TRUNCATIONS AND EXTENSIONS OF THE BRAUER-CHEN ALGEBRA

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To the memory of Kay Magaard

Abstract. The Brauer-Chen algebra is a generalization of the algebra of Brauer diagrams to arbitrary complex reflection groups, that admits a natural monodromic deformation. We determine the generic representation theory of the first non trivial quotient of this algebra. We also define natural extensions of this algebra and prove that they similarly admit natural monodromic deformations.

CONTENTS

1. INTRODUCTION

The algebra of Brauer diagrams was introduced by Brauer in [4] in order to understand the commutant of the orthogonal (or symplectic) groups of the n-fold tensor powers of a quadratic space V, in the same way as the (group algebra of the) symmetric group $\mathfrak{S}_n$ captures the commutant of the general linear group on the n-fold tensor powers of a vector space V. Its

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structure has been determined by Wenzl in [21]. Combinatorially, it can be described as an extension of the group algebra of $\mathfrak{S}_n$.

It has been shown that natural generalizations of the Brauer algebra exists for other finite reflection groups. In [8], Cohen Frenk and Wales attached one to any Coxeter group of type ADE, the case of type A_n corresponding to the original case. A complete structural description was obtained in this case.

Later, Chen proposed in [6] a much larger generalization to *arbitrary* (complex) reflection groups, which is isomorphic to the Cohen-Frenk-Wales algebra for real reflection groups of type ADE.

By contrast with the ADE case, the structural description of this algebra in general is still open, and even the determination of its dimension is still an open problem in general. In this paper we provide a partial description, and more precisely we decompose its first non-trivial quotient.

We introduce a natural series of two-sided ideals $I_1 \supset I_2 \supset \ldots$ and call representation of level r of the Brauer-Chen algebra $Br(W)$ any representation factorizing through $Br_r(W) = Br(W)/I_{r+1}$ but not through $Br_{r-1}(W)$. This series of ideals satisfies that $Br_0(W)$ coincides with the group algebra of W, so $Br_1(W)$ is the first non-trivial quotient, and the first new representations are the 'level 1 representations' arising there.

The very first prototype of these representations were implicitly described in [16], as they served as a guide to Chen to construct his algebra. These representations can be deformed using monodromy means to the generalized Krammer representations of the braid group of W in type ADE as defined in [10, 12], see [15]. The explicit (algebraic, combinatorial) construction of these generalized Krammer representation for arbitrary complex reflection groups is quite an ongoing challenge. To my knowledge, the first successful attempt in this direction in the non-real case has been made by Neaime in [20], were it was constructed for $W = G(3,3,3)$, using a deformation of the Brauer-Chen algebra. It appears in this case that this representation admits non-trivial Galois conjugates. These conjugate representations are monodromy of other level 1 representations of the Brauer-Chen representation, and this provides another motivation for the present work, namely to describe infinitesimally the Galois conjugates of the generalized Krammer representation that we introduced in [16]. Note that Neaime also constructed a conjectural matrix model for $G(4,4,3)$.

Our first main result is then the following one.

Theorem 1.1. *Let κ be a field of characteristic 0, $\mathbf{k} = \kappa(\delta)$. Let W be (pseudo-)reflection group, with set of reflection hyperplanes $\mathcal{A}$, and $Br(W)$ the generic Brauer-Chen algebra defined over $\mathbf{k}$, as in section 2.1. Let us pick a system of representatives H_0 for each $\mathcal{A}_0 \in \mathcal{A}/W$. Then $Br_1(W)$ is a semisimple algebra, whose irreducible representations not factoring through $\overline{\mathbb{Q}}W$ are in 1-1 correspondence with the pairs $(\mathcal{A}_0, \theta)$ where $\mathcal{A}_0 \in \mathcal{A}/W$ and θ is an irreducible representation of $N(W_{H_0})/W_{H_0}$ over $\mathbf{k}$. The restriction of such a representation to $W \subset Br_1(W)$ is the induced representation $\mathrm{Ind}_{N(W_{H_0})}^{W}\theta$.*

The dimension of $Br_1(W)$ is equal to

$$|W| + \sum_{\mathcal{A}_0 \in \mathcal{A}/W} |\mathcal{A}_0| \times |W|/|W_{H_0}|$$

The above result provides a complete description of $Br(W)$ exacly when $Br(W) = Br_1(W)$, that is when there is no couple of transverse reflecting hyperplanes. This is the case for all irreducible rank 2 groups, but also for a couple of groups in rank 3. Specifically, this is the

case for the groups $G(e, e, 3)$ with $e \geq 3$ (see [20], proposition 5.3.2) as well as the primitive reflection groups G_{24} and G_{27} (but it is *not* the case for the Coxeter group H_3). Therefore, we get the following

Corollary 1.2. *(see section 7.2) When $W = G(e, e, 3)$ and e is odd, $Br(W)$ is semisimple and has dimension $3e^2(2 + 3e)$.*

In particular, this provides additional evidence towards conjecture 5.4.2 in [20] about the structure of a generalized BMW algebra in type $G(e, e, 3)$.

A major obstacle to extend the above result to the whole algebra $Br(W)$ is that a good understanding in the general case is still missing of what was called in [8], for type ADE, *admissible collections* of transverse hyperplanes, and described there in terms of the associated root system.

Our second main result (see theorem 2.6) is the following one. We proved in [17, 18] that Hecke algebras admit natural extensions by the Möbius algebra $\mathbb{C}\mathcal{L}$ of the lattice $\mathcal{L}$ of the reflection subgroups of W, and that these algebras are monodromic deformation of $\mathbb{C}W \ltimes \mathbb{C}\mathcal{L}$ in the same way as the Hecke algebra is a monodromic deformation of $\mathbb{C}W$. Here we prove that the same phenomenon occur for the Brauer-Chen algebra. In particular, there is a natural KZ-type connection on $Br(W) \ltimes \mathbb{C}\mathcal{L}$ that 'covers' in some sense these two different constructions, and which should be related, when $W = \mathfrak{S}_n$, with the tied-BWM algebra introduced by Aicardi and Juyumaya in [1].

All this suggests that there should exist a general definition of a generalized (tied-)BMW algebra for arbitrary complex reflection groups, that should be a module of finite type over its natural ring of definition.

2. The Brauer-Chen algebra and its extensions

2.1. **Presentations.** Let W be a complex (pseudo-)reflection group, $\mathcal{R}$ its set of reflections, and $\mathcal{A}$ the set of reflecting hyperplanes. Two hyperplanes H_1, H_2 are called transverse if $H_1 \neq H_2$ and $\{H \in \mathcal{A} \mid H \supset H_1 \cap H_2\} = \{H_1, H_2\}$. In this case we denote $H_1 \pitchfork H_2$. A *transverse collection* of hyperplanes is a subset $\underline{H} = \{H_1, \dots H_r\}$ of $\mathcal{A}$ with the property that every two hyperplanes in the collection are transverse. Let $\mathbf{k}$ be a commutative ring with 1, and $\delta \in \mathbf{k}^\times$ an invertible scalar. We fix the choice of one scalar parameter $\mu_s \in \mathbf{k}$ per reflection s, subject to the condition $\mu_s = \mu_t$ as soon as s and t are conjugates. We denote $\underline{\mu}$ the collection of such parameters.

By definition, the Brauer-Chen algebra $Br(W) = Br_{\underline{\mu}}(W)$ attached to $\underline{\mu}$ is defined by generators $w \in W$, $e_H, H \in \mathcal{A}$ together with the relations of W, the semi-direct type relations $w e_H = e_{w(H)} w$ and the following relations

(1) $\forall H \in \mathcal{A}$ $e_H^2 = \delta e_H$ & $e_H s = s e_H = e_H$ whenever $\mathrm{Ker}(s - 1) = H$
(2) $H_1 \pitchfork H_2 \Rightarrow e_{H_1} e_{H_2} = e_{H_2} e_{H_1}$
(3) If H_1 and H_2 are distinct and not transverse, then

$$e_{H_1} e_{H_2} = \left(\sum_{s \in \mathcal{R} \ \mid \ s(H_2) = H_1} \mu_s s \right) e_{H_2} = e_{H_1} \left(\sum_{s \in \mathcal{R} \ \mid \ s(H_2) = H_1} \mu_s s \right)$$

Remark 2.1. *A natural attempt to a further generalization would be to consider relations of the form $e_H^2 = \delta_H e_H$ for scalars δ_H depending on H. By the semi-direct type relations such a scalar should depend only on the orbit of H under W. But then, up to rescaling the coefficients $\underline{\mu}$ we can assume all such δ_H's are the same, at least when all of them are nonzero. Moreover, if some of them are zero this provides quotients of the orginial algebra by relations of the form e_H^2 so this also fits inside the original framework.*

2.2. **Chen's additional relations.** In the relations above, we removed one relation from Chen's original definition, the relation called (1)' in [6].

$$(1)' \quad we_H = e_H w = e_H \quad \text{if } w(H) = H$$

$$\text{and } \exists H_1, H_2 \in \mathcal{A} \quad \mathrm{Ker}(w-1) \cap H = H_1 \cap H_2 \text{ and } H_1, H_2 \text{ are not transverse.}$$

It is claimed without proof in [6] that this condition is equivalent to the following one

$(1)'' \quad se_H = e_H s = e_H \quad \text{if } s \in \mathcal{R}, s(H) = H \text{ and } H \text{ and } \mathrm{Ker}(s-1) \text{ are (distinct and) not transverse.}$

Actually, the preprint version of [6] on the arxiv (`arXiv:1102.4389v1`) is the version of (1)' given here, and the claim that (1)' and (1)'' are equivalent can be found only in the published version. But on the other hand, in the published version of (1)' the element w is additionally assumed to be a reflection, which makes (1)' not only *equivalent*, but *formally equal* to (1)''. So we have to interprete this addition to be a typo (corroborated by some examples in Chen's paper).

The implication $(1)' \Rightarrow (1)''$ is because, if $s \in \mathcal{R}$ is such that $s(H) = H$ and H and $H_s = \mathrm{Ker}(s-1)$ are not transverse, then, setting $H_1 = H_s$, $H_2 = H$ and $w = s$, we get $\mathrm{Ker}(w-1) \cap H = H_1 \cap H_2$ and by (1)' the conclusion. Conversely, if w, H_1, H_2 are as in the assumption of (1)', then w belongs to the parabolic subgroup fixing $H_1 \cap H_2$. If this rank 2 parabolic subgroup is a dihedral group, then we have the conclusion because

(1) either w is a reflection, and we have the conclusion by (1)''
(2) otherwise, setting s the reflection w.r.t. H, we have that ws is a reflection satisfying the same assumptions, and writing $w = ws.s$ we get the conclusion by applying (1)' twice.

Therefore, (1)' and (1)'' are equivalent in a number of cases, including all Coxeter groups. It is however not true, in general that (1)'' implies (1)'. In order to check this, we consider the complex reflection group W of type $G(4, 2, 2)$ (see section 7.1 below), and apply to the given presentations a Gröbner basis algorithms, using the GAP4 package GBNP (see [11]), for a given value of $\delta \in \mathbf{k} = \mathbb{Q}$. We obtain for dimensions 28 and 40 depending whether we add (1)' or (1)'', and 64 without both of them.

In [6] §9, it is argued that a reason for relation (1)' to be added is that it may be closer to a previously introduced algebra in Coxeter type B and more generally in type $G(d, 1, n)$ (see [13]). Our purpose here being to consider the largest possible finite-dimensional algebra we consider it better to eliminate this. As noted by Chen, this condition is void in the usual (type A) case, so we get indeed a generalization of the usual Brauer algebra.

2.3. **Flat connection.** In any case, we reprove proposition 5.1 of [6] in order to make it clear that these additional relations are not needed for the associated connection to be flat. Actually, we notice that our relation (1) is not needed either. Therefore, we let $Br^0(W)$ denote the algebra defined as $Br(W)$ but with relation (1) removed. We remark that this algebra

is actually defined over $\mathbb{Z}[\underline{\mu}]$ and has infinite rank. To every H hyperplane we associate the logarithmic 1-form $(1/\pi i)d\alpha_H/\alpha_H$ where α_H is an arbitrary linear form with kernel H.

Proposition 2.2. *(Chen) Assume $\mathbf{k} = \mathbb{C}$. Then, the following 1-form*

$$\omega = \sum_{H \in \mathcal{A}} \left(\left(\sum_{Ker(s-1)=H} \mu_s s \right) - e_H \right) \omega_H \in \Omega^1(X) \otimes Br^0(W)$$

is integrable and W-equivariant.

Proof. We let $t_H = \left(\sum_{Ker(s-1)=H} \mu_s s \right) - e_H$. It is clear that $w t_H w^{-1} = t_{w(H)}$ hence we only need to prove that Kohno's holonomy relations of [14] are satisfied. We recall these relations now. Let Z be a codimension 2 flat, and $t_Z = \sum_{H \supset Z} t_H$. One needs to prove $[t_Z, t_H] = 0$ for all $H \supset Z$. If Z is what Chen calls a crossing edge, that is if it is contained in exactly two hyperplanes H_1 and H_2, then it is clear that all the elements involved in t_{H_1} and t_{H_2} commute with each other, whence $[t_{H_1}, t_{H_2}] = 0$ hence $[t_Z, t_H] = 0$ for all $H \supset Z$.

If not, letting $\varphi_H = \sum_{Ker(s-1)=H} \mu_s s$ and $\varphi_Z = \sum_{H \supset Z} \varphi_H$, we first notice that $[\varphi_H, \varphi_Z] = 0$ from the integrability of the Cherednik connection (see e.g. [5]). Let us pick $H_0 \supset Z$. We have

$$[t_{H_0}, t_Z] = [\varphi_{H_0} - e_{H_0}, \varphi_Z - \sum_H e_H] = -[\varphi_{H_0}, \sum_H e_H] - [e_{H_0}, \varphi_Z - \sum_H e_H].$$

But, for all $s \in \mathcal{R}$ with $Ker(s-1) \supset Z$, we have $s(\sum_H e_H) = (\sum_H e_H)s$ hence $[\varphi_{H_0}, \sum_H e_H] = 0$. It remains to compute $[e_{H_0}, \varphi_Z - \sum_H e_H]$. We have

$$e_{H_0} \sum_H e_H = \sum_H e_{H_0} e_H = \sum_H e_{H_0} \sum_{s(H)=H_0} \mu_s s = e_{H_0} \sum_{s \in \mathcal{R}; Ker(s-1) \supset Z} \mu_s s = e_{H_0} \varphi_Z$$

and similarly $(\sum_H e_H) e_{H_0} = \varphi_Z e_{H_0}$ hence

$$[e_{H_0}, \varphi_Z - \sum_H e_H] = 0$$

and this proves the claim.

$\square$

2.4. **Finiteness of dimension.** Chen proved that his algebra has finite rank as a $\mathbf{k}$-module. Actually, following the same lines of proof, one can prove the following more general statement.

Proposition 2.3. *Let $\mathbf{k}$ be a commutative ring with 1, $\underline{Q} = (Q_H)_{H \in \mathcal{A}} \in k[X]$ such that $H_1 \sim H_2 \Rightarrow Q_{H_1} = Q_{H_2}$. Then the quotient $Br^{\underline{Q}}(W)$ of $Br^0(W)$ by the relations $e_H Q_H(e_H) = 0$ has finite rank over $\mathbf{k}$.*

Proof. By the semidirect product relations it is clear that every element $Br^0(W)$ is a linear combinations of terms of the form $w e_{H_1} \ldots e_{H_r}$ for $w \in W$. If m is the maximum of the degrees of the Q_H, we claim that we need no term with r larger than $m \, rk(W)$. We argue by contradiction, and consider a term with r minimal but greater than $rk(W) \times m$ which can not be rewritten using smaller r's. First of all, for every $i < r$ we have that the H_i and H_{i+1} are either equal or transverse, by minimality of r and relation (3). In particular, H_i commutes

with H_{i+1}, and actually (by induction) with every H_j for $j > i$ for the same reason. Therefore we can assume that

$$(H_1, \ldots, H_r) = (\underbrace{J_1, \ldots, J_1}_{u_1}, \ldots, \underbrace{J_k, \ldots, J_k}_{u_k})$$

with $\{J_1, \ldots, J_k\}$ of cardinality k, and in particular a transverse collection of hyperplanes. By the polynomial relation on the e_H's and the minimality of r, we have $u_i \le m$ for all i, hence $r \le m \times k$. Let us pick for each i a nonzero vector in the orthogonal of J_i (with respect to some unitary form preserved by W). For $i \ne j$ we have that J_i and J_j are transverse, and thus v_i and v_j are orthogonal. Therefore the v_i form an orthogonal family of cardinality k, thus $k \le \mathrm{rk}(W)$ and $r \le m \times \mathrm{rk}(W)$, a contradiction. This proves the claim. $\square$

When **k** is a field, it is clear that all finite-dimensional representation of $Br^0(W)$ factorize through $Br^{\underline{Q}}(W)$ for some $\underline{Q}$. Moreover, note that, picking one root of Q_H for each conjugacy class of hyperplanes provide a surjective morphism from $Br^{\underline{Q}}(W)$ to $Br(W)$ with parameter(s) corresponding to the root(s). Actually, from the proof of the classification of the irreducible representations of $Br_1(W)$ given below, it will be clear that all the irreducible representations of $Br_{\underline{1}}^{\underline{Q}}(W)$ (over an algebraically closed field) factor through one of them.

2.5. **A generalized Vogel algebra.** Finally, we prove here that, when W is a 2-reflection group, these algebras appear as quotients of the algebra $\mathfrak{V}(W)$ defined by generators $t_H, H \in \mathcal{A}$, $w \in W$, and relations the relations of W together with the W-invariance and holonomy relations

$$wt_H = t_{w(H)}w, \quad \left[t_H, \sum_{H' \supset Z} t_{H'}\right] = 0$$

where Z runs among the codimension 2 flats, and

$$\begin{cases} t_H s_H = s_H t_H = t_H \\ t_H^2 - (\alpha + \beta)t_H + \frac{\alpha\beta}{2}(1 + s_H) = 0 \end{cases}$$

for all $H \in \mathcal{A}$. This algebra is a generalization of an algebra introduced by P. Vogel in the framework of Vassiliev invariants, see [19].

An intriguing open question is whether these algebras are finite dimensional in general. Is is conjectured to be the case when $W = \mathfrak{S}_n$, and known to be true for $n \le 5$ by [19].

When W is a finite Coxeter group with generating set S, another presentation of $\mathfrak{Q}(W)$ is easily seen to be given by generators $t_H, H \in \mathcal{A}$, $s \in S$, together with the Coxeter relations, the holonomy relations, $st_u = t_{sus}s$ for $s \in S$ and $u \in \mathcal{R}$, and

$$\begin{cases} t_s s = s t_s = t_s \\ t_s^2 - (\alpha + \beta)t_s + \frac{\alpha\beta}{2}(1 + s) = 0 \end{cases}$$

for all $s \in S$.

Since W is a 2-reflection group, in the definition of the Brauer-Chen algebra one can set $\lambda_H = \mu_{s_H}$ for s_H the only reflection with hyperplane H. Let $t_H = \lambda_H(1 + s_H) - e_H$. Then it is readily checked that $t_H s_H = t_H$, and $t_H^2 = 2\lambda_H^2(1 + s_H) + xe_H - 4\lambda_H e_H = 2\lambda_H^2(1 + s_H) + (x - 4\lambda_H)e_H$. Then $t_H^2 + (x - 4\lambda_H)t_H = 2\lambda_H^2(1 + s_H) + (x - 4\lambda_H)e_H + (x - 4\lambda_H)\lambda_H(1 + s_H) - (x - 4\lambda_H)e_H = (x - 2\lambda_H)\lambda_H(1 + s_H)$, hence $t_H^2 - (\alpha + \beta)t_H + \frac{\alpha\beta}{2}(1 + s_H) = 0$ with $\alpha = 2\lambda_H - x$ and $\beta = 2\lambda_H$.

Therefore we get the following.

Proposition 2.4. *If W is a 2-reflection group and $\mu_{s_H} = \lambda_H$, $\alpha = 2\lambda_H - x$ and $\beta = 2\lambda_H$, then there exists a surjective morphism $\mathfrak{Q}(W) \twoheadrightarrow Br(W)$ which is the identity on W and maps $t_H \mapsto \lambda_H(1 + s_H) - e_H$.*

2.6. **Lattice extensions.** We operate a mixture of these ideas together with the ones of [17, 18]. Let $\mathcal{L}$ denote an admissible lattice in the sense of [18], that is a W-invariant sublattice of the lattice of all full reflection subgroups of W, where full means that if $s \in W$ is a reflection, then W contains all the reflections fixing $\mathrm{Ker}(s-1)$, with the following properties of containing the cyclic (full) reflection subgroups and the trivial subgroups.

We consider its Möbius algebra $\mathbf{k}\mathcal{L}$. The following proposition was proved in [17] under the additionnal unnecessary assumption that $\mathcal{L}$ is the lattice of all parabolic subgroups. Here we provide the general proof.

Proposition 2.5. *Assume $\mathbf{k} = \mathbb{C}$. Let us choose a collection of scalars $\lambda_H, H \in \mathcal{A}$ such that $\lambda_{w(H)} = \lambda_H$ for all $w \in W$. Then the following 1-form*

$$\omega = \sum_{H \in \mathcal{A}} \left(\lambda_H + \sum_{Ker(s-1)=H} \mu_s s \right) f_H \omega_H \in \Omega^1(X) \otimes \mathbb{C}W \ltimes \mathbb{C}\mathcal{L}$$

is integrable and W-equivariant.

Proof. Let us assume that we have picked a 'distinguished' reflection s_H for each $H \in \mathcal{A}$ with the property that $w s_H w^{-1} = s_{w(H)}$ for all $w \in \mathcal{A}$ and $\langle s_H \rangle = W_H$. We denote $\lambda_H^{(0)} = \lambda_H$ and $\lambda_H^{(i)} = \mu_{s_H}^i$ for $i > 0$ and $H \in \mathcal{A}$. We set $t_H = \sum_{0 \leq i < m_H} \lambda_H^{(i)} s_H^i f_H$, with m_H the order of s_H. Let Z be a codimension 2 flat, and $t_Z = \sum_{H \supset Z} t_H$. One needs to prove $[t_Z, t_{H_0}] = 0$ for all $H \supset Z$. For this we only need to prove that $[s f_{H_0}, t_Z] = 0$ for $s \in \langle s_H \rangle$. We do this. We have

$$s f_{H_0} t_Z = \sum_{\substack{H \supset Z \\ 0 \leq i < m_H}} s f_{H_0} \lambda_H^{(i)} s_H^i f_H = \sum_{\substack{H \supset Z \\ 0 \leq i < m_H}} \lambda_H^{(i)} s s_H^i f_{s_H^{-i}(H_0)} f_H$$

and

$$t_Z s f_{H_0} = \sum_{\substack{H \supset Z \\ 0 \leq i < m_H}} \lambda_H^{(i)} s_H^i f_H s f_{H_0} = \sum_{\substack{H \supset Z \\ 0 \leq i < m_H}} \lambda_H^{(i)} s (s^{-1} s_H s)^i f_{s^{-1}(H)} f_{H_0}$$

$$= \sum_{\substack{H \supset Z \\ 0 \leq i < m_H}} \lambda_{s^{-1}(H)}^{(i)} s s_{s^{-1}(H)}^i f_{s^{-1}(H)} f_{H_0} = \sum_{\substack{H \supset Z \\ 0 \leq i < m_H}} \lambda_H^{(i)} s s_H^i f_H f_{H_0}$$

and so we only need to check that $f_H f_{H_0} = f_{s_H^{-i}(H_0)} f_H$. It is sufficient to prove this for the maximal admissible lattice $\mathcal{L} = \mathcal{L}_\infty$ of all full reflection subgroups. In this case, $f_H f_{H_0} = f_G$ for G the smallest full reflection subgroup containing $\langle s_H, s_{H_0} \rangle$. Since $\langle s_{s_H^{-i}(H_0)}, s_H \rangle = \langle s_H^{-i} s_{H_0} s_H^i, s_H \rangle = \langle s_{H_0}, s_H \rangle$ we indeed get $f_H f_{H_0} = f_{s_H^{-i}(H_0)} f_H$, and this proves the claim, W-invariance being obvious.

$\square$

We now denote $Br^0(W, \mathcal{L})$ the algebra presented by generators $w \in W$, $e_H, H \in \mathcal{A}$, $f_L, L \in \mathcal{L}$, together with the relations

- $w = w_1 w_2$ if $w = w_1 w_2$ inside W
- $w e_H = e_{w(H)} w$
- $H_1 \pitchfork H_2 \Rightarrow e_{H_1} e_{H_2} = e_{H_2} e_{H_1}$

- If H_1 and H_2 are distinct and not transverse, then

$$e_{H_1}e_{H_2} = \left(\sum_{s \in \mathcal{R} \ | \ s(H_2)=H_1} \mu_s s f_{H_s} \right) e_{H_2} = e_{H_1} \left(\sum_{s \in \mathcal{R} \ | \ s(H_2)=H_1} \mu_s s f_{H_s} \right)$$

- $e_H f_L = f_L e_H$
- $e_H f_H = f_H e_H = e_H$
- $f_{L_1} f_{L_2} = f_{L_1 \vee L_2}$

The following then provides an upgrading of both propositions 2.2 and 2.5.

Theorem 2.6. *Assume* $\mathbf{k} = \mathbb{C}$. *Let us choose a collection of scalars* $\lambda_H \in \mathbb{C}$, $H \in \mathcal{A}$ *such that* $\lambda_{w(H)} = \lambda_H$ *for all* $w \in W$. *Then the following 1-form*

$$\omega = \sum_{H \in \mathcal{A}} \left(\left(\lambda_H + \sum_{Ker(s-1)=H} \mu_s s \right) - e_H \right) f_H \omega_H \in \Omega^1(X) \otimes Br^0(W, \mathcal{L})$$

is integrable and W-*equivariant.*

Proof. As in the proof of proposition 2.2, we start by setting $\varphi_H = \lambda_H + \sum_{Ker(s-1)=H} \mu_s s$, $t_H = (\varphi_H - e_H) f_H$. It is clear that $w t_H w^{-1} = t_{w(H)}$ hence we only need to prove that Kohno's relations are satisfied. Let Z be a codimension 2 flat, and $t_Z = \sum_{H \supset Z} t_H$. One needs to prove $[t_Z, t_H] = 0$ for all $H \supset Z$. If Z is a crossing edge, that is it is contained in exactly two hyperplanes H_1 and H_2, then it is clear that all the elements involved in t_{H_1} and t_{H_2} commute with each other, whence $[t_{H_1}, t_{H_2}] = 0$ hence $[t_Z, t_H] = 0$ for all $H \supset Z$.

If not, letting $\psi_Z = \sum_{H \supset Z} \varphi_H f_H$, let us pick $H_0 \supset Z$. We first notice that $[\varphi_H f_H, \psi_Z] = 0$ from proposition 2.5.

We have

$$[t_{H_0}, t_Z] = [\varphi_{H_0} f_{H_0} - e_{H_0}, \psi_Z - \sum_H e_H] = -[\varphi_{H_0} f_{H_0}, \sum_H e_H] - [e_{H_0}, \psi_Z - \sum_H e_H]$$

But, for all $s \in \mathcal{R}$ with $H_0 = Ker(s-1) \supset Z$, we have

$$s f_{H_0}(\sum_H e_H) = f_{H_0} s(\sum_H e_H) = f_{H_0}(\sum_H e_H)s = (\sum_H e_H)s f_{H_0}$$

hence $[\varphi_{H_0}, \sum_H e_H] = 0$. It remains to compute $[e_{H_0}, \psi_Z - \sum_H e_H]$. We have

$$e_{H_0} \sum_H e_H = \sum_H e_{H_0} e_H = \sum_H e_{H_0} \sum_{s(H)=H_0} \mu_s f_{H_s} s = e_{H_0} \sum_{s \in \mathcal{R}; Ker(s-1) \supset Z} \mu_s f_{H_s} s = e_{H_0} \left(\psi_Z - \sum_{H \supset Z} \lambda_H f_H \right)$$

and similarly $(\sum_H e_H) e_{H_0} = (\psi_Z - \sum_{H \supset Z} \lambda_H f_H) e_{H_0}$ hence

$$[e_{H_0}, \psi_Z - \sum_H e_H] = 0$$

and this proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that the quotient of the $Br^0(W, \mathcal{L})$ by the relations $e_H = 0$ provides the semidirect product $\mathbf{k}W \ltimes k\mathcal{L}$, while the quotient by the relations $f_H = 1$ provides $Br^0(W)$. We can similarly introduce the algebras $Br^{\underline{Q}}(W, \mathcal{L})$ and in particular $Br(W, \mathcal{L})$ by imposing the relations

$e_H^2 = \delta e_H$ for $H \in \mathcal{A}$. By a straightforward adaptation of its proof, one gets the following analog of proposition 2.3.

Proposition 2.7. *Let* **k** *be a commutative ring with* 1, $\underline{Q} = (Q_H)_{H \in \mathcal{A}} \in k[X]$ *such that* $H_1 \sim H_2 \Rightarrow Q_{H_1} = Q_{H_2}$. *Then the quotient* $Br^{\underline{Q}}(W, \mathcal{L})$ *of* $Br^0(W, \mathcal{L})$ *by the relations* $e_H Q_H(e_H) = 0$ *has finite rank over* **k**.

The existence of this flat connection raises the following question :

Question 2.8. *When* $W = \mathfrak{S}_n$ *and* $\mathcal{L}$ *is the lattice of all reflection subgroups of* W, *does this monodromy representation of the braid group over* $Br(W, \mathcal{L})$ *provide the braid group representations factoring through the tied-BMW algebra of Aicardi and Juyumaya (see* [1]*) ?*

3. The ideals I_r

To any transverse collection $\underline{H} = \{H_1, \dots H_r\}$ we associate $e_{\underline{H}} = e_{H_1} \dots e_{H_r} \in Br(W)$. We denote I_r for $r \geq 1$ the left ideal of $Br(W)$ generated by the $e_{\underline{H}}$ for $\underline{H}$ of cardinality r (or equivalently, at least r). We first prove

Lemma 3.1. *Each* I_r *is a two-sided ideal of* $Br(W)$.

Proof. Let $\underline{H}$ be a transverse collection of cardinality r. One needs to prove that $e_{\underline{H}} w \in I_r$ and $e_{\underline{H}} e_K \in I_r$ for all $w \in W$ and $K \in \mathcal{A}$. One readily checks that $e_{\underline{H}} w = e_{w^{-1}(\underline{H})}$ and that $w^{-1}(\underline{H})$ is again a transverse collection, so this proves $e_{\underline{H}} w \in I_r$ and more generally $e_{\underline{H}} \mathbb{Q}[x] w \subset I_r$. We now consider $e_{\underline{H}} e_K$. If $K \in \underline{H}$ then clearly $e_{\underline{H}} e_K = x e_{\underline{H}} \in I_r$. Otherwise, if K is transverse to every hyperplane in $\underline{H}$ we have $e_{\underline{H}} e_K = e_{\underline{H} \cup \{K\}} = e_K e_{\underline{H}} \in I_r$. If not, then $\underline{H} = \underline{J} \cup \{L\}$ with L not transverse to K. But then $e_L e_K \in e_L . \mathbb{Q}[x] W$ hence $e_{\underline{H}} e_K \subset e_{\underline{H}} \mathbb{Q}[x] W \subset I_r$ as we already proved. This proves the claim. $\square$

We denote $Br_r(W) = Br(W)/I_{r+1}$, so in particular $Br_0(W) = \mathbb{Q}[\delta] w$.

For any transverse collection $\underline{H}$, let $W_{\underline{H}}$ denote the subgroup generated by all the reflections w.r.t. an hyperplane in $\underline{H}$. For $w \in W_{\underline{H}}$, we have $w e_{\underline{H}} = e_{\underline{H}}$, hence for arbitrary $w \in W$ the element $w e_{\underline{H}}$ depends only on the class of w modulo $W_{\underline{H}}$. Therefore, $c e_{\underline{H}}$ is well-defined for $c \in W/W_{\underline{H}}$.

Proposition 3.2. *The collection of the* $cW_{\underline{H}}$ *for* $\underline{H}$ *a (possibly empty) transverse collection of hyperplanes and* $c \in W/W_{\underline{H}}$ *is a spanning set for* $Br(W)$.

Proof. By the above remark it is equivalent to say that the collection of the $w e_{\underline{H}}$ for $w \in W$ and $\underline{H}$ a transverse collection form a spanning set. For this we need to prove that multiplying on the left such elements by the $g \in W$ and e_K for $K \in \mathcal{A}$ can be written as a linear combination of such elements. Since this is clear for $g \in W$, we look at $e_K w e_{\underline{H}} = w e_{w^{-1}(K)} e_{\underline{H}}$ and we finally need to prove that $e_J e_{\underline{H}}$ is equal to such a linear combination. The proof is then similar to the one of the previous lemma : if K is transverse to all hyperplanes inside $\underline{H}$ then $e_K e_{\underline{H}} = e_{\underline{H} \cup \{K\}}$ and we are done, the case $K \in \mathcal{H}$ is also clear, and otherwise we can write $\underline{H} = \{L\} \cup \underline{J}$ with L not transverse to K and

$$e_K e_{\underline{H}} = e_K e_L e_{\underline{J}} \in \mathbb{Q}[\delta] W e_L e_{\underline{J}} \subset \mathbb{Q}[\delta] W e_{\underline{H}}$$

and we are done. $\square$

As a corollary, when W is a 2-reflection group admitting a single conjugacy class of reflections, then $Br_1(W)$ is spanned by a family of cardinality

$$|W| + |\mathcal{R}| \times |W|/2$$

Now consider the case of a Coxeter group with generating set S. One could consider the ideal J_2 generated by the $e_r e_s$ for $r, s \in S$ being non-adjacent nodes of the Coxeter diagram. It is clear that $J_2 \subset I_2$. Conversely, if $\underline{H} = (H_1, H_2)$ is a transverse collection, the parabolic subgroup W_Z fixing $Z = H_1 \cap H_2$ is generated by s_{H_1} and s_{H_2}. It has for conjugate a standard parabolic $\langle r_1, r_2 \rangle$ with $r_1, r_2 \in S$, $s_{H_i} = w r_i w^{-1}$ for $i = \{1, 2\}$ and some $w \in W$. Since $\mathrm{Ker}(r_1 - 1)$ and $\mathrm{Ker}(r_2 - 1)$ are also transverse we have that r_1 and r_2 are non-adjacent in the Coxeter graph hence $e_{r_1} e_{r_2} \in J_2$. But then $e_{H_1} e_{H_2} = w e_{r_1} e_{r_2} w^{-1} \in J_2$ hence $I_2 \subset J_2$ and this proves that $I_2 = J_2$. This provides a sometimes more handy description of $Br_1(W)$ in the Coxeter case.

4. Connexions with the Cohen-Frenk-Wales algebra

In [8], Cohen, Frenk and Wales associated a Brauer algebra to any Coxeter group of type ADE as follows. If (W, S) is such a Coxeter system, it is defined by generators $r \in S$, $e_r, r \in S$ and the Coxeter relations on S together with the following ones :

- (RSre) $r e_r = e_r$
- (RSer) $e_r r = e_r$
- (HSee) $e_r^2 = \delta e_r$
- (HCee) if $r \not\sim s$ then $e_r e_s = e_s e_r$
- (HCer) if $r \not\sim s$ then $e_r s = s e_r$
- (RNrre) if $r \sim s$ then $r s e_r = e_s e_r$
- (HNrer) if $r \sim s$ then $s e_r s = r e_s r$

We denote this algebra by $BrCFW(W)$. There is a natural injective map $W \to BrCFW(W)$ mapping S to S identically.

For every $t \in \mathcal{R}$ there is $s \in S$ and $w \in W$ such that $t = w s w^{-1}$.

We replace (RNrre) and (HNrer) by the following two conditions

- (RNrre') if $r \sim s$ then $r s r e_s = e_r e_s$
- (HNrer') if $r \sim s$ then $r s r e_s = e_r s r s$

Under the other conditions, it is readily checked that (RNrre) is equivalent to (RNrre') and that (HNrer) is equivalent to (HNrer'). From this it can be shown (see [6]) that this algebra is isomorphic to the Brauer-Chen algebra $Br(W)$.

The irreducible representations on which I_2 vanishes which are described in [8] are indexed by a W-orbit of positive roots – which can be identified to a W-orbit of reflecting hyperplanes – together with an irreducible character of a subgroup (called $W(C)$ in [8]) of W, which is shown ([8], proposition 4.7) to be a complement of the parabolic subgroup W_0 fixing a given hyperplane H_0 inside the normalizer $N_W(W_0) = \{w \in W \mid w(H_0) = H_0\}$ of W_0, and is therefore isomorphic to $N(W_0)/W_0$. This complement is described as the reflection subgroup generated by the reflections associated to the roots orthogonal to the highest one in the W-orbit under consideration. The construction of the representations is based on the root system. We show in the next section that these constructions can be made and generalized in a way independent of the chosen root system to arbitrary complex reflection groups.

5. Representations of $Br_1(W)$

In this section we denote $\boldsymbol{\kappa}$ a field of characteristic 0, and we assume $\mathbf{k} = \boldsymbol{\kappa}(\delta)$ is the field of rational fractions in δ.

5.1. Preliminaries.

We will need the following easy lemmas, for which we could not find a convenient reference.

Lemma 5.1. *Let $K_0 \subset K$ be a field extension. Then all elements of $K(\delta)$ which are algebraic over K_0 belong to K.*

Proof. Let $x \in K(\delta)$, that we write $x = p/q$ with $p, q \in K_0[\delta]$ coprime. Assume that there exists $P = a_0 + a_1 X + \cdots + X^n \in K_0[X] \setminus \{0\}$ such that $P(x) = 0$ with $a_i \in K_0$. We can assume that P is irreducible, and that $a_0 \neq 0$. This means

$$a_0 q^n + a_1 p q^{n-1} + a_2 p^2 q^{n-2} + \cdots + p^n = 0$$

hence p divides q^n, contradicting the coprimality of p and q, except if p or q belong to $\mathbb{Q}$. But then this forces both to belong to K_0 by considering the degree of the above expression. This proves the claim. $\qquad\square$

Lemma 5.2. *Let K be a field of characteristic 0, G a finite group. Then the map $M \mapsto M \otimes_K K(t)$ induces a bijection between isomorphism classes of KG-modules and $K(t)G$-modules.*

Proof. Recall (from e.g. prop. 1 of [3], ch. V, annexe) the very general fact that $M_1 \simeq M_2 \Leftrightarrow M_1 \otimes_K K(t) \simeq M_2 \otimes_K K(t)$, so this map is injective. Since both KG and $K(t)G$ are semisimple algebras over their base field, it is then sufficient to prove that every irreducible representation $\rho : G \to \mathrm{GL}_n(K(t))$ is isomorphic to the extension $\tilde{\rho}_0$ to $K(t)$ of a representation $\rho_0 : G \to \mathrm{GL}_n(K)$. Since G is finite and K is infinite there exists $t_0 \in K$ such that all entries of the $\rho(g)$, $g \in G$ can be specialized at t_0 and such that t_0 is not a root of the $\det \rho(g)$, $g \in G$. Denote $\rho_0 : G \to \mathrm{GL}_n(K)$ the corresponding specialization. Now notice that the $\chi(g) = \mathrm{tr}\rho(g) \in K(t)$ are algebraic over $\mathbb{Q} \subset K$ hence belong to K by the previous lemma. Therefore the character of ρ is equal to the character of $\tilde{\rho}_0$ hence $\rho \simeq \tilde{\rho}_0$ and this proves the claim. $\qquad\square$

5.2. A direct presentation for $Br_1(W)$.

The algebra $Br_1(W) = Br(W)/I_2$ admits a more tractable presentation with the same generators, and for relations the relations of W, the semidirect type relations, and the following ones :

- (1) $\forall H \in \mathcal{A}$ $e_H^2 = \delta e_H$ & $e_H s = s e_H = e_H$ whenever $\mathrm{Ker}(s-1) = H$
- (3_1) If H_1 and H_2 are distinct, then

$$e_{H_1} e_{H_2} = \left(\sum_{s \in \mathcal{R} \ \mid \ s(H_2)=H_1} \mu_s s \right) e_{H_2} = e_{H_1} \left(\sum_{s \in \mathcal{R} \ \mid \ s(H_2)=H_1} \mu_s s \right)$$

In other terms, the defining relations (2) and (3) of $Br(W)$ together with the defining relations of I_2 are replaced by (3_1).

Indeed, relation (2) modulo I_2 means $H_1 \pitchfork H_2 \Rightarrow e_{H_1} e_{H_2} = 0$, and this is equivalent to asserting (3_1) for H_1 and H_2, since there are no reflection mapping H_1 to H_2 when H_1 and H_2 are transverse (see [16], lemma 3.1).

5.3. **The $Br_1(W)$-modules of the form $\tilde{M}$: definition.** Let $H_0 \in \mathcal{A}$, $W_0 = W_{\underline{H_0}} = \langle s_0 \rangle$ be the pointwise stabilizer of H_0. We set

$$N_0 = \{w \in W; w(H_0) = H_0\} = N_W(W_0)$$

the normalizer of W_0 in W. Let M be a $\boldsymbol{\kappa} N_0$-module on which W_0 acts trivially. Then $\tilde{M} = \mathbf{k}W \otimes_{\mathbf{k}N_0} M_{\mathbf{k}}$ with $M_{\mathbf{k}} = \mathbf{k} \otimes_{\boldsymbol{\kappa}} M$ is a $\mathbf{k}W$-module which is the induced representation of $M_{\mathbf{k}}$. Let $\mathcal{A}_0 = W.H_0 \subset \mathcal{A}$ denote the orbit of H_0 under W. For all $H \in \mathcal{A}_0$ we pick $g_H \in W$ such that $g_H(H_0) = H$. The set $\mathcal{G}_0$ of all such g_H is a set of representatives of W/N_0, hence $\tilde{M}$ admits a direct sum decomposition

$$\tilde{M} = \bigoplus_{H \in \mathcal{A}_0} V_H$$

with $V_H = g_H \otimes M_{\mathbf{k}}$ such that $wV_H \subset V_{w(H)}$ for all $w \in W$. Notice that $V_0 = V_{H_0}$ is canonically identified to M as a N_0-module, under the natural inclusion $N_0 \subset W$.

We define $p_0 = p_{H_0} \in \mathrm{End}(\tilde{M})$ by

$$p_{H_0}(x) = \sum_{u(H)=H_0} \mu_u u.x$$

when the u's are understood to be reflections and $x \in V_H$, except when $H = H_0$ in which case $p_{H_0}(x) = \delta x$.

Lemma 5.3.

(1) For all $w \in N_0$ and $x \in \tilde{M}$ we have $wp_0.x = p_0 w.x$
(2) $p_0(\tilde{M}) \subset V_{H_0}$
(3) $p_0^2 = \delta p_0$

Proof. We first prove (1). For $x \in V_{H_0}$ this is immediate. For $H \in \mathcal{A} \setminus \{H_0\}$ we have, for $w \in N_0$ and $y \in M$,

$$wp_0.(g_H \otimes y) = w \sum_{u(H)=H_0} \mu_u u.(g_H \otimes y) = \sum_{u(H)=H_0} \mu_u wug_H \otimes y$$

while

$$p_0 w.g_H \otimes y = p_0.(wg_s) \otimes y = \sum_{vw(H)=H_0} \mu_v (vwg_H) \otimes y = \sum_{wv^w(H)=H_0} \mu_v (wv^w g_H) \otimes y$$

$$= \sum_{v^w(H)=H_0} \mu_{v^w} wv^w g_H \otimes y$$

and this proves the claim, as $v \mapsto v^w$ is a bijection of $\mathcal{R}$. We now prove (2). For all u with $u(H) = H_0$ we have $u.(g_H \otimes y) = ug_H \otimes y$ and $ug_H(H_0) = u(H) = H_0$ hence $ug_H \in \mathbb{C}N_0$ and this proves the claim. (3) is then an immediate consequence. $\square$

Lemma 5.4. *For $w \in W$, and $H = w(H_0)$, then $p_H \in \mathrm{End}(\tilde{M})$ defined by $p_H.x = wp_0w^{-1}.x$ depends only on H. If $H \in \mathcal{A} \setminus \mathcal{A}_0$ is not a conjugate of H_0 then we set $p_H = 0$. Then, if $H_1, H_2 \in \mathcal{A}$ satisfy $H_2 = w(H_1)$ for some $w \in W$, then $wp_{H_1}w^{-1}.x = p_{H_2}.x$ for all $x \in \tilde{M}$.*

Proof. Assume that $w_1, w_2 \in W$ satisfy $w_1(H_0) = w_2(H_0)$. Then $w = w_2^{-1} w_1 \in N_0$ hence $w p_0 w^{-1}.x = x$ for all $x \in \tilde{M}$ by the previous lemma (1) and this implies $w_1 p_0 w_1^{-1}.x = w_2 p_0 w_2^{-1}.x$ for all $x \in \tilde{M}$ and this proves the first part of the lemma. For the second part, if $H_2 \notin \mathcal{A}_0$ then $H_1 \notin \mathcal{A}_0$ too, hence $p_{H_1} = p_{H_2} = 0$ satisfy the property. If not, let $w_0 \in W$ such that $H_1 = w_0(H_0)$. Then $H_2 = w(H_1) = (w w_0)(H_0)$. It follows that, for all $x \in \tilde{M}$, we have $p_{H_2}.x = (w w_0) p_0 (w w_0)^{-1}.x = w(w_0 p_0 w_0^{-1}) w^{-1}.x = w p_{H_1} w^{-1}.x$ and this proves the claim. $\qquad\square$

A consequence of the definition is that, for $w(H_0) = H_1$, we have

$$p_{H_1}(\tilde{M}) = w p_{H_0} w^{-1}(\tilde{M}) = w p_{H_0}(\tilde{M}) = w V_{H_0} = V_{w(H_0)} = V_{H_1}.$$

Since, on V_{H_2}, p_0 coincides with the action of $\sum_{u(H_2)=H_0} \mu_u u$, this implies that

$$p_0 p_{H_2} = \left(\sum_{u(H_2)=H_0} \mu_u u \right) p_{H_2}$$

for all $H_2 \in \mathcal{A}$. Then, since $H = w(H_0)$, we have

$$p_{H_1} p_{H_2} = p_{w(H_0)} p_{H_2} = w p_{H_0} w^{-1} p_{H_2} = w p_{H_0} p_{w^{-1}(H_2)} w^{-1}$$

which is equal by the above to

$$w \left(\sum_{v w^{-1}(H_2)=H_0} \mu_v v p_{w^{-1}(H_2)} \right) w^{-1} = \sum_{v w^{-1}(H_2)=H_0} \mu_v w v w^{-1} p_{H_2}$$

$$= \sum_{{}^w v(H_2)={}^w(H_0)} \mu_{{}^w v}{}^w v p_{H_2} = \sum_{u(H_2)=H_1} \mu_u u p_{H_2}$$

where as usual u, v are assumed to belong to $\mathcal{R}$, and this proves

$$p_{H_1} p_{H_2} = \left(\sum_{u(H_2)=H_1} \mu_u u \right) p_{H_2}$$

for all $H_1, H_2 \in \mathcal{A}_0$.

We now want to prove that

$$p_{H_1} p_{H_2} = p_{H_1} \left(\sum_{u(H_2)=H_1} \mu_u u \right).$$

For this we note that

$$p_{H_0} p_{H_1} = \left(\sum_{v(H_1)=H_0} \mu_v v \right) p_{H_1} = \sum_{v(H_1)=H_0} v p_{H_1} v^{-1} \mu_v v = \sum_{v(H_1)=H_0} p_{H_0} \mu_v v = p_{H_0} \left(\sum_{v(H_1)=H_0} \mu_v v \right)$$

and we conclude as before.

Finally, we need to check that $p_0 s = s p_0 = p_0$ whenever $\mathrm{Ker}(s-1) = H$. We have $p_0(\tilde{M}) = V_{H_0} = 1 \otimes M$, and $s.(1 \otimes y) = s \otimes y = 1 \otimes s.y$. But W_0 acts trivially on M by assumption, hence $s p_0 = p_0$. Then $p_0 s = s(s^{-1} p_0 s) = s p_{s^{-1}(H_0)} = s p_0 = p_0$, and through W-conjugation we get $p_H s = s p_H = p_s$ for all $H \in \mathcal{A}_0$, and $\mathrm{Ker}(s-1) = H$, the case $H \notin \mathcal{A}_0$ being trivial.

This proves that $e_H \mapsto p_H$ extends the $\mathbf{k}W$-module structure on $\tilde{M}$ to a $Br_1(W)$-module structure.

5.4. The $Br_1(W)$-modules of the form $\tilde{M}$: properties.

Proposition 5.5. *Let M be a $\boldsymbol{\kappa}N_0$-module on which W_0 acts trivially. If M is irreducible, then $\tilde{M}$ is an irreducible $Br_1(W)$-module. Moreover, if M_1 and M_2 are two such irreducible modules, then $\tilde{M}_1 \simeq \tilde{M}_2$ iff $M_1 \simeq M_2$.*

Proof. For the irreducibility, we adapt the arguments of [8], proposition 5.3. Assume that M is irreducible, and let $U \subset \tilde{M}$ a nonzero $Br_1(W)$-invariant subspace, and $q \in U \setminus \{0\}$.

We have $q = \sum_H \lambda_H g_H \otimes y_H$ for some $y_H \in M_{\mathbf{k}}$ and a collection of $\lambda_H \in \mathbf{k} = \boldsymbol{\kappa}(\delta)$. Each $y_H \in M_{\mathbf{k}}$ can be written $y_H = \sum_i \nu_{H,i} m_{H,i}$ for some $m_{H,i} \in M$ and $\nu_{H,i} \in \mathbf{k}$. Up to chasing denominators, we can assume $\lambda_H, \nu_{H,i} \in \boldsymbol{\kappa}[\delta]$. Let us pick H_1 and i_0 with $\lambda_{H_1}\nu_{H,i_0}$ being of maximal degree in δ among the terms $\lambda_H\nu_{H,i} \in \boldsymbol{\kappa}[\delta]$ with $y_H \neq 0$. Then $e_{H_1}.q = \sum_H \sum_i \lambda'_H \nu'_{H,i} g_{H_1} \otimes m'_{H,i}$ for some $m'_{H,i} \in M$ and $\lambda'_H, \nu'_{H,i} \in \boldsymbol{\kappa}[\delta]$ with $\deg \lambda'_H \nu'_{H,i} \leq \deg \lambda_H \nu_{H,i} \leq \lambda_{H_1}\nu_{H_1,i}$ whenever $H \neq H_1$. On the other hand, we have $\lambda'_{H_1}\nu'_{H_1,i} = \delta\lambda_{H_1}\nu_{H_1,i}$ and $m'_{H_1,i} = m_{H_1,i}$ for all i. It follows that $e_1.q$ can be written as

$$g_{H_1} \otimes \left(\delta\lambda_{H_1} \left(\sum_i \nu_{H_1,i} m_{H_1,i} \right) + \sum_j \omega_j m''_j \right) = g_{H,1} \otimes \left(\delta g_H \otimes y_{H_1} + \sum_j \omega_j m''_j \right)$$

with the $m''_j \in M$, $\omega_j \in \boldsymbol{\kappa}[\delta]$ with degree (strictly) lower than maximal degree of the $\lambda_{H_1}\nu_{H_1,i}$. Since $y_{H_1} \neq 0$, this implies that $e_{H_1}.q \neq 0$. On the other hand, $e_{H_1}.q \in U \cap V_{H_1}$ hence $U \cap V_{H_1} \neq \{0\}$. We can thus assume $q = g_{H_1} \otimes y \in U \cap V_{H_1} \setminus \{0\}$.

Since M is irreducible, by Lemma 5.2 we have that $M_{\mathbf{k}}$ is irreducible as a $\mathbf{k}N_0$-module. It follows that $V_{H_1} \subset U$. Since W acts transitively on the V_H, $H \in \mathcal{A}_0$, it follows that U contains all the V_H hence $U = \tilde{M}$ and $\tilde{M}$ is an irreducible $Br_1(W)$-module.

Now assume that a given $Br_1(W)$-module $\tilde{M}$ that is obtained from two simple $\boldsymbol{\kappa}N_0$-modules with the property that the action of W_0 is trivial. We thus have two a priori distinct direct sum decompositions

$$\tilde{M} = \bigoplus_{H \in \mathcal{A}_0} V_H^{(1)} = \bigoplus_{H \in \mathcal{A}_0} V_H^{(2)}$$

with the property that the action of N_0 on $V_{H_0}^{(i)}$ is (isomorphic to) M_i for $i \in \{1, 2\}$. But since V_{H_0} is the image of p_{H_0} by lemma 5.3 it follows that $(M_1)_{\mathbf{k}} \simeq (M_2)_{\mathbf{k}}$ as $\mathbf{k}N_0$-modules. But this implies that $M_1 \simeq M_2$ as $\boldsymbol{\kappa}N_0$-modules and this proves the second part of the statement. $\square$

Remark 5.6. *For the second part of the argument, we are not replicating the argument of [8], proposition 5.4, because it looks incorrect to us. Indeed, it is claimed there that the restriction of $\tilde{M}$ to $W(C_{\mathcal{B}})$, which is equivalent to the restriction to our N_0, is isomorphic to $|\mathcal{R}|M$, while it is actually isomorphic to $\mathrm{Res}_{N_0}\mathrm{Ind}_{N_0}^W M$. Actually, already for D_4 one can find non-isomorphic modules M_1, M_2 for which the restriction of $\tilde{M}_1$ and $\tilde{M}_2$ to W are isomorphic.*

Proposition 5.7. *Every irreducible representation of $Br_1(W)$ has the form $\tilde{M}$ for some irreducible M.*

Proof. Let Q be an irreducible representation of $Br_1(W)$ not factoring through W. This means that there exists some $H_0 \in \mathcal{A}$ for which e_{H_0} acts nontrivially. Denote $e_0 = e_{H_0}$, and $\mathcal{A}_0$ the orbit of H_0 under W. Since $e_0^2 = \delta e_0$ and e_0 acts non trivially, there exists $x_0 \in Q$

such that $e_0.x_0 = \delta x_0$. Let W_0 denote the subgroup fixing H_0 and N_0 its normalizer. For $s \in W_0$ we have $s.x_0 = (1/\delta)se_0.x_0 = (1/\delta)e_0.x_0 = (\delta/\delta)x_0 = x_0$ hence W_0 acts trivially on x_0. Now we set $V = \mathbf{k}N_0.x_0 \subset Q$. Note that V is a $\mathbf{k}N_0$-module factoring through $\mathbf{k}N_0/W_0$. Moreover, for all $w \in N_0$, we have $e_0w.x_0 = we_0x_0 = \delta w.x_0$ hence e_0 acts as $\delta.\mathrm{Id}$ on V.

Let $H \in \mathcal{A}_0$, and $w \in W$ such that $H = w(H_0)$. Then $w.V$ depends only on H. Indeed, if $w_1^{-1}w_2 \in N_0$ we have $w_1^{-1}w_2 V = V$ hence $w_2 V = w_1 V$. We set $V_H = w.V$. For $g \in W$ we have $g.V_H = gw.V = V_{H'}$ for $H' = gw(H)$. Clearly $\sum_H V_H$ is a $\mathbf{k}W$-submodule of Q. Then, for $x \in V_H$ and $H = w(H_0)$, we have $e_H.x = we_0w^{-1}.x = we_0.(w^{-1}.x) = \delta w.(w^{-1}.x) = \delta.x$. But this implies that, either $e_0.x = (1/\delta)e_0e_H.x = 0$ or

$$e_0.x = (1/\delta)e_0e_H.x = (1/\delta) \sum_{u \in \mathcal{R}; u(H)=H_0} \mu_u u e_H.x = \sum_{u \in \mathcal{R}; u(H)=H_0} \mu_u u.x$$

hence e_0 maps x inside V. In particular $\sum V_H$ is stable under e_0. Finally, if $H' \in \mathcal{A} \setminus \mathcal{A}_0$, we have $e_{H'}.x = (1/\delta)e_{H'}e_0.x = 0$ for all $x \in V$. This implies similarly that $e_{H'}.wx = w.e_{w^{-1}(H')}.x = 0$ for all $x \in V$, hence $e_{H'}$ acts by 0 on $\sum_H V_H$. This implies that $\sum_H V_H$ is stable under $Br_1(W)$, hence $\sum_H V_H = Q$.

Let now assume that V is not irreducible as a $\mathbf{k}N_0$-module, and contains a proper irreducible submodule U. Then one proves similarly that $\sum_H U_H$ is stable under $Br_1(W)$, where $U_H = w.U$ for $w(H_0) = H$, hence $\sum_H U_H = \sum_H V_H = Q$. Finally, remark that, if $x \in \sum_H V_H$ satisfies $e_0.x = \delta.x$, then $x \in V$, since we know that $e_0.(\sum_H V_H) \subset V$. Similarly, we check that e_0 maps $\sum_H U_H$ to U, hence U and V can be both identified with the nullspace of $e_0 - \delta$ on the same space, hence $U = V$, which proves the irreducibility of V as a $\mathbf{k}N_0$-module.

Finally, for $H = w(H_0)$ we have that $e_H = we_0w^{-1}$ maps Q to $V_H = w.V$ and acts by δ on V_H. It follows that V_H is the nullspace of $e_H - \delta$ and this proves that $Q = \bigoplus_H V_H$. This proves that, as a $\mathbf{k}W$-module, Q is the induced representation of V. But, by Lemma 5.2, for any $\boldsymbol{\kappa}(\delta)N_0$-module V there exists a $\boldsymbol{\kappa}N_0$-module V_0 such that $V \simeq V_0 \otimes_{\boldsymbol{\kappa}} \mathbf{k}$, hence $Q \simeq \tilde{V}_0$, and this proves the claim.

$\square$

Remark 5.8. *The determination of the representations of $Br_1(W)$ for type H_3 was essentially done in* [6].

6. Proof of theorem 1.1

From proposition 3.2 we know that $Br_1(W)$ admits a spanning set formed by the $w \in W$ and the $we_H, H \in \mathcal{H}$, where $w \in W/W_H$ and W_H is the parabolic subgroup fixing H. Therefore the dimension of $Br_1(W)$ is at most $|W| + \sum_{\mathcal{A}_0 \in \mathcal{A}/W} |\mathcal{A}_0| \times |W|/|W_{H_0}|$ where W_{H_0} is the fixer of a representative $H_0 \in \mathcal{A}_0$ of $\mathcal{A}_0$.

On the other hand, from the previous section we know that $Br_1(W)$ has irreducible representations of dimension $|\mathcal{A}_0| \times \dim \theta$ where θ is an irreducible representation of $N_W(W_{H_0})/W_{H_0}$. Therefore its dimension is at least

$$\sum_{\mathcal{A}_0 \in \mathcal{A}/W} |\mathcal{A}_0|^2 \times \sum_{\theta \in \mathrm{Irr}N_W(W_{H_0})/W_{H_0}} (\dim \theta)^2 = \sum_{\mathcal{A}_0 \in \mathcal{A}/W} |\mathcal{A}_0|^2 \frac{|N_W(W_{H_0})|}{|W_{H_0}|}$$

But since $W/N_W(W_{H_0})$ is in bijection with $|\mathcal{A}_0|$, this is equal to the previous quantity. Therefore we get the formula for the dimension given by the theorem, and semisimplicity as well.

7. Examples

7.1. Example : $G(4,2,2)$. We consider the group $W = G(4,2,2)$ of rank 2, made of monomial matrices with entries in $\mu_4 = \{1,-1,i,-i\}$ and whose product of the nonzero entries belongs to $\mu_2 = \{-1,1\}$. It admits the presentation

$$W = \langle s,t,u \mid stu = tus = ust, \ s^2 = t^2 = u^2 = 1 \rangle$$

and therefore an automorphism of order 3 mapping $s \mapsto t \mapsto u \mapsto s$. It has order 16, and 6 reflections, s,t,u,s',t',u', forming 3 conjugacy classes of two elements, $\{s,s'\}, \{t,t'\}, \{u,u'\}$. The reflection s' is equal to $tst = t\underline{stuu} = tt\underline{usu} = usu$, hence $t' = utu = sts$, $u' = sus = tut$.

The normaliser N_0 of $W_0 = \langle s \rangle$ is abelian of order 8, isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/4$. The element $z = stu$ is a generator of the center, and has order 4, and its image generates N_0/W_0.

Therefore N_0/W_0 is naturally identified with $Z(W)$.

The irreducible representations of N_0 satisfying the property that W_0 acts trivially are therefore uniquely determined by a choice of $\zeta \in \mu_4$ and given by the formula $R_\zeta : z \mapsto \zeta$. We now consider the induced module $V_s(\zeta) = \mathrm{Ind}_{N_0}^W R_\zeta$. It admits for basis $v_s, v_{s'} = t.v_s$, where N_0 acts on $\mathbb{C}v_s$ by R_ζ. From this we get $u.v_s = ts.stu.v_s = \zeta ts.v_s = \zeta t.v_s = \zeta v_{s'}$, hence $u.v_{s'} = u^{-1}.v_{s'} = \zeta^{-1}.v_s$, and similarly $s.v_s = v_s$, $s.v_{s'} = u.ust.v_s = \zeta.u.v_s = \zeta^2.v_s$, and $t.v_s = v_{s'}$, $t.v_{s'} = v_s$. Therefore we get the matrices

$$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta^2 \end{pmatrix} t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u \mapsto \begin{pmatrix} 0 & \zeta^{-1} \\ \zeta & 0 \end{pmatrix} s' \mapsto \begin{pmatrix} \zeta^2 & 0 \\ 0 & 1 \end{pmatrix} t' \mapsto \begin{pmatrix} 0 & \zeta^2 \\ \zeta^2 & 0 \end{pmatrix} u' \mapsto \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix}$$

Notice that, for $\zeta \in \{-i, i\}$, we get reflection representations for W. We have $s, s' \in N_0$, while $xsx = s'$ for $x \in \{t, u, t', u'\}$. Letting $s_0 = s$, we get $p_s.v_s = \delta v_s$ and

$$p_s.v_{s'} = \sum_{xsx=s'} x.v_{s'} = (t + u + t' + u').v_{s'} = (1 + \zeta + \zeta^2 + \zeta^3).v_{s'}$$

Since $s' = tst$ we get

$$p_s = \begin{pmatrix} \delta & 0 \\ 0 & 1 + \zeta + \zeta^2 + \zeta^3 \end{pmatrix}, \quad p_{s'} = \begin{pmatrix} 1 + \zeta + \zeta^2 + \zeta^3 & 0 \\ 0 & \delta \end{pmatrix}$$

and $p_x = 0$ for $x \in \{t, t', u, u'\}$. Note that $1 + \zeta + \zeta^2 + \zeta^3 = 0$ unless $\zeta = 1$.

Condition (1)'' is always fulfilled on such representations when $p_H = 0$, so we consider only the case where it is the reflecting hyperplane of s or s'. But in which case the reflection under consideration has to be the other one, and therefore we need to check whether $sp_{s'} = p_{s'}s = p_{s'}$ and $s'p_s = p_s s' = p_s$. This is the case if and only if $\zeta^2 = 1$, that is $\zeta \in \{-1, 1\}$.

In order that condition (1)' is fulfilled, we need first of all that (1)'' is fulfilled, so we consider only the case $\zeta^2 = 1$. Let $w \in W$ satisfying ${}^w s = s$, that is $w \in N_0$. Since $Z(W)$ is a complement to W_0 inside W, and since it is generated by z, the condition is then whether $zp_s = p_s z = p_s$, which would imply $zp_{s'} = p_{s'}z = p_{s'}$ after conjugation by t. But $zp_s = p_s z = \zeta p_s$, hence the conditionn is fulfilled only if $\zeta = 1$.

These two facts prove that conditions (1)' and (1)'' are not equivalent, and that they are genuine additional conditions.

The other representations $V_t(\zeta)$, $V_u(\zeta)$ are deduced from $V_s(\zeta)$ by applying φ, as it is readily extended to an automorphism of order 3 of $Br(M)$.

7.2. **Example :** $G(e,e,3)$. For $W = G(e,e,3)$, there is a single conjugacy class of reflections. Let us choose $s_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then N_0 is easily determined to be equal the image of $\langle \mu_e, (1,2) \rangle < \mathrm{GL}_2(\mathbb{C})$ under the map $M \mapsto \mathrm{diag}(M, q(M)^{-1})$ where $q(M)$ is the product of the entries of M. Therefore $N_0/W_0 \simeq \mu_e$. Now, W has order $e^2 \times 3! = 6e^2$, N_0 has order $2e$, $\mathcal{R}$ has cardinality $3e$, hence $Br(M) = Br_1(M)$ has dimension $6e^2 + 9e^3 = 3e^2(2 + 3e)$.

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