A NOTE ON THE CENTER CONJECTURE FOR ARTIN GROUPS

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ABSTRACT. In this note we provide an alternative shorter proof of a theorem of K. Jankiewicz and K. Schreve, stating that a positive solution of the $K(\pi,1)$ conjecture for Artin groups implies that the center of any Artin group without spherical factors is trivial. We extend this result to the case of an infinite number of generators.

1. THE CENTER CONJECTURE

Let (W,S) be a Coxeter system, that is S is a generating set for the group W, and W admits for presentation over S the relations $s^2=1, s\in S$ together with the so-called braid relations

$$\underbrace{sts\dots}_{m_{s,t}} = \underbrace{tst\dots}_{m_{s,t}} \tag{1}$$

for $m_{s,t} \geq 2$ the (possibly infinite) order of $st \in W$ for $s \neq t$. We do not assume that S is finite. The Artin group A associated to this Coxeter system is defined by the presentation with generators $a_s, s \in S$ with relations (1), that is $a_s a_t a_s \cdots = a_t a_s a_t \dots$. It is called *spherical* if W is finite. The Coxeter graph Γ is defined with vertices $s\in S$ and a edge between s and t if $m_{st}>2$, and this edge is labelled by m_{st}. The corresponding Coxeter and Artin groups are called irreducible if Γ is connected.

For any $S_0 \subset S$, we have a so-called parabolic subgroups $W_{S_0} = \langle S_0\rangle$, which is associated to the Coxeter system (W_{S_0}, S_0), and has a corresponding Artin group A_{S_0} mapping homomorphically to A. We refer to [5] for a survey of basic results on Coxeter and Artin groups. The natural morphism $A_{S_0} \to A_S$ is always injective, but the classical references unnecessarily assume that S is finite, so we provide a proof here, as the general case is useful to us.

Proposition 1.1. *For $S_0 \subset S$, the natural morphism $A_{S_0} \to A_S$ is injective.*

Proof. The case when S is finite is due to Van der Lek, see Theorem 5.4 in [5]. In general, let us consider x in the kernel of $\varphi_{S_0,S} : A_{S_0} \to A_S$. Choose an expression of x as a product of the generators and their inverses. It involves only a finite subset $S_0' \subset S_0$ of generators. Since x can be written as $\varphi_{S_0',S_0}(x')$ for some $x' \in A_{S_0'}$ and $\varphi_{S_0',S} = \varphi_{S_0,S}\circ\varphi_{S_0',S_0}$, we get $\varphi_{S_0',S}(x')=1$. But this means that x has the same image as a (finite !) product of relators, each of them involving a finite number of generators, so there is a finite subset $S' \subset S$, that can be assumed to contain S_0', and such that $\varphi_{S_0',S'}(x')=1$. By Van der Lek's result this implies $x'=1$ hence $x=1$. □

If Γ is a disjoint union $\Gamma_1 \sqcup \Gamma_2$, and $S = S_1 \sqcup S_2$ is the corresponding partition of its vertices, then $W \simeq W_{S_1} \times W_{S_2}$ and $A \simeq A_{S_1} \times A_{S_2}$, and each A_{S_i} (resp. $\langle S_i \rangle$) is called a factor of A (resp. of W). The Artin group A is said to be without spherical factors it there does not exist such a decomposition with $\langle S_1\rangle$ (non trivial and) finite. The so-called *Center conjecture* for Artin groups is the following one.

Conjecture 1.2. *If A is an Artin group without any spherical factor, then $Z(A)=\{1\}$.*

We shall prove in this paper that this conjecture can be reduced to the case where S is finite (see Proposition 3.2).

In case S is finite, there is a unique maximal partition $S = S_1 \sqcup \dots \sqcup S_r$, called the canonical decomposition and yielding $W \simeq W_{S_1}\times\dots\times W_{S_r}$ and $A \simeq A_{S_1}\times\dots\times A_{S_r}$, such that each W_{S_i}, A_{S_i} is irreducible. Moreover, we can associate to any such Coxeter system (W,S) with S finite the complement $\mathcal{H}(W)$ in some open cone of a complexified hyperplane arrangement, whose fundamental group can be identified with the corresponding Artin group A. We refer to [5] and [1] for a precise definition of $\mathcal{H}(W)$ and for the context of the following conjecture, called the $K(\pi,1)$ conjecture.

Conjecture 1.3. *Assume that S is finite. The orbit space $\mathcal{H}(W)/W$ of the hyperplane complement associated to the Coxeter system (W,S) is a $K(\pi,1)$-space for A.*

Part (1) of the following theorem is due to Jankiewicz and Strebe.

Theorem 1.4.

(1) *If Conjecture 1.3 holds for some (W,S) with S finite, then Conjecture 1.2 holds for the corresponding Artin group.*
(2) *If Conjecture 1.3 holds for every (W,S) with S finite, then Conjecture 1.2 holds in general.*

The first aim of this paper is to provide an alternative shorter proof of the result of Jankiewicz and Strebe. We use their idea to study the cohomological dimension and get a contradiction with the presence of a $A_{S_0}\times\mathbf{Z}$-subgroup inside A, but replace all their careful examination of a geometric realization of A_S inside a mapping class group, as well as all the reduction arguments needed to evacuate the ∞-labelled edges of the Coxeter graph, with a more direct study of the Hecke algebra representation.

A secondary aim is to show part (2) of the Theorem, involving the case where S is infinite. This is a consequence of the already mentionned Proposition 3.2, that we prove using the same tool (Lemma 3.1) we needed to shorten the argument of [3].

2. Hecke algebra representation

Let $\mathbf{Z}$ be the ring of integers and q an indeterminate. By definition, the Hecke algebra of the Coxeter system (W,S) is a free $\mathbf{Z}[q,q^{-1}]$-module with 'standard' basis the $T_w, w\in W$ whose algebra structure is determined by the fact that T_1 is the unit and by the formulas $T_sT_w=T_{sw}$ if $\ell(sw)=1+\ell(w)$, and $T_s^2=(q-1)T_s+qT_1$, with $\ell : W\to\mathbf{N}=\mathbf{Z}_{\geq 0}$ the classical length function on W with respect to S. We refer to [2] for basic results on this structure, including the fact that it is a quotient of the group algebra of A, under the map $a_s\mapsto T_s$, by the relations $a_s^2=(q-1)a_s+q$. Another, almost immediate, useful property, is that $\ell(ww')=\ell(w)+\ell(w')\Rightarrow T_{ww'}=T_wT_{w'}$.

We denote by $\mathbf{Z}[[h]]$ the ring of formal power series and use the ring embedding $\mathbf{Z}[q,q^{-1}]\to\mathbf{Z}[[h]]$ mapping q to $1+h$ to consider the Hecke algebra H as defined over $\mathbf{Z}[[h]]$. We have in particular $T_s^2=hT_s+(1+h)T_1$.

We set $B=\mathbf{N}[[h]]$ and $B^+=B\setminus\{0\}$. We have $B+B=B$ and $B+B^+=B^+$, $B^+.B^+=B^+$. This provides a partial ordering on $\mathbf{Z}[[h]]$, defined by $a\geq b\Leftrightarrow a-b\in B$, so that $a>b\Leftrightarrow a-b\in B^+$. It has the property that, for $c>0$, $a\geq b\Rightarrow ac\geq bc$ and $a>b\Rightarrow ac>bc$.

For $s\in S$ and $w\in W$ we get that, either $\ell(sw)=\ell(w)+1$ and $T_sT_w=T_{sw}$, or $w=sw'$ with $\ell(w)=1+\ell(w')$, in which case $T_sT_w=T_s^2T_{w'}=hT_w+(1+h)Tw'$. Notice that, in both cases, the coefficients of T_sT_w are ≥ 0. Let $H_{\geq 0}=\sum_{w\in W}BT_w=\{x\in H\mid \forall w\in W\ T_w^*(x)\geq 0\}$ with $(T_w^*)_{w\in W}$ the dual basis of the standard basis. We thus get $T_sH_{\geq 0}\subset H_{\geq 0}$ for every $s\in S$, hence $T_wH_{\geq 0}\subset H_{\geq 0}$ for every $w\in W$.

If W is finite, it admits a unique longest element w_0 with respect to ℓ. We claim that,

$$r\geq 1\Rightarrow T_{w_0}^r\in B^+T_{w_0}+H_{\geq 0} \tag{2}$$

Indeed, this is obvious for $r=1$, and, for every $s\in S$, we have $T_sT_{w_0}=T_sT_sT_{sw_0}=hT_sT_{sw_0}+(1+h)T_{sw_0}\in hT_{w_0}+H_{\geq 0}$. Writing $T_{w_0}^{r-1}$ as a (possibly empty) product of the generators $T_s, s\in S$, this proves that $T_{w_0}^r=T_{w_0}^{r-1}T_{w_0}\in B^+T_{w_0}+H_{\geq 0}$.

3. Proof of the main theorem

We use the preceding results to prove the following lemma, in which we do not assume that S is finite.

Lemma 3.1. *Let $S_0\subset S$ such that $\langle S_0\rangle$ is finite, and such that no factor of A_{S_0} is a factor of A_S. Then, for every $y\in Z(A_{S_0})\setminus\{1\}$ there exists $s\in S\setminus S_0$ such that $ya_s\neq a_sy$.*

Before proving the lemma, we introduce the following notation. For $s_1s_2\dots s_r$ a reduced expression of w, we denote $a_w=a_{s_1}a_{s_2}\dots a_{s_r}$ the corresponding element of A_S.

Proof. Assume by contradiction $y \neq 1$, and write $S_0 = S_1 \sqcup \cdots \sqcup S_r$ the canonical decomposition inducing $A_{S_0} \simeq A_{S_1} \times \cdots \times A_{S_r}$, with W_{S_i} irreducible. Then $y = a_{w_0(S_1)}^{m_1} a_{w_0(S_2)}^{m_2} \dots a_{w_0(S_r)}^{m_r}$ for some integers $m_1, \dots, m_r$, where $w_0(S_i)$ is the longest element of $\langle S_i \rangle$. Since $y \neq 1$, w.l.o.g. we can assume $m_1 \neq 0$ and, up to considering y^{-1} instead of y, we can assume $m_1 > 0$. We consider the image of this element inside H, denoted $Y = T_{w_0(S_1)}^{m_1} T_{w_0(S_2)}^{m_2} \dots T_{w_0(S_r)}^{m_r}$. We proved earlier – see Equation (2) – that the coefficient of $T_{w_0(S_1)}$ in $T_{w_0(S_1)}^{m_1}$ is nonzero. Also, the elements $T_{w_0(S_i)}^{m_i}$ are nonzero for $i \geq 2$, as they are invertible. Let $w_i \in \langle S_i \rangle$ be an element of maximal length such that the coefficient of T_{w_i} in the expansion of $T_{w_0(S_i)}^{m_i}$ is nonzero (there may be several such w_i). We get that the coefficient of $T_{w_0(S_1)w_2w_3\dots w_r}$ in Y is nonzero, and that $w = w_0(S_1)w_2w_3 \dots w_r$ has the maximal possible length for this property. Now, since $\langle S_1 \rangle$ is not a factor of W, there exists $s_1 \in S \setminus S_1$ which does not commute with at least one element of S_1. Since all elements of $S_0 \setminus S_1$ commute with S_1, we have $s_1 \in S \setminus S_0$, and in particular $\ell(ws_1) = \ell(s_1 w) = \ell(w)+1$.

But then, $a_{s_1} y = y a_{s_1}$ with $s_1 \notin S_0$ implies $T_{s_1} Y = Y T_{s_1}$, hence $T_{s_1} T_w = T_{s_1 w_0(S_1) w_2 w_3 \dots w_r}$, which appears with a nonzero coefficient in the expansion (in the standard basis) of $T_{s_1} Y$, has to be equal to some $T_{w' s_1} = T_{w'} T_{s_1}$ with $w' \in \langle S_0 \rangle$ of the same length, appearing in the expansion of $Y T_{s_1}$. Such a w' necessarily has the form $w_0(S_1) w_2' w_3' \dots w_r'$ with $w_i' \in \langle S_i \rangle$ and $\ell(w_i') = \ell(w_i)$. Recall that a so-called reduced expression for an element of W is an expression as a product of the generators which has minimal length. Considering a reduced expression of each of the $w_0(S_1), w_i, w_i'$, we get two reduced expressions of $s_1 w_0(S_1) w_2 w_3 \dots w_r = w_0(S_1) w_2' w_3' \dots w_r' s_1$, and Matsumoto's theorem (see [2] §1.2) tells us that these can be deduced from each other by using only braid relations (1). But s_1 appears only once in each expression, and does not commute with at least one of the elements of S_1. Since they all appear in the chosen expression for $w_0(S_1)$, we get a contradiction proving $y = 1$. □

Now, as in [3], consider a maximal subset $S_f \subset S$ such that $\langle S_f \rangle$ is finite. It is known that the orbit space $\mathcal{H}(W)/W$ is homotopically equivalent to a CW-complex having cells of dimension at most $|S_f|$ (see e.g. [5] Lemma 3.2 and Corollary 3.4). Therefore the $K(\pi, 1)$ conjecture implies $\mathrm{cd}(A) \leq |S_f|$ and, since $\mathrm{cd}(A_T) \geq |T|$ whenever W_T is finite (for instance because the kernel P_T of $A_T \twoheadrightarrow W_T$ has non trivial rational cohomology in degree $|T|$, see [4] Cor. 6.62), it implies $\mathrm{cd}(A) = \mathrm{cd}(A_{S_f})$. As in [3] Theorem 12 in [3], we prove that this implies $Z(A) = \{1\}$.

Write $S_f = S_1 \sqcup \cdots \sqcup S_r$ the canonical decomposition inducing $A_{S_f} \simeq A_{S_1} \times \cdots \times A_{S_r}$, such that each $\langle S_i \rangle$ is irreducible. Since $A = A_S$ has no spherical factor, for each i there exists $s_i \in S \setminus S_f$ not commuting with at least one element of S_i, as otherwise A_{S_i} would be a spherical factor of A.

Let then $y \in Z(A)$, and assume by contradiction $y \neq 1$. Since we assumed $\mathrm{cd}(A) < \infty$, we know that y has infinite order. Then, there must be some $k \neq 0$ such that $y^k \in A_{S_f}$, for otherwise A contains $\mathbf{Z} \times A_{S_f}$, of cohomological dimension $1 + \mathrm{cd}(A_{S_f}) = 1 + \mathrm{cd}(A) > \mathrm{cd}(A)$. W.l.o.g. we can assume $k = 1$, and $y \in Z(A) \cap A_{S_f} \subset Z(A_{S_f})$. But then Lemma 3.1 implies $y = 1$, and this proves part (1) of Theorem 1.4.

We now prove part (2). This is a consequence of the following proposition.

Proposition 3.2. *If Conjecture 1.2 holds whenever S is finite, then it holds in general.*

Proof. Let $y \in A$ belonging to $Z(A)$. Then it belongs to A_{S_0} for some finite $S_0 \subset S$. If A_{S_0} has no spherical factor we are done, as then $y \in Z(A_{S_0}) = \{1\}$. Otherwise, we can write $A_{S_0} \simeq A_{S_1} \times A_{S_2}$ with $S_0 = S_1 \sqcup S_2$, $\langle S_1 \rangle$ finite and A_{S_2} without any spherical factor. Since $Z(A_{S_2}) = \{1\}$ we get $y \in Z(A_{S_1})$. But since A has no spherical factor, if $y \neq 1$ our Lemma 3.1 implies that there exists $s' \in S \setminus S_1$ with $a_{s'} y \neq y a_{s'}$, and this contradiction proves $y = 1$. □

References

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