_1^{-1}s_3^{-1}s_4s_3s_2\ s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_4(s_3s_2^{-1}s_3^{-1})s_1^{-1}s_4s_3s_2\ s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_4s_2^{-1}s_3^{-1}s_2s_1^{-1}s_4s_3s_2\ s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4s_3s_2\ s_1^{-1}s_3s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4(s_3s_2\ s_3)s_1^{-1}s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4s_2s_3\ s_2s_1^{-1}s_4s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2(s_4s_3s_4)\ s_2s_1^{-1}s_2s_3^{-1}s_4 \\ & = & s_1^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3\ s_2s_1^{-1}s_2s_3^{-1}s_4 \\ & \subset & A_4s_4s_3^{-1}s_2s_1^{-1}s_2s_3s_4s_3\ s_2s_1^{-1}s_2s_3^{-1}s_4 \end{array}$$

and this proves the claim. ∎

Lemma 6.5.

$$u_4u_3u_2u_1u_2u_3u_4u_2u_3u_1u_2u_3u_4 \subset A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4$$

Proof. We consider the expression $u_4s_3^{\alpha}u_2u_1u_2s_3^{\beta}u_4u_2u_3u_1u_2u_3u_4$ and we first assume $\alpha = \beta$; by applying if necessary Φ, we can then assume $\alpha = \beta = -1$. Since $u_2u_1u_2 \subset u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we have

$$\begin{array}{rcl} u_4s_3^{-1}u_2u_1u_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 & \subset & u_4s_3^{-1}u_1s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 + u_4s_3^{-1}u_1u_2u_1s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 \\ & \subset & u_1u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 + u_1u_4s_3^{-1}u_2u_1s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 \end{array}$$

and we are reduced to $u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4$ by lemmas 6.3 and 6.4. Now

$$u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_2u_3u_1u_2u_3u_4 = u_4(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_2u_1u_4u_3u_2u_3u_4$$

and $(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_2u_1 \subset A_3(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})+A_3u_3A_3+A_3s_3s_2^{-1}s_3A_3$ by lemma 4.4. We then have

$$\begin{array}{rcl} u_4(A_3u_3A_3+A_3s_3s_2^{-1}s_3A_3)u_4u_3u_2u_3u_4 & \subset & A_3u_4u_3A_3u_4u_3u_2u_3u_4+A_3u_4s_3s_2^{-1}s_3A_3u_4u_3u_2u_3u_4 \\ & \subset & A_3u_4u_3(u_1u_2u_1u_2)u_4u_3u_2u_3u_4 \\ & & +A_3u_4s_3s_2^{-1}s_3(u_1u_2u_1u_2)u_4u_3u_2u_3u_4 \\ & \subset & A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 \\ & & +A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4(u_2u_3u_2u_3)u_4 \\ & \subset & A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 \\ & & +A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4u_3u_2u_3u_2u_4 \\ & \subset & A_3u_4u_3u_1u_2u_1u_2u_4u_3u_2u_3u_4 \\ & & +A_3u_4s_3s_2^{-1}s_3u_1u_2u_1u_4u_3u_2u_3u_4u_2 \\ & \subset & A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 \\ & & +A_4u_4A_4u_4A_4 \end{array}$$

by lemmas 6.3 and 6.4, and $u_4A_3(s_3^{-1}s_2s_1^{-1}s_2s_3^{-1})u_4u_3u_2u_3u_4 = A_3u_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}u_4u_3u_2u_3u_4 \subset A_4u_4A_4u_4A_4$ by proposition 6.3, so this solves the case $\alpha=\beta$.

We can thus assume $\alpha=-\beta$, that is we consider the expression $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}u_2u_4u_1u_3u_2u_3u_4$, that we split in two cases $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}s_2^{\gamma}u_4u_1u_3u_2u_3u_4$ for $\gamma\in\{-1,1\}$. Up to applying Φ, we can restrict to $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}s_2^{\gamma}s_4u_1u_3u_2u_3s_4^{\alpha}$ for some $\alpha\in\{-1,1\}$, and using $u_3u_2u_3\subset s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}u_2+u_2u_3u_2$ we can restrict to $u_4s_3^{\beta}u_2u_1u_2s_3^{-\beta}s_2^{\gamma}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha}$ by proposition 6.3.

First assume $\gamma=-1$. Using again $u_2u_1u_2\subset u_1s_2s_1^{-1}s_2+u_1u_2u_1$ we can restrict to

$$u_4s_3^{\beta}s_2s_1^{-1}s_2s_3^{-\beta}s_2^{-1}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha}.$$

If $\beta=1$, then we get

$$\begin{array}{rcl} u_4s_3s_2s_1^{-1}s_2s_3^{-1}s_2^{-1}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} & \subset & u_4s_3s_2s_1^{-1}(s_2s_3^{-1}s_2^{-1})s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4s_3s_2s_1^{-1}s_3^{-1}s_2^{-1}s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4(s_3s_2s_3^{-1})s_1^{-1}s_2^{-1}s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4s_2^{-1}s_3s_2s_1^{-1}s_2^{-1}s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & s_2^{-1}u_4s_3s_2s_1^{-1}s_2^{-1}s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3. For the case $\beta=-1$, we can restrict to an expression of the form

$$u_4s_3^{-1}s_2s_1^{-1}s_2s_3s_2^{-1}s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha},$$

and we get

$$\begin{array}{rcl} u_4s_3^{-1}s_2s_1^{-1}(s_2s_3s_2^{-1})s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} & \subset & u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_3s_4u_1s_3^{\alpha}s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2(s_3s_4s_3^{\alpha})u_1s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_4^{\alpha}s_3s_4u_1s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha} \\ & \subset & u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_4^{\alpha}s_3u_1s_2^{-\alpha}s_4s_3^{\alpha}s_4^{\alpha}) \\ & \subset & u_4s_3^{-1}s_2s_1^{-1}s_3^{-1}s_2s_4^{\alpha}s_3u_1s_2^{-\alpha}s_3^{\alpha}s_4^{\alpha}s_3 \\ & \subset & A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4+A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3 and lemma 6.4.

Now assume $\gamma = 1$. Using again $u_2u_1u_2 \subset u_1s_2^{-1}s_1s_2^{-1} + u_1u_2u_1$ we can restrict to the form $u_4s_3^{\beta}s_2^{-1}s_1s_2^{-1}s_3^{-\beta}s_2u_4u_1u_3u_2u_3u_4$. If $\beta = 1$ we get

$$\begin{array}{lcl} u_4s_3s_2^{-1}s_1(s_2^{-1}s_3^{-1}s_2)u_4u_1u_3u_2u_3u_4 & \subset & u_4s_3s_2^{-1}s_1s_3s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & u_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & u_4u_2(s_3^{-1}s_2s_3^{-1})s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & & +u_4u_2u_3u_2s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & u_2u_4s_3^{-1}s_2s_3^{-1}s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4+ \\ & & u_2u_4u_3u_2s_1s_2^{-1}s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & u_2u_4s_3^{-1}s_2s_1(s_3^{-1}s_2^{-1}s_3^{-1})u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\ & \subset & u_2u_4s_3^{-1}(s_2s_1s_2^{-1})s_3^{-1}s_2^{-1}u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\ & \subset & u_2u_4s_3^{-1}s_1^{-1}s_2s_1s_3^{-1}s_2^{-1}u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\ & \subset & u_2s_1^{-1}u_4s_3^{-1}s_2s_1s_3^{-1}s_2^{-1}u_4u_1u_3u_2u_3u_4 + A_4u_4A_4u_4A_4 \\ & \subset & A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_4 + A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3 and lemma 6.4.

If $\beta = -1$ we get

$$\begin{array}{lcl} u_4s_3^{-1}s_2^{-1}s_1(s_2^{-1}s_3s_2)u_4u_1u_3u_2u_3u_4 & \subset & u_4s_3^{-1}s_2^{-1}s_1s_3s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & u_4(s_3^{-1}s_2^{-1}s_3)s_1s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & u_4s_2s_3^{-1}s_2^{-1}s_1s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & s_2u_4s_3^{-1}s_2^{-1}s_1s_2s_3^{-1}u_4u_1u_3u_2u_3u_4 \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3. This concludes the proof of the lemma. □

Proposition 6.6. *$u_4A_4u_4A_4u_4 \subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_4 + A_4u_4A_4u_4A_4$, and thus $A_5^{(3)} \subset A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_4 + A_4u_4A_4u_4A_4$.*

Proof. We will actually prove

$$\begin{array}{lcl} u_4A_4u_4A_4u_4 & \subset & A_4u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3 + A_4u_4A_4u_4A_4 \\ & & +A_3u_4u_3u_2u_3u_1u_2u_4u_3u_2u_1u_2u_3u_4A_3 + A_3u_4u_3u_2u_1u_2u_3u_4u_2u_1u_3u_2u_3u_4A_3 \end{array}$$

and the statement will then follow by lemmas 6.4 and 6.5.

By theorem 4.1 we have $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ and $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + u_3u_2u_1u_2u_3A_3$, whence

$$\begin{array}{lcl} u_4A_4u_4A_4u_4 & \subset & u_4A_3u_3A_3u_4A_3u_3A_3u_4 + u_4A_3u_3A_3u_4A_3u_3u_2u_3A_3u_4 \\ & & +u_4A_3u_3A_3u_4u_3u_2u_1u_2u_3A_3u_4 + u_4A_3u_3u_2u_3A_3u_4A_3u_3A_3u_4 \\ & & +u_4A_3u_3u_2u_3A_3u_4A_3u_3u_2u_3A_3u_4 + u_4A_3u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3A_3u_4 \\ & & +u_4A_3u_3u_2u_1u_2u_3u_4A_3u_3A_3u_4 + u_4A_3u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3A_3u_4 \\ & & +u_4A_3u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3A_3u_4 \\ & \subset & A_3u_4u_3A_3u_4u_3u_4A_3 + A_3u_4u_3A_3u_4u_3u_2u_3u_4A_3 \\ & & +A_3u_4u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 + A_3u_4u_3u_2u_3u_4A_3u_3u_4A_3 \\ & & +A_3u_4u_3u_2u_3A_3u_4u_3u_2u_3u_4A_3 + A_3u_4u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 \\ & & +A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_4A_3 + A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3u_4A_3 \\ & & +A_3u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3A_3u_4u_3u_4A_3 + A_3u_4u_3A_3u_4u_3u_2u_3u_4A_3 \\ & & +A_3u_4u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 + A_3u_4u_3u_2u_3u_4A_3u_3u_4A_3 \\ & & +A_3u_4u_3u_2u_3A_3u_4u_3u_2u_3u_4A_3 + A_3u_4u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 \\ & & +A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_4A_3 + A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3u_4A_3 \\ & & +A_3u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3 \end{array}$$

We have

(1) $A_3u_4u_3A_3u_4u_3u_4A_3 \subset A_3u_4u_3(u_2u_1u_2u_1)u_4u_3u_4A_3 \subset A_4u_4A_4u_4A_4$ by proposition 6.3.
(2) $A_3u_4u_3A_3u_4u_3u_2u_3u_4A_3 \subset A_3u_4u_3u_2u_1u_2u_1u_4u_3u_2u_3u_4A_3 \subset A_4u_4A_4u_4A_4$ by proposition 6.3.

(3) We have

$$\begin{array}{rcl} A_3u_4u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 & \subset & A_3u_4u_3u_2u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_4u_3(u_1u_2u_1u_2)u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_4u_3u_2u_1u_2u_1u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3.

(4) $A_3u_4u_3u_2u_3u_4A_3u_3u_4A_3 \subset A_3u_4u_3u_2u_3u_4u_2u_1u_2u_1u_3u_4A_3 \subset A_4u_4A_4u_4A_4$ by proposition 6.3.

(5) Using $A_3 = u_2u_1u_2u_1$ we get

$$\begin{array}{rcl} A_3u_4u_3u_2u_3A_3u_4u_3u_2u_3u_4A_3 & \subset & A_3u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_3u_4A_3 \\ & \subset & A_4(u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4)A_3 + A_4u_4A_4u_4A_4 \end{array}$$

by lemma 6.4.

(6) Using $A_3 = u_2u_1u_2u_1$ we get

$$\begin{array}{rcl} A_3u_4u_3u_2u_3A_3u_4u_3u_2u_1u_2u_3u_4A_3 & \subset & A_3u_4u_3u_2u_3u_2u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_3u_1u_2u_1u_4u_3u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_3u_1u_2u_4u_3u_1u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_3u_1u_2u_4u_3(u_1u_2u_1u_2)u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_3u_1u_2u_4u_3u_2u_1u_2u_1u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_3u_1u_2u_4u_3u_2u_1u_2u_3u_4A_3 \end{array}$$

(7) Using $A_3 = u_1u_2u_1u_2$ we get

$$\begin{array}{rcl} A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_4A_3 & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_1u_3u_4u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3(u_2u_1u_2u_1)u_3u_4u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_1u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_2u_1u_2u_3u_4A_3 \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3.

(8) Using $A_3 = u_1u_2u_1u_2$ we get

$$\begin{array}{rcl} A_3u_4u_3u_2u_1u_2u_3u_4A_3u_3u_2u_3u_4A_3 & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_2u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1(u_2u_3u_2u_3)u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_3u_2u_3u_2u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_1u_2u_1u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_1u_3u_4u_2u_1u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3(u_2u_1u_2u_1)u_3u_4u_2u_1u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_1u_2u_1u_2u_3u_4u_2u_1u_3u_2u_3u_4A_3 \\ & \subset & A_3u_4u_3u_2u_1u_2u_3u_4u_2u_1u_3u_2u_3u_4A_3 \end{array}$$

(9) the case $A_3u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4A_3$ is clear.

□

6.2. **The A_4-bimodule $A_5^{(3)}/A_5^{(2)}$: a smaller set of generators.**

Lemma 6.7. *For all* $\alpha, \beta, \gamma, \dots \in \{-1, 1\}$,

$$\begin{array}{lcl} s_4^{\alpha} s_3^{\beta} A_3 s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} A_3 s_3^{\zeta} s_4^{\eta} & \subset & u_1 s_4^{\alpha} s_3^{\beta} (s_2 s_1^{-1} s_2) s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} (s_2 s_1^{-1} s_2) s_3^{\zeta} s_4^{\eta} u_1 + A_5^{(2)} \\ s_4^{\alpha} s_3^{\beta} A_3 s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} A_3 s_3^{\zeta} s_4^{\eta} & \subset & u_1 s_4^{\alpha} s_3^{\beta} (s_2^{-1} s_1 s_2^{-1}) s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} (s_2 s_1^{-1} s_2) s_3^{\zeta} s_4^{\eta} u_1 + A_5^{(2)} \\ s_4^{\alpha} s_3^{\beta} A_3 s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} A_3 s_3^{\zeta} s_4^{\eta} & \subset & u_1 s_4^{\alpha} s_3^{\beta} (s_2 s_1^{-1} s_2) s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} (s_2^{-1} s_1 s_2^{-1}) s_3^{\zeta} s_4^{\eta} u_1 + A_5^{(2)} \\ s_4^{\alpha} s_3^{\beta} A_3 s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} A_3 s_3^{\zeta} s_4^{\eta} & \subset & u_1 s_4^{\alpha} s_3^{\beta} (s_2^{-1} s_1 s_2^{-1}) s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} (s_2^{-1} s_1 s_2^{-1}) s_3^{\zeta} s_4^{\eta} u_1 + A_5^{(2)} \end{array}$$

Proof. This is an easy consequence of the decompositions $A_3 = u_1 u_2 u_1 + u_1 s_2 s_1^{-1} s_2 = u_1 u_2 u_1 + s_2 s_1^{-1} s_2 u_1 = u_1 u_2 u_1 + u_1 s_2^{-1} s_1 s_2^{-1} = u_1 u_2 u_1 + s_2^{-1} s_1 s_2^{-1} u_1$ of theorem 3.2 and of proposition 6.3. □

Lemma 6.8. *For* $i, j, k, \alpha, \beta, \gamma \in \{-1, 1\}$,

1. $s_4^i s_3^{\alpha} A_3 s_3^{-\alpha} s_4^j s_3^{\beta} A_3 s_3^{\gamma} s_4^k \subset A_5^{(2)}$ *unless* $i = j = k$
2. $s_4^i s_3^{\alpha} A_3 s_3^{\beta} s_4^j s_3^{\gamma} A_3 s_3^{-\gamma} s_4^k \subset A_5^{(2)}$ *unless* $i = j = k$
3. $s_4^i s_3^{\alpha} A_3 s_3^{-\alpha} s_4^j s_3^{\beta} A_3 s_3^{\beta} s_4^k \subset A_5^{(2)}$
4. $s_4^i s_3^{\alpha} A_3 s_3^{\alpha} s_4^j s_3^{\beta} A_3 s_3^{-\beta} s_4^k \subset A_5^{(2)}$
5. $s_4^i s_3^{\alpha} A_3 s_3^{-\alpha} s_4^j s_3^{\alpha} A_3 s_3^{-\alpha} s_4^k \subset A_5^{(2)}$

Proof. We use the formulas $s_3^{-1}(s_2 s_1^{-1} s_2) s_3 = s_2 s_1 (s_3 s_2^{-1} s_3) s_1^{-1} s_2^{-1}$ and $s_3 (s_2 s_1^{-1} s_2) s_3^{-1} = s_2^{-1} s_1^{-1} (s_3 s_2^{-1} s_3) s_1 s_2$ which are easy to prove and which already hold in the braid group B_4, and can be summarized as $s_3^{-\alpha}(s_2 s_1^{-1} s_2) s_3^{\alpha} = s_2^{\alpha} s_1^{\alpha} (s_3 s_2^{-1} s_3) s_1^{-\alpha} s_2^{-\alpha}$ for $\alpha \in \{-1, 1\}$. We also use the fact that s_2 (and thus s_2^{-1}) commutes with $s_3 s_2 s_1^{-1} s_2 s_3$ (already in the braid group B_4), and similarly s_2^{-1} (and thus s_2) commutes with $s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3^{-1}$. Together with lemma 6.7, this yields

$$\begin{array}{lcl} s_4^i s_3^{\alpha} A_3 s_3^{-\alpha} s_4^j s_3^{\beta} A_3 s_3^{\beta} s_4^k & \subset & s_4^i s_3^{\alpha} (s_2 s_1^{-1} s_2) s_3^{-\alpha} s_4^j s_3^{\beta} (s_2^{\beta} s_1^{-\beta} s_2^{\beta}) s_3^{\beta} s_4^k + A_5^{(2)} \\ & \subset & s_4^i (s_3^{\alpha} s_2 s_1^{-1} s_2 s_3^{-\alpha}) s_4^j (s_3^{\beta} s_2^{\beta} s_1^{-\beta} s_2^{\beta} s_3^{\beta}) s_4^k + A_5^{(2)} \\ & \subset & s_4^i s_2^{-\alpha} s_1^{-\alpha} (s_3 s_2^{-1} s_3) s_1^{\alpha} s_2^{\alpha} s_4^j (s_3^{\beta} s_2^{\beta} s_1^{-\beta} s_2^{\beta} s_3^{\beta}) s_4^k + A_5^{(2)} \\ & \subset & s_2^{-\alpha} s_1^{-\alpha} s_4^i s_3 s_2^{-1} s_3 s_1^{\alpha} s_4^j s_2^{\alpha} (s_3^{\beta} s_2^{\beta} s_1^{-\beta} s_2^{\beta} s_3^{\beta}) s_4^k + A_5^{(2)} \\ & \subset & A_3 s_4^i s_3 s_2^{-1} s_3 s_1^{\alpha} s_4^j (s_3^{\beta} s_2^{\beta} s_1^{-\beta} s_2^{\beta} s_3^{\beta}) s_2^{\alpha} s_4^k + A_5^{(2)} \\ & \subset & A_3 s_4^i s_3 s_2^{-1} s_3 s_1^{\alpha} s_4^j (s_3^{\beta} s_2^{\beta} s_1^{-\beta} s_2^{\beta} s_3^{\beta}) s_4^k A_3 + A_5^{(2)} \\ & \subset & A_5^{(2)} \end{array}$$

by proposition 6.3, and this proves (3), as well as the symmetric case (4). This also proves (1) in case $\beta = \gamma$. We thus deal with

$$\begin{array}{lcl} s_4^i s_3^{\alpha} A_3 s_3^{-\alpha} s_4^j s_3^{\beta} A_3 s_3^{-\beta} s_4^k & \subset & s_4^i s_3^{\alpha} (s_2 s_1^{-1} s_2) s_3^{-\alpha} s_4^j s_3^{\beta} (s_2 s_1^{-1} s_2) s_3^{-\beta} s_4^k + A_5^{(2)} \\ & \subset & s_4^i (s_3^{\alpha} s_2 s_1^{-1} s_2 s_3^{-\alpha}) s_4^j (s_3^{\beta} s_2 s_1^{-1} s_2 s_3^{-\beta}) s_4^k + A_5^{(2)} \\ & \subset & s_4^i s_2^{-\alpha} s_1^{-\alpha} (s_3 s_2^{-1} s_3) s_1^{\alpha} s_2^{\alpha} s_4^j s_2^{-\beta} s_1^{-\beta} (s_3 s_2^{-1} s_3) s_1^{\beta} s_2^{\beta} s_4^k + A_5^{(2)} \\ & \subset & s_2^{-\alpha} s_1^{-\alpha} s_4^i s_3 s_2^{-1} s_3 s_1^{\alpha} s_2^{\alpha} s_2^{-\beta} s_4^j s_1^{-\beta} (s_3 s_2^{-1} s_3) s_4^k s_1^{\beta} s_2^{\beta} + A_5^{(2)} \\ & \subset & A_5^{(2)} \end{array}$$

if $\alpha = \beta$ by proposition 6.3, and we get (5). Otherwise, $\alpha = -\beta$, and

$$\begin{array}{lcl} s_4^i s_3^{\alpha} A_3 s_3^{-\alpha} s_4^j s_3^{\beta} A_3 s_3^{-\beta} s_4^k & \subset & s_2^{-\alpha} s_1^{-\alpha} s_4^i s_3 s_2^{-1} s_3 s_1^{\alpha} s_2^{2\alpha} s_1^{\alpha} s_4^j (s_3 s_2^{-1} s_3) s_4^k s_1^{-\alpha} s_2^{-\alpha} + A_5^{(2)} \\ & \subset & A_5^{(2)} \end{array}$$

unless $i = j = k$ by proposition 6.3, and we get (1). (2) is proved symmetrically.

□

Corollary 6.9.

1. $s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3 s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 \in A_4 u_4 A_4 u_4 A_4$
2. $s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3 s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 \in A_4 u_4 A_4 u_4 A_4$

Lemma 6.10. $s_4^{-1} w^+ s_4^{-1} w^+ s_4^{-1} \in A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 u_4 A_4 u_4 A_4$

Proof. We first use $s_2^{-1}s_1s_2^{-1} \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ and $s_2^{-1}s_1s_2^{-1} \in s_2s_1^{-1}s_2u_1 + u_1u_2u_1$ together with proposition 6.3 to get

$$\begin{array}{rcl} s_4^{-1}w^+s_4^{-1}w^+s_4^{-1} & = & s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} \\ & \subset & u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} + u_1s_4^{-1}s_3u_2u_1s_3s_4^{-1}s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{-1} \\ & \subset & u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1} + A_4u_4A_4u_4A_4 \\ & \subset & u_1s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}u_1 + A_4u_4A_4u_4A_4 \\ & \subset & A_4(s_3^{-1}s_4^{-1}s_3)s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2(s_3s_4^{-1}s_3^{-1})A_4 + A_4u_4A_4u_4A_4 \\ & \subset & A_4s_4s_3^{-1}s_4^{-1}s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_4^{-1}s_3^{-1}s_4A_4 + A_4u_4A_4u_4A_4 \\ & \subset & A_4s_4s_3^{-1}s_2s_1^{-1}s_2(s_4^{-1}s_3s_4^{-1}s_3s_4^{-1})s_2s_1^{-1}s_2s_3^{-1}s_4A_4 + A_4u_4A_4u_4A_4 \end{array}$$

By lemma 3.5 $s_4^{-1}s_3s_4^{-1}s_3s_4^{-1}$ is a linear combination of terms of several kinds

(1) elements x of u_3u_4 or u_4u_3, for which we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by a direct application of proposition 6.3.
(2) elements x that can be put in the the form $s_4^{\alpha}s_3^{\beta}s_4^{\gamma}$ with $\alpha = -1$ or $\gamma = -1$, in which case we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ through one application of the equation $s_4s_3^{\cdot\cdot}s_4^{-1} \in u_3u_4u_3$ or $s_4^{-1}s_3^{\cdot\cdot}s_4 \in u_3u_4u_3$, and proposition 6.3.
(3) the element $s_3^{-1}s_4s_3^{-1}$, which provides $s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4 = s_4w^-s_4w^-s_4w^-$.
(4) the element $x = s_3s_4^{-1}s_3$, for which we get $s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 \subset A_4u_4A_4u_4A_4$ by corollary 6.9 (1).
(5) the element $x = s_4^{-1}s_3s_4^{-1}s_3$, for which we get

$$\begin{array}{rcl} s_4s_3^{-1}s_2s_1^{-1}s_2xs_2s_1^{-1}s_2s_3^{-1}s_4 & = & s_4s_3^{-1}s_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\ & = & (s_4s_3^{-1}s_4^{-1})s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\ & = & s_3^{-1}s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\ & \subset & A_4s_4^{-1}s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_4 \\ & \subset & A_4u_4A_4u_4A_4 \end{array}$$

by corollary 6.9 (2).

This proves the inclusion.

∎

Lemma 6.11.

(1) $u_4A_4u_4u_3u_4 \subset A_4u_4A_4u_4A_4$
(2) $u_4u_3u_4A_4u_4 \subset A_4u_4A_4u_4A_4$
(3) $s_4^{\beta}u_3u_2u_1u_2s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha}u_2u_1u_2u_3s_4^{\beta} \subset A_4u_4A_4u_4A_4$
(4) $s_4^{\alpha}s_3^{\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha}u_2u_1u_2u_3s_4^{\beta} \subset A_4u_4A_4u_4A_4$
(5) $s_4^{\beta}u_3u_2u_1u_2s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha}u_2u_1u_2s_3^{-\alpha}s_4^{-\alpha} \subset A_4u_4A_4u_4A_4$
(6) $u_4s_3^{\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\alpha}s_3^{\alpha}u_2u_1u_2s_3^{\alpha}u_4 \subset A_4u_4A_4u_4A_4$
(7) $s_4w^+s_4^{-1}w^+s_4^{-1} \in A_4u_4A_4u_4A_4$
(8) $s_4w^-s_4w^-s_4^{-1} \in A_4u_4A_4u_4A_4$

Proof. Since $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + A_3u_3u_2u_1u_2u_3$ we have $u_4A_4u_4u_3u_4 \subset A_3u_4u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_3A_3u_4u_3u_4 + A_3u_4u_3u_2u_1u_2u_3u_4u_3u_4$. We have $u_4u_3A_3u_4u_3u_4 \subset u_4u_3u_1u_2u_1u_2u_4u_3u_4 \subset A_4u_4A_4u_4A_4$ by proposition 6.3,

$$\begin{array}{rcl} u_4u_3u_2u_3A_3u_4u_3u_4 & \subset & u_4(u_3u_2u_3u_2)u_1u_2u_1u_4u_3u_4 \\ = u_4u_2u_3u_2u_3u_1u_2u_1u_4u_3u_4 & = & u_2u_4u_3u_2u_3u_1u_4u_2u_1u_3u_4 \subset A_4u_4A_4u_4A_4 \end{array}$$

by proposition 6.3, and $u_4u_3u_2u_1u_2u_3u_4u_3u_4 \subset A_4u_4A_4u_4A_4$ by proposition 6.3. This proves (1). (2) is deduced from (1) by applying Ψ. We turn to (3). Since $s_4^{\beta}u_3u_2u_1u_2(s_3^{\alpha}s_4^{\gamma}s_3^{-\alpha})u_2u_1u_2u_3s_4^{\beta} = s_4^{\beta}u_3u_2u_1u_2s_4^{-\alpha}s_3^{\gamma}s_4^{\alpha}u_2u_1u_2u_3s_4^{\beta} = s_4^{\beta}u_3s_4^{-\alpha}u_2u_1u_2s_3^{\gamma}u_2u_1u_2s_4^{\alpha}u_3s_4^{\beta}$ and either $s_4^{\beta}u_3s_4^{-\alpha} \subset u_3u_4u_3$ or $s_4^{\alpha}u_3s_4^{\beta} \subset u_3u_4u_3$. In both cases we get an element of $A_4u_4A_4u_4u_3u_4A_4 \subset A_4u_4A_4u_4A_4$ or $A_4u_4u_3u_4A_4u_4A_4 \subset A_4u_4A_4u_4A_4$ by (1) or (2), and this proves (3). (4) and (5) are similar and left to the reader.

Now

$$\begin{array}{rcl} u_4 s_3^\alpha u_2 u_1 u_2 s_3^\alpha s_4^\alpha s_3^\alpha u_2 u_1 u_2 s_3^\alpha u_4 & = & u_4 s_3^\alpha u_2 u_1 u_2 s_4^\alpha s_3^\alpha s_4^\alpha u_2 u_1 u_2 s_3^\alpha u_4 \\ & = & (u_4 s_3^\alpha s_4^\alpha) u_2 u_1 u_2 s_3^\alpha u_2 u_1 u_2 (s_4^\alpha s_3^\alpha u_4) \\ & \subset & u_3 u_4 u_3 u_2 u_1 u_2 u_3 u_2 u_1 u_2 u_3 u_4 u_3 \quad \subset \quad A_4 u_4 A_4 u_4 A_4 \end{array}$$

and this proves (6).

To prove (7), we compute, using b, b' for elements in $u_2 u_1 u_2$,

$$\begin{array}{rcl} s_4 w^+ s_4^{-1} w^+ s_4^{-1} & \subset & s_4 s_3 b s_3 s_4^{-1} s_3 b' s_3 s_4^{-1} \\ & \subset & s_3^{-1}(s_3 s_4 s_3) b s_3 s_4^{-1} s_3 b' s_3 s_4^{-1} \\ & \subset & s_3^{-1} s_4 s_3 s_4 b s_3 s_4^{-1} s_3 b' s_3 s_4^{-1} \\ & \subset & s_3^{-1} s_4 s_3 b (s_4 s_3 s_4^{-1}) s_3 b' s_3 s_4^{-1} \\ & \subset & s_3^{-1} s_4 s_3 b s_3^{-1} s_4 s_3^2 b' s_3 s_4^{-1} \end{array}$$

Now $s_3^2 \in R + R s_3 + R s_3^{-1}$, and $s_3^{-1} s_4 s_3 b s_3^{-1} s_4 b' s_3 s_4^{-1} \in A_4 u_4 A_4 u_4 A_4$ by proposition 6.3,

$$s_3^{-1} s_4 s_3 b s_3^{-1} s_4 s_3 b' s_3 s_4^{-1} \in A_4 u_4 A_4 u_4 A_4$$

by lemma 6.8 (3), and $s_3^{-1} s_4 s_3 b s_3^{-1} s_4 s_3^{-1} b' s_3 s_4^{-1} \subset A_5^{(2)}$ by lemma 6.8 (1). The proof of (8) is similar :

$$\begin{array}{rcl} s_4 w^- s_4 w^- s_4^{-1} & \subset & s_4 s_3^{-1} b s_3^{-1} s_4 s_3^{-1} b' s_3^{-1} s_4^{-1} \\ & \subset & s_4 s_3^{-1} b s_3^{-1} s_4 s_3^{-1} b' (s_3^{-1} s_4^{-1} s_3^{-1}) \\ & \subset & s_4 s_3^{-1} b s_3^{-1} s_4 s_3^{-1} b' s_4^{-1} s_3^{-1} s_4^{-1} \\ & \subset & s_4 s_3^{-1} b s_3^{-1} (s_4 s_3^{-1} s_4^{-1}) b' s_3^{-1} s_4^{-1} \\ & \subset & s_4 s_3^{-1} b s_3^{-1} s_3^{-1} s_4^{-1} s_3 b' s_3^{-1} s_4^{-1} \\ & \subset & s_4 s_3^{-1} b s_3^{-2} s_4^{-1} s_3 b' s_3^{-1} s_4^{-1} \end{array}$$

Now $s_3^{-2} \in R + R s_3 + R s_3^{-1}$, and we conclude similarly.

□

Lemma 6.12.

(1) $s_4 s_3^{-1} A_3 s_3 s_4 s_3 A_3 s_3^{-1} s_4 \subset u_3 s_4^- w^+ s_4 w^- s_4 + A_5^{(2)}$
(2) $(s_4 w^+ s_4^{-1} w^+ s_4) s_3^{-1} \in s_3^{-1}(s_4 w^+ s_4^{-1} w^+ s_4) + A_5^{(2)}$
(3) $s_4 w^+ s_4^{-1} w^+ s_4 \in A_3^\times s_4 (s_3 s_2^{-1} s_3)(s_1 s_2^{-1} s_1) s_4 (s_3 s_2^{-1} s_3) s_4 A_4^\times + A_5^{(2)}$
(4) $s_4 w^- s_4 w^+ s_4^{-1} \in A_4^\times s_4^{-1} w^+ s_4 w^- s_4 A_4^\times + A_5^{(2)}$
(5) $s_4 w^- s_4^{-1} w^+ s_4^{-1} \in A_4^\times s_4^{-1} w^+ s_4^{-1} w^- s_4 A_4^\times + A_5^{(2)}$
(6) $s_4 s_3 A_3 s_3^{-1} s_4 s_3^{-1} A_3 s_3 s_4 \subset u_3 s_4 w^+ s_4^{-1} w^+ s_4 u_3 + A_5^{(2)}$

Proof. We first prove (1). By lemma 6.7 we need to prove $s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \subset u_3 s_4^- w^+ s_4 w^- s_4 + A_5^{(2)}$, and we get, using proposition 6.3

$$\begin{array}{rcl} s_4 s_3^{-1} s_2 s_1^{-1} s_2 (s_3 s_4 s_3) s_2 s_1^{-1} s_2 s_3^{-1} s_4 & = & s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\ & = & (s_4 s_3^{-1} s_4) s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\ & \subset & u_3 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\ & & + u_3 u_4 u_3 s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \\ & \subset & u_3 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\ & \subset & u_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 (s_4^{-1} s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\ & \subset & u_3 s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_3 s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\ & \subset & u_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \\ & \subset & u_3 s_4^{-1} w^+ s_4 w^- s_4 + A_5^{(2)}. \end{array}$$

We now prove (2). We have, using $s_3 s_4^{-1} s_3 s_4^{-1} \in R s_4 s_3^{-1} s_4 s_3^{-1} + u_3 u_4 u_3 + u_4 u_3 u_4$ and $s_3 s_4^{-1} s_3 s_4^{-1} - s_4^{-1} s_3 s_4^{-1} s_3 \in u_3 u_4 + u_4 u_3$ by lemma 3.6, we get

$$
\begin{array}{rcl}
s_4 w^+ s_4^{-1} w^+ s_4 . s_3^{-1} & = & s_4 w^+ s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} (s_3 s_4 s_3^{-1}) \\
& = & s_4 w^+ s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_4^{-1} s_3 s_4 \\
& = & s_4 w^+ s_4^{-1} s_3 s_4^{-1} s_2^{-1} s_1 s_2^{-1} s_3 s_4 \\
& = & s_4 s_3 s_2^{-1} s_1 s_2^{-1} (s_3 s_4^{-1} s_3 s_4^{-1}) s_2^{-1} s_1 s_2^{-1} s_3 s_4 \\
& \in & s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_4^{-1} s_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 + A_5^{(2)} \\
& \subset & (s_4 s_3 s_4^{-1}) s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 + A_5^{(2)} \\
& \subset & s_3^{-1} s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1} s_3 s_2^{-1} s_1 s_2^{-1} s_3 s_4 + A_5^{(2)} \\
& \subset & s_3^{-1} . s_4 w^+ s_4^{-1} w^+ s_4 + A_5^{(2)}
\end{array}
$$

We now prove (3). We have

$$
\begin{array}{rcl}
s_4 w^+ s_4^{-1} w^+ s_4 & = & s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4 \\
& \in & s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4 + A_5^{(2)} \\
& \subset & s_3^{-1} (s_3 s_4 s_3) s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 (s_3 s_4 s_3) s_3^{-1} + A_5^{(2)} \\
& \subset & s_3^{-1} s_4 s_3 s_4 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_4 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3 (s_4^{-1} s_3 s_4) s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 (s_3^2) s_4 s_3^{-1}) s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4^2 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
& & + R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 (s_4 s_3 s_4) s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
& & + R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 (s_3 s_4 s_3) s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
& & + R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4 s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} \\
& & + R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R^\times s_3^{-1} s_4 s_3 s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R^\times s_3^{-1} (s_4 s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R^\times s_3^{-1} s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R^\times s_4 (s_3 s_2 s_1^{-1} s_2 s_3^{-1}) s_4 (s_3^{-1} s_2 s_1^{-1} s_2 s_3) s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & R^\times s_4 s_2^{-1} s_1^{-1} (s_3 s_2^{-1} s_3) s_1 s_2 s_4 s_2 s_1 (s_3 s_2^{-1} s_3) s_1^{-1} s_2^{-1} s_4 s_3^{-1} + A_5^{(2)} \\
& \subset & A_3^\times s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_2 s_1 s_4 (s_3 s_2^{-1} s_3) s_4 s_1^{-1} s_2^{-1} s_3^{-1} + A_5^{(2)}
\end{array}
$$

and then $s_1 s_2 s_2 s_1 = s_1 s_2^2 s_1 \in R^\times\ s_1 s_2^{-1} s_1 + R s_1 s_2 s_1 + R s_1^2$. Since $s_4 (s_3 s_2^{-1} s_3) s_1 s_2 s_1 s_4 (s_3 s_2^{-1} s_3) s_4 = s_4 (s_3 s_2^{-1} s_3) s_2 s_1 s_2 s_4 (s_3 s_2^{-1} s_3) s_4 \subset s_4 (u_3 u_2 u_3 u_2) s_1 s_2 s_4 u_3 u_2 u_3 s_4 = s_4 u_2 u_3 u_2 u_3 s_1 s_2 s_4 u_3 u_2 u_3 s_4 = u_2 s_4 u_3 u_2 u_3 s_1 s_2 s_4 u_3 u_2 u_3 s_4 \subset A_5^{(2)}$ by proposition 6.3 and similarly $s_4 (s_3 s_2^{-1} s_3) s_1 s_1 s_4 (s_3 s_2^{-1} s_3) s_4 \in s_4 (s_3 s_2^{-1} s_3) u_1 s_4 (s_3 s_2^{-1} s_3) s_4 \subset A_5^{(2)}$, this proves

$$
s_4 w^+ s_4^{-1} w^+ s_4 \in A_3^\times s_4 (s_3 s_2^{-1} s_3)(s_1 s_2^{-1} s_1) s_4 (s_3 s_2^{-1} s_3) s_4 A_4^\times + A_5^{(2)}.
$$

We now prove (4).

$$
\begin{array}{rcl}
s_4 w^- s_4 w^+ s_4^{-1} & = & s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1} s_3 s_4^{-1} \\
& = & s_3^{-1} (s_3 s_4 s_3^{-1})(s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1} (s_3 s_4^{-1} s_3^{-1}) s_3 \\
& = & s_3^{-1} (s_4^{-1} s_3 s_4)(s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1} (s_4^{-1} s_3^{-1} s_4) s_3 \\
& = & s_3^{-1} s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 s_3^{-1} s_4 s_3 s_4^{-1} (s_2^{-1} s_1 s_2^{-1} s_3^{-1} s_4 s_3
\end{array}
$$

We now prove (4).

$$
\begin{array}{rcl}
s_4 w^- s_4 w^+ s_4^{-1} & = & s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} \\
& = & s_4 s_3^{-1} (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_3 s_4^{-1} \\
& = & s_3^{-1} (s_3 s_4 s_3^{-1})(s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1})(s_3 s_4^{-1} s_3^{-1}) s_3 \\
& = & s_3^{-1} s_4^{-1} s_3 s_4 (s_2 s_1^{-1} s_2) s_3^{-1} s_4 s_3 (s_2^{-1} s_1 s_2^{-1}) s_4^{-1} s_3^{-1} s_4 s_3 \\
& = & s_3^{-1} s_4^{-1} s_3 (s_2 s_1^{-1} s_2) s_4 s_3^{-1} s_4 s_3 s_4^{-1} (s_2^{-1} s_1 s_2^{-1}) s_3^{-1} s_4 s_3
\end{array}
$$

We have $s_4s_3^{-1}(s_4s_3s_4^{-1}) = s_4s_3^{-1}s_3^{-1}s_4s_3) = s_4s_3^{-2}s_4s_3 \in R^{\times}s_4s_3s_4s_3 + u_4s_3 + Rs_4s_3^{-1}s_4s_3 = R^{\times}(s_4s_3s_4)s_3 + u_4s_3 + Rs_4s_3^{-1}s_4s_3 = R^{\times}s_3s_4s_3^2 + u_4s_3 + Rs_4s_3^{-1}s_4s_3 \subset R^{\times}s_3s_4s_3^{-1} + Rs_3s_4s_3 + Rs_3s_4 + u_4s_3 + Rs_4s_3^{-1}s_4s_3$. We have

$$s_4^{-1}s_3(s_2s_1^{-1}s_2)(u_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 \subset A_5^{(2)}$$

by lemma 6.11 (2) ;

$$s_4^{-1}s_3(s_2s_1^{-1}s_2)(s_3s_4)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 \subset A_5^{(2)}$$

by lemma 6.11 (1) ;

$$s_4^{-1}s_3(s_2s_1^{-1}s_2)(s_3s_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 \subset A_5^{(2)}$$

by lemma 6.8 (4) ;

$$\begin{array}{rcl} s_4^{-1}s_3(s_2s_1^{-1}s_2)(s_4s_3^{-1}s_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 & = & s_4^{-1}s_3(s_2s_1^{-1}s_2)s_4(s_3^{-1}s_4s_3)(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 \\ & = & s_4^{-1}s_3(s_2s_1^{-1}s_2)s_4(s_4s_3s_4^{-1})(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 \\ & = & s_4^{-1}s_3(s_2s_1^{-1}s_2)s_4^2s_3s_4^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4 \\ & = & s_4^{-1}s_3s_4^2(s_2s_1^{-1}s_2)s_3(s_2^{-1}s_1s_2^{-1})(s_4^{-1}s_3^{-1}s_4) \\ & = & s_4^{-1}s_3s_4^2(s_2s_1^{-1}s_2)s_3(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}s_3^{-1} \\ & \in & A_5^{(2)} \end{array}$$

by lemma 6.11. It follows that $s_4w^-s_4w^+s_4^{-1} \in u_3^{\times}s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4u_3^{\times} + A_5^{(2)}$ hence $s_4w^-s_4w^+s_4^{-1} \in u_3^{\times}s_4^{-1}w^+s_4w^-s_4u_3^{\times} + A_5^{(2)}$

The proof of (5) is similar : one first gets

$$s_4w^-s_4^{-1}w^+s_4^- = s_3^{-1}s_4^{-1}s_3(s_2s_1^{-1}s_2)s_4s_3^{-1}s_4^{-1}s_3s_4^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4s_3$$

and then writes down $(s_4s_3^{-1}s_4^{-1})s_3s_4^{-1} = s_3^{-1}s_4^{-1}s_3^2s_4^{-1} \in R^{\times}s_3^{-1}(s_4^{-1}s_3^{-1}s_4^{-1}) + R(s_3^{-1}s_4^{-1}s_3)s_4^{-1} + Rs_3^{-1}s_4^{-2} = R^{\times}s_3^{-2}s_4^{-1}s_3^{-1} + Rs_4s_3^{-1}s_4^{-2} + Rs_3^{-1}s_4^{-2} \subset R^{\times}s_3s_4^{-1}s_3^{-1} + Rs_3^{-1}s_4^{-1}s_3^{-1} + Rs_4^{-1}s_3^{-1} + Rs_4s_3^{-1}u_4 + Rs_3^{-1}s_4^{-2}$; one then shows using the same arguments as before that all terms but $R^{\times}s_3s_4^{-1}s_3^{-1}$ provide an element of $A_5^{(2)}$, thus

$$\begin{array}{rcl} s_4w^-s_4^{-1}w^+s_4^- & \in & u_3^{\times}s_4^{-1}s_3(s_2s_1^{-1}s_2)s_3s_4^{-1}s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3^{-1}s_4u_3^{\times} + A_5^{(2)} \\ & \in & u_3^{\times}s_4^{-1}w^+s_4^{-1}w^-s_4u_3^{\times} + A_5^{(2)} \end{array}$$

We prove (6).

$$\begin{array}{rcll} s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3s_3s_4 & \subset & s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3(s_3s_4s_3)s_3^{-1} & \\ & \subset & s_4s_3A_3s_3^{-1}s_4s_3^{-1}A_3s_4s_3s_4s_3^{-1} & \\ & \subset & s_4s_3A_3(s_3^{-1}s_4s_3^{-1}s_4)A_3s_3s_4s_3^{-1} & \\ & \subset & s_4s_3A_3s_4^{-1}s_3s_4^{-1}s_3A_3s_3s_4s_3^{-1} + A_5^{(2)} & \text{(lemmas 3.6 and 6.11 (1))} \\ & \subset & (s_4s_3s_4^{-1})A_3s_3s_4^{-1}s_3A_3s_3s_4s_3^{-1} + A_5^{(2)} & \\ & \subset & s_3^{-1}s_4s_3A_3s_3s_4^{-1}s_3A_3s_3s_4s_3^{-1} + A_5^{(2)} & \\ & \subset & u_3s_4w^+s_4^{-1}w^+s_4u_3 + A_5^{(2)} & \text{(lemma 6.7)} \end{array}$$

□

Proposition 6.13.

$$\begin{array}{rcl} u_4u_3u_2u_1u_2u_3u_4u_3u_2u_1u_2u_3u_4 & \subset & A_4s_4w^-s_4w^-s_4A_4 + A_4s_4w^+s_4^{-1}w^+s_4A_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}A_4 \\ & & +A_4s_4w^-s_4w^+s_4^{-1}A_4 + A_4s_4w^-s_4^{-1}w^+s_4^{-1}A_4 + A_5^{(2)} \end{array}$$

Proof. We first note that, by lemma 6.10 and lemma 6.12 (4) and (5), the right-hand side (RHS) of the statement is invariant under Φ and Ψ. We now consider an expression of the form $s_4^{?}s_3^{\alpha}u_2u_1u_2s_3^{\beta}s_4^{?}s_3^{\gamma}u_2u_1u_2s_3^{\delta}s_4^{?}$ with $\alpha,\beta,\gamma \in \{-11\}$. By lemma 6.8 we can assume $\alpha = \beta$ and$\gamma = \delta$, except for the expression $s_4^{\varepsilon}s_3^{\alpha}u_2u_1u_2s_3^{-\alpha}s_4^{\varepsilon}s_3^{-\alpha}u_2u_1u_2s_3^{\alpha}s_4^{\varepsilon}$. Up to applying Φ, we can moreover assume $\varepsilon = 1$, and we get the conclusion by lemma 6.12 (6) for $\alpha = 1$, by lemma 6.12 (1) and (4) for $\alpha = -1$.

We can now assume $\alpha = \beta$, $\gamma = \delta$, and still $\varepsilon = 1$. By lemma 6.7 this reduces our examination to expressions $x = s_3w^{\alpha}s_4^{\varepsilon}w^{\beta}s_4^{\eta}$ for new parameters $\alpha,\varepsilon,\beta,\eta \in \{-1,1\}$. If $\alpha = \beta = \varepsilon$, we have

$x \in A_5^{(2)}$ by lemma 6.11 (6) ; if $\alpha = \beta = -\varepsilon$, we get $x \in A_5^{(2)}$ if in addition $\eta = -1$, by lemma 6.11 (7) and (8), and $x = s_4 w^{\alpha} s_4^{-\alpha} w^{\alpha} s_4 \in RHS$ otherwise. As a consequence, we can reduce to the case $\alpha = -\beta$, that is $x = s_4 w^{\alpha} s_4^{\varepsilon} w^{-\alpha} s_4^{\eta}$. If $\alpha = 1$, $x \in A_5^{(2)}$ by lemma 6.11 (4). If $\alpha = -1$, all the possibilities for x clearly lie in the RHS, except for $s_4 w^- s_4^{-1} w^+ s_4$, which belongs to $A_5^{(2)}$ by lemma 6.11 (3). This concludes the proof. □

6.3. **Image of the center of the braid group in $A_5^{(3)}/A_5^{(2)}$.** Recall that the center of the braid group B_n is infinite cyclic, generated for $n \geq 3$ by $c_n = (s_1 \dots s_{n-1})^n$, and that this generator can be written as $c_n = c_{n-1} y_n = y_n c_{n-1} = y_n y_{n-1} \dots y_3 y_2$ where the $y_n \in B_n \setminus B_{n-1}$ under the usual inclusions $B_2 \subset B_3 \subset \dots \subset B_{n-1}$ form another family of commuting elements defined by $y_2 = s_1^2$ and $y_{n+1} = s_n y_n s_n = s_n s_{n-1} \dots s_2 s_1^2 s_2 \dots s_{n-1} s_n$.

We let $c = c_5 = (s_1 s_2 s_3 s_4)^5 = (s_4 s_3 s_2 s_1)^5$. The center of G_{32} is cyclic of order 6 and is generated by the image of c. We let $w_0 = y_4 = s_3 s_2 s_1^2 s_2 s_3 = c_4 c_3^{-1}$, which by definition commutes with B_3, and $\delta = y_5 = s_4 s_3 s_2 s_1^2 s_2 s_3 s_4 = c_5 c_4^{-1}$ which commutes with B_4.

We first need a preparatory lemma.

Lemma 6.14.

1. *In A_4,* $s_4^{\alpha} w_0^2 s_4^{\beta} w_0 s_4^{\gamma} \in A_3^{\times} s_4^{\alpha} w_0^{-1} s_4^{\beta} w_0 s_4^{\gamma} + A_3 s_4^{\alpha} w_0 s_4^{\beta} w_0 s_4^{\gamma} + A_5^{(2)}$
2. *For all $\alpha, \beta, \gamma, \delta, \varepsilon \in \{-1, 1\} = \{-, +\}$,*

$$\begin{array}{rcl}
s_4^{\alpha} w^{\beta} s_4^{\gamma} w^{\delta} s_4^{\varepsilon} & \in & s_4^{\alpha} w^{\beta} s_4^{\gamma} w_0^{\delta} s_4^{\varepsilon} A_3^{\times} + s_4^{\alpha} w^{\beta} s_4^{\gamma} u_1 u_3 u_2 u_3 s_4^{\varepsilon} A_3 \\
& \subset & s_4^{\alpha} w^{\beta} s_4^{\gamma} w_0^{\delta} s_4^{\varepsilon} A_3^{\times} + A_5^{(2)} \\
s_4^{\alpha} w^{\beta} s_4^{\gamma} w^{\delta} s_4^{\varepsilon} & \in & A_3^{\times} s_4^{\alpha} w_0^{\beta} s_4^{\gamma} w^{\delta} s_4^{\varepsilon} + A_3 s_4^{\alpha} u_3 u_2 u_3 u_1 s_4^{\gamma} w^{\delta} s_4^{\varepsilon} \\
& \subset & A_3^{\times} s_4^{\alpha} w_0^{\beta} s_4^{\gamma} w^{\delta} s_4^{\varepsilon} + A_5^{(2)} \\
s_4^{\alpha} w^{\beta} s_4^{\gamma} w^{\delta} s_4^{\varepsilon} & \in & A_3^{\times} s_4^{\alpha} w_0^{\beta} s_4^{\gamma} w_0^{\delta} s_4^{\varepsilon} + A_3 s_4^{\alpha} u_3 u_2 u_3 u_1 s_4^{\gamma} w^{\delta} s_4^{\varepsilon} + A_3 s_4^{\alpha} w_0^{\alpha} s_4^{\gamma} u_3 u_2 u_3 u_1 s_4^{\varepsilon} \\
& \subset & A_3^{\times} s_4^{\alpha} w_0^{\beta} s_4^{\gamma} w_0^{\delta} s_4^{\varepsilon} + A_5^{(2)}
\end{array}$$

Proof. (1) is a straightforward consequence of lemma 4.9 and of the fact that $s_4^{\alpha} U_0 s_4^{\beta} w_0 s_4^{\gamma} \subset s_4^{\alpha} A_3 u_3 A_3 s_4^{\beta} w_0 s_4^{\gamma} + s_4^{\alpha} A_3 u_3 u_2 u_3 s_4^{\beta} w_0 s_4^{\gamma} = A_3 s_4^{\alpha} u_3 s_4^{\beta} w_0 s_4^{\gamma} A_3 + A_3 s_4^{\alpha} u_3 u_2 u_3 s_4^{\beta} w_0 s_4^{\gamma} A_3 \subset A_5^{(2)}$ by proposition 6.3. (2) follows from an easy variation in the proof of lemma 6.7 and from lemma 4.6. □

We are then in position to prove the following.

Lemma 6.15.

1. $s_4 w^- s_4 w^+ s_4^{-1} \in A_4^{\times} s_4 w^+ s_4^{-1} w^+ s_4 + A_5^{(2)}$
2. $s_4 w^- s_4^{-1} w^+ s_4^{-1} \in s_4^{-1} w^- s_4 w^- s_4^{-1} A_4^{\times} + A_5^{(2)}$

Proof. We have $s_4 w^- s_4 w^+ s_4^{-1} \in A_3^{\times} s_4 w_0^{-1} s_4 w_0 s_4^{-1} + A_5^{(2)}$ by lemma 6.14 (2). Since $s_4 w_0^{-1} s_4 w_0 s_4 \in A_3^{\times} s_4 w^- s_4 w^+ s_4 \subset A_5^{(2)}$ by lemma 6.11 (5) and since $s_4^{-1} \in R^{\times} s_4^2 + R s_4 + R$, we have $s_4 w_0^{-1} s_4 w_0 s_4^{-1} \equiv s_4 w_0^{-1} s_4 w_0 s_4^2 \mod A_5^{(2)}$. Then $s_4 w_0^{-1} s_4 w_0 s_4^2 \in A_3^{\times} s_4 w_0^2 s_4 w_0 s_4^2 + A_3 s_4 w_0 s_4 w_0 s_4^2 + A_5^{(2)}$ by lemma 6.14 (1), and $s_4 w_0 s_4 w_0 s_4^2 \in R s_4 w_0 s_4 w_0 s_4 + R s_4 w_0 s_4 w_0 s_4^{-1} + A_5^{(2)}$. Then

$$\begin{array}{rrcccl}
& s_4 w_0 s_4 w_0 s_4^{-1} & \in & A_3^{\times} s_4 w^+ s_4 w^+ s_4^{-1} & \subset & A_5^{(2)} \\
\text{and} & s_4 w_0 s_4 w_0 s_4 & \in & A_3^{\times} s_4 w^+ s_4 w^+ s_4 & \subset & A_5^{(2)}
\end{array}$$

by lemmas 6.14 (2) and 6.11 (6). It follows that $s_4 w_0^{-1} s_4 w_0 s_4^2 \in A_3^{\times} s_4 w_0^2 s_4 w_0 s_4^2 + A_5^{(2)}$.

Now $s_4 w_0^2 s_4 w_0 s_4^2 = s_4 w_0 (w_0 (s_4 w_0 s_4)) s_4$ and $w_0 (s_4 w_0 s_4) = c_2^{-1} c_4 \in A_3^{\times} c_4$ commutes with w_0 and s_4. Thus $s_4 w_0^2 s_4 w_0 s_4^2 = (w_0 (s_4 w_0 s_4)) s_4 w_0 s_4 \in A_4^{\times} s_4 w_0 s_4^2 w_0 s_4$. Now $s_4 w_0 s_4^2 w_0 s_4 \in R^{\times} s_4 w_0 s_4^{-1} w_0 s_4 + R s_4 w_0 s_4 w_0 s_4 + A_5^{(2)}$; moreover we already noticed $s_4 w_0 s_4 w_0 s_4 \in A_5^{(2)}$, hence $s_4 w_0 s_4^2 w_0 s_4 \in R^{\times} s_4 w_0 s_4^{-1} w_0 s_4 + A_5^{(2)} \subset A_3^{\times} s_4 w^+ s_4^{-1} w^+ s_4 + A_5^{(2)}$ by lemma 6.14 (2) , and this proves (1).

Now we have $s_4 w^- s_4^{-1} w^+ s_4^{-1} = \Psi(s_4 w^- s_4 w^+ s_4^{-1}) \in \Psi(A_4^{\times} s_4 w^+ s_4^{-1} w^+ s_4) + \Psi(A_5^{(2)}) = s_4^{-1} w^- s_4 w^- s_4^{-1} A_4^{\times} + A_5^{(2)}$, and this proves (2). □

By a direct computation, we will prove the following lemma, which will turn out to be crucial in the proof of the main theorem. We postpone this (lengthy) calculation to section 7.

Lemma 6.16. *In A_5, δ^3 belongs to*

$$A_4^{\times} s_4 w^- s_4 w^- s_4 A_3^{\times} + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$$

6.4. **Right actions are left actions.**

Lemma 6.17.

(1) *For all $\alpha, \beta, \gamma, \dots \in \{-1, 1\}$, $x, y \in A_3$,*

$$s_1(s_4^{\alpha} s_3^{\beta} x s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} y s_3^{\zeta} s_4^{\eta}) \equiv (s_4^{\alpha} s_3^{\beta} x s_3^{\gamma} s_4^{\delta} s_3^{\varepsilon} y s_3^{\zeta} s_4^{\eta}) s_1 \mod A_5^{(2)}$$

(2) *For all $x \in A_4$, $(s_4 w^+ s_4^{-1} w^+ s_4) x \in A_4 (s_4 w^+ s_4^{-1} w^+ s_4) \mod A_5^{(2)}$*
(3) *For all $x \in A_4$, $(s_4^{-1} w^- s_4 w^- s_4^{-1}) x \in A_4 (s_4^{-1} w^- s_4 w^- s_4^{-1}) \mod A_5^{(2)}$*
(4) $(s_4 w^- s_4 w^+ s_4^{-1}) s_3^{-1} \in s_3^{-1} (s_4 w^- s_4 w^+ s_4^{-1}) + A_5^{(2)}$
(5) $(s_4 w^- s_4^{-1} w^+ s_4^{-1}) s_3^{-1} \in u_3 s_4^{-1} w^+ s_4^{-1} w^- s_4 + A_5^{(2)}$.

Proof. We first prove (1). By lemma 6.7 and because s_1 commutes with u_1 we can assume $x = y = s_2^{-1} s_1 s_2^{-1}$. Since $(s_2^{-1} s_1 s_2^{-1}) s_1 \in s_1 (s_2^{-1} s_1 s_2^{-1}) + u_1 u_2 u_1$ by lemma 2.3 (and even $(s_2^{-1} s_1 s_2^{-1}) s_1 \in s_1 (s_2^{-1} s_1 s_2^{-1}) + u_1 u_2 + u_2 u_1$, see lemma 3.6), by proposition 6.3 we get the conclusion.

We then prove (2). Because of (1), and because we have the result for $x = s_3^{-1}$ by lemma 6.12 (2), we need only consider $x = s_2$. For $x = s_2$, we first use that

$$s_4 w^+ s_4^{-1} w^+ s_4 \in u_1^{\times} s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} \ s_2 s_3 s_4 u_1^{\times} + A_5^{(2)}$$

by lemma 6.7 ; then, because of (1) we get that

$$u_1^{\times} s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} \ s_2 s_3 s_4 u_1^{\times} \subset u_1^{\times} s_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3 s_2 s_1^{-1} \ s_2 s_3 s_4 + A_5^{(2)}.$$

Then

$$\begin{array}{rcl} s_4 w^+ s_4^{-1} w^+ s_4 . s_2 & \in & u_1^{\times} s_4 (s_3 s_2 s_1^{-1} s_2 s_3) s_4^{-1} (s_3 s_2 s_1^{-1} \ s_2 s_3) s_4 s_2 + A_5^{(2)} \\ & \subset & u_1^{\times} s_2 s_4 (s_3 s_2 s_1^{-1} s_2 s_3) s_4^{-1} (s_3 s_2 s_1^{-1} \ s_2 s_3) s_4 + A_5^{(2)} \end{array}$$

because s_2 commutes with both s_4 and $s_3 s_2 s_1^{-1} s_2 s_3$ and this proves (2). One gets (3) by applying Φ to (2).

We prove (4). One easily gets $(s_4 w^- s_4 w^+ s_4^{-1}) s_3^{-1} = s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3 s_4^{-2} s_2 s_1^{-1} s_2 s_3^{-1} s_4$. Now $s_4^{-2} \in R^{\times} s_4 + R s_4^{-1} + R$, and it is easily checked that the terms originating from $R s_4^{-1}$ and R belong to $A_5^{(2)}$. We thus get $s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3 s_4^{-2} s_2 s_1^{-1} s_2 s_3^{-1} s_4 \in s_4 s_3^{-1} s_2 s_1^{-1} s_2 (s_4 s_3 s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)} \subset s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 + A_5^{(2)}$, and

$$s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_3 s_4 s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_4 \in u_3 s_4^{-} w^+ s_4 w^- s_4 + A_5^{(2)}$$

by lemma 6.12 (1). We prove (5). One easily gets

$$(s_4 w^- s_4^{-1} w^+ s_4^{-1}) s_3^{-1} = s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4^{-2} s_2 s_1^{-1} s_2 s_3^{-1} s_4,$$

and $s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3^{-1} x s_2 s_1^{-1} s_2 s_3^{-1} s_4 \in A_5^{(2)}$ for $x \in 1, s_4^{-1}$, hence $(s_4 w^- s_4^{-1} w^+ s_4^{-1}) s_3^{-1}$ belongs to

$$\begin{array}{rlcl} & s_4 s_3^{-1} s_2 s_1^{-1} s_2 s_4 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} & = & s_4 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} \\ \subset & u_3 s_4^{-1} s_3 s_4^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} & = & u_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 (s_4^{-1} s_3^{-1} s_4) s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} \\ = & u_3 s_4^{-1} s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1} s_4 A_5^{(2)} & \subset & u_3 s_4^{-1} w^+ s_4^{-1} w^- s_4 + A_5^{(2)} \end{array}$$

□

Remark 6.18. *Another proof of item (2). It is easily checked that $s_4 w^+ s_4^{-1} w^+ s_4 \equiv (s_4 s_3 s_2 s_1^2 s_2 s_3 s_4)^2$ mod $A_5^{(2)}$, and the element $s_4 s_3 s_2 s_1^2 s_2 s_3 s_4$ of the braid group B_5 is well-known to centralize B_4.*

Proposition 6.19.

$$\begin{array}{rcl} A_5^{(3)} & = & A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} \\ A_5^{(3)} & = & s_4 w^- s_4 w^- s_4 A_4 + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)} \end{array}$$

Proof. Clearly the RHS are included in $A_5^{(3)}$. By propositions 6.6 and 6.13 we have $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_4 s_4 w^- s_4 w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4^{-1} w^+ s_4^{-1} A_4 + A_5^{(2)}$. Lemma 6.15 then implies $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$. By lemma 6.17 this implies $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)}$ and $A_5^{(3)} \subset A_4 s_4 w^- s_4 w^- s_4 A_4 + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$. Now, by lemma 6.16, $s_4 w^- s_4 w^- s_4$ belongs to $A_4^{\times} \delta^3 A_3^{\times} + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$, hence $A_4 s_4 w^- s_4 w^- s_4 A_4 \subset A_4 \delta^3 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)} = A_4 c^3 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)} = A_4 c^3 A_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 + A_5^{(2)}$ and, since c^3 is central and by lemma 6.17, this latter expression can be written as $A_4 c^3 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} = A_4 \delta^3 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} = A_4 s_4 w^- s_4 w^- s_4 A_3^{\times} + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)} = A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} + A_5^{(2)}$. The other expression is deduced from this one by application of $\Phi \circ \Psi$.

□

Proposition 6.20. $A_5 = A_5^{(3)}$.

Proof. One only needs to prove $A_5^{(4)} \subset A_5^{(3)}$, that is $u_4 a u_4 b u_4 c u_4 \subset A_5^{(3)}$ for all $a, b, c \in A_4$. We have $u_4 a u_4 b u_4 c \in A_5^{(3)} = A_5^{(2)} A_4 s_4 w^- s_4 w^- s_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1}$ hence

$$u_4 a u_4 b u_4 c u_4 \subset A_5^{(2)} u_4 A_4 s_4 w^- s_4 w^- s_4 u_4 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 u_4 + A_4 s_4^{-1} w^- s_4 w^- s_4^{-1} u_4 \subset A_5^{(3)}.$$

This proves the claim. □

This proves theorem 6.1, and actually the following refinement :

Theorem 6.21.

$$\begin{array}{rcl} A_5 & = & A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4 A_4 \\ & & + A_4 s_4^{-1} w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4 A_4 + A_4 s_4^{-1} w^- s_4^{-1} A_4 + A_4 s_4 w^+ s_4 A_4 + s_4 w^- s_4 w^- s_4 A_4 \\ & & + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4 \end{array}$$

6.5. A_5 **as a** A_4**-module.** We need the following lemma on A_3 :

Lemma 6.22.

(1) $u_2 u_1 u_2 \subset u_1 s_2 s_1^2 s_2 + u_1 u_2 u_1$
(2) $u_2 u_1 u_2 \subset u_1 s_2^{-1} s_1^{-2} s_2^{-1} + u_1 u_2 u_1$

Proof. (2) is a consequence of (1) by using Φ, so it is enough to prove (1). We have $u_2 u_1 u_2 \subset u_1 u_2 u_1 + \sum_{\alpha \in \{-1,1\}} R s_2^{\alpha} s_1^{-\alpha} s_2^{\alpha}$ because u_i is R-spanned by $1, s_i, s_i^{-1}$ and because of lemma 2.2. Moreover $s_2^{-1} s_1 s_2^{-1} \in u_1 s_2 s_1^{-1} s_2 + u_1 u_2 u_1$ by lemmas 2.4 and 2.3, hence $u_2 u_1 u_2 \subset u_1 s_2 s_1^{-1} s_2 + u_1 u_2 u_1$. Since $s_1^{-1} \in R s_1^2 + R s_1 + R$ we get $s_2 s_1^{-1} s_2 \in R s_2 s_1^2 s_2 + R s_2 s_1 s_2 + R s_2^2 \subset R s_2 s_1^2 s_2 + u_1 u_2 u_1$ hence $u_2 u_1 u_2 \subset u_1 s_2 s_1^2 s_2 + u_1 u_2 u_1$. □

We introduce or re-introduce the following submodules of A_5 :

$$\begin{array}{rcl} A_5^{(1)} & = & A_4 u_4 A_4 = A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 \\ A_5^{(1\frac{1}{4})} & = & A_5^{(1)} + A_4 s_4 s_3^{-1} s_4 A_4 (= A_5^{(1)} + A_4 sh^2(A_3) A_4) \\ A_5^{(1\frac{1}{2})} & = & A_5^{(1\frac{1}{4})} + A_4 u_4 u_3 u_2 u_3 u_4 A_4 \\ & = & A_5^{(1\frac{1}{4})} + A_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} A_4 \\ A_5^{(2)} & = & A_4 u_4 A_4 u_4 A_4 = A_5^{(1\frac{1}{2})} + \sum_{\alpha,\beta \in \{1,-1\}} A_4 s_4^{\alpha} w^{\beta} s_4^{\alpha} A_4 \end{array}$$

and

$A_5 = A_5^{(3)} = A_4u_4A_4u_4A_4u_4A_4 = A_5^{(2)} + A_4s_4w^-s_4w^-s_4 + A_4s_4w^+s_4^{-1}w^+s_4 + A_4s_4^{-1}w^-s_4w^-s_4^{-1}$

We let $\mathcal{B}$ denote the family of elements defined in corollary 5.12, which span A_4 as a left B-module, $\mathcal{A}$ the family spanning A_4 as a A_3-module defined in proposition 4.8, and $\mathcal{A}'$ its image under the automorphism $\mathrm{Ad}\,\Delta$ of A_4 (that is $s_1 \leftrightarrow s_3$, $s_2 \leftrightarrow s_2$). We prove the following.

Lemma 6.23.

(1) $A_5^{(1)} = A_4 + \sum_{x\in\mathcal{A}} A_4s_4x + \sum_{x\in\mathcal{A}} A_4s_4^{-1}x$
(2) $A_5^{(1\frac{1}{4})} = A_5^{(1)} + \sum_{x\in\mathcal{B}} A_4s_4s_3^{-1}s_4x$

Proof. (1) is a consequence of $A_3s_4^{\pm 1} = s_4^{\pm 1}A_3$, because

$$A_5^{(1)} = A_4+A_4s_4A_4+A_4s_4^{-1}A_4 = A_4+A_4s_4\sum_{x\in\mathcal{A}} A_3x+A_4s_4^{-1}\sum_{x\in\mathcal{A}} A_3x = A_4+\sum_{x\in\mathcal{A}} A_4s_4x+\sum_{x\in\mathcal{A}} A_4s_4^{-1}x$$

We prove (2). We have $(s_4s_3^{-1}s_4)s_1 = s_1(s_4s_3^{-1}s_4)$ and $(s_4s_3^{-1}s_4)s_3^{-1} \in s_3^{-1}(s_4s_3^{-1}s_4)+u_3u_4+u_4u_3$ by lemma 3.6, hence $(s_4s_3^{-1}s_4)B \subset B(s_4s_3^{-1}s_4) + A_5^{(1)}$, where we recall $B = \langle s_1, s_3^{-1}\rangle = \langle s_1, s_3\rangle$. Thus

$$A_5^{(1\frac{1}{4})} = A_5^{(1)} + A_4s_4s_3^{-1}s_4\sum_{x\in\mathcal{B}} Bx = A_5^{(1)} + \sum_{x\in\mathcal{B}} A_4s_4s_3^{-1}s_4Bx = A_5^{(1)} + \sum_{x\in\mathcal{B}} A_4s_4s_3^{-1}s_4x$$

□

Lemma 6.24.

(1) $A_5^{(1\frac{1}{2})} \subset A_5^{(1\frac{1}{4})} + A_4s_4s_3s_2^2s_3s_4A_4 + A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}A_4$
(2) $A_5^{(1\frac{1}{2})} \subset A_5^{(1\frac{1}{4})} + \sum_{x\in\mathcal{A}'} A_4s_4s_3s_2^2s_3s_4x + \sum_{x\in\mathcal{A}'} A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}x$

Proof. We have $s_3^{-1}s_2s_3^{-1} \subset u_2s_3s_2^2s_3+u_2u_3u_2$ by lemma 6.22 hence $s_4s_3^{-1}s_2s_3^{-1}s_4 \subset s_4u_2s_3s_2^2s_3s_4+s_4u_2u_3u_2s_4 \subset u_2s_4s_3s_2^2s_3s_4 + u_2s_4u_3s_4u_2 \subset u_2s_4s_3s_2^2s_3s_4 + A_5^{(1\frac{1}{4})}$. Applying Φ this implies $s_4^{-1}s_3s_2^{-1}s_3s_4^{-1} \subset u_2s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1} + A_5^{(1\frac{1}{4})}$ which proves (1). Let $A_3' = \langle s_2, s_3\rangle$. We have $A_4 = \sum_{x\in\mathcal{A}'} x$. Since $s_4s_3s_2^2s_3s_4$ commutes with s_2 and s_3 hence to A_3', we get

$$A_4s_4s_3s_2^2s_3s_4A_4 \subset \sum_{x\in\mathcal{A}'} A_4s_4s_3s_2^2s_3s_4A_3'x \subset \sum_{x\in\mathcal{A}'} A_4s_4s_3s_2^2s_3s_4x$$

and similarly $s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1} = (s_4s_3s_2^2s_3s_4)^{-1}$ commutes with s_2 and s_3 hence

$$A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}A_4 \subset \sum_{x\in\mathcal{A}'} A_4s_4^{-1}s_3^{-1}s_2^{-2}s_3^{-1}s_4^{-1}x$$

which proves (2). □

Lemma 6.25.

$$A_5^{(2)} = A_5^{(1\frac{1}{2})} + \sum_{\alpha\in\{-1,1\}} A_4s_4^\alpha w^\alpha s_4^\alpha + \sum_{\alpha\in\{-1,1\}}\sum_{x\in\mathcal{A}} A_4s_4^\alpha w^{-\alpha}s_4^\alpha x$$

Proof. By lemma 4.6 (1), we have $w_0 \in A_3^\times w^+ + U_0$, $w_0^{-1} \in A_3^\times w^- + U_0$, hence $w^+ \in A_3^\times w_0 + U_0$, $w^- \in A_3^\times w_0^{-1} + U_0$, with $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3 \subset A_4$. As a consequence, for $\alpha,\beta \in \{-1,1\}$, we have $A_4s_4^\alpha w^\beta s_4^\alpha A_4 \subset A_4s_4^\alpha A_3^\times w_0^\beta s_4^\alpha A_4 + A_4s_4^\alpha U_0s_4^\alpha A_4$ Moreover, $s_4^\alpha U_0s_4^\alpha = s_4^\alpha A_3u_3A_3s_4^\alpha + s_4^\alpha A_3u_3u_2u_3A_3s_4^\alpha = A_3s_4^\alpha u_3s_4^\alpha A_3 + A_3s_4^\alpha u_3u_2u_3s_4^\alpha A_3 \subset A_5^{(1\frac{1}{4})} + A_5^{(1\frac{1}{2})} = A_5^{(1\frac{1}{2})}$, hence $A_4s_4^\alpha w^\beta s_4^\alpha A_4 \subset A_4s_4^\alpha w_0^\beta s_4^\alpha A_4 + A_5^{(1\frac{1}{2})}$. Since w_0 and s_4 commute with s_1 and s_2, we have

$$A_4s_4^\alpha w_0^\beta s_4^\alpha A_4 \subset \sum_{x\in\mathcal{A}} A_4s_4^\alpha w_0^\beta s_4^\alpha A_3x \subset \sum_{x\in\mathcal{A}} A_4s_4^\alpha w_0^\beta s_4^\alpha x.$$

If moreover $\alpha = \beta$, $s_4^\alpha w_0^\alpha s_4^\alpha = (s_4s_3s_2s_1^2s_2s_3s_4)^\alpha$ commutes with $\langle s_1, s_2, s_3\rangle = A_4$, hence $A_4s_4^\alpha w_0^\alpha s_4^\alpha A_4 = A_4s_4^\alpha w_0^\alpha s_4^\alpha$, and this concludes the proof. □

From this one can conlude the following.

Theorem 6.26.

(1) $A_5 = A_5^{(3)}$ *is generated as a* A_4*-module by* 240 *elements.*
(2) $A_5 = A_5^{(3)}$ *is generated as a* R*-module by* $155,520$ *elements.*

Proof. By lemma 6.22, $A_5^{(1)}$ is generated as an A_4-module by $1+2\times 27 = 55$ elements, $A_5^{(1\frac{1}{4})}$ by $A_5^{(1)}$ and $|\mathcal{B}| = 72$ elements, $A_5^{(1\frac{1}{2})}$ after lemma 6.24 by $A_5^{(1\frac{1}{4})}$ and $2\times|\mathcal{A}'| = 2\times 27 = 54$ elements, $A_5^{(2)}$ by $A_5^{(1\frac{1}{2})}$ and $2+2\times|\mathcal{A}| = 56$ elements (lemma 6.25), and $A_5^{(3)}$ by $A_5^{(2)}$ and 3 elements. It follows that A_5 is A_4-generated by $55+72+54+56+3 = 240$ elements, which proves (1). Since A_4 is R-generated by 648 elements, we get that A_5 is R-generated by $240\times 648 = 155,520$ elements, which proves (2).

□

7. Proof of lemma 6.16

For the sake of concision we denote $V_0 = A_5^{(2)}$ and $V^+ = A_5^{(2)} + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 = V_0 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4$. We will prove that $X \in A_4^\times s_4 w^- s_4 w^- s_4 A_4^\times + V^+$, starting from $X = \delta^3 = s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4$ to $X = \delta^3 = s_4 w^- s_4 w^- s_4$ (for which the statement is trivial) through a sequence of reductions of the type $X \to X'$ where $X' \in A_4^\times X A_3^\times + V^+$

7.1. **Reduction to $s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4$.** Using $s_4^2 \in R^\times s_4^{-1} + R s_4 + R$ we get $s_4 w_0 s_4^2 w_0 s_4^2 w_0 s_4 \in R^\times s_4 w_0 s_4^2 w_0 s_4^{-1} w_0 s_4 + R s_4 w_0 s_4^2 w_0 s_4 w_0 s_4 + R s_4 w_0 s_4^2 w_0^2 s_4$. The fact that $s_4 w_0 s_4^2 w_0 s_4 w_0 s_4, s_4 w_0 s_4^2 w_0^2 s_4$ belongs to V^+ is proved in the following lemma 7.1

Lemma 7.1.

(1) $s_4 w_0 s_4^2 w_0^2 s_4 \in V^+$.
(2) $s_4 w_0 s_4^2 w_0 s_4 w_0 s_4 \in V^+$
(3) $s_4 w_0^2 s_4^{-1} w_0 s_4 \in V^+$
(4) $s_4 w_0 s_4 w_0 s_4^{-1} w_0 s_4 \in V_0$

Proof. We prove (1). By lemma 4.9 we have $s_4 w_0 s_4^2 w_0^2 s_4 \in A_3^\times s_4 w_0 s_4^2 w_0^{-1} s_4 + A_3 s_4 w_0 s_4^2 w_0 s_4 + A_3 s_4 w_0 s_4^3$. Clearly $s_4 w_0 s_4^3 \in V_0$, hence, expanding s_4^2,

$$s_4 w_0 s_4^2 w_0^2 s_4 \in A_3 s_4 w_0 s_4^{-1} w_0^{-1} s_4 + A_3 s_4 w_0 s_4 w_0^{-1} s_4 + A_3 s_4 w_0 s_4^{-1} w_0 s_4 + A_3 s_4 w_0 s_4 w_0 s_4 + V_0.$$

We already know $s_4 w_0 s_4 w_0 s_4 \in V_0$ by lemma 6.11 (6). Moreover $s_4 w_0 s_4^{-1} w_0^{-1} s_4 \in A_3 s_4 w^+ s_4^{-1} w^- s_4 + V_0 \subset V_0$ and $s_4 w_0 s_4 w_0^{-1} s_4 \in A_3 s_4 w^+ s_4 w^- s_4 + V_0 \subset V_0$ by lemma 6.12 (4), and finally

$$s_4 w_0 s_4^{-1} w_0^{-1} s_4 \in A_3 s_4 w^+ s_4^{-1} w^+ s_4 + V_0 \subset V_+.$$

We now prove (2). We have

$$s_4 w_0 s_4^2 (w_0 s_4 w_0 s_4) = (w_0 s_4 w_0 s_4) s_4 w_0 s_4^2 \subset A_4 s_4 w_0 s_4^2 w_0 s_4^2,$$

as $w_0 s_4 w_0 s_4 = c_5 c_3^{-1}$ commutes with s_4 and w_0. The term $s_4 w_0 s_4^2 w_0 s_4^2$ is a linear combinations of terms of the form $s_4 w_0 s_4^\alpha w_0 s_4^\beta$ for $\alpha, \beta \in \{0, -1, 1\}$, and we have (lemma 6.11 (6) and (7)) $s_4 w_0 s_4^\alpha w_0 s_4^\beta \in A_3 s_4 w^+ s_4^\alpha w^+ s_4^\beta + V_0 \subset V_0$, unless $(\alpha,\beta) = (-1,1)$, in which case $s_4 w_0 s_4^{-1} w_0 s_4 \in A_3 s_4 w^+ s_4^{-1} w^+ s_4 + V_0 \subset V^+$. For proving (3) we use that $s_4 w_0^2 s_4^{-1} w_0 s_3 \in V_0 + A_3 \sum_{\alpha\in\{\pm\}} s_4 w^\alpha s_4^- w^+ s_4$, and that $s_4 w^- s_4^{-1} w^+ s_4 \in V_0$ by lemma 6.11 (5). We prove (4). We have $(s_4 w_0 s_4 w_0) s_4^{-1} w_0 s_4 = s_4^{-1} w_0 s_4 (s_4 w_0 s_4 w_0) \in s_4^{-1} w_0 s_4^2 w_0 s_4 A_4$ and $s_4^{-1} w_0 s_4^2 w_0 s_4 \in R s_4^{-1} w_0^2 s_4 + R s_4^{-1} w_0 s_4^{-1} w_0 s_4 + R s_4^{-1} w_0 s_4 w_0 s_4$. Now, $s_4^{-1} w_0^2 s_4 \in V_0$, $s_4^{-1}(w_0 s_4 w_0 s_4) = (w_0 s_4 w_0 s_4) s_4^{-1} \in V_0$, and $s_4^{-1} w_0 s_4^{-1} w_0 s_4 \in A_3 s_4^{-1} w^+ s_4^{-1} w^+ s_4 + V_0$. Since $\Phi(s_4^{-1} w^+ s_4^{-1} w^+ s_4) \in V_0$ by lemma 6.11 (8) we get (4). □

7.2. **Reduction to $s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3^2(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4$.** Using $s_4^2 \in R^\times s_4^{-1} + Rs_4 + R$ we get $s_4w_0s_4^2w_0s_4^{-1}w_0s_4 \in R^\times s_4w_0s_4^{-1}w_0s_4^{-1}w_0s_4 + Rs_4w_0^2s_4^{-1}w_0s_4 + Rs_4w_0s_4w_0s_4^{-1}w_0s_4$. The fact that $Rs_4w_0^2s_4^{-1}w_0s_4 + Rs_4w_0s_4w_0s_4^{-1}w_0s_4 \subset V^+$ has been proved in lemma 7.1. Finally, $s_4w_0s_4^{-1}w_0s_4^{-1}w_0s_4 = s_3^{-1}.s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3^2(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4$ is easily checked to hold in the braid group B_5.

Before going further, we first need to establish several lemmas.

Lemma 7.2.
(1) *For all* $\alpha, \beta \in \mathbf{Z}$, $s_4w_0s_4^\alpha s_3^\beta w_0s_4 \in V^+$.
(2) *For all* $\alpha, \beta \in \mathbf{Z}$, $s_4w_0s_3^\beta s_4^\alpha w_0s_4 \in V^+$.

Proof. Since $s_4w_0s_4^\alpha s_3^\beta w_0s_4 = s_4w_0s_4^\alpha s_3^{\beta+1}s_2s_1^2s_2s_3s_4$, we need to consider $s_4w_0s_4^\alpha s_3^\beta s_2s_1^2s_2s_3s_4$, where we can assume $\alpha, \beta \in \{1, -1\}$, the cases $\alpha = 0$ and $\beta = 0$ being obvious by proposition 6.3. If $\beta = 1$ we have $s_4w_0s_4^\alpha w_0s_4 \in A_3s_4w^+s_4^\alpha w^+s_4 + V_0 \subset V^+$, so we can assume $\beta = -1$. Expanding s_1^2 we get a linear combinations of $s_4w_0s_4^\alpha s_3^{-1}s_2s_1^{-1}s_2s_3s_4$, $s_4w_0s_4^\alpha s_3^{-1}s_2s_1s_2s_3s_4$ and $s_4w_0s_4^\alpha s_3^{-1}s_2^2s_3s_4$. We have $s_4w_0s_4^\alpha s_3^{-1}s_2^2s_3s_4 \in V_0$ by 6.3 and $s_4w_0s_4^\alpha s_3^{-1}(s_2s_1s_2)s_3s_4 = s_4w_0s_4^\alpha s_3^{-1}s_1s_2s_1s_3s_4 = s_4w_0s_4^\alpha s_1s_3^{-1}s_2s_3s_4s_1 \in V_0$ by proposition 6.3. There remains to consider $s_4w_0s_4^\alpha s_3^{-1}(s_2s_1^{-1}s_2)s_3s_4 = s_4w_0s_4^\alpha s_2s_1(s_3s_2^{-1}s_3)s_1^{-1}s_2^{-1}s_4 = s_2s_1s_4w_0s_4^\alpha(s_3s_2^{-1}s_3)s_4s_1^{-1}s_2^{-1} \in V_0$ by proposition 6.3. This concludes the proof of (1). Then (2) is an immediate consequence of (1) by application of $\Phi \circ \Psi$.

□

Lemma 7.3.
(1) $s_4w_0s_4^{-1}s_3s_4^{-1}w_0s_4 \in V^+$
(2) $s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4w_0s_4^{-1}w_0s_4 \in V^+$

Proof. By using braid relations one gets $s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4w_0s_4^{-1}w_0s_4 = s_3(s_2s_1^2s_2)s_4w_0s_4^{-1}s_3s_4^{-1}w_0s_4$, hence (2) reduces to (1). We now prove (1). Expanding s_4^{-1} as a linear combination of s_4^2, s_4 and 1, we get that $s_4w_0(s_4^{-1})s_3s_4^{-1}w_0s_4$ is a linear combination of $s_4w_0s_3s_4^{-1}w_0s_4$, $s_4w_0s_4s_3s_4^{-1}w_0s_4$ and $s_4w_0s_4^2s_3s_4^{-1}w_0s_4$. We have $s_4w_0s_3s_4^{-1}w_0s_4 \in V_0$ by lemma 7.2 (2), $(s_4w_0s_4)s_3s_4^{-1}w_0s_4 = s_3(s_4w_0s_4)s_4^{-1}w_0s_4 = s_3s_4w_0^2s_4 \in V_0$ because $s_4w_0s_4 = c_5c_4^{-1}$ commutes with B_4 and in particular s_3, and similarly $s_4w_0s_4^2s_3s_4^{-1}w_0s_4 = s_4w_0s_4(s_4s_3s_4^{-1})w_0s_4 = (s_4w_0s_4)s_3^{-1}s_4s_3w_0s_4 = s_3^{-1}(s_4w_0s_4)s_4s_3w_0s_4 \in A_4s_4w_0s_4^2s_3w_0s_4 \in V^+$ by lemma 7.2 (2). □

Lemma 7.4.
(1) *For all* β, $s_4A_4s_4s_3^\beta w_0s_4 \subset V^+$
(2) $s_4s_3s_2^{-1}s_3u_1u_2s_4u_3w_0s_4 \subset V^+$
(3) $s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \in V^+$
(4) $u_4u_3u_2u_3u_4A_4s_4 \subset V^+$ *; moreover* $s_4^\alpha u_3u_2u_3s_4^\beta A_4s_4 \subset V_0$ *when* $\alpha, \beta \in \{-1, 1\}$ *with* $(\alpha, \beta) \neq (1, 1)$
(5) $u_4U_0u_4A_4s_4 \subset V^+$
(6) $u_4s_3^\pm A_3s_3^\mp u_4A_4s_4 \subset V^+$

Proof. From theorem 4.1 one easily deduces $A_4 = A_3u_3A_3 + A_3s_3s_2^{-1}s_3A_3 + A_3w_0 + A_3w_0^{-1}$. Then $s_4(A_3u_3A_3)s_4s_3^\beta w_0s_4 = A_3s_4u_3s_4A_3s_3^\beta w_0s_4 \subset V_0$ by lemma 6.11 (2) ; $s_4(A_3w_0)s_4s_3^\beta w_0s_4 = A_3s_4w_0s_4s_3^\beta w_0s_4 \subset V^+$ by lemma 7.2 (2) ; $s_4(A_3w_0^{-1})s_4s_3^\beta w_0s_4 = A_3s_4w_0^{-1}s_4s_3^\beta w_0s_4$ and $s_4w_0^{-1}s_4s_3^\beta w_0s_4$ is a linear combination of $s_4w_0^{-1}s_4s_3^{\beta'}s_2s_1^2s_2s_3s_4$ for $\beta' \in \{0, 1, -1\}$. For $\beta' = 1$, $s_4w_0^{-1}s_4s_3s_2s_1^2s_2s_3s_4 = s_4w_0^{-1}s_4w_0s_4 \in A_3s_4w^-s_4w^+s_4 + V_0 \subset V_0$ by lemma 6.11 (5). For $\beta' = -1$, $s_4w_0^{-1}s_4s_3^{-1}s_2s_1^2s_2s_3s_4 \in V_0$ by lemma 6.8 (4), and the case $\beta = 0$ also lies in V_0 by proposition 6.3. It then remains to prove $s_4A_3s_3s_2^{-1}s_3A_3s_4s_3^\beta w_0s_4 \subset V^+$, that is $s_4s_3s_2^{-1}s_3A_3s_4s_3^\beta w_0s_4 \subset V^+$. We use $A_3 \subset s_2^{-1}s_1s_2^{-1}u_1 + u_1u_2u_1$ to get $s_4s_3s_2^{-1}s_3A_3s_4s_3^\beta w_0s_4 \subset s_4s_3s_2^{-1}s_3s_2^{-1}s_1s_2^{-1}u_1s_4s_3^\beta w_0s_4 + s_4s_3s_2^{-1}s_3u_1u_2u_1s_4s_3^\beta w_0s_4$. Now, for $s_4s_3s_2^{-1}s_3u_1u_2u_1s_4s_3^\beta w_0s_4 = s_4s_3s_2^{-1}s_3u_1u_2s_4s_3^\beta w_0s_4u_1$ we are reduced to proving (2), while the expression $s_4(s_3s_2^{-1}s_3s_2^{-1})s_1s_2^{-1}u_1s_4s_3^\beta w_0s_4 = s_4(s_3s_2^{-1}s_3s_2^{-1})s_1s_2^{-1}s_4s_3^\beta w_0s_4u_1$ is

$$s_4s_2^{-1}s_3s_2^{-1}s_3s_1s_2^{-1}s_4s_3^\beta w_0s_4u_1 + s_4u_2u_3s_1s_2^{-1}u_1s_4s_3^\beta w_0s_4 + s_4u_3u_2s_1s_2^{-1}s_4s_3^\beta w_0s_4u_1$$

by lemma 3.6. We have $s_4u_2u_3s_1s_2^{-1}s_4s_3^{\beta}w_0s_4 + s_4u_3u_2s_1s_2^{-1}s_4s_3^{\beta}w_0s_4 \subset V_0$ by proposition 6.3, while $s_4s_2^{-1}s_3s_2^{-1}s_3s_1s_2^{-1}s_4s_3^{\beta}w_0s_4 = s_2^{-1}s_4s_3s_2^{-1}s_3s_1s_2^{-1}s_4s_3^{\beta}w_0s_4$ $s_4s_3s_2^{-1}s_3s_1s_2^{-1}s_4s_3^{\beta}w_0s_4$ is a linear combination of $s_4s_3s_2^{-1}s_3s_1s_2^{-1}s_4s_3^{\beta'}s_2s_1^2s_2s_3s_4$ for $\beta' \in \{0,1,-1\}$. Now

$$\begin{array}{rlcl} & s_4s_3s_2^{-1}s_3s_1s_2^{-1}s_4s_3^{\beta'}s_2s_1^2s_2s_3s_4 & = & s_4s_3s_2^{-1}s_3s_1s_4(s_2^{-1}s_3^{\beta'}s_2)s_1^2s_2s_3s_4 \\ = & s_4s_3s_2^{-1}s_3s_1s_4s_3s_2^{\beta'}s_3^{-1}s_1^2s_2s_3s_4 & = & s_4s_3s_2^{-1}s_3s_1s_4s_3s_2^{\beta'}s_1^2(s_3^{-1}s_2s_3)s_4 \\ = & s_4s_3s_2^{-1}s_3s_1s_4s_3s_2^{\beta'}s_1^2s_2s_3s_2^{-1}s_4 & = & s_4s_3s_2^{-1}s_3s_1s_4s_3s_2^{\beta'}s_1^2s_2s_3s_4s_2^{-1} \in V_0 \end{array}$$

by proposition 6.3. This proves (1). For proving (2), we can reduce to an expression of the form $s_4s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_4s_3^{\gamma}s_2s_1^2s_2s_3s_4$ (with $\alpha,\beta,\gamma \in \{0,1,-1\}$). Using only braid relations, one gets

$$\begin{array}{rlcl} & s_4s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_4s_3^{\gamma}s_2s_1^2s_2s_3s_4 & = & s_4s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_4s_3^{\gamma}s_2s_1^2s_2(s_3s_4s_3^{-1})s_3 \\ = & s_4s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}s_4s_3^{\gamma}s_2s_1^2s_2s_4^{-1}s_3s_4s_3 & = & s_4s_3s_2^{-1}s_3s_1^{\alpha}s_2^{\beta}(s_4s_3^{\gamma}s_4^{-1})s_2s_1^2s_2s_3s_4s_3 \\ = & s_4s_3s_2^{-1}s_1^{\alpha}(s_3s_2^{\beta}s_3^{-1})s_4^{\gamma}s_3s_2s_1^2s_2s_3s_4s_3 & = & s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3^{\beta}s_2s_4^{\gamma}s_3s_2s_1^2s_2s_3s_4s_3 \\ = & s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3^{\beta}s_4^{\gamma}(s_2s_3s_2)s_1^2s_2s_3s_4s_3 & = & s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3^{\beta}s_4^{\gamma}s_3s_2s_3s_1^2s_2s_3s_4s_3 \\ = & s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3^{\beta}s_4^{\gamma}s_3s_2s_1^2(s_3s_2s_3)s_4s_3 & = & s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3^{\beta}s_4^{\gamma}s_3s_2s_1^2s_2s_3s_2s_4s_3 \\ = & s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3^{\beta}s_4^{\gamma}s_3s_2s_1^2s_2s_3s_4.s_2s_3 & & \end{array}$$

which belongs to V_0 by lemma 6.8 (3) as soon as $\beta = -1$. If $\beta = 0$, it is equal to

$$s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_4^{\gamma}s_3s_2s_1^2s_2s_3s_4.s_2s_3 = s_4s_3s_4^{\gamma}s_2^{-1}s_1^{\alpha}s_2^{-1}s_3s_2s_1^2s_2s_3s_4.s_2s_3 \in V_0$$

by lemma 6.11 (2). Otherwise, considering all possibilities for α and applying lemmas 6.7 and 6.8 it lies inside $V_0 + s_4s_3s_2^{-1}s_1s_2^{-1}s_3s_4^{\gamma}s_3s_2s_1^2s_2s_3s_4A_4 \subset s_4w^+s_4^{\gamma}w^+s_4A_4 + V_0$. In cases $\gamma \in \{-1,0\}$ this clearly belongs to V^+, while $s_4w^+s_4w^+s_4 \in V_0$ by lemma 6.11 (6).

We now prove (3). We have

$$\begin{array}{rlcl} & s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 & = & s_4s_3(s_2s_1^2s_2)s_3^{-1}(s_2s_1^2s_2)(s_4s_3s_4^{-1})w_0s_4 \\ = & s_4s_3(s_2s_1^2s_2)s_3^{-1}(s_2s_1^2s_2)s_3^{-1}s_4s_3w_0s_4 & \in & s_4A_4s_4s_3w_0s_4 \subset V^+ \end{array}$$

by (1).

We prove (4), considering an expression of the form $s_4^{\alpha}u_3u_2u_3s_4^{\beta}A_4s_4$ for $\alpha,\beta \in \{-1,1\}$, the case $\alpha = 0$ or $\beta = 0$ being obvious. We use the decomposition $A_4 = A_3u_3A_3 + A_3u_3u_2u_3A_3 + u_3u_2u_1u_2u_3A_3$. We have

$$\begin{array}{rlcl} & s_4^{\alpha}u_3u_2u_3s_4^{\beta}A_3u_3A_3s_4 & = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}A_3u_3s_4A_3 \\ = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_2u_1u_2u_1u_3s_4A_3 & = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_2u_1u_2u_3s_4u_1A_3 \subset V_0 \end{array}$$

by proposition 6.3, and $s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_3u_2u_1u_2u_3A_3s_4 = s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_3u_2u_1u_2u_3s_4A_3 \subset V_0$ by proposition 6.3. There remains to consider

$$\begin{array}{rlcl} & s_4^{\alpha}u_3u_2u_3s_4^{\beta}A_3u_3u_2u_3A_3s_4 & = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}A_3u_3u_2u_3s_4A_3 \\ = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_1u_2u_1(u_2u_3u_2u_3)s_4A_3 & = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_1u_2u_1u_3u_2u_3u_2s_4A_3 \\ = & s_4^{\alpha}u_3u_2u_3s_4^{\beta}u_1u_2u_1u_3u_2u_3s_4u_2A_3 & \subset & V_0 \end{array}$$

by proposition 6.3, unless $\alpha = \beta = 1$. In that case the proof of lemma 6.4, lemma 6.12 (1) and lemma 6.15 (2) together yield $s_4u_3u_2u_3s_4u_1u_2u_1u_3u_2u_3s_4 \in V^+$.

Now (5) is a consequence of (4) and proposition 6.3, as $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3$ and

$$\begin{array}{rcl} u_4U_0u_4A_4s_4 & \subset & u_4A_3u_3A_3u_4A_4s_4 + u_4A_3u_3u_2u_3A_3u_4A_4s_4 \\ & = & A_3u_4u_3u_4A_4s_4 + u_4A_3u_3u_2u_3u_4A_4s_4 \end{array}$$

and both terms belong to V^+, by lemma 6.11 (2) and by (4). Then (6) is an immediate consequence of (5) and lemma 4.6 (3).

□

7.3. **Reduction to $s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4$.** Expanding s_3^2, we get

$$\begin{array}{rcl} s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3^2(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 & \in & R^\times s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \\ & & +Rs_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \\ & & +Rs_4s_3(s_2s_1^2s_2)s_3^{-1}s_4(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4. \end{array}$$

We have

$$\begin{array}{ll} s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \in V^+ & \text{by lemma 7.4 (3)} \\ s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \in V^+ & \text{by lemma 7.3 (2).} \end{array}$$

Lemma 7.5.

(1) *For all* $\alpha \in \mathbf{Z}$ $s_4s_3^{-1}s_2^\alpha s_3^{-1}s_4^{-1}s_3w_0s_4 \in V^+$
(2) $s_4s_3^{-1}s_2^2s_3^{-1}s_4^2s_3w_0s_4 \in V^+$
(3) $s_4s_3^{-1}s_2s_4s_3^{-1}s_2^2s_3s_4^{-1}w_0s_4 \in V^+$
(4) $s_4s_3^{-1}s_2s_4s_3^{-1}s_2s_1s_2s_3s_4^{-1}w_0s_4 \in V^+$

Proof. We prove (1). We get

$$\begin{array}{rlcl} & s_4s_3^{-1}s_2^\alpha(s_3^{-1}s_4^{-1}s_3)w_0s_4 & = & s_4s_3^{-1}s_2^\alpha s_4(s_3^{-1}s_4^{-1}s_3)s_2s_1^2s_2s_3s_4 \\ = & s_4s_3^{-1}s_2^\alpha s_4s_4s_3^{-1}s_4^{-1}s_2s_1^2s_2s_3s_4 & = & s_4s_3^{-1}s_2^\alpha s_4^2s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4) \\ = & s_4s_3^{-1}s_4^2s_2^\alpha s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} & \in & V_0 \end{array}$$

by lemma 6.11 (2). Part (2) is obtained by expanding s_4^2 and using (1) and lemma 7.4 (1). For (3), we use

$$\begin{array}{rlcl} & s_4s_3^{-1}s_2s_4(s_3^{-1}s_2^2s_3)s_4^{-1}w_0s_4 & = & s_4s_3^{-1}s_2s_4s_2s_3^2s_2^{-1}s_4^{-1}w_0s_4 \\ = & s_4s_3^{-1}s_2^2(s_4s_3^2s_4^{-1})w_0s_4s_2^{-1} & = & s_4s_3^{-1}s_2^2s_3^{-1}s_4^2s_3w_0s_4s_2^{-1} \in V^+ \end{array}$$

because of (2). For (4) we use $s_4s_3^{-1}s_2s_4s_3^{-1}(s_2s_1s_2)s_3s_4^{-1}w_0s_4 = s_4s_3^{-1}s_2s_4s_3^{-1}s_1s_2s_1s_3s_4^{-1}w_0s_4 = s_4s_3^{-1}s_2s_4s_1(s_3^{-1}s_2s_3)s_4^{-1}w_0s_4s_1 = s_4s_3^{-1}s_2s_4s_1s_2s_3s_2^{-1}s_4^{-1}w_0s_4s_1 = s_4s_3^{-1}s_2s_1s_2s_4s_3s_4^{-1}w_0s_4s_2^{-1}s_1 = s_4s_3^{-1}s_2s_1s_2s_3^{-1}s_4s_3w_0s_4s_2^{-1}s_1 \in V^+$ by lemma 7.4 (1).

□

Lemma 7.6.

(1) $s_4s_3^{-1}A_3s_4u_3s_4^{-1}w_0s_4 \subset V^+$
(2) $s_4s_3^{-1}s_2s_4s_3^{-1}u_1u_2u_1s_3s_4^{-1}w_0s_4 \subset V^+$
(3) $s_4s_3^{-1}s_2s_4s_3^{-1}(s_2s_1^{-1}s_2)s_3s_4^{-1}w_0s_4 \in V^+$

Proof. We consider $s_4s_3^{-1}A_3s_4s_3^\alpha s_4^{-1}w_0s_4$ for $\alpha \in \{-1,0,1\}$. When $\alpha = 0$ this expression clearly belongs to V_0, when $\alpha = 1$ we get $s_4s_3^{-1}A_3(s_4s_3s_4^{-1})w_0s_4 = s_4s_3^{-1}A_3s_3^{-1}s_4s_3w_0s_4 \subset V^+$ by lemma 7.4 (1). When $\alpha = -1$, we get $s_4s_3^{-1}A_3s_4(s_3^{-1}s_4^{-1}s_3)s_2s_1^2s_2s_3s_4 = s_4s_3^{-1}A_3s_4s_4s_3^{-1}s_4^{-1}s_2s_1^2s_2s_3s_4 = s_4s_3^{-1}A_3s_4^2s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4) = s_4s_3^{-1}s_4^2A_3s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} \subset V_0$ by lemma 6.11 (1). This proves (1). We consider now $s_4s_3^{-1}s_2s_4s_3^{-1}u_1u_2u_1s_3s_4^{-1}w_0s_4 = s_4s_3^{-1}s_2u_1s_4(s_3^{-1}u_2s_3)s_4^{-1}w_0s_4u_1 = s_4s_3^{-1}s_2u_1s_4s_2u_3s_2^{-1}s_4^{-1}w_0s_4u_1 = s_4s_3^{-1}s_2u_1s_2(s_4u_3s_4^{-1})w_0s_4s_2^{-1}u_1 = s_4s_3^{-1}s_2u_1s_2s_3^{-1}u_4s_3w_0s_4s_2^{-1}u_1$ which is a linear combination of the $Y = s_4s_3^{-1}s_2u_1s_2s_3^{-1}s_4^\alpha s_3w_0s_4s_2^{-1}u_1$ for $\alpha \in \{-1,0,1\}$. When $\alpha = 0$ clearly $Y \subset V_0$, when $\alpha = 1$ we have $Y \subset V^+$ by lemma 7.4 (1), and when $\alpha = -1$ we get $s_4s_3^{-1}s_2u_1s_2(s_3^{-1}s_4^{-1}s_3)w_0s_4s_2^{-1}u_1 = s_4s_3^{-1}s_2u_1s_2s_4s_3^{-1}s_4^{-1}w_0s_4s_2^{-1}u_1 \subset V^+$ by (1). This proves (2). We consider now $s_4s_3^{-1}s_2s_4s_3^{-1}(s_2s_1^{-1}s_2)s_3s_4^{-1}w_0s_4 = s_4s_3^{-1}s_2s_4s_2s_1(s_3s_2^{-1}s_3)s_1^{-1}s_2^{-1}s_4^{-1}w_0s_4 = s_4s_3^{-1}s_2^2s_1s_4(s_3s_2^{-1}s_3)s_4^{-1}w_0s_4s_1^{-1}s_2^{-1}$. By lemma 2.4 it belongs to

$$s_4s_3^{-1}s_2^2s_1s_4s_3^{-1}s_2s_3^{-1}u_2s_4^{-1}w_0s_4s_1^{-1}s_2^{-1} + s_4s_3^{-1}s_2^2s_1s_4u_2u_3u_2s_4^{-1}w_0s_4s_1^{-1}s_2^{-1}.$$

We have

$$s_4s_3^{-1}s_2^2s_1s_4u_2u_3u_2s_4^{-1}w_0s_4s_1^{-1}s_2^{-1} = s_4s_3^{-1}s_2^2s_1u_2s_4u_3s_4^{-1}w_0s_4u_2s_1^{-1}s_2^{-1} \subset V^+$$

by (1), and

$$\begin{array}{rcl} s_4s_3^{-1}s_2^2s_1s_4s_3^{-1}s_2s_3^{-1}u_2s_4^{-1}w_0s_4s_1^{-1}s_2^{-1} & = & s_4s_3^{-1}s_2^2s_1s_4s_3^{-1}s_2(s_3^{-1}s_4^{-1}s_3)s_2s_1^2s_2s_3s_4u_2s_1^{-1}s_2^{-1} \\ & = & s_4s_3^{-1}s_2^2s_1s_4s_3^{-1}s_2s_4s_3^{-1}s_4^{-1}s_2s_1^2s_2s_3s_4u_2s_1^{-1}s_2^{-1} \\ & = & s_4s_3^{-1}s_2^2s_1s_4s_3^{-1}s_2s_4s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4)u_2s_1^{-1}s_2^{-1} \\ & = & (s_4s_3^{-1}s_4)s_2^2s_1s_3^{-1}s_2s_4s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1}u_2s_1^{-1}s_2^{-1} \end{array}$$

belongs to

$u_3s_4^{-1}s_3s_4^{-1}s_2^2s_1s_3^{-1}s_2s_4s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1}u_2s_1^{-1}s_2^{-1}+u_3u_4u_3s_2^2s_1s_3^{-1}s_2s_4s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1}u_2s_1^{-1}s_2^{-1}$.

We have

$$\begin{array}{rcl} & s_4^{-1}s_3s_4^{-1}s_2^2s_1s_3^{-1}s_2s_4s_3^{-1}s_2s_1^2s_2s_3s_4 & = & s_4^{-1}s_3s_2^2s_1(s_4^{-1}s_3^{-1}s_4)s_2s_3^{-1}s_2s_1^2s_2s_3s_4 \\ = & s_4^{-1}s_3s_2^2s_1s_3s_4^{-1}s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4 & = & s_4^{-1}s_3s_2^2s_3s_4^{-1}s_1s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4 \in u_4u_3u_2u_3u_4A_4s_4 \end{array}$$

and also $u_4u_3s_2^2s_1s_3^{-1}s_2s_4s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1}u_2s_1^{-1}s_2^{-1} \subset u_4u_3u_2u_3u_4A_4s_4A_4$. The conclusion follows from 7.4 (4).

∎

7.4. **Reduction to** $s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4$**.** Expanding s_1^2, we get

$$\begin{array}{rcl} s_4s_3(s_2s_1^2s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 & \in & R^{\times}s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \\ & & +Rs_4s_3(s_2s_1s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \\ & & +Rs_4s_3s_2^2s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \end{array}$$

Since

$$\begin{array}{rcl} s_4s_3(s_2s_1s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 & = & s_4s_3s_1s_2s_1s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \\ & = & s_1s_4s_3s_2s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4s_1 \end{array}$$

(as s_1 commutes with $s_2s_1^2s_2 = c_3c_2^{-1}$), the latter two terms belong to

$$\begin{array}{rcl} A_2s_4(s_3u_2s_3^{-1})s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4A_2 & = & A_2s_4s_2^{-1}u_3s_2s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4A_2 \\ & \subset & A_3s_4u_3s_4s_2s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4A_2 \end{array}.$$

We thus only need to prove that the $s_4s_3^{\alpha}s_4s_2s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4$ belong to V^+ for $\alpha \in \{-1,0,1\}$. When $\alpha = 0$ this is a consequence of lemma 7.4 (6) ; when $\alpha = 1$ we get

$$\begin{array}{rcl} (s_4s_3s_4)s_2s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 & = & s_3s_4(s_3s_2s_3^{-1})(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 \\ & = & s_3s_4s_2^{-1}s_3(s_2^2s_1^2s_2)s_3s_4^{-1}w_0s_4. \end{array}$$

Since $s_2^2s_1^2s_2 \in u_1s_2^{-1}s_1s_2^{-1} + u_1u_2u_1$ the conclusion follows from proposition 6.3. When $\alpha = -1$, expanding s_1^2 we only need to consider the $s_4s_3^{-1}s_4s_2s_3^{-1}s_2s_1^{\beta}s_2s_3s_4^{-1}w_0s_4$ for $\beta \in \{-1,0,1\}$. The case $\beta = -1$ is a consequence of lemma 7.6 (3), while the other two cases follow from lemma 7.5 (3) and (4).

Lemma 7.7.

(1) $s_4u_3s_2u_1s_2s_3^{-1}u_4s_3w_0s_4 \subset V^+$
(2) $s_4s_3s_2s_1^{-1}u_3u_2u_4s_3w_0s_4 \subset V_0$
(3) $s_4s_3s_2s_1^{-1}u_2u_3u_4s_3w_0s_4 \subset V^+$

Proof. For proving (1), we consider the expression $s_4s_3^{\alpha}s_2s_1^{\beta}s_2s_3^{\gamma}u_4s_3w_0s_4$ for $\alpha,\beta,\gamma \in \{-1,0,1\}$. If one of these is 0, it lies in V_0 by proposition 6.3. If $\beta = 1$, using $s_2s_1s_2 = s_1s_2s_1$ we get the same conclusion, so we can assume $\beta = -1$. By expanding if necessary s_3^2, we are then reduced to considering expressions of the form $s_4s_3^{\alpha}s_2s_1^{\beta}s_2s_3^{\gamma}u_4s_3^{\delta}s_2s_1^2s_2s_3s_4$ for $\delta \in \{0,1,-1\}$, the case $\delta = 0$ being again trivial. We then get the conclusion from lemmas 6.8 and 6.12 (6).

We now prove (2). We have

$$\begin{array}{rcl} & s_4s_3s_2s_1^{-1}u_3u_2u_4s_3w_0s_4 & = & s_4(s_3s_2u_3)s_1^{-1}u_2u_4s_3w_0s_4 \\ \subset & s_4u_2s_3s_2s_1^{-1}u_2u_4s_3w_0s_4 & = & u_2s_4s_3s_2s_1^{-1}u_2u_4s_3w_0s_4 \quad \subset \quad V_0 \end{array}$$

by proposition 6.3. Finally, (3) is similar to (1) : considering $s_4s_3s_2s_1^{-1}s_2^{\alpha}s_3^{\beta}s_4^{\gamma}s_3^{\delta}s_2s_1^2s_2s_3s_4$, if one of the exponents is zero we get trivially the conclusion by proposition 6.3 ; if $\alpha = -1$ it lies inside V_0 by $s_2s_1^{-1}s_2^{-1} = s_1^{-1}s_2^{-1}s_1$ and proposition 6.3, so we can assume $\alpha = 1$. By studying separately the cases $\beta = -1$ and $\beta = 1$ one easily gets the conclusion from lemmas 6.8, 6.11 and 6.12.

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7.5. **Reduction to** $s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}w_0s_4$**.** Using $s_2s_1^2s_2 \in s_2^{-1}s_1s_2^{-1}u_1^{\times} + u_1u_2u_1$ we get

$$\begin{array}{rcl} s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2s_1^2s_2)s_3s_4^{-1}w_0s_4 & \in & R^{\times}s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}s_2^{-1}s_1s_2^{-1}s_3s_4^{-1}w_0s_4u_1^{\times} \\ & & +Rs_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}u_1u_2u_1s_3s_4^{-1}w_0s_4. \end{array}$$

We have

$$\begin{array}{rcl} & s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}u_1u_2u_1s_3s_4^{-1}w_0s_4 & = & s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_4(s_3^{-1}u_2s_3)s_4^{-1}w_0s_4u_1 \\ = & s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_4s_2u_3s_2^{-1}s_4^{-1}w_0s_4u_1 & = & s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_2(s_4u_3s_4^{-1})w_0s_4s_2^{-1}u_1 \\ = & s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4s_2^{-1}u_1. & & \end{array}$$

Using $(s_2s_1^{-1}s_2)u_1 \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we get that $s_4s_3(s_2s_1^{-1}s_2)u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4$ belongs to

$$u_1s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4 + s_4s_3u_1u_2u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4.$$

Now

$$\begin{array}{rcl} & s_4s_3u_1u_2u_1s_3^{-1}s_2s_3^{-1}u_4s_3w_0s_4 & = & u_1s_4(s_3u_2s_3^{-1})u_1s_2s_3^{-1}u_4s_3w_0s_4 \\ = & u_1s_4s_2^{-1}u_3s_2u_1s_2s_3^{-1}u_4s_3w_0s_4 & = & u_1s_2^{-1}s_4u_3s_2u_1s_2s_3^{-1}u_4s_3w_0s_4 \subset V^+ \end{array}$$

by lemma 7.7 (1), and $s_4s_3s_2s_1^{-1}(s_2s_3^{-1}s_2s_3^{-1})u_4s_3w_0s_4$ belong to

$$s_4s_3s_2s_1^{-1}s_3^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 + s_4s_3s_2s_1^{-1}u_2u_3u_4s_3w_0s_4 + s_4s_3s_2s_1^{-1}u_3u_2u_4s_3w_0s_4$$

by lemma 3.6. The latter two terms belong to $V+$ by lemma 7.7 (1) and (2), and

$$\begin{array}{rcl} & s_4s_3s_2s_1^{-1}s_3^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 & = & s_4(s_3s_2s_3^{-1})s_1^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 \\ = & s_4s_2^{-1}s_3s_2s_1^{-1}s_2s_3^{-1}s_2u_4s_3w_0s_4 & = & s_2^{-1}s_4s_3s_2s_1^{-1}s_2s_3^{-1}u_4s_2s_3w_0s_4 \subset V^+ \end{array}$$

by lemma 7.4 (6).

Lemma 7.8. $s_4(s_3s_2^{-1}\ s_3)u_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \subset V^+$

Proof. Using braid relations one gets

$$\begin{array}{rcl} & s_4(s_3s_2^{-1}\ s_3)u_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 & = & s_4s_3s_2^{-1}\ u_1(s_3s_4s_3^{-1})s_2s_3^{-1}s_4^{-1}w_0s_4 \\ = & s_4s_3s_2^{-1}\ u_1s_4^{-1}s_3s_4s_2s_3^{-1}s_4^{-1}w_0s_4 & = & (s_4s_3s_4^{-1})s_2^{-1}\ u_1s_3s_2(s_4s_3^{-1}s_4^{-1})w_0s_4 \\ = & s_3^{-1}s_4s_3s_2^{-1}\ u_1(s_3s_2s_3^{-1})s_4^{-1}s_3w_0s_4 & = & s_3^{-1}s_4s_3s_2^{-1}\ u_1s_2^{-1}s_3s_2s_4^{-1}s_3w_0s_4 \end{array}$$

which is a linear combination of $s_3^{-1}s_4s_3s_2^{-1}s_1^{\alpha}s_2^{-1}s_3s_2s_4^{-1}s_3w_0s_4$ for $\alpha \in \{-1,0,1\}$. For $\alpha = -1$ we get

$$\begin{array}{rcl} s_3^{-1}s_4s_3(s_2^{-1}s_1^{-1}s_2^{-1})s_3s_2s_4^{-1}s_3w_0s_4 & = & s_3^{-1}s_4s_3s_1^{-1}s_2^{-1}s_1^{-1}s_3s_2s_4^{-1}s_3w_0s_4 \\ & = & s_3^{-1}s_1^{-1}s_4s_3s_2^{-1}s_3s_4^{-1}s_1^{-1}s_2s_3w_0s_4 \in V^+ \end{array}$$

by lemma 7.4 (4) ; for $\alpha = 0$ we get $s_3^{-1}s_4s_3s_2^{-2}s_3s_2s_4^{-1}s_3w_0s_4 \in V_0$ by proposition 6.3 ; for $\alpha = 1$ it remains to consider $s_3^{-1}s_4s_3(s_2^{-1}s_1s_2^{-1})s_3s_2s_4^{-1}s_3w_0s_4$. Using $s_2^{-1}s_1s_2^{-1} \in u_1s_2s_1^{-1}s_2 + u_1u_2u_1$ we get $s_4s_3(s_2^{-1}s_1s_2^{-1})s_3s_2s_4^{-1}s_3w_0s_4 \in u_1s_4s_3s_2s_1^{-1}s_2s_3s_2s_4^{-1}s_3w_0s_4 + u_1s_4s_3u_2u_1s_3s_2s_4^{-1}s_3w_0s_4$. Now $s_4s_3u_2u_1s_3s_2s_4^{-1}s_3w_0s_4 = s_4s_3u_2s_3s_4^{-1}u_1s_2s_3w_0s_4 \subset V^+$ by lemma 7.4 (4) while

$$\begin{array}{rcl} & s_4s_3s_2s_1^{-1}(s_2s_3s_2)s_4^{-1}s_3w_0s_4 & = & s_4s_3s_2s_1^{-1}s_3s_2s_3s_4^{-1}s_3w_0s_4 \\ = & s_4(s_3s_2s_3)s_1^{-1}s_2s_3s_4^{-1}s_3w_0s_4 & = & s_4s_2s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3w_0s_4 \\ = & s_2s_4s_3s_2s_1^{-1}s_2s_3s_4^{-1}s_3w_0s_4 & & \end{array}$$

lies in V^+ by lemma 6.8. This concludes the proof. □

Lemma 7.9. $s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_2u_3s_4^{-1}w_0s_4 \subset V^+$

Proof. We have

$$\begin{array}{rcl} s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_2u_3s_4^{-1}w_0s_4 & = & s_4(s_3s_2^{-1}s_3)s_2s_1u_2(s_4u_3s_4^{-1})w_0s_4 \\ & \subset & s_4s_3(s_2^{-1}s_3s_2)s_1u_2s_3^{-1}u_4s_3w_0s_4 \\ & = & s_4s_3^2s_2s_3^{-1}s_1u_2s_3^{-1}u_4s_3w_0s_4, \end{array}$$

whose elements are linear combinations of the $s_4s_3^2s_2s_3^{-1}s_1s_2^{\alpha}s_3^{-1}u_4s_3w_0s_4$ for $\alpha \in \{0,1,-1\}$. When $\alpha = 0$, such an element belongs to V_0 by proposition 6.3 ; when $\alpha = -1$, we have $s_4s_3^2s_2s_3^{-1}s_1s_2^{-1}s_3^{-1}u_4s_3w_0s_4 = s_4s_3^2s_2s_1(s_3^{-1}s_2^{-1}s_3^{-1})u_4s_3w_0s_4 = s_4s_3^2(s_2s_1s_2^{-1})s_3^{-1}s_2^{-1}u_4s_3w_0s_4 =$

$s_4 s_3^2 s_1^{-1} s_2 s_1 s_3^{-1} s_2^{-1} u_4 s_3 w_0 s_4 = s_1^{-1} s_4 s_3^2 s_2 s_3^{-1} u_4 s_1 s_2^{-1} s_3 w_0 s_4 \in V^+$ by lemma 7.4 (4) ; when $\alpha = 1$, expanding s_3^2 we get a linear combination of $s_4 s_3^{\beta} s_2 s_3^{-1} s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4$ for $\beta \in \{0, 1, -1\}$. When $\beta = 0$ such an element lies in V_0 by commutation relations and proposition 6.3 ; when $\beta = 1$ we get $s_4(s_3 s_2 s_3^{-1}) s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = s_4 s_2^{-1} s_3 s_2 s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = s_2^{-1} s_4 s_3 (s_2 s_1 s_2) s_3^{-1} u_4 s_3 w_0 s_4 = s_2^{-1} s_4 s_3 s_1 s_2 s_1 s_3^{-1} u_4 s_3 w_0 s_4 = s_2^{-1} s_1 s_4 s_3 s_2 s_1 s_3^{-1} u_4 s_3 w_0 s_4 \in V_0$ by proposition 6.3 ; when $\beta = -1$ we get that $s_4 s_3^{-1} s_2 s_3^{-1} s_1 s_2 s_3^{-1} u_4 s_3 w_0 s_4 = s_4 s_3^{-1} s_2 s_1 (s_3^{-1} s_2 s_3^{-1}) u_4 s_3 w_0 s_4$ belongs to

$$s_4 s_3^{-1} s_2 s_1 u_2 s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 + s_4 s_3^{-1} s_2 s_1 u_2 u_3 u_2 u_4 s_3 w_0 s_4$$

by lemma 2.4. Now

$$\begin{array}{rcl} s_4 s_3^{-1}(s_2 s_1 u_2) u_3 u_2 u_4 s_3 w_0 s_4 & \subset & s_4 s_3^{-1} u_1 s_2 s_1 u_3 u_2 u_4 s_3 w_0 s_4 \\ & \subset & u_1 s_4 s_3^{-1} s_2 u_3 u_4 s_1 u_2 s_3 w_0 s_4 \subset V^+ \end{array}$$

by lemma 7.4 (4) and

$$\begin{array}{rcl} s_4 s_3^{-1}(s_2 s_1 u_2) s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 & = & s_4 s_3^{-1} u_1 s_2 s_1 s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 \\ = \ u_1 s_4 s_3^{-1} s_2 s_1 s_3 s_2^{-1} s_3 u_4 s_3 w_0 s_4 & = & u_1 s_4 (s_3^{-1} s_2 s_3) s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4 \\ = \ u_1 s_4 s_2 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4 & = & u_1 s_2 s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4. \end{array}$$

Since $s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4$ is spanned by the $s_4 w^+ s_4^{\alpha} s_3^{\beta} s_2 s_1^2 s_2 s_3 s_4$ one readily gets

$$s_4 s_3 s_2^{-1} s_1 s_2^{-1} s_3 u_4 s_3 w_0 s_4 \subset V^+$$

and the conclusion. □

Lemma 7.10.

(1) $s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_3 u_2 s_4^{-1} w_0 s_4 \subset V_0$
(2) $u_4 u_3 u_2 u_1 s_3^{-1} s_2 s_3^{-1} s_4^{-1} w_0 s_4 \subset V^+$

Proof. We prove (1).

$$\begin{array}{rcl} s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_3 u_2 s_4^{-1} w_0 s_4 & = & s_4(s_3 s_2^{-1} s_3) s_2 s_1 (s_4 u_3 s_4^{-1}) w_0 s_4 u_2 \\ \subset \ s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_3^{-1} u_4 s_3 w_0 s_4 u_2 & = & s_4 s_3 s_2^{-1}(s_3 s_2 s_3^{-1}) s_1 u_4 s_3 w_0 s_4 u_2 \\ = \ s_4 s_3 s_2^{-2} s_3 s_2 s_1 u_4 s_3 w_0 s_4 u_2 & \subset & s_4 u_3 u_2 u_3 u_2 s_1 u_4 s_3 w_0 s_4 u_2 \\ = \ s_4 u_2 u_3 u_2 u_3 s_1 u_4 s_3 w_0 s_4 u_2 & = & u_2 s_4 u_3 u_2 u_3 s_1 u_4 s_3 w_0 s_4 u_2 \subset V_0 \end{array}$$

by proposition 6.3. We now prove (2). We have

$$\begin{array}{rcl} u_4 u_3 u_2 u_1 s_3^{-1} s_2 s_3^{-1} s_4^{-1} w_0 s_4 & = & u_4 (u_3 u_2 s_3^{-1}) u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 \\ & \subset & u_4 u_2 s_3 s_2^{-1} s_3 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 + u_4 u_2 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4, \end{array}$$

and $u_4 u_2 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 = u_2 u_4 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4$ with $u_4 u_3 u_2 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 \subset V_0 + A_4 s_4 w^- s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^- s_4^{-1} w^+ s_4 A_4 \subset V_0$ by lemma 6.11 (3) and (4). Moreover

$$\begin{array}{rcl} u_4 u_2 s_3 s_2^{-1} s_3 u_1 s_2 s_3^{-1} s_4^{-1} w_0 s_4 & = & u_2 u_4 s_3 s_2^{-1} u_1 (s_3 s_2 s_3^{-1}) s_4^{-1} w_0 s_4 \\ = \ u_2 u_4 s_3 s_2^{-1} u_1 s_2^{-1} s_3 s_2 s_4^{-1} w_0 s_4 & = & u_2 u_4 s_3 s_2^{-1} u_1 s_2^{-1} s_3 s_4^{-1} w_0 s_4 s_2, \end{array}$$

so we are reduced to studying

$$\begin{array}{rcl} u_4 s_3 (s_2^{-1} u_1 s_2^{-1}) s_3 s_4^{-1} w_0 s_4 & \subset & u_4 s_3 u_1 s_2 s_1^{-1} s_2 s_3 s_4^{-1} w_0 s_4 + u_4 s_3 u_1 u_2 u_1 s_4^{-1} w_0 s_4 \\ & \subset & u_1 u_4 s_3 s_2 s_1^{-1} s_2 s_3 s_4^{-1} w_0 s_4 + u_1 u_4 s_3 u_2 u_1 s_4^{-1} w_0 s_4 \\ & \subset & V_0 + A_4 s_4 w^+ s_4^{-1} w^+ s_4 A_4 + A_4 s_4^{-1} w^+ s_4^{-1} w^+ s_4 A_4. \end{array}$$

Since $s_4^{-1} w^+ s_4^{-1} w^+ s_4 \in V_0$ by lemma 6.11 (8) (after applying Φ), this concludes the proof of (2). □

Lemma 7.11. $s_4(s_3 s_2^{-1} s_3) s_1 s_2 s_1 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 \in V^+$

We have $s_4(s_3 s_2^{-1} s_3)(s_1 s_2 s_1) s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 = s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_2 s_4 (s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4 = s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 (s_2 s_3^{-1} s_2 s_3^{-1}) s_4^{-1} w_0 s_4$ which belongs to $s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 s_3^{-1} s_2 s_3^{-1} s_2 s_4^{-1} w_0 s_4 + s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_2 u_3 s_4^{-1} w_0 s_4 + s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_3 u_2 s_4^{-1} w_0 s_4$ by lemma 3.6. Now

$$s_4(s_3 s_2^{-1} s_3) s_2 s_1 s_4 u_2 u_3 s_4^{-1} w_0 s_4 \subset V^+$$

by lemma 7.9 , while $s_4(s_3s_2^{-1}s_3)s_2s_1s_4u_3u_2s_4^{-1}w_0s_4 \subset V^+$ by lemma 7.10 (1). We are thus reduced to considering

$$\begin{array}{rcl} s_4(s_3s_2^{-1}s_3)s_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 & \in & s_4s_2(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 \\ & & +s_4u_2u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 \end{array}$$

by lemma 2.3. We have

$$s_4s_2(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 = s_2s_4(s_3s_2^{-1}s_3)s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4s_2 \in V^+$$

by lemma 7.8. We have $s_4u_2u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_2s_4^{-1}w_0s_4 = u_2s_4u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4s_2$ and $s_4u_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ is a linear combination of the $s_4s_3^{\alpha}u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4$ for $\alpha \in \{0,1,-1\}$. When $\alpha = 0$ we get $s_4u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 = u_2s_1s_4^2s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$ by proposition 6.3 ; for $\alpha = 1$ we have

$$\begin{array}{rlcl} & s_4s_3u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 & = & (s_4s_3s_4)u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \\ = & s_3s_4s_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 & = & s_3s_4(s_3u_2s_3^{-1})s_1s_2s_3^{-1}s_4^{-1}w_0s_4 \\ = & s_3s_4s_2^{-1}u_3s_2s_1s_2s_3^{-1}s_4^{-1}w_0s_4 & = & s_3s_2^{-1}s_4u_3(s_2s_1s_2)s_3^{-1}s_4^{-1}w_0s_4 \\ = & s_3s_2^{-1}s_4u_3s_1s_2s_1s_3^{-1}s_4^{-1}w_0s_4 & = & s_3s_2^{-1}s_1s_4u_3s_2s_3^{-1}s_4^{-1}w_0s_4s_1 \subset V^+ \end{array}$$

by proposition 6.3 ; for $\alpha = -1$ we have

$$\begin{array}{rcl} s_4s_3^{-1}u_2s_1s_4s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 & = & (s_4s_3^{-1}s_4)u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \\ & \subset & u_3^{\times}s_4^{-1}s_3s_4^{-1}u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \\ & & +u_3u_4u_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \end{array}$$

by lemma 2.4, and $u_4u_3u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 \subset V^+$ by lemma 7.10 (2). Finally,

$$\begin{array}{rlcl} & s_4^{-1}s_3s_4^{-1}u_2s_1s_3^{-1}s_2s_3^{-1}s_4^{-1}w_0s_4 & = & s_4^{-1}s_3u_2s_1s_4^{-1}s_3^{-1}s_2(s_3^{-1}s_4^{-1}s_3)s_2s_1^2s_2s_3s_4 \\ = & s_4^{-1}s_3u_2s_1s_4^{-1}s_3^{-1}s_2s_4s_3^{-1}s_4^{-1}s_2s_1^2s_2s_3s_4 & = & s_4^{-1}s_3u_2s_1(s_4^{-1}s_3^{-1}s_4)s_2s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4) \\ = & s_4^{-1}s_3u_2s_1s_3s_4^{-1}s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} & = & s_4^{-1}s_3u_2s_3s_4^{-1}s_1s_3^{-1}s_2s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} \subset V^+ \end{array}$$

by lemma 7.4 (4).

7.6. **Reduction to $s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4$.** We apply the following relations of B_4 :

$$\left\{ \begin{array}{rcl} s_3(s_2s_1^{-1}\ s_2)s_3^{-1} & = & s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_1s_2 \\ s_3^{-1}(s_2^{-1}s_1\ s_2^{-1})s_3 & = & s_2s_1(s_3^{-1}s_2s_3^{-1})s_1^{-1}s_2^{-1} \end{array} \right.$$

This yields

$$\begin{array}{rl} & s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4s_3^{-1}(s_2^{-1}s_1s_2^{-1})s_3s_4^{-1}w_0s_4 \\ = & s_4s_2^{-1}s_1^{-1}(s_3s_2^{-1}s_3)s_1s_2s_4s_2s_1(s_3^{-1}s_2s_3^{-1})s_1^{-1}s_2^{-1}s_4^{-1}w_0s_4 \\ = & s_2^{-1}s_1^{-1}s_4(s_3s_2^{-1}s_3)s_1s_2^2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4s_1^{-1}s_2^{-1}. \end{array}$$

Expanding s_2^2 we get

$$\begin{array}{rcl} s_4(s_3s_2^{-1}s_3)s_1s_2^2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 & \in & R^{\times}s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \\ & & +Rs_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \\ & & +Rs_4(s_3s_2^{-1}s_3)s_1^2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \end{array}$$

We have $s_4(s_3s_2^{-1}s_3)s_1^2s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$ by lemma 7.8, and

$$s_4(s_3s_2^{-1}s_3)s_1s_2s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 \in V^+$$

by lemma 7.11.

Lemma 7.12.

(1) $s_4s_3s_1^{-1}s_2s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
(2) $s_4s_3s_2^{-1}s_1^{-1}s_2s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V_0$

Proof. We prove (1). Using braid relations we get

$$\begin{array}{lll} & s_4s_3s_1^{-1}s_2s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 & = & s_1^{-1}s_4(s_3s_2s_3^{-1})\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ = & s_1^{-1}s_4s_2^{-1}s_3s_2\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 & = & s_1^{-1}s_2^{-1}(s_4s_3\ s_4^{-1})\ s_2^2s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ = & s_1^{-1}s_2^{-1}s_3^{-1}s_4\ s_3\ s_2^2s_3s_4^{-1}s_1s_2s_3w_0s_4 \end{array}$$

and this latter term $s_1^{-1}\ s_2^{-1}s_3^{-1}s_4s_3s_2^2s_3s_4^{-1}s_1s_2s_3w_0s_4$ belongs to V^+ by lemma 7.4 (4). We now prove (2). By using braid relations we get

$$\begin{array}{lll} & s_4s_3(s_2^{-1}s_1^{-1}s_2)s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 & = & s_4s_3s_1s_2^{-1}s_1^{-1}s_3^{-1}\ s_4^{-1}\ s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ = & s_1s_4(s_3s_2^{-1}s_3^{-1})\ s_4^{-1}\ s_1^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 & = & s_1s_4s_2^{-1}s_3^{-1}s_2\ s_4^{-1}\ s_1^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ = & s_1s_2^{-1}(s_4s_3^{-1}\ s_4^{-1})\ s_2s_1^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 & = & s_1s_2^{-1}s_3^{-1}s_4^{-1}\ s_3\ s_2s_1^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ = & s_1s_2^{-1}s_3^{-1}s_4^{-1}\ s_3\ s_2(s_1^{-1}s_2s_1)s_3s_4^{-1}s_2s_3w_0s_4 & = & s_1s_2^{-1}s_3^{-1}s_4^{-1}\ s_3\ s_2s_2s_1s_2^{-1}s_3s_4^{-1}s_2s_3w_0s_4 \\ = & s_1s_2^{-1}s_3^{-1}s_4^{-1}\ s_3\ s_2s_2s_1s_2^{-1}s_3s_2s_4^{-1}s_3w_0s_4 & = & s_1s_2^{-1}s_3^{-1}s_4^{-1}\ s_3\ s_2s_2s_1s_2^{-1}s_3s_2s_4^{-1}s_3^2s_2s_1^2s_2s_3s_4 \end{array}$$

and $s_4^{-1}s_3s_2^2s_1s_2^{-1}s_3s_2s_4^{-1}s_3^2s_2s_1^2s_2s_3s_4$ is a linear combination of $s_4^{-1}s_3s_2^2s_1s_2^{-1}s_3s_2s_4^{-1}s_3^{\alpha}s_2s_1^2s_2s_3s_4$ for $\alpha \in \{-1,1,0\}$. When $\alpha = 1$, we get $s_4^{-1}s_3s_2^2s_1s_2^{-1}s_3s_2s_4^{-1}s_3s_2s_1^2s_2s_3s_4 = s_4^{-1}s_3s_2^2s_1s_2^{-1}s_3s_2s_4^{-1}w_0s_4 = s_4^{-1}s_3s_2^2s_1s_2^{-1}s_3s_4^{-1}w_0s_4s_2$ which clearly (by expanding s_2^2) belongs to $V_0 + A_4s_4^{-1}w^+s_4^{-1}w_0s_4A_4 = V_0 + A_4s_4^{-1}w^+s_4^{-1}w^+s_4A_4 \subset V^+$ by lemma 6.11 (8) (take the image by Φ of the identity there). When $\alpha \in \{0,-1\}$ we write $s_4^{-1}s_3s_2^2s_1(s_2^{-1}s_3s_2)s_4^{-1}s_3^{\alpha}s_2s_1^2s_2s_3s_4 = s_4^{-1}s_3s_2^2s_1s_3s_2s_3^{-1}s_4^{-1}s_3^{\alpha}s_2s_1^2s_2s_3s_4$. When $\alpha = 0$, we get $s_4^{-1}s_3s_2^2s_1s_3s_2s_3^{-1}s_4^{-1}(s_2s_1^2s_2)s_3s_4 = s_4^{-1}s_3s_2^2s_1s_3s_2s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4) = s_4^{-1}s_3s_2^2s_1s_3s_2s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} \in V_0$, so we can assume $\alpha = -1$. Then

$$\begin{array}{lll} s_4^{-1}s_3s_2^2s_1s_3s_2(s_3^{-1}s_4^{-1}s_3^{-1})s_2s_1^2s_2s_3s_4 & = & s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}s_4^{-1}s_2s_1^2s_2s_3s_4 \\ & = & s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}s_2s_1^2s_2(s_4^{-1}s_3s_4) \\ & = & s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}s_2s_1^2s_2s_3s_4s_3^{-1} \end{array}$$

is a linear combination of $s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}s_2s_1^{\beta}s_2s_3s_4s_3^{-1}$ for $\beta \in \{-1,0,1\}$. When $\beta = 0$ we get $s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}s_2^2s_3s_4s_3^{-1} \in V_0$ by lemma 7.4 (4) (taking the image by Ψ of the second identity there). When $\beta = 1$ we get

$$\begin{array}{lll} s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}(s_2s_1s_2)s_3s_4s_3^{-1} & = & s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}s_1s_2s_1s_3s_4s_3^{-1} \\ & = & s_4^{-1}s_3s_2^2s_1s_3s_2s_1s_4^{-1}s_3^{-1}s_2s_3s_4s_3^{-1}s_1 \in V_0 \end{array}$$

by the same argument. When $\beta = -1$ we get

$$\begin{array}{lll} s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}s_3^{-1}(s_2s_1^{-1}s_2)s_3s_4s_3^{-1} & = & s_4^{-1}s_3s_2^2s_1s_3s_2s_4^{-1}(s_2s_1)(s_3s_2^{-1}s_3)(s_1^{-1}s_2^{-1})s_4s_3^{-1} \\ & = & s_4^{-1}s_3s_2^2s_1s_3s_2(s_2s_1)s_4^{-1}(s_3s_2^{-1}s_3)s_4(s_1^{-1}s_2^{-1})s_3^{-1} \in V_0 \end{array}$$

again by the same argument, and this concludes the proof. □

7.7. **Reduction to** $s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4$**.** We have

$$\begin{array}{lll} s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4(s_3^{-1}s_2s_3^{-1})s_4^{-1}w_0s_4 & = & s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4s_3^{-1}s_2(s_3^{-1}s_4^{-1}s_3^{-1})\ s_3w_0s_4 \\ & = & s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_4s_3^{-1}s_2s_4^{-1}s_3^{-1}s_4^{-1}\ s_3w_0s_4 \\ & = & s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1(s_4s_3^{-1}s_4^{-1})s_2s_3^{-1}s_4^{-1}\ s_3w_0s_4 \\ & = & s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_3^{-1}s_4^{-1}(s_3s_2s_3^{-1})s_4^{-1}\ s_3w_0s_4 \\ & = & s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_1s_3^{-1}s_4^{-1}s_2^{-1}s_3s_2s_4^{-1}\ s_3w_0s_4 \\ & = & s_4s_3s_2^{-1}s_1(s_3s_2^{-1}s_3^{-1})s_1s_4^{-1}s_2^{-1}s_3s_4^{-1}\ s_2s_3w_0s_4 \\ & = & s_4s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}s_2s_1s_4^{-1}s_2^{-1}s_3s_4^{-1}\ s_2s_3w_0s_4 \\ & = & s_4s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}(s_2s_1s_2^{-1})s_4^{-1}s_3s_4^{-1}\ s_2s_3w_0s_4 \\ & = & s_4s_3s_2^{-1}s_1s_2^{-1}s_3^{-1}s_1^{-1}s_2s_1s_4^{-1}s_3s_4^{-1}\ s_2s_3w_0s_4 \\ & = & s_4s_3s_2^{-1}(s_1s_2^{-1}s_1^{-1})s_3^{-1}s_2s_4^{-1}s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \\ & = & s_4s_3s_2^{-1}s_2^{-1}s_1^{-1}s_2s_3^{-1}s_2s_4^{-1}s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \\ & = & s_4s_3s_2^{-2}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \end{array}$$

and, expanding s_2^{-2}, we get

$$\begin{array}{lcl} s_4s_3s_2^{-2}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 & \in & R^{\times}s_4s_3s_2s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \\ & & +Rs_4s_3s_2^{-1}s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \\ & & +Rs_4s_3s_1^{-1}s_2s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}\ s_1s_2s_3w_0s_4 \end{array}$$

and the last two terms belong to V_0 by lemma 7.12 (1) and (2).

Lemma 7.13.

(1) $s_4(s_3s_2^{-1}s_3)s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
(2) $s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$
(3) $s_4s_3s_2^{-1}s_3s_1s_2s_4^{-1}s_3s_4^{-1} \in A_4^{\times}s_4s_3s_2^{-1}s_3s_4^{-2}A_3^{\times}$
(4) $s_4u_2u_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \in V_0$

Proof. We prove (1). $s_4s_3s_2^{-1}s_3s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 = s_4s_3s_2^{-1}s_1(s_3s_4^{-1}s_3s_4^{-1})s_1s_2s_3w_0s_4$ belongs to

$$s_4s_3s_2^{-1}s_1s_4^{-1}s_3s_4^{-1}s_3s_1s_2s_3w_0s_4 + s_4s_3s_2^{-1}s_1u_3u_4s_1s_2s_3w_0s_4 + s_4s_3s_2^{-1}s_1u_4u_3s_1s_2s_3w_0s_4$$

by lemma 3.6. We have $s_4s_3s_2^{-1}s_1u_3u_4s_1s_2s_3w_0s_4 = s_4s_3s_2^{-1}u_3u_4s_1^2s_2s_3w_0s_4 \subset V^+$ by lemma 7.4 (4), and $s_4s_3s_2^{-1}s_1u_4u_3s_1s_2s_3w_0s_4 = s_4s_3u_4s_2^{-1}s_1u_3s_1s_2s_3w_0s_4 \subset V_0$ by lemma 6.11 (2). Moreover $s_4s_3s_2^{-1}s_1s_4^{-1}s_3s_4^{-1}s_3s_1s_2s_3w_0s_4 = (s_4s_3s_4^{-1})s_2^{-1}s_1s_3s_4^{-1}s_3s_1s_2s_3w_0s_4 = s_3^{-1}s_4s_3s_2^{-1}s_3s_4^{-1}s_3s_1^2s_2s_3w_0s_4 = s_3^{-1}s_4s_3s_2^{-1}s_3s_4^{-1}s_3s_1^2s_2s_3w_0s_4 \in V^+$ by lemma 7.4 (4). This proves (1). Using only braid relations we get

$$\begin{array}{lllll} & s_4s_3s_2^{-1}s_3s_1s_2s_4^{-1}s_3s_4^{-1} & = & s_1s_1^{-1}s_4s_3s_2^{-1}s_1s_3s_2s_4^{-1}s_3s_4^{-1} \\ = & s_1s_4s_3(s_1^{-1}s_2^{-1}s_1)s_3s_2s_4^{-1}s_3s_4^{-1} & = & s_1s_4s_3s_2s_1^{-1}(s_2^{-1}s_3s_2)s_4^{-1}s_3s_4^{-1} \\ = & s_1s_4s_3s_2s_1^{-1}s_3s_2s_3^{-1}s_4^{-1}s_3s_4^{-1} & = & s_1s_4(s_3s_2s_3)s_1^{-1}s_2s_3^{-1}s_4^{-1}s_3s_4^{-1} \\ = & s_1s_4s_2s_3s_2s_1^{-1}s_2s_3^{-1}s_4^{-1}s_3s_4^{-1} & = & s_1s_2s_4s_3s_2s_1^{-1}s_2(s_3^{-1}s_4^{-1}s_3)s_4^{-1} \\ = & s_1s_2s_4s_3s_2s_1^{-1}s_2s_4s_3^{-1}s_4^{-1}s_4^{-1} & = & s_1s_2(s_4s_3s_4)s_2s_1^{-1}s_2s_3^{-1}s_4^{-2} \\ = & s_1s_2s_3s_4(s_3(s_2s_1^{-1}s_2)s_3^{-1})s_4^{-2} & = & s_1s_2s_3s_4(s_2^{-1}s_1^{-1})(s_3s_2^{-1}s_3)(s_1s_2)s_4^{-2} \\ = & (s_1s_2s_3s_2^{-1}s_1^{-1})s_4s_3s_2^{-1}s_3s_4^{-2}(s_1s_2) & & \end{array}$$

which proves (3). From this we deduce that $s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in A_4s_4u_3u_2u_3u_4A_4s_4 \subset V^+$ by lemma 7.4 (4). This proves (2). Finally

$$\begin{array}{lcl} s_4u_2u_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 & = & u_2s_1^{-1}s_4u_3s_4^{-1}s_2s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ & \subset & u_2s_1^{-1}s_3^{-1}u_4(s_3s_2s_3)s_4^{-1}s_1^2s_2s_3w_0s_4 \\ & \subset & u_2s_1^{-1}s_3^{-1}u_4s_2s_3s_2s_4^{-1}s_1^2s_2s_3w_0s_4 \\ & \subset & u_2s_1^{-1}s_3^{-1}s_2u_4s_3s_4^{-1}s_2s_1^2s_2s_3w_0s_4 \subset V_0 \end{array}$$

by lemma 6.11 (2). This proves (4).

□

Lemma 7.14.

(1) $u_4A_4s_4^{-1}s_3^{\beta}\ w_0s_4 \subset V^+$
(2) $s_4u_3u_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \subset V^+$

Proof. We prove (1). Using $A_4 = A_3u_3A_3 + A_3u_3u_2u_3 + A_3w^+ + A_3w^-$ we get that $u_4A_4s_4^{-1}s_3^{\beta}\ w_0s_4$ is the sum of the following abelian groups :

- $u_4A_3u_3A_3s_4^{-1}s_3^{\beta}\ w_0s_4 = A_3u_4u_3s_4^{-1}A_3s_3^{\beta}\ w_0s_4 \subset V_0$ by lemma 6.11 (2).
- $u_4A_3w^+s_4^{-1}s_3^{\beta}\ w_0s_4 = A_3u_4w^+s_4^{-1}s_3^{\beta}\ w_0s_4$, which is included in $V_0 + A_4u_4w^+s_4^{-1}w^+s_4A_4$ by lemma 6.8 (4) and proposition 6.3. Now $u_4w^+s_4^{-1}w^+s_4 \subset V_0 + Rs_4w^+s_4^{-1}w^+s_4 + Rs_4^{-1}w^+s_4^{-1}w^+s_4$ and we have $s_4^{-1}w^+s_4^{-1}w^+s_4 \in V_0$ by lemma 6.11 (7) (apply $\Phi \circ \Psi$ to the identity there), so $u_4A_3w^+s_4^{-1}s_3^{\beta}\ w_0s_4 \subset V^+$
- $u_4A_3w^-s_4^{-1}s_3^{\beta}\ w_0s_4 = A_3u_4w^-s_4^{-1}s_3^{\beta}\ w_0s_4$, which is included in $V_0 + A_4u_4w^-s_4^{-1}w^+s_4A_4$ by lemma 6.8 (4) and proposition 6.3. Since $u_4w^-s_4^{-1}w^+s_4 \subset V_0$ by lemma 6.11 (5) we get $u_4A_3w^-s_4^{-1}s_3^{\beta}\ w_0s_4 \subset V^+$
- $u_4A_3u_3u_2u_3A_3s_4^{-1}s_3^{\beta}\ w_0s_4 = A_3u_4u_3u_2u_3s_4^{-1}A_3s_3^{\beta}\ w_0s_4 \subset V^+$ by lemma 7.4 (4).

This proves (1). Since $s_4u_3u_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 = s_4u_3s_4^{-1}u_2s_1^{-1}s_2s_3s_4^{-1}s_1^2s_2s_3w_0s_4 = s_3^{-1}u_4s_3u_2s_1^{-1}s_2s_3s_1^2s_2s_4^{-1}s_3w_0s_4 \subset s_3^{-1}u_4A_4s_4^{-1}s_3w_0s_4 \subset V^+$ by (1), and this proves (2).

∎

7.8. **Reduction to $s_4w^-s_4w_0^2s_4$.** We have

$$\begin{aligned} s_4s_3(s_2s_1^{-1}s_2)s_3^{-1}s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 &= s_4(s_2^{-1}s_1^{-1})(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_2s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ &= (s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_2^2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \end{aligned}$$

and, expanding s_2^2, we get that this last element belongs to

$$\begin{aligned} & R^\times(s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ + \ & R(s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \\ + \ & R(s_2^{-1}s_1^{-1})s_4(s_3s_2^{-1}s_3)s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4. \end{aligned}$$

Now $s_4(s_3s_2^{-1}s_3)s_1s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$ by lemma 7.13 (1) and

$$s_4(s_3s_2^{-1}s_3)s_1s_2s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 \in V^+$$

by lemma 7.13 (2). We are thus reduced to

$$\begin{aligned} s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_4^{-1}s_3s_4^{-1}s_1s_2s_3w_0s_4 &= s_4(s_3s_2^{-1}s_3)s_1s_2^{-1}s_4^{-1}s_3s_4^{-1}s_1^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_4(s_3s_2^{-1}s_3)(s_1s_2^{-1}s_1^{-1})s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_4(s_3s_2^{-1}s_3s_2^{-1})s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \end{aligned}$$

which, by lemma 3.6, lies in

$$\begin{aligned} & R^\times s_4s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ + \ & s_4u_2u_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ + \ & s_4u_3u_2s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4. \end{aligned}$$

The two latter terms lie in V^+ by lemma 7.13 (4) and 7.14 (2), so we are reduced to

$$\begin{aligned} s_4s_2^{-1}s_3s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 &= s_2^{-1}s_4s_3s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}(s_3s_4s_3)s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_4s_3s_4s_2^{-1}s_3s_1^{-1}s_2s_4^{-1}s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_4s_3s_2^{-1}(s_4s_3s_4^{-1})s_1^{-1}s_2s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_4(s_3s_2^{-1}s_3^{-1})s_4s_3s_1^{-1}s_2s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_4s_2^{-1}s_3^{-1}s_2s_4s_3s_1^{-1}s_2s_3s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4(s_3s_2s_3)s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_4s_2s_3s_2s_4^{-1}s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2(s_4s_3s_4^{-1})s_2s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_2^{-1}s_4s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}s_4s_3s_2s_1^2s_2s_3w_0s_4 \\ &= s_2^{-1}s_3^{-1}s_2^{-1}s_4w^-s_4w_0^2s_4 \end{aligned}$$

hence to $s_4w^-s_4w_0^2s_4$.

7.9. **Conclusion of the computation.** By lemma 4.9, we have $w_0^2 \in A_3^\times w_0^{-1} + U^+ = w_0^{-1}A_3^\times + U^+$, and $U^+ = A_3w_0 + U_0 = w_0A_3 + U_0$. We then have $s_4w^-s_4w_0^2s_4 \in s_4w^-s_4w_0^{-1}s_4A_3^\times + s_4w^-s_4w_0s_4A_3 + s_4w^-s_4U_0s_4$.

On the one hand, we know that $s_4w^-s_4A_3u_3A_3s_4 = s_4w^-s_4A_3u_3s_4A_3 = s_4w^-s_4u_2u_1u_2u_1u_3s_4A_3 = s_4w^-s_4u_2u_1u_2u_3s_4u_1A_3 \subset V_0$ by proposition 6.3, and that

$$\begin{aligned} s_4w^-s_4A_3u_3u_2u_3A_3s_4 &= s_4w^-s_4A_3u_3u_2u_3s_4A_3 \\ &= s_4w^-s_4u_1u_2u_1(u_2u_3u_2u_3)s_4A_3 \\ &= s_4w^-s_4u_1u_2u_1u_3u_2u_3u_2s_4A_3 \\ &= s_4w^-s_4u_1u_2u_1u_3u_2u_3s_4u_2A_3 \quad \subset \quad V^+ \end{aligned}$$

by lemma 7.4 (4) (apply $\Phi \circ \Psi$ to the identity there). From $U_0 = A_3u_3A_3 + A_3u_3u_2u_3A_3$ one thus gets $s_4w^-s_4U_0s_4 \subset V^+$.

On the other hand, we have $s_4w^-s_4w_0s_4 \in V_0 + s_4w^-s_4w^+s_4A_3 \subset V_0$ by lemma 6.11 (5). We are thus reduced to $s_4w^-s_4w_0^{-1} \in s_4w^-s_4w^-s_4A_3^\times + V_0$, which concludes the proof.

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