THE CENTER OF PURE COMPLEX BRAID GROUPS

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ABSTRACT. Broué, Malle and Rouquier conjectured in [2] that the center of the pure braid group of an irreducible finite complex reflection group is cyclic. We prove this conjecture, for the remaining exceptional types, using the analogous result for the full braid group due to Bessis, and we actually prove the stronger statement that any finite index subgroup of such braid group has cyclic center.

1. INTRODUCTION

Let W denote a (finite) complex reflection group of rank $n \geq 1$, that is a finite subgroup of $\mathrm{GL}_n(\mathbf{C})$ generated by pseudo-reflections, that is elements of $\mathrm{GL}_n(\mathbf{C})$ which fix an hyperplane. We assume W to be irreducible, in order to simplify statements. To such a group is associated an hyperplane arrangement $\mathcal{A}$, made of the collection of the fixed hyperplanes associated to the pseudo-reflections in W. Let $X = \mathbf{C}^n \setminus \bigcup \mathcal{A}$ denote the hyperplane complement. The groups $P = \pi_1(X)$ and $B = \pi_1(X/W)$ are known as the pure braid group and braid group associated to W. There is a short exact sequence $1 \to P \to B \to W \to 1$. In the case W is a real reflection group, the group B is an Artin-Tits group, with prototype the usual braid group B associated to $W = \mathfrak{S}_n$. An extension of many results on Artin-Tits groups to the more general setup introduced here has been proposed in [2]. Several of them are still conjectural, such as the determination of the center of these groups. The goal of this note is to clarify the status of this question. Recall that in [2] are introduced (infinite order) elements $\beta \in Z(B)$, $\pi \in Z(P)$, with $\beta^{|Z(W)|} = \pi$ such that the image of β in W generates $Z(W)$.

Our purpose is to summarize what can be stated on this topic, as follows:

Theorem 1.1. *The center $Z(B)$ is infinite cyclic, and generated by β.*

Theorem 1.2. *The center $Z(P)$ is infinite cyclic, and generated by π.*

Theorem 1.3. *There exists a short exact sequence*
$$1 \to Z(P) \to Z(B) \to Z(W) \to 1$$

These three results were conjectured in [2], and proved there for the infinite series $G(de, e, n)$, as well as for the (easy) case of groups of rank 2. Actually, for a given group W, theorem 1.2 implies by general arguments theorem 1.3 and theorem 1.1, as every element in $Z(W)$ can be lifted to an element in $Z(B)$. The remaining cases were the exceptional groups $G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}$ in Shephard-Todd notation. As noted in [2], theorem 1.1 for the so-called 'Shephard' groups G_{25}, G_{26}, G_{32} is true because they have the same B as some Coxeter group (however, contrary to what is claimed in [2], this does not prove the other two results !). Note also that in [2, proposition 2.23] the bottom-right square in the diagram is not commutative and the bottom sequence is not exact.

Theorem 1.1 was proved by D. Bessis in [1] for the well-generated groups G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{32}, G_{33}, G_{34}. The remaining case of G_{31} can be obtained by a short argument due to D. Bessis (personal communication) that we reproduce here for the convenience of the reader (see section 3).

Once theorem 1.1 is known, theorem 1.3 reduces to theorem 1.2, and more precisely to the statement $Z(P) \subset Z(B)$. Actually we prove the following stronger theorem in the subsequent sections.

Theorem 1.4. *If U is a finite index subgroup of B, then $Z(U) \subset Z(B)$.*

2. Garside theory

A Garside monoid is a cancellative monoid such that any two elements have a least common right-multiple and a greatest common left-divisor and such that there exist an element Δ (a "Garside element") whose left- and right-divisors are the same and generate the monoid. Here a left-divides b, which will be denoted by $a \preccurlyeq b$, means that there exists c with $b = ac$ and similarly for right-divisibility. We refer to [7] or [11] for the basic notions on Garside theory. We will use in particular the following property: if M is a Garside monoid, let $\alpha(x)$ denote the left gcd of an element $x \in M$ with Δ, then $\alpha(xy) = \alpha(x\alpha(y))$ for any $y \in M$.

In this section, we consider a Garside monoid M which satisfies in addition the following properties:

(i) There is an additive length function on M such that elements of length 1 form a generating set and the only element of length 0 is 1.

This implies that an element of length 1 is an atom, that is has no proper divisor.

(ii) For every couple of atoms $s \neq t$, their lcm $\Delta_{s,t}$ is balanced (meaning that its left- and right-divisors are the same).

(iii) For any couple of atoms $s \neq t$ and any positive integer n, the left gcd of $\Delta_{s,t}$ and s^n is equal to s.

Note that many of the monoids which have been studied for complex braid groups and for Artin-Tits groups satisfy the above conditions. In particular we have

Proposition 2.1. *Let M be one of the following monoids:*

(M1) *The classical monoid of positive elements in the Artin-Tits groups associated to finite Coxeter groups (see [7]).*

(M2) *The "dual" monoid of [1].*

(M3) *The "parachute" monoid of [6].*

(M4) *The "dual" monoids for Artin-Tits groups of type $\tilde{A}$ and $\tilde{C}$ of [9] and [10].*

(M5) *The monoids $f(h, m)$ (for $h, m \geq 1$) presented by generators $x_1, \ldots, x_m$ and relations*

$$\underbrace{x_1 x_2 \ldots x_m x_1 \ldots}_{h \ terms} = \underbrace{x_2 x_3 \ldots x_m x_1 \ldots}_{h \ terms} = \ldots$$

Then M satisfies (i), (ii) and (iii).

Proof. In case (M1) since in the classical monoid we have $\Delta_{s,t} = \underbrace{sts\ldots}_{e} = \underbrace{tst\ldots}_{e}$ for some $e \geq 2$ and these are the only decompositions of Δ_{st}, we get that Δ_{st} is

balanced. Moreover the only divisors of s^n are smaller powers of s, of which only s divides $\Delta_{s,t}$.

In cases (M2) and (M4) all divisors of Δ are balanced, in particular the lcm of two atoms. Moreover in any decomposition of Δ into a product of atoms, all atoms are different, hence the same property holds for divisors of Δ. If s is an atom we have thus $s^2 \npreceq \Delta$, so that $\alpha(s^2) \neq s^2$. Since $\alpha(s^2)$ has length at most 2 and is different from s^2 it has to be of length 1, hence equal to s and by induction we get $\alpha(s^n) = \alpha(s\alpha(s^{n-1})) = \alpha(s^2) = s$. This implies that the gcd of s^n with $\Delta_{s,t}$, which divides $\alpha(s^n)$, is equal to s.

In the monoid M for $G(e,e,r)$ introduced in [6], any pair of dictinct atoms can be embedded in a submonoid M' which is of type (M1) or (M2) (see [4, section 6.3]). This embedding maps lcms in M' to lcms in M and left-divisibility in M' is the restriction of left-divisibility in M (see [4, lemmas 5.1 and 5.3]).

This implies that for properties (ii) and (iii) we are either in the first or second situation, depending on the choice of the couple of atoms .

The monoids (M5) have been investigated by M. Picantin in his thesis [19]. Property (i) is clear, as the presentation is homogeneous. It is readily checked that the gcd of two distinct atoms is the Garside element $\Delta = x_1 x_2 \ldots x_m x_1 \ldots$ (h terms), which proves (ii). Finally, (iii) follows from the fact that every divisor of Δ but itself has a unique decomposition as a product of atoms.

$\square$

As pointed out to us by E. Godelle, there exist Garside monoids satisfying (i) and (ii) but not (iii), such as $M = < a, b \mid a^2 = b^2 >$, which provides a counterexample to the next proposition. In [13, proposition 4.4.3] the next proposition is proved for a classical Artin-Tits monoid (see also [14, lemma 2.2]).

Proposition 2.2. *Let M be a Garside monoid satisfying (i), (ii) and (iii) above. Let r be an atom of M and $b, z \in M$ be such that $r^j b = bz$ for some $j \geq 1$; then $z = t^j$ for some atom t with $rb = bt$.*

Proof of the proposition. Note that by properties (i) and (iii), given two atoms $r \neq t$, the element $\delta_{r,t}$ defined by $\Delta_{r,t} = r\delta_{r,t}$ satisfies $\delta_{r,t} \neq 1$ and $r \npreceq \delta_{r,t}$.

We first prove the following lemma.

Lemma 2.3. *Let r be an atom in M and $b \in M$ such that $r \npreceq b$ and let $j \geq 1$ be such that $r^j b$ is divisible by some atom different from r; then there exists an atom $s \neq r$ such that $\delta_{r,s} \preceq b$.*

Proof. The proof is by induction on $j \geq 1$. Let $u \neq r$ be an atom with $u \preceq r^j b$. Then $\Delta_{r,u} \preceq r^j b$; writing $\Delta_{r,u} = r\delta_{r,u}$, we get $\delta_{r,u} \preceq r^{j-1}b$. If $j = 1$ the conclusion holds with $s = u$, and this covers the case $j = 1$.

In the case $j > 1$, since $\delta_{r,u} \neq 1$ and $r \npreceq \delta_{r,u}$, we have $v \preceq \delta_{r,u}$ for some atom $v \neq r$ hence $v \preceq r^{j-1}b$ and we can apply the induction assumption to $j - 1 < j$ and get the result. $\square$

We also need the following lemma.

Lemma 2.4. *Let $b \in M$ be a balanced element. Then, for any atom $r \preceq b$ there exists an atom $t \preceq b$ with $rb = bt$, and $r \mapsto t$ provides a permutation of the atoms dividing b.*

Proof. For $r \preccurlyeq b$ we write $b = rb'$. Since b is balanced we have $b' \preccurlyeq b$ hence $b = b't$ for some $t \in M$ (which obviously divides b). Thus $rb = rb't = bt$ and $r \mapsto t$ is well-defined and clearly injective from the set of divisors of b to itself. Because of the length function it maps atoms to atoms, and it is surjective by the obvious reverse construction. $\qquad\square$

We show now the proposition by induction on the length of b, the case of length 0 being trivial. If $r \preccurlyeq b$ then, by simplifying b by r, we get the result by induction. We thus can assume $b \neq 1$ and $r \not\preccurlyeq b$. By lemma 2.3 there is an atom $s \neq r$ such that $\delta_{r,s} \preccurlyeq b$. Writing $b = \delta_{r,s}b'$ we get $r^j\delta_{r,s}b' = \delta_{r,s}b'z$. As $\Delta_{r,s}$ is balanced, it conjugates r to some atom u, by lemma 2.4, hence, cancelling a power of r, we get $r\delta_{r,s} = \delta_{r,s}u$, whence $\delta_{r,s}u^jb' = \delta_{r,s}b'z$. Cancelling $\delta_{r,s}$ and applying the induction assumption we get $z = t^j$ for some atom t with $ub' = b't$, so $rb = r\delta_{r,s}b' = \delta_{r,s}ub' = \delta_{r,s}b't = bt$. $\qquad\square$

We first prove a general corollary for a Garside group, that is for the group of fractions G of a Garside monoid M. It is a property of Garside monoids that G is generated as a monoid by M and the powers of Δ.

Corollary 2.5. *Let G be the group of fractions of a Garside monoid M satisfying properties (i), (ii) and (iii) then if U is a subgroup of G containing a nontrivial power of each atom, the center of U is contained in the center of G.*

Proof. If $x \in Z(U)$, then for any atom s, there exists a positive integer N such that x commutes with s^N. Let i be large enough so that $x\Delta^i \in M$. Since conjugation by Δ permutes the atoms by lemma 2.4, we have $s^Nx\Delta^i = xs^N\Delta^i = x\Delta^iu^N$ for some atom u such that $s\Delta^i = \Delta^iu$. Proposition 2.2 applied with $b = x\Delta^i$, $r = s$, $j = N$ implies that $sx\Delta^i = x\Delta^it$, for some atom t, hence we have $t^N = u^N$, which gives $t = u$ by property (iii), since the gcd of $t^N = u^N$ with $\Delta_{t,u}$ has to be equal to t and to u. Hence $sx\Delta^i = x\Delta^iu$ and $sx = xs$. $\qquad\square$

A particular case of the above corollary is the following, using the fact that for monoids of type (M4) in proposition 2.1, the squares of the atoms lie in the pure group.

Corollary 2.6. *The center of pure Artin-Tits groups of type $\tilde{A}$ or $\tilde{C}$ is trivial.*

Another application of proposition 2.2 gives theorem 1.4 for all finite complex reflection groups except G_{31}.

Corollary 2.7. *Assume W is a complex reflection group different from G_{31} and let U be a finite index subgroup of B. Then $Z(U) \subset Z(B)$.*

Proof. It is well-known that all possible braid groups B can be obtained from a 2-reflection group, so we can restrict to this case. Corollary 2.5 gives the result for all complex braid group for which a monoid satisfying (i), (ii) and (iii) is known. This covers all of them except for a complex reflection group of type G_{31} and the infinite series $G(de, e, r)$ for $d > 1$ and $e > 1$. Indeed, one can use

- the classical monoid for Coxeter groups and Shephard groups (groups which have X/W as a Coxeter group), as well as for G_{13}, which has the same braid group as the Coxeter group $I_2(6)$
- the parachute monoid for the $G(e, e, r)$

- the dual monoid for the $G(e,e,r)$, $G(d,1,r)$ and the exceptional groups of rank at least 3 which are not G_{31}
- the monoids $f(4,3)$ and $f(5,3)$ for G_{12} and G_{22}.

Since the only 2-reflection exceptional groups are G_{12}, G_{13} and G_{22} this indeed covers everything but G_{31} and the $G(de,e,r)$, for $d > 1, e > 1$.

But the braid group associated with $G(de,e,r)$ is a subgroup of finite index in the braid group associated to $G(de,1,r)$ (and thus of the classical braid group on r strands), whence the result in this case. $\qquad\square$

3. Springer theory and G_{31}

We let B_n denote the braid group associated to G_n, and $P_n = \mathrm{Ker}(B_n \to G_n)$ the corresponding pure braid group. In particular B_{37} denotes the Artin-Tits group of type E_8. By Springer theory (see [20]), G_{31} appears as the centralizer of a regular element c of order 4 in G_{37}, and, as a consequence of [1, thm. 12.5 (iii)], B_{31} can be identified with the centralizer of a lift $\tilde{c} \in B_{37}$ of c, in such a way that the natural diagram

$$
\begin{array}{ccc}
B_{31} & \longrightarrow & B_{37} \\
\downarrow & & \downarrow \\
G_{31} & \longrightarrow & G_{37}
\end{array}
$$

commutes. The proof of theorem 1.1 for G_{31} was communicated to us by D. Bessis. For the convenience of the reader we reproduce it here. We let π denote the positive generator of $Z(P_{37})$. We have $\tilde{c}^4 = \pi$. On the other hand, it can be checked that G_{37} has a regular element of order 24, and as a consequence of [3] there exists $\tilde{d} \in B_{37}$ with $\tilde{d}^{24} = \pi$ whose image in G_{37} is a regular element d of order 24. Moreover, by [1, 12.5 (ii)], $\tilde{d}^6$ is conjugated to $\tilde{c}$, so up to conjugating $\tilde{d}$ we can assume $\tilde{d}^6 = \tilde{c}$, and in particular $\tilde{d} \in B_{31}$. So $Z(B_{31})$ lies inside the centralizer of $\tilde{d}$, which is by another application of [1, 12.5 (iii)] the braid group of the centralizer in G_{37} of d. Since 24 divides exactly one reflection degree of G_{37}, Springer's theory says that d generates its centralizer, as its centralizer is a reflection group whose single reflection degree is 24. This implies that the centralizer of $\tilde{d}$ is the cyclic group generated by $\tilde{d}$, thus any $x \in Z(B_{31})$ is a power $\tilde{d}^a$ of $\tilde{d}$. Since its image in G_{37} should lie in $Z(G_{31}) = <d^6>$, $Z(B_{31}) = <\tilde{d}^6>$ is a cyclic group, isomorphic to $\mathbf{Z}$ as it is infinite, for instance because $\beta_{31} \in Z(B_{31})$ has infinite order. But $\beta_{31} \in Z(B_{31}) = <\tilde{d}^6>$ satisfies $\beta_{31}^4 = \pi_{31} = \pi_{37} = (\tilde{d}^6)^4$, hence $\tilde{d}^6 = \beta_{31}$ and $Z(B_{31}) = <\beta_{31}> \simeq \mathbf{Z}$. This concludes the proof of theorem 1.1.

In order to prove theorem 1.4, we will need an explicit description of this embedding $B_{31} \hookrightarrow B_{37}$. Using the algorithms of [12] included in the GAP3 package CHEVIE (see [21]), we find that B_{31}, considered as the centralizer of β_{31} in B_{37}, is generated by

$$
\begin{aligned}
x_1 &= (s_2 s_3 s_1 s_5)^{-1} s_1 s_4 (s_2 s_3 s_1 s_5) \\
x_2 &= (s_4 s_2 s_3 s_5 s_6 s_5 s_7)^{-1} s_2 s_5 (s_4 s_2 s_3 s_5 s_6 s_5 s_7) \\
x_3 &= (s_5 s_6 s_7)^{-1} s_1 s_4 (s_5 s_6 s_7) \\
x_4 &= (s_2 s_5 s_6)^{-1} s_4 s_6 (s_2 s_5 s_6) \\
x_5 &= (s_3 s_1 s_5 s_6)^{-1} s_4 s_8 (s_3 s_1 s_5 s_6)
\end{aligned}
$$

These generators satisfy the defining relations of the presentations for B_{31} conjectured in [2] (and proved in [1]). Here the presentations taken for B_{37} and B_{31} correspond respectively to the Coxeter-like diagrams
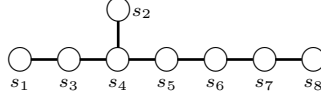
and
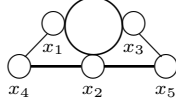
where the circle means $x_1 x_2 x_3 = x_2 x_3 x_1 = x_3 x_1 x_2$.

One can check $\beta_{31} = (s_4 s_2 s_3 s_1 s_4 s_3 s_5 s_6 s_7 s_8)^6 = (x_4 x_1 x_2 x_3 x_5)^6$.

4. KRAMMER REPRESENTATIONS AND G_{31}

We use the generalized Krammer representation $\hat{R} : B_{37} \hookrightarrow \mathrm{GL}_{120}(\mathbf{Q}[q, q^{-1}, t, t^{-1}])$ for the Artin-Tits group of type E_8 defined in [8] and [5]. We use the definitions of [8] (note however the erratum given in [15]). One has $\beta_{37} = \beta_{31}^2$ and $\hat{R}(\beta_{37}) = q^{30}t$. The standard generators s_i of B_{37} are mapped to semisimple matrices with eigenvalues $q^2 t$ (once), $-q$ (28 times) and 1 (91 times). We embed $\mathbf{Q}[q^{\pm 1}, t^{\pm 1}]$ into $\mathbf{Q}[q^{\pm 1}, u^{\pm 1}]$ under $t \mapsto u^2$. Then $R(\beta_{31})$ can be diagonalized, has two 60-dimensional eigenspaces, corresponding to the eigenvalues $\pm q^{15}u$. An explicit base change, that we choose in $\mathrm{GL}_{120}(\mathbf{Q}[q^{\pm 1}, u^{\pm 1}])$ so that it specializes to the identity when $q \mapsto 1, u \mapsto 1$, provides another faithful representation $R : B_{37} \to \mathrm{GL}_{120}([q^{\pm 1}, u^{\pm 1}])$ with $R(B_{31}) \subset \mathrm{GL}_{60}(\mathbf{Q}[q^{\pm 1}, u^{\pm 1}]) \times \mathrm{GL}_{60}(\mathbf{Q}[q^{\pm 1}, u^{\pm 1}])$. Let U be a finite index subgroup of B_{31}, and let $N \geq 1$ such that $x_i^{2N} = (x_i^2)^N \in U$ for all i. Note that, since the images of the x_i's in G_{31} have order 2, we have $x_i^2 \in P_{31}$.

We prove that the centralizer of $R(U)$ in $\mathrm{GL}_{60} \times \mathrm{GL}_{60}$ is equal to the centralizer of $R(B_{31})$. This is the case as soon as $R_j(KU) = \mathrm{Mat}_{60}(K)$, where $j \in \{1, 2\}$, where K is an arbitrary extension of $\mathbf{Q}(q, u)$, $R = R_1 \times R_2$ is the obvious decomposition, and KU is the group algebra of U. Actually, it is clearly enough to prove this for a *specialization* of R_j to given values of t, u. More precisely, if A is a unital ring with field of fractions K and if we have a morphism $\mathbf{Q}[q^{\pm 1}, u^{\pm 1}] \to A$, letting $R_j' : B_{31} \to \mathrm{GL}_{60}(A)$ denote the induced representation we have that $R_j'(KU) = \mathrm{Mat}_{60}(K)$ implies that $R_j(\mathbf{Q}(q, u)U) = \mathrm{Mat}_{60}(\mathbf{Q}(q, u))$.

Let $j \in \{1, 2\}$. We use the morphism $\mathbf{Q}[q^{\pm 1}, u^{\pm 1}] \to \mathbf{Q}[[h]]$ which maps $q \mapsto e^h$, $u \mapsto e^{7h}$ (the choice of 7 being rather random), and denote $R_j' : P_{31} \to \mathrm{Mat}_{60}(\mathbf{Q}[[h]])$ the induced representation. We prove that the unital algebra generated over $\mathbf{Q}((h))$ by the $R_j'(x_i^{2N})$ full $\mathrm{Mat}_{60}(\mathbf{Q}((h)))$. As $x_i^2 \in P_{31} \subset P_{37}$ we have $R_j'(x_i^2) \equiv 1$ modulo h (see e.g. [15]) hence $R_j'(x_i^2)^N \equiv 1$ modulo h. We define $y_i \in \mathrm{Mat}_{60}(\mathbf{Q})$ by $h^{-1}(R_j'(x_i^2) - 1) \equiv y_i$ modulo h and thus $h^{-1}(R_j'(x_i^2)^N - 1) \equiv N y_i$. By Nakayama's lemma it is now sufficient to prove that the y_i generate $\mathrm{Mat}_{60}(\mathbf{Q})$. It turns out that the y_i belong to $\mathrm{Mat}_{60}(\mathbf{Z})$. By another application of Nakayama's lemma it is sufficient to check that the reduction mod p of the y_i generates $\mathrm{Mat}_{60}(\mathbf{F}_p)$ for some prime p. For a given p, the determination by computer of the dimension of the subalgebra of $\mathrm{Mat}_{60}(\mathbf{F}_p)$ generated by elements $y_1, \ldots, y_5$ is easy : starting from the line $F_0 = \mathbf{F}_p \mathrm{Id} \subset \mathrm{Mat}_{60}(\mathbf{F}_p)$ we compute the vector

space $F_{r+1} = F_r + F_r y_1 + \ldots F_r y_5$ until $\dim F_r = \dim F_{r+1}$. For $p = 37$ we get $3600 = \dim \mathrm{Mat}_{60}(\mathbf{F}_p)$, thus concluding the argument.

By faithfulness of R this proves that $Z(U)$ commutes with all B_{31}, thus $Z(U) \subset Z(B_{31})$, and this concludes the proof of theorem 1.4.

5. Miscellaneous consequences

We notice that the representations R_1 and R_2 are deduced one from the other under the field automorphism of $\mathbf{Q}(q, u)$ defined by $q \mapsto q$, $u \mapsto -u$. Since R is faithful, this implies that the representations R_j are faithful representations of B_{31}. We note that the R_j have dimension 60, which is the number of reflections in G_{31}, and thus the dimension of the representation involved in conjecture 1 of [18].

This enables us to prove this conjecture for G_{31}, that is the following theorem:

Theorem 5.1. B_{31} *can be embedded into* $\mathrm{GL}_{60}(K)$ *as a Zariski-dense subgroup, for K a field of characteristic 0, and P_{31} is residually torsion-free nilpotent.*

Proof. Since P_{31} embeds in P_{37}, the statement about residual torsion-free nilpotence is a consequence of the corresponding statement for P_{37}, proved in [15], [17]. Embedding $\mathbf{Q}(q, u)$ into $\mathbf{Q}((h))$ through $q \mapsto e^h$, $u \mapsto e^{\alpha h}$, for α a transcendent number, we can assume $R_j(B_{31}) \subset \mathrm{GL}_{60}(\mathbf{Q}[\alpha]((h)))$.

In addition we know that $R_j(x_i^2) = \exp(h a_i)$ for some $a_i \in \mathfrak{gl}_{60}(\mathbf{Q}[\alpha][[h]])$ with $a_i \equiv \tilde{y}_i \mod h$ for some $\tilde{y}_i \in \mathfrak{gl}_{60}(\mathbf{Q}[\alpha])$ which specialize to y_i under $\alpha \mapsto 7$. A computer calculation similar to the above shows that the Lie algebra generated by the y_i is $\mathfrak{gl}_{60}(\mathbf{Q})$, and thus that the Lie algebra generated by the a_i over $\mathbf{C}((h))$ is $\mathfrak{gl}_{60}(\mathbf{C}((h)))$. By [16] lemme 21 this proves that the Zariski closure of $R_j(B_{31})$ contains $\mathrm{GL}_{60}(\mathbf{C}((h)))$, which proves the theorem. $\qquad \square$

We remark that this conjecture, if true for all braid groups, would immediately imply theorem 1.4. Note however that the representations R_j have same dimension as, but are *not* isomorphic to the representation constructed in [18]. This can be seen from the eigenvalues of the generators : since $R(x_1)$ is conjugated to $R(s_1 s_4)$ and since $s_1 s_4 = s_4 s_1$, the 3 eigenvalues of the $R(s_i)$ provide at most 9 eigenvalues for $R(x_1)$, and by checking each of them we find that $R(x_1)$ has for eigenvalues $q^2 t$ (twice), q^2 (6 times), $-q$ (44 times) and 1 (68 times). As a consequence, the $R_j(x_1)$ both have 4 eigenvalues, to be compared with the 3 eigenvalues the generators have in the construction of [18].

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