ARTIN GROUPS AND YOKONUMA-HECKE ALGEBRAS

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Abstract. We attach to every Coxeter system (W, S) an extension $\mathcal{C}_W$ of the corresponding Iwahori-Hecke algebra. We construct a 1-parameter family of (generically surjective) morphisms from the group algebra of the corresponding Artin group onto $\mathcal{C}_W$. When W is finite, we prove that this algebra is a free module of finite rank which is generically semisimple. When W is the Weyl group of a Chevalley group, $\mathcal{C}_W$ naturally maps to the associated Yokonuma-Hecke algebra. When $W = \mathfrak{S}_n$ this algebra can be identified with a diagram algebra called the algebra of 'braids and ties'. The image of the usual braid group in this case is investigated. Finally, we generalize our construction to finite complex reflection groups, thus extending the Broué-Malle-Rouquier construction of a generalized Hecke algebra attached to these groups.
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Contents

1. Introduction

Let (W, S) be a Coxeter system, as in [4], and $m_{st} \in \mathbf{Z}_{\geq 2} \cup \{\infty\}$ denote the order of st for $s, t \in S$. Three objects are classically attached to it : another group, the Artin group B defined

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by a presentation made of 'braid relations',

$$\langle S \mid \underbrace{sts\dots}_{m_{st}} = \underbrace{tst\dots}_{m_{st}} \quad \forall s,t \in S\rangle,$$

a monoid B^+ of positive braids defined by the same presentation, and an algebra, called the Iwahori-Hecke algebra. This algebra H_W is defined over a ring $\mathbf{k}$ containing elements $u_s, s \in S$ subject to the condition $u_s = u_t$ if s,t both lie in the same conjugacy class, as the quotient of the monoid algebra $\mathbf{k}B^+$ by the relations $(s-1)(s+u_s) = 0$ for $s \in S$. It is a deformation of the group algebra of W, obtained by the specialization at $u_s = 1$. When W is the Weyl group of some reductive group, H_W admits a natural interpretation as a convolution algebra. The specialization at $u_s = -1$ of H_W admits a natural central extension which is also a quotient of $\mathbf{k}B$, recently defined in [28].

In this paper we define another natural object, a $\mathbf{k}$-algebra $\mathcal{C}_W$ which is an extension of H_W, and admits a 1-parameter family of morphisms $B \to \mathcal{C}_W$. This algebra admits generators $g_s, e_s, s \in S$ and is defined by generators and relations in section 3.1. We prove (see theorem 3.4) that it is a free $\mathbf{k}$-module. When W is finite, we show that $\mathcal{C}_W$ has rank $|W|.Bell(W)$, where $Bell(W)$ is a natural generalization of the Bell number $Bell_n$ of partitions of a set of n elements, namely the number of reflection subgroups of W. Precisely, in the general case a basis of $\mathcal{C}_W$ is naturally indexed by couples (w, W_0) for $w \in W$ and W_0 a finitely-generated reflection subgroup of W.

The original motivation for this algebra comes from an analysis of the so-called Yokonuma-Hecke algebra associated to a Chevalley group G and its unipotent radical $\mathfrak{U}$, namely the Hecke convolution ring $\mathcal{H}(G,\mathfrak{U})$, defined by Yokonuma in [38]. Assume W is the Weyl group of G, with generating set S. Part of the natural generators of this algebra are directly connected to the structure of the torus, while the other ones are in 1-1 correspondence with S and satisfy braid relations, together with a quadratic relation also involving elements of the torus. In [17], using a Fourier transform construction, J. Juyumaya introduced other natural 'braid' generators $g_s, s \in S$, for which the quadratic relation now involves some idempotent e_s (in which is 'hidden' a linear combinations of elements of the torus). Therefore, there is a natural subalgebra generated by the g_s, e_s, and a natural question is to find a presentation for this subalgebra, at least when the field of definition of G is generic enough. The algebra $\mathcal{C}_W$ that we introduce provides an answer to that question. More precisely, a better answer is a natural quotient $\mathcal{C}_W^R$ of $\mathcal{C}_W$ where reflection subgroups, in natural 1-1 correspondence with root subsystems, are identified if they have the same closure (see section 3.4).

Although one is, at least since Tits's classical article [37], somewhat accustomed to such a phenomenon, it remains surprising that once again such an object arising from reductive groups admits a natural generalization to arbitrary Coxeter groups. This algebra $\mathcal{C}_W$ can be viewed as a deformation of the semidirect product $\mathcal{C}_W(1)$ of the group algebra of W with a commutative algebra spanned by the collection of finitely generated reflection subgroups of W. We show in theorem 3.9 that, when W is finite and under obvious conditions on the characteristic, this algebra $\mathcal{C}_W(1)$ is semisimple, and therefore $\mathcal{C}_W$ is generically semisimple. For $W = \mathfrak{S}_n$ this generalizes and provides a more direct proof of a result of [3]. Actually, we show that in the case $W = \mathfrak{S}_n$ and in characteristic 0, the algebra $\mathcal{C}_W(1)$ is *split* semisimple. The question about a similar statement for other Weyl groups raises new problems on the normalizers of reflection and parabolic subgroups in finite Weyl groups (see section 3.7).

In section 4 we introduce a family of morphisms $\Psi_{\underline{\lambda}} : \mathbf{k}B \to \mathcal{C}_W(\underline{u})$ and we exhibit an unexpected connection between the quotient of the group algebra of the braid group appearing inside the Yokonuma-Hecke algebra of type A and (a specialization of) the one connected with the Links-Gould polynomial invariant of knots and links. We are then able to deduce from Ishii's work on the Links-Gould invariant a new relation inside the Yokonoma-Hecke algebra. Amusingly enough, we notice that Ishii's work and Juyumaya's work on these previously unrelated topics appeared following each other in the same issue of the same journal (see [18, 14]).

A natural question is whether the natural map $B \to \mathcal{C}_W(\underline{u})$ is injective. Since there is a natural (surjective) map $\mathcal{C}_W(\underline{u}) \to H_W(\underline{u})$, this would be the case if the induced map $B \to H_W(\underline{u})$ was itself injective. Right know, this is an open question, settled (positively) only in rank 2, by work of

Squier [34], and an alternative proof can be found in [23]. Our question of whether $B \to \mathcal{C}_W(\underline{u})$ is injective therefore may or may not be a consequence of the solution of this one. A possibly easier question is whether the (restriction to B of the) maps $\Psi_{\underline{\lambda}}$ are injective for generic $\underline{\lambda}$. We show in section 4.4 that a simpleminded application of the existing methods does not suffice to conclude on this point. They however incite to define and look at a new *monoid* representation $B^+ \to \mathcal{C}_W$ with positive coefficients.

In the last section, we show that the natural quotient $\mathcal{C}_W^p$ of $\mathcal{C}_W$, where reflection subgroups are identified if they have the same parabolic closure, can be generalized to the setting where W is a finite complex reflection group, in such a way that $\mathcal{C}_W^p$ is a natural extension of the generalized Hecke algebra H_W associated to W by Broué, Malle and Rouquier in [5]. The main conjecture on H_W, that H_W is a free module of finite rank, is naturally extended to an a priori stronger conjecture on $\mathcal{C}_W^p$, that we prove to be true for a couple of cases. In particular we prove this conjecture for W the complex reflection group of monomial $n \times n$ matrices with coefficients d-th roots of 1, which provides a natural extension of the so-called Ariki-Koike algebra.

As a conclusion, we wonder whether other classical objects attached to Iwahori-Hecke algebras, like Kazhdan-Lusztig bases and Soergel bimodules, can be naturally extended to this setting. In particular it would be interesting to construct an extension of Lusztig's isomorphism of [24] to $\mathcal{C}_W$. We also consider very likely that the whole machinery of Cherednik algebras, including the so-called KZ functor, can be generalized in a natural way to our 'extended' setting. We leave this to future work.

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2. Preliminaries

2.1. **Yokonuma-Hecke algebra.** Following Yokonuma's original paper [38], we use Chevalley's notation as in [7]. Let G be the Chevalley group associated to a semi-simple complex Lie algebra $\mathfrak{g}$ and to a finite field $K = \mathbf{F}_q$ and $\mathfrak{H}, \mathfrak{W}, \mathfrak{U} \subset G$ as in [7]. To each root α of $\mathfrak{g}$ we let $\varphi_\alpha : SL_2(K) \to G$ denote the associated morphism, and

$$h_{\alpha,t} = \varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \omega_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Choosing a system $\alpha_1, \dots, \alpha_l$ of simple roots, we let $\omega_i = \omega_{\alpha_i}$. There is a short exact sequence $1 \to \mathfrak{H} \to \mathfrak{W} \to W \to 1$, where W is the corresponding Weyl group. Each ω_α is mapped in W to the reflection s_α associated to α. The Weyl group admits a presentation as Coxeter system (W, S) with $S = \{s_1, \dots, s_l\}$ in 1-1 correspondence with the set of simple roots under $s_i = s_{\alpha_i} \leftrightarrow \alpha_i$. The subgroup $\mathfrak{H}$ is generated by the $h_{\alpha,t}$. For short, we let $h_{i,t} = h_{\alpha_i,t}$. In [7], Chevalley denotes h_α the coroot $\check{\alpha}$ associated to α. In order to facilitate cross-references between [7] and [4] we will use both notations : $h_\alpha = \check{\alpha}$. The maximal torus $\mathfrak{H}$ is described in [7] as the image of $\mathrm{Hom}(L, K^\times)$, where L is the root lattice, under the map $\chi \mapsto h(\chi)$ where $h(\chi)$ is an automorphism of the associated complex Lie algebra $\mathfrak{g}$ acting trivially on the Cartan subalgebra and by $h(\chi)X_r = \chi(r)X_r$ on the generator associated to the root r. With these notations, $h_{\alpha,t} = h(\chi_{\alpha,t})$ where $\chi_{\alpha,t}(r) = t^{r(h_\alpha)} = t^{r(\check{\alpha})}$.

In [38], théorème 3, T. Yokonuma proves that the Hecke ring $\mathcal{H}(G, \mathfrak{U})$ over $\mathbf{Z}$ admits a presentation by generators $a(h), h \in \mathfrak{H}$, $a_1, \dots, a_l$ and relations

(1) $a(h_1)a(h_2) = a(h_1h_2)$ for all $h_1, h_2 \in \mathfrak{H}$
(2) $a_i a(h) = a(h')a_i$, where $h' = \omega_i h \omega_i^{-1}$
(3) $a_i^2 = qa(h_i) + \sum_{t \in K^\times} a(h_{i,t})a_i$ where $h_i = \omega_i^2$
(4) $\underbrace{a_i a_j a_i \dots}_{m_{ij}} = \underbrace{a_j a_i a_j \dots}_{m_{ij}}$ for $1 \leq i, j \leq l$

Let $\tilde{e}_i = \sum_{t \in K^\times} a(h_{i,t})$ and, in general, $\tilde{e}_\alpha = \sum_{t \in K^\times} a(h_{\alpha,t})$.

The following proposition is crucial for us. Parts (1) and (2) are standard, parts (3) and (4) appear to be new, at least in the general case.

Proposition 2.1.

(1) *For every root* α*, we have* $\tilde{e}_\alpha^2 = (q-1)\tilde{e}_\alpha$*, and* $\tilde{e}_{-\alpha} = \tilde{e}_\alpha$
(2) *For every two roots* α, β*, we have* $\tilde{e}_\alpha \tilde{e}_\beta = \tilde{e}_\beta \tilde{e}_\alpha$
(3) *For every two roots* α, β*, we have* $\tilde{e}_\alpha \tilde{e}_\beta = \tilde{e}_\alpha \tilde{e}_{s_\alpha(\beta)}$
(4) *For every two roots* α, β*, if* γ *is a root such that* $\check{\gamma} = \check{\alpha} + \check{\beta}$*, then* $\tilde{e}_\alpha \tilde{e}_\gamma = \tilde{e}_\alpha \tilde{e}_\beta$*.*

Proof. We have $\tilde{e}_{-\alpha} = \sum_{t \in K^\times} a(h_{-\alpha,t}) = \sum_{t \in K^\times} a(h_{s_\alpha(\alpha),t}) = \sum_{t \in K^\times} a(h_{s_\alpha(\alpha),t})$. Since, after [7], $h_{s_\alpha(\alpha),K^\times} = \omega_\alpha h_{\alpha,K^\times} \omega_\alpha^{-1}$ we have $\tilde{e}_{-\alpha} = \sum_{t \in K^\times} a(\omega_\alpha h_{\alpha,t} \omega_\alpha^{-1})$. From the definitions we have $\omega_\alpha h_{\alpha,t} \omega_\alpha^{-1} = \varphi_\alpha \begin{pmatrix} -t^{-1} & 0 \\ 0 & -t \end{pmatrix} = h_{\alpha,-t^{-1}}$. Since $t \mapsto -t^{-1}$ is a bijection from $K^\times$ to $K^\times$ this proves $\tilde{e}_{-\alpha} = \tilde{e}_\alpha$. Now,

$$\tilde{e}_\alpha^2 = \sum_{t,u \in K^\times} a(h_{\alpha,t} h_{\alpha,u}) = \sum_{t,u \in K^\times} a(h_{\alpha,tu}) = \sum_{v \in J^\times} (\#\{(t,u) \in K^\times; tu = v\}) a(h_{\alpha,v}) = (q-1)\tilde{e}_\alpha.$$

and this proves (1). Since $\mathfrak{H}$ is commutative (2) is obvious. We now prove (3), considering two roots α, β. If $\beta \in \{\alpha, -\alpha\}$ we get the conclusion from $\tilde{e}_{-\alpha} = \tilde{e}_\alpha$. Otherwise, β and α are linearly independant. Then, with obvious notations, $s_{\check{\alpha}}(\check{\beta})$ is the coroot associated to $s_\alpha(\beta)$ in the dual root system. By the elementary properties of root systems we have $s_{\check{\alpha}}(\check{\beta}) = \check{\beta} + m\check{\alpha}$ for some $m \in \mathbf{Z}$. Then, $\chi_{\alpha,t} : x \mapsto t^{x(h_{s_\alpha(\beta)})} = t^{x(h_\beta)} t^{mx(h_\alpha)}$ hence

$$\tilde{e}_\alpha \tilde{e}_{s_\alpha(\beta)} = \sum_{t,u \in K^\times} a(h_{\alpha,t} h_{s_\alpha(\beta),u})$$

and by definition (see [7]) $h_{\alpha,t} h_{s_\alpha(\beta),u}$ corresponds to the element of $\mathrm{Hom}_{\mathbf{Z}}(L, K^\times)$ which is given by $x \mapsto t^{x(h_\alpha)} u^{x(h_\beta) + mx(h_\alpha)} = (tu^m)^{x(h_\alpha)} u^{x(h_\beta)}$. Therefore $h_{\alpha,t} h_{s_\alpha(\beta),u} = h_{\alpha,(tu)^m} h_{\beta,u}$ and

$$\tilde{e}_\alpha \tilde{e}_{s_\alpha(\beta)} = \sum_{t,u \in K^\times} a(h_{\alpha,(tu)^m}) a(h_{\beta,u}) = \sum_{t,u \in K^\times} a(h_{\alpha,t}) a(h_{\beta,u}) = \tilde{e}_\alpha \tilde{e}_\beta$$

since $(t,u) \mapsto (tu^m, u)$ is a bijection from $(K^\times)^2$ to itself. This proves (3). The proof of (4) is similar : we get $\tilde{e}_\alpha \tilde{e}_\gamma = \sum_{t,u \in K^\times} a(h_{\alpha,t} h_{\gamma,u})$ and $h_{\alpha,t} h_{\gamma,u}$ corrresponds to $x \mapsto t^{x(h_\alpha)} u^{x(h_\alpha + h_\beta)} = (tu)^{x(h_\alpha)} u^{h_\beta}$ and we conclude as before. This proves the claim. ☐

The maximal torus $\mathfrak{H}$ can be identified with $(K^\times)^l$ through the identification with $\mathrm{Hom}(L, K^\times) = \mathrm{Hom}(\bigoplus_{i=1}^l \mathbf{Z}\alpha_i, K^\times) = \prod_{i=1}^l \mathrm{Hom}(\mathbf{Z}\alpha_i, K^\times) \simeq (K^\times)^l$. If $\beta_1, \dots, \beta_k$ are roots, and $t_1, \dots, t_k \in K^\times$,then $a(h_{\beta_1,t_1}) a(h_{\beta_2,t_2}) \dots a(h_{\beta_k,t_k}) \in \mathfrak{H}$ is identified with the l-tuple

$$(t_1^{\alpha_i(\check{\beta}_1)} t_2^{\alpha_i(\check{\beta}_2)} \dots t_k^{\alpha_i(\check{\beta}_k)})_{1 \le i \le l} \in (K^\times)^l.$$

Choosing a generator a of $K^\times$, and therefore an isomorphism $K^\times \simeq \mathbf{Z}/(q-1)\mathbf{Z}$, it is identified with the l-tuple

$$\left(a^{\alpha_i(m_1\check{\beta}_1 + \dots + m_k\check{\beta}_k)} \right)_{1 \le i \le l} \in (K^\times)^l$$

where $t_j = a^{m_j}, m_j \in \mathbf{Z}/(q-1)\mathbf{Z}$, and therefore with the l-tuple $(\alpha_i(m_1\check{\beta}_1 + \dots + m_k\check{\beta}_k))_{1 \le i \le l} \in (\mathbf{Z}/(q-1)\mathbf{Z})^l$. Let us now assume that $\beta_1, \dots, \beta_k$ forms a basis of a root subsystem. Then $\tilde{e}_{\beta_1} \dots \tilde{e}_{\beta_k}$ is mapped inside $\mathbf{Z}\mathfrak{H} \simeq \mathbf{Z}[(\mathbf{Z}/(q-1)\mathbf{Z})^l]$ to $\sum_{m_1,\dots m_k \in \mathbf{Z}/(q-1)\mathbf{Z}} [\alpha_i(m_1\check{\beta}_1 + \dots + m_k\check{\beta}_k)]_{1 \le i \le l}$. We consider the map $\Phi : (\mathbf{Z}/(q-1)\mathbf{Z})^k \to (\mathbf{Z}/(q-1)\mathbf{Z})^l$ given by $(m_1, \dots, m_k) \mapsto [\alpha_i(m_1\check{\beta}_1 + \dots + m_k\check{\beta}_k)]_{1 \le i \le l}$. It is a $\mathbf{Z}$-module homomorphism, with kernel the set of $m_1, \dots, m_k$ such that $m_1\check{\beta}_1 + \dots + m_k\check{\beta}_k$ lies in the kernel of all α_i's modulo $q-1$.

Therefore $\tilde{e}_{\beta_1} \dots \tilde{e}_{\beta_k}$ is mapped to

$$(\#\mathrm{Ker}\,\Phi) \sum_{\underline{v} \in \mathrm{Im}(\Phi)} \underline{v}.$$

Let F denote the sub-lattice of the co-root lattice spanned by $\check{\beta}_1, \dots, \check{\beta}_k$, and C the Cartan matrix of the root system. The values obtained as $\underline{v} \in \mathrm{Im}(\Phi)$ are exactly the image of F under C modulo $q-1$, and $\mathrm{Ker}\,\Phi$ depends only on $q-1$, F and C. Let r be a prime dividing $q-1$ and not dividing $\det(C)$. We let $\Phi_r : \mathbf{F}_r^k \to \mathbf{F}_r^l$ denote the reduction of Φ modulo r. Then, under the map $\mathbf{Z}[(\mathbf{Z}/(q-1)\mathbf{Z})^l] \to \mathbf{Z}[\mathbf{F}_r^l]$, $\tilde{e}_{\beta_1} \dots \tilde{e}_{\beta_k}$ is mapped to

$$(\#\mathrm{Ker}\,\Phi_r) \sum_{\underline{v} \in \mathrm{Im}(\Phi_r)} [\alpha_i(\underline{v})]_{1 \le i \le l}.$$

Since C is invertible modulo r, the image $\mathrm{Im}\Phi_r$ of the lattice $F \mod rL$ under C determines $F \mod r$. Since there is a finite number of possible lattices F, there exists r_0 such that, for all prime $r \geq r_0$, the knowledge of $F \mod rL$ determines F. Let us choose such a prime number. By the Dirichlet prime number theorem there exists a prime $p = q$ such that $p \equiv 1 \mod r$, that is $r | q-1$. Therefore, the subalgebra generated by the $\tilde{e}_\alpha$ is 'generically' freely spanned by a family indexed by the collection of all *closed* symmetric subsystems of (the dual of) our original subsystem. Recall that there exists reduced root systems with proper closed symmetric subsystems of the same rank, for instance the long roots in type G_2 form a subsystem of type A_2 with this property.

2.2. **Juyumaya's generators.** In [17], Juyumaya introduced new generators L_i's of $\mathcal{H}(G,U)$ in replacement of the a_i's, keeping the $a(h)$ as they are. Choosing a non trivial additive character ψ of $(K,+)$, and using some kind of Fourier transform, he defines for every root α the element $\psi_\alpha = \sum_{r \in K^\times} \psi(r) h_{\alpha,r}$. Then, letting $L_i = q^{-1}(\tilde{e}_{\alpha_i} + a_i \psi_{\alpha_i})$ he shows, in collaboration with S. Kannan ([16], theorem 2) that $\mathcal{H}(G,U)$ admits a presentation with generators $L_1, \dots, L_l$, $a(h), h \in \mathfrak{H}$ and relations

(1) $a(h_1)a(h_2) = a(h_1h_2)$ for all $h_1, h_2 \in \mathfrak{H}$
(2) $L_i a(h) = a(h')L_i$ where $h' = \omega_i h \omega_i^{-1}$
(3) $L_i^2 = 1 - q^{-1}(\tilde{e}_{\alpha_i} - L_i \tilde{e}_{\alpha_i})$
(4) $\underbrace{L_iL_jL_i\dots}_{m_{ij}} = \underbrace{L_jL_iL_j\dots}_{m_{ij}}$ for $1 \le i,j \le l$

Then, letting $u = q^{-1}$, $e_\alpha = (q-1)^{-1}\tilde{e}_\alpha$, $e_i = e_{\alpha_i}$ and $g_i = -L_i$, this presentation becomes the following one :

(1) $a(h_1)a(h_2) = a(h_1h_2)$ for all $h_1, h_2 \in \mathfrak{H}$
(2) $g_i a(h) = a(h')g_i$ where $h' = \omega_i h \omega_i^{-1}$
(3) $g_i^2 = 1 + (u-1)e_i(1+g_i)$
(4) $\underbrace{g_ig_jg_i\dots}_{m_{ij}} = \underbrace{g_jg_ig_j\dots}_{m_{ij}}$ for $1 \le i,j \le l$

2.3. **Yokonuma-Hecke algebras of type** A. A particularly studied variation of the above construction mimics the situation above for the (non-semisimple !) reductive group $\mathrm{GL}_n(K)$ with K a 'field of order $d+1$'. Let us fix a commutative ring $\mathbf{k}$ (with 1), $u \in \mathbf{k}$, $d \in \mathbf{Z}_{>0}$. We assume that $d.1$ and u are invertible in $\mathbf{k}$. The literature on the subject, see e.g. [8], denotes $Y_{d,n}(u)$ and calls the Yokonuma-Hecke algebra of type A the $\mathbf{k}$-algebra generated with generators $g_1, \dots, g_{n-1}$, $t_1, \dots, t_n$ and relations

(1) $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$, $g_ig_j = g_jg_i$ if $|j-i| \geq 2$ (braid relations),
(2) $t_it_j = t_jt_j$, $g_it_j = t_{s_i(j)}g_i$ for all i,j, where s_i is the transposition $(i,i+1)$;
(3) $t_i^d = 1$ for all i,
(4) $g_i^2 = 1 + (u-1)e_i(1+g_i)$

where, by definition $e_i = e_{i,i+1}$ with

$$e_{i,j} = \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_j^{-s}.$$

whenever $i \neq j$ and $1 \le i,j \le n$. The elements g_i are invertible, with inverse $g_i^{-1} = g_i + (u^{-1} - 1)e_i + (u^{-1}-1)e_ig_i$. It can be easily proved that the following relations hold :

(5) $e_{ij} = e_{ji}$ for all $i \neq j$
(6) $e_{i,j}e_{k,l} = e_{k,l}e_{i,j}$ for all $i \neq j$, $k \neq l$
(7) $g_ie_{j,k} = e_{s_i(j),s_i(k)}g_i$ for all i,j,k with $k \neq j$

(8) $e_{ij}^2 = e_{ij}$ for all $i \neq j$.

The subalgebra of $Y_{d,n}(u)$ generated by the g_i's and e_i's has been called the algebra of braids and ties by J. Juyumaya and F. Aicardi, according to [31], and provided a diagrammatic description. A Markov trace was subsequently constructed on this algebra, see [1]. This subalgebra is efficiently studied in [31], where S. Ryom-Hansen provides a faithful module for this algebra, and uses it to show that the algebra has dimension $n! Bell_n$, where $Bell_n$ is the n-th Bell number. Theorem 3.4 below generalizes this last statement.

Now we notice that, in [9], M. Chlouveraki and L. Poulain d'Andecy introduce other generators $g_i' = g_i + (v^{-1}-1)e_i g_i$, with $u = v^2$. The relation between g_i' and e_i is then $(g_i')^2 = 1+(v-v^{-1})e_i g_i'$. They notice that these generators also satisfy the braid relations. We will give a general explanation for this phenomenon in section 4.1.

3. Construction of the algebra $\mathcal{C}_W$

3.1. General construction.

Here $\mathbf{k}$ is a commutative ring (with 1). Let W denote a Coxeter group, with generating set S. We let $\mathcal{R} \supset S$ denote its set of reflections. If W is finite this set can be defined as the geometric reflections of W in its natural representation, and in the general case this is the set of conjugates of S. We denote $\mathcal{P}_f(\mathcal{R})$ the set of all finite subsets of $\mathcal{R}$, and by $\mathcal{P}(\mathcal{R})$ the set of all its subsets. We recall that a reflection subgroup of W is a subgroup generated by a subset of $\mathcal{R}$.

We also recall that a Coxeter group W given by the Coxeter system (W, S) is finitely generated as a group if and only if S is finite. Indeed, if $W = \langle x_1, \dots, x_n \rangle$ for some $x_1, \dots, x_n$, we can write the x_i's as a product of a finite number of elements of S, hence W is equal to its standard parabolic subgroup (W_X, X) for some finite $X \subset S$. Since $W_X \cap S = X$ ([4], IV §1 No. 8, corollaire 2) this proves that $S = X$ is finite.

Finally, we recall from Dyer's thesis the following basic fact, extending a well-known property of finite Coxeter groups to general ones :

Proposition 3.1. *(Dyer, PhD thesis, theorem 1.8; see also* [13] *corollary 3.11 (ii) and Deodhar* [12]*) Let W_0 a reflection subgroup of W. Then W_0 is a Coxeter group (W_0, S_0) with $S_0 \subset \mathcal{R}$ and $W_0 \cap \mathcal{R} = \mathcal{R}_0$, with $\mathcal{R}_0$ the set of reflections of (W_0, S_0). Moreover, if W_0 is generated by $J \subset \mathcal{R}$, then every element of $\mathcal{R}_0$ is a conjugate inside W_0 of an element of J.*

For every $s \in S$, we choose $u_s \in \mathbf{k}$ such that $s_1 \sim s_2 \Rightarrow u_{s_1} = u_{s_2}$, where $a \sim b$ means that $a, b \in S$ lie in the same conjugacy class. We set $\underline{u} = (u_s)_{s \in S}$ and define $\mathcal{C}_W(\underline{u})$ be the $\mathbf{k}$-algebra (with 1) defined by generators $g_s, s \in S$, $e_t, t \in \mathcal{R}$, and relations

(1) $\underbrace{g_s g_t g_s \dots}_{m_{st}} = \underbrace{g_t g_s g_t \dots}_{m_{st}}$ for $s, t \in S$

(2) $e_t^2 = e_t$ for all $t \in \mathcal{R}$

(3) $e_{t_1} e_{t_2} = e_{t_2} e_{t_1}$ for all $t_1, t_2 \in \mathcal{R}$

(4) $e_t e_{t_1} = e_t e_{t t_1 t^{-1}}$ for all $t, t_1, t_2 \in \mathcal{R}$

(5) $g_s e_t = e_{sts} g_s$ for all $s \in S$, $t \in \mathcal{R}$

(6) $g_s^2 = 1 + (u_s - 1) e_s (1 + g_s)$ for all $s \in S$.

Note that $\mathcal{C}_W(\underline{u})$ is actually finitely generated as soon as S is finite, by the following elementary proposition.

Proposition 3.2. *The algebra $\mathcal{C}_W(\underline{u})$ is generated by the g_s, e_s for $s \in S$.*

Proof. Let A be the subalgebra of $\mathcal{C}_W(\underline{u})$ generated by the g_s, e_s for $s \in S$. It is sufficient to show that $e_t \in A$ for all $t \in \mathcal{R}$. By definition such a t can be written as $w^{-1} s_0 w$ for some $s_0 \in S$ and $w \in W$. Writing $w = s_1 \dots s_r$ with $s_1, \dots, s_r \in S$, we need to prove $e_{s_r s_{r-1} \dots s_1 s_0 s_1 \dots s_r} \in A$ for all $s_0, s_1, \dots, s_r \in S$. By induction on r this results from the relation $g_{s_r} e_{s_{r-1} \dots s_1 s_0 s_1 \dots s_{r-1}} g_{s_r}^{-1} = e_{s_r s_{r-1} \dots s_1 s_0 s_1 \dots s_{r-1} s_r}$. □

For $w \in W$, we let $g_w = g_{s_1} \dots g_{s_r}$ if $s_1 \dots s_r$ is a reduced expression of w. Since the g_s's satisfy the braid relations this does not depend on the chosen expression by Iwahori-Matsumoto's theorem.

For $J \in \mathcal{P}_f(\mathcal{R})$, we set $e_J = \prod_{t\in J} e_t$. In order to study these elements we define an equivalence relation on $\mathcal{P}_f(\mathcal{R})$ by taking the equivalence relation generated by $J \sim K$ when J contains $\{s,t\}$ and $K = J \cup \{sts\}$.

From this we get that this equivalence relation can be restated as follows. Say that J and K are equivalent if $\langle J\rangle = \langle K\rangle$, that is if the reflection subgroups generated by J and K are the same.

It is clear that, if $J \sim K$, then $\langle J\rangle = \langle K\rangle$. Conversely, let us assume $\langle J\rangle = \langle K\rangle$. If $J = \emptyset$ or $K = \emptyset$ then clearly $J \sim K$. Otherwise, let us set $W_0 = \langle J\rangle = \langle K\rangle$. By proposition 3.1 every reflection of W_0 is conjugated to an element of J, and also to an element of K. In particular every element of K is conjugated to an element of J by some element of $\langle J\rangle$. Writing such an element as $s_r s_{r-1}\dots s_1 s_0 s_1 \dots s_{r-1} s_r$ for $s_0,\dots,s_r \in J$, we easily get by induction on r that $K \subset J'$, for some $J' \in \mathcal{P}_f(\mathcal{R})$ with $J' \sim J$. Therefore, we can assume $K \subset J$. By the same argument, every element of J can be written as $s_r s_{r-1}\dots s_1 s_0 s_1 \dots s_{r-1} s_r$ for $s_0,\dots,s_r \in K \subset J$. By induction on $|J \setminus K|$ we get from this that $J \sim K$.

Therefore, the set of equivalence classes is in natural bijection with the collection $\mathcal{W}$ of finitely generated reflection subgroups of W. In particular, when W is finite, the number of equivalence classes can be identified with the number of reflection subgroups of W. Notice that, when W is the Weyl group of some root system R, then reflection subgroups are in 1-1 correspondence with root subsystems (in the sense of a subset of R satisfying the axioms of root systems, as in [4]).

By relations (2) and (4) above, we have $e_s e_t = e_s e_t e_t = e_s e_{sts} e_t$ and thus $J \sim K$ implies $e_J = e_K$. Therefore, we can define e_{W_0} for every finitely generated reflection subgroup W_0 of W, by letting $e_{W_0} = e_J$ for any $J \in \mathcal{P}_f(\mathcal{R})$ with $\langle J\rangle = W_0$. Notice that, when W is finite, there is a distinguished representative of the class of $J \in \mathcal{P}_f(\mathcal{R}) = \mathcal{P}(\mathcal{R})$, namely $\overline{J} := \langle J\rangle \cap \mathcal{R}$. In the general case, one can make a different choice, taking for $\overline{J}$ Dyer's canonical set of Coxeter generators for $\langle J\rangle$ (since such set can be infinite only if the Coxeter group is not finitely generated). In the sequel, we will denote $\overline{J} \in \mathcal{P}_f(\mathcal{R})$ the chosen representative of the class of $J \in \mathcal{P}_f(\mathcal{R})$.

3.2. **An extension of the Iwahori-Hecke algebra.** The algebra $\mathcal{C}_W(\underline{u})$ is an extension of the Iwahori-Hecke algebra $H_W(\underline{u})$. We let $T_s, s \in S$ denote the natural generators of W, and $T_w = T_{s_1}\dots T_{s_r}$ when $w = s_1 \dots s_r$ is a reduced expression of $w \in W$.

Proposition 3.3. *Let (W,S) denote a Coxeter system.*

(1) The map $g_s \mapsto T_s$, $e_s \mapsto 1$ induces a surjective $\mathbf{k}$-algebra morphism $\mathfrak{p} : \mathcal{C}_W(\underline{u}) \twoheadrightarrow H_W(\underline{u})$. For $w \in W$, it maps g_w to T_w and each e_J to 1. Its kernel is the two-sided ideal generated by the $e_s - 1$, $s \in S$.

(2) If S is finite, then $\mathfrak{p}$ is split. A splitting is given by $T_w \mapsto g_w e_S$, with $e_S = e_W = \prod_{s\in S} e_s$.

Proof. One gets that the map $g_s \mapsto T_s$, $e_s \mapsto 1$ induces a morphism of (unital) $\mathbf{k}$-algebra $\mathfrak{p}$: $\mathcal{C}_W(\underline{u}) \to H_W(\underline{u})$, by checking that the defining relations of $\mathcal{C}_W(\underline{u})$ hold inside $H_W(\underline{u})$. This is immediate for relations (1)-(5), and (6) is mapped to the defining relation $T_s^2 = u_s + (u_s - 1)T_s$ of $H_W(\underline{u})$. This morphism is surjective because the T_s's generate $H_W(\underline{u})$ as a unital $\mathbf{k}$-algebra. By definition of g_w and T_w it is clear that $\mathfrak{p}(g_w) = T_w$ for all $w \in W$, and similarly that $\mathfrak{p}(e_J) = 1$ for all J's. It is clear that $\mathcal{C}_W(\underline{u})$ is spanned by the $g_w e_J$, with $w \in W$ and $J \in \mathcal{P}_f(\mathcal{R})$. An element $x \in \operatorname{Ker}\mathfrak{p}$ can be written $\sum_{w,J} a_{w,J} g_w e_J$ with $a_{w,J} \in \mathbf{k}$ almost all zero, such that $0 = \sum_{w,J} a_{w,J} T_w = \sum_w \left(\sum_J a_{w,J}\right) T_w$. Let us fix $w \in W$, and let $b_J = a_{w,J}$. We have $\sum b_J = 0$ since the T_w's form a basis of $H_W(\underline{u})$, so it is sufficient to prove that every element in $x \in \mathfrak{p}$ of the form $\sum_J b_J e_J$ belongs to the ideal $\mathfrak{I}$ generated by the $e_s - 1, s \in S$. This amounts to saying that $e_J - 1 \in \mathfrak{I}$ for all J. Letting $r(W_0)$ denotes the minimal number of reflections needed for generating W_0, we prove this by induction on $r(\langle J\rangle)$. The case $r(\langle J\rangle) = 0$ is obvious, the case $r(\langle J\rangle) = 1$ is a consequence of $g_w(e_s - 1)g_w^{-1} = e_{wsw^{-1}} - 1$ for all $w \in W$ and $s \in S$. Now, if $r(\langle J\rangle) > 1$, there exists $t \in J$ such that $r(\langle K\rangle) < r(\langle J\rangle)$, where $K = J \setminus \{t\}$. Again because $g_w(e_J - 1)g_w^{-1} = e_{wJw^{-1}} - 1$, we can assume $s \in S$. Then, $e_J = e_K e_s$ and $e_J - 1 = e_K(e_s - 1) + e_K - 1 \in e_K - 1 + \mathfrak{I}$, so we get $e_J - 1 \in \mathfrak{I}$ by the induction assumption. This completes the proof of (1).

In order to prove (2), we first note that e_W is central and idempotent. We prove that $T_s \mapsto g_s e_W$, $1 \mapsto e_W$ induces an algebra morphism. Since e_W is central, the braid relations $T_s T_t T_s \dots = T_t T_s T_t \dots$ are mapped to $e_W^{m_{st}} g_s g_t g_s \dots = e_W^{m_{st}} g_t g_s g_t \dots$ and this holds true inside $\mathcal{C}_W(\underline{u})$. The quadratic relation $T_s^2 = (u_s-1)T_s + u_s$ is mapped to $g_s^2 e_W = (u_s-1)g_s e_W + u_s e_W$. We prove that this holds true, because the relation $g_s^2 = 1+(u_s-1)e_s(1+g_s)$ implies $g_s^2 e_s = e_s+(u_s-1)e_s(1+g_s) = u_s e_s + (u_s-1)g_s$ and therefore, since $e_s e_W = e_W$, we get $g_s^2 e_W = u_s e_W + (u_s-1)g_s e_W$. Therefore there exists a $\mathbf{k}$-algebra morphism $\mathfrak{q} : H_W(\underline{u}) \to \mathcal{C}_W(\underline{u})$, which maps T_w to $g_w e_W$ as is readily checked by induction on $\ell(w)$. We have $\mathfrak{p}(\mathfrak{q}(T_w)) = \mathfrak{p}(g_w e_W) = T_w$, and this proves (2).

□

3.3. **Description as a module.**

Theorem 3.4. *The algebra $\mathcal{C}_W(\underline{u})$ is a free $\mathbf{k}$-module with basis the $e_{\bar{J}} g_w$, for $w \in W$ and $J \in \mathcal{P}_f(\mathcal{R})$. In particular, if W is finite then it has for rank the order $|W|$ of W multiplied by the number $|\mathcal{W}|$ of reflection subgroups of W.*

We will see in section 3.6 that $|\mathcal{W}|$ may be called the Bell number of type W.

Proof. We denote by ℓ the classical length function on the Coxeter group W. To each $J \in \mathcal{P}_f(\mathcal{R})$ we associate $\varepsilon_J = \prod_{t\in J} e_t$. Let us consider $J \in \mathcal{P}_f(\mathcal{R})$, $w \in W$ and $s \in S$. Then $g_s e_J g_w = e_{sJs^{-1}} g_s g_w$. If $\ell(sw) = \ell(w)+1$ we have $g_s g_w = g_{sw}$ and we get $g_s.e_J g_w = e_{sJs^{-1}} g_{sw}$. Otherwise w can be written $w = sw'$ with $\ell(w') = \ell(w)-1$. Then $g_s g_w = g_s^2 g_{w'} = g_{w'} + (u_s-1)e_s(1+g_s)g_{w'} = g_{w'} + (u_s-1)e_s g_{w'} + (u_s-1)e_s g_s g_{w'} = g_{w'} + (u_s-1)e_s g_{w'} + (u_s-1)e_s g_w$. It follows that $g_s.e_J g_w = e_{sJs} g_{w'} + (u_s-1)e_{sJs} e_s g_{w'} + (u_s-1)e_{sJs} e_s g_w = e_{sJs} g_{sw} + (u_s-1)e_{sJs\cup\{s\}} g_{sw} + (u_s-1)e_{sJs\cup\{s\}} g_w$. Finally, in all cases we have $e_s.(e_J g_w) = e_{J\cup\{s\}} g_w$. Since $\mathcal{C}_W(\underline{u})$ is generated as a unital algebra by the g_s and e_s, $s \in S$ this proves that the set of the $e_J g_w$ for $J \in \mathcal{P}_f(\mathcal{R})$, $w \in W$, and therefore of the $e_{\bar{J}} g_w$ for $J \in \mathcal{P}_f(\mathcal{R})$, $w \in W$, is a spanning set for $\mathcal{C}_W(\underline{u})$.

We notice that $(e_J g_w)e_s = e_J e_{wsw^{-1}} g_w = e_{J\cup\{wsw^{-1}\}} g_w$ and, if $\ell(ws) = \ell(w)+1$, then $(e_J g_w)g_s = e_J g_{ws}$. If $\ell(ws) = \ell(w)-1$, then $e_J g_w g_s = e_J g_{ws} g_s^2 = e_J g_{ws}(1+(u_s-1)e_s(1+g_s)) = e_J g_{ws} + (u_s-1)e_J g_{ws} e_s + (u_s-1)e_J g_{ws} e_s g_s = e_J g_{ws} + (u_s-1)e_J e_{ws.s.(ws)^{-1}} g_{ws} + (u_s-1)e_J e_{ws.s.(ws)^{-1}} g_{ws} g_s = e_J g_{ws} + (u_s-1)e_J e_{wsw^{-1}} g_{ws} + (u_s-1)e_J e_{wsw^{-1}} g_w = e_J g_{ws} + (u_s-1)e_{J\cup\{wsw^{-1}\}} g_{ws} + (u_s-1)e_{J\cup\{wsw^{-1}\}} g_w$.

We now consider a free $\mathbf{k}$-module V with basis $v_{J,w}$ for $J \in \mathcal{P}_f(\mathcal{R})$, $w \in W$, with the convention $v_{J,w} = v_{K,w}$ if $J \sim K$. We introduce $\mathbf{k}$-linear endomorphisms $G_s, E_s, G'_s, E'_s \in \mathrm{End}(V)$ defined by the formulas $E_s.v_{J,w} = v_{J\cup\{s\},w}$ and

$$\begin{array}{rcll} G_s.v_{J,w} & = & v_{sJs,sw} & \text{if } \ell(sw)=\ell(w)+1 \\ & = & v_{sJs,sw} + (u_s-1)v_{sJs\cup\{s\},sw} + (u_s-1)v_{sJs\cup\{s\},w} & \text{if } \ell(sw)=\ell(w)-1 \\ E_s.v_{J,w} & = & v_{J\cup\{s\},w} & \\ E'_s.v_{J,w} & = & v_{J\cup\{wsw^{-1}\},w} & \\ G'_s.v_{J,w} & = & v_{J,ws} & \text{if } \ell(ws)=\ell(w)+1 \\ & = & v_{J,ws} + (u_s-1)v_{J\cup\{wsw^{-1}\},ws} + (u_s-1)v_{J\cup\{wsw^{-1}\},w} & \text{if } \ell(ws)=\ell(w)-1 \end{array}$$

We easily check on these formulas that $E_s^2 = E_s$, $(E'_s)^2 = E'_s$. Moreover

$$E_s E'_t.v_{J,w} = E_s.v_{J\cup\{wtw^{-1}\},w} = v_{J\cup\{wtw^{-1}\}\cup\{s\},w},$$

while $E'_t E_s v_{J,w} = E'_t.v_{J\cup\{s\},w} = v_{J\cup\{s\}\cup\{wtw^{-1}\},w}$. This proves $E_s E'_t = E'_t E_s$ for all s,t. Similarly, if $\ell(wt) = \ell(w)+1$, we have $E_s G'_t.v_{J,w} = E_s v_{J,wt} = v_{J\cup\{s\},wt}$. Otherwise $G'_t E_s.v_{J,w} = G'_t.v_{J\cup\{s\},w} = v_{J\cup\{s\},wt}$; if $\ell(wt) = \ell(w)-1$, we have $E_s G'_t.v_{J,w} = E_s.(v_{J,wt} + (u_t-1)v_{J\cup\{wtw^{-1}\},wt} + (u_t-1)v_{J\cup\{wtw^{-1}\},w}) = v_{J\cup\{s\},wt} + (u_t-1)v_{J\cup\{wtw^{-1}\}\cup\{s\},wt} + (u_t-1)v_{J\cup\{wtw^{-1}\}\cup\{s\},w}$ and $G'_t E_s v_{J,w} = G'_t v_{J\cup\{s\},w} = v_{J\cup\{s\},wt} + (u_t-1)v_{J\cup\{wtw^{-1}\}\cup\{s\},wt} + (u_t-1)v_{J\cup\{s\}\cup\{wtw^{-1}\},w}$, which proves $G'_t E_s = E_s G'_t$ for all s,t. By a similar computation we get $G_t E'_s = E'_s G_t$ for all s,t.

We now want to check that $G_s G'_t = G'_t G_s$. We first recall the following classical fact, of which we recall a proof for the convenience of the reader:

Lemma 3.5. *Let (W,S) be a Coxeter system. For $s,t \in S$ and $w \in W$, the equalities $\ell(swt) = \ell(w)$ and $\ell(sw) = \ell(wt)$ imply $sw = wt$.*

Proof. $\ell(swt) = \ell(w)$ implies that, either $\ell(sw) = \ell(w)+1$ and $\ell(swt) = \ell(sw)-1$, or $\ell(sw) = \ell(w)-1$ and $\ell(swt) = \ell(sw)+1$. We start dealing with the first case. Let $n = \ell(w)$ and $s_1 \dots s_n = w$ a reduced expression. Since $\ell(wt) = \ell(sw) = \ell(w)+1$ we get that $wt = s_1 \dots s_n t$ is again a reduced expression. Since $\ell(s.wt) < \ell(wt)$ we get from the exchange lemma that, either $swt = s_1 \dots s_{j-1} s_{j+1} \dots s_n t$ for some $j \in \{1, \dots, n\}$, or $swt = s_1 \dots s_n$. In the first case we would have $sw = s_1 \dots s_{j-1} s_{j+1} \dots s_n$, contradicting $\ell(sw) = n+1$. Therefore $swt = s_1 \dots s_n = w$ and $sw = wt$.

Now, if $\ell(sw) = \ell(w) - 1$ and $\ell(swt) = \ell(sw) + 1$, letting $w' = sw$ we can apply the previous discussion and get $sw' = w't$, that is $sw = wt$.

□

If $\ell(sw) = \ell(w) + 1$ and $\ell(wt) = \ell(w) + 1$ then, either $\ell(swt) = \ell(wt) + 1 = \ell(sw) + 1$, or $\ell(swt) = \ell(wt) - 1 = \ell(w)$ in which case $sw = wt$. In the first case, we have $G_s G'_t v_{J,w} = G_s v_{J,wt} = v_{sJs,swt}$ and $G'_t G_s v_{J,w} = G'_t v_{sJs,sw} = v_{sJs,swt}$; in the second case, we have $G_s G'_t v_{J,w} = G_s v_{J,wt} = G_s v_{J,sw} = v_{sJs,w} + (u_s - 1) v_{sJs \cup \{s\},w} + (u_s - 1) v_{sJs \cup \{s\},sw}$ and $G'_t G_s v_{J,w} = G'_t v_{sJs,sw} = G'_t v_{sJs,wt} = v_{sJs,w} + (u_t - 1) v_{sJs \cup \{wtw^{-1}\},w} + (u_t - 1) v_{sJs \cup \{wtw^{-1}\},wt}$. Since the condition $sw = wt$ implies $wtw^{-1} = s$ and in particular $s \sim t$, whence $u_s = u_t$. Therefore, $G_s G'_t . v_{J,w} = G'_t G_s . v_{J,w}$.

If $\ell(sw) = \ell(w) + 1$ and $\ell(wt) = \ell(w) - 1$, then we have $\ell(swt) = \ell(w)$, for otherwise $\ell(swt) = \ell(w) - 2$ and $\ell(sw) < \ell(w)$. Then $G_s G'_t v_{J,w} = G_s (v_{J,wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},w}) = G_s (v_{J,wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},w}) = v_{sJs,swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}s\},swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}s\},sw}$ while $G'_t G_s v_{J,w} = G'_t v_{sJs,sw} = v_{sJs,swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}\}s,swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}s\},sw}$ hence $G'_t G_s v_{J,w} = G_s G'_t v_{J,w}$.

If $\ell(sw) = \ell(w) - 1$ and $\ell(wt) = \ell(w) + 1$, then we have $\ell(swt) = \ell(w)$ for the same reason as in the preceding case. Then $G_s G'_t v_{J,w} = G_s v_{J,wt} = v_{sJs,swt} + (u_s - 1) v_{sJs \cup \{s\},swt} + (u_s - 1) v_{sJs \cup \{s\},wt}$ while $G'_t G_s v_{J,w} = G'_t (v_{sJs,sw} + (u_s - 1) v_{sJs \cup \{s\},sw} + (u_s - 1) v_{sJs \cup \{s\},w}) = v_{sJs,swt} + (u_s - 1) v_{sJs \cup \{s\},swt} + (u_s - 1) v_{sJs \cup \{s\},wt}$ hence $G'_t G_s v_{J,w} = G_s G'_t v_{J,w}$.

If $\ell(sw) = \ell(w) - 1 = \ell(wt)$, then

- either $\ell(swt) = \ell(wt) - 1 = \ell(sw) - 1$, in which case

$$\begin{array}{rcl} G_s G'_t v_{J,w} & = & G_s (v_{J,wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},w}) \\ & = & v_{sJs,swt} + (u_s - 1) v_{sJs \cup \{s\},swt} + (u_s - 1) v_{sJs \cup \{s\},wt} + (u_t - 1)(v_{sJs \cup \{swtw^{-1}s\},swt} \\ & + & (u_s - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},swt} + (u_s - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},wt}) \\ & + & (u_t - 1)(v_{sJs \cup \{swtw^{-1}s\},sw} + (u_s - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},sw} \\ & + & (u_s - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},w}) \end{array}$$

and

$$\begin{array}{rcl} G'_t G_s v_{J,w} & = & G'_t (v_{sJs,sw} + (u_s - 1) v_{sJs \cup \{s\},sw} + (u_s - 1) v_{sJs \cup \{s\},w}) \\ & = & v_{sJs,swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}s\},swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}s\},sw} \\ & + & (u_s - 1)(v_{sJs \cup \{s\},swt} + (u_t - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},swt} + (u_t - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},sw}) \\ & + & (u_s - 1)(v_{sJs \cup \{s\},wt} + (u_t - 1) v_{sJs \cup \{s\} \cup \{wtw^{-1}\},wt} + (u_t - 1) v_{sJs \cup \{s\} \cup \{wtw^{-1}\},w}) \end{array}$$

Therefore, these terms are equal as soon as we have

$$v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},wt} + v_{sJs \cup \{s\} \cup \{swtw^{-1}s\},w} = v_{sJs \cup \{s\} \cup \{wtw^{-1}\},wt} + v_{sJs \cup \{s\} \cup \{wtw^{-1}\},w}.$$

Since

$$\overline{sJs \cup \{s\} \cup \{swtw^{-1}s\}} = \overline{sJs \cup \{s\} \cup \{wtw^{-1}\}}$$

this holds true.

- or $\ell(swt) = \ell(w)$. But since $\ell(sw) = \ell(wt)$ this implies $sw = wt$. Then

$$\begin{array}{rcl} G_s G'_t v_{J,w} & = & G_s (v_{J,wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},wt} + (u_t - 1) v_{J \cup \{wtw^{-1}\},w}) \\ & = & v_{sJs,swt} + (u_t - 1) v_{sJs \cup \{swtw^{-1}s^{-1}\},swt} + (u_t - 1)(v_{sJs \cup \{swtw^{-1}s^{-1}\},sw} \\ & & + (u_s - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s^{-1}\},sw} + (u_s - 1) v_{sJs \cup \{s\} \cup \{swtw^{-1}s^{-1}\},w}) \end{array}$$

and

$$\begin{array}{rcl} G'_t G_s v_{J,w} & = & G'_t (v_{sJs,sw} + (u_s - 1) v_{sJs \cup \{s\},sw} + (u_s - 1) v_{sJs \cup \{s\},w}) \\ & = & v_{sJs,swt} + (u_s - 1) v_{sJs \cup \{s\},swt} + (u_s - 1) G'_t v_{sJs \cup \{s\},w} \\ & = & v_{sJs,swt} + (u_s - 1) v_{sJs \cup \{s\},swt} + (u_s - 1)(v_{sJs \cup \{s\},wt} \\ & & + (u_s - 1) v_{sJs \cup \{s\} \cup \{wtw^{-1}\},wt} + (u_t - 1) v_{sJs \cup \{s\} \cup \{wtw^{-1}\},w}) \end{array}$$

Since $sw = wt$ implies $swt = w$, $s = wtw^{-1}$, $swtw^{-1}s^{-1} = s$ and $u_s = u_t$, these two expressions are equal.

We thus proved that the G_s, E_s commute with the G'_t, E'_t for $s, t \in S$.

We finally define, for $K \subset \mathcal{R}$, the endomorphism $E_K \in \mathrm{End}(V)$ by $E_K.v_{J,w} = v_{J\cup K,w}$. For $s \in S$ we have $G_s E_K.v_{J,w} = G_s.v_{J\cup K,w} = v_{sJs\cup sKs,sw}$ and $E_{sKs}G_s.v_{J,w} = E_{sKs}v_{sJs,sw} = v_{sKs\cup sJs,sw}$ if $\ell(sw) = \ell(w)+1$, while $G_sE_K.v_{J,w} = G_s.v_{J\cup K,w} = v_{sJs\cup sKs,sw} + (u_s-1)v_{sJs\cup sKs\cup\{s\},sw} + (u_s-1)v_{sJs\cup sKs\cup\{s\},w}$ and $E_{sKs}G_s v_{J,w} = E_{sKs}(v_{sJs,w} + (u_s-1)v_{sJs\cup\{s\},sw} + (u_s-1)v_{sJs\cup\{s\},w}) = v_{sJs\cup sKs,sw} + (u_s-1)v_{sJs\cup sKs\cup\{s\},sw} + (u_s-1)v_{sJs\cup sKs\cup\{s\},w}$ it $\ell(sw) = \ell(w)-1$. This proves $G_sE_K = E_{s(K)}G_s$ for all $s \in S$.

Now, for $s, t \in S$, we denote m_{st} the order of st in W. We let

$$\omega = \underbrace{sts\dots}_{m_{st}} = \underbrace{tst\dots}_{m_{st}} \in W.$$

Then,

$$\underbrace{G_sG_tG_s\dots}_{m_{st}} v_{\emptyset,1} = v_{\emptyset,\underbrace{sts\dots}_{m_{st}}} = v_{\emptyset,\underbrace{tst\dots}_{m_{st}}} = \underbrace{G_tG_sG_t\dots}_{m_{st}} v_{\emptyset,1}$$

hence, writing w as $t_1\dots t_r$ with $t_i \in S$, we have

$$\begin{aligned}
\underbrace{G_sG_tG_s\dots}_{m_{st}} v_{J,w} &= \underbrace{G_sG_tG_s\dots}_{m_{st}} G'_{t_r}\dots G'_{t_2}G'_{t_1}v_{J,1} \\
&= G'_{t_r}\dots G'_{t_2}G'_{t_1}\underbrace{G_sG_tG_s\dots}_{m_{st}} v_{J,1} \\
&= G'_{t_r}\dots G'_{t_2}G'_{t_1}\underbrace{G_sG_tG_s\dots}_{m_{st}} E_J v_{J,1} \\
&= G'_{t_r}\dots G'_{t_2}G'_{t_1}E_{\omega J\omega^{-1}}\underbrace{G_sG_tG_s\dots}_{m_{st}} v_{\emptyset,1} \\
&= E_{\omega J\omega^{-1}}G'_{t_r}\dots G'_{t_2}G'_{t_1}\underbrace{G_sG_tG_s\dots}_{m_{st}} v_{\emptyset,1} \\
&= E_{\omega J\omega^{-1}}G'_{t_r}\dots G'_{t_2}G'_{t_1}\underbrace{G_tG_sG_t\dots}_{m_{st}} v_{\emptyset,1} \\
&= E_{\omega J\omega^{-1}}\underbrace{G_tG_sG_t\dots}_{m_{st}} G'_{t_r}\dots G'_{t_2}G'_{t_1}v_{\emptyset,1} \\
&= E_{\omega J\omega^{-1}}\underbrace{G_tG_sG_t\dots}_{m_{st}} v_{\emptyset,w} \\
&= \underbrace{G_tG_sG_t\dots}_{m_{st}} E_J v_{\emptyset,w} \\
&= \underbrace{G_tG_sG_t\dots}_{m_{st}} E_J v_{J,w}
\end{aligned}$$

From this we get that the map $g_s \mapsto G_s$, $e_J \mapsto E_J$ induces a $\mathbf{k}$-algebra homomorphism $\mathcal{C}_W(\underline{u}) \to \mathrm{End}(V)$. We let A denote its image. Since the $e_{\overline{J}}g_w$ span $\mathcal{C}_W(\underline{u})$ and their image maps $v_{\emptyset,1}$ to $v_{\overline{J},w}$ we get that this homomorphism is injective, and that its image surjects onto the free $\mathbf{k}$-module V under the map $a \mapsto a.v_{\emptyset,1}$. This proves the claim. □

3.4. **Meaningful quotients.** We recall that $\mathcal{W}$ denotes the collection of finitely generated reflection subgroups of W, endowed with the conjugation action of W. If $J \in \mathcal{P}_f(\mathcal{R})$, we let $e_{\langle J\rangle} = e_{\overline{J}} = e_J$. The algebra $\mathcal{C}_W(\underline{u})$ is spanned by elements $e_{\langle J\rangle}g_w$ for $w \in W$ and $\langle J\rangle \in \mathcal{W}$. Let $\mathcal{F}$ be a W-set and $p = \mathcal{W} \to \mathcal{F}$ be a surjective map which is W-equivariant. Such a map can be seen as an equivalence relation on $\mathcal{W}$ compatible with the action of W. We also assume that $p(\langle J\rangle) = p(\langle K\rangle)$ implies $p(\langle J, s\rangle) = p(\langle K, s\rangle)$ for all $s \in S$.

Proposition 3.6. *Let $p : \mathcal{W} \twoheadrightarrow \mathcal{F}$ be as above, and I_p the ideal of $\mathcal{C}_W(\underline{u})$ generated by the $e_J - e_K$ for $p(J) = p(K)$. The quotient algebra $\mathcal{C}^{\mathcal{F}}_W(\underline{u}) = \mathcal{C}_W(\underline{u})/I_p$ is a free module, of rank $|W|.|\mathcal{F}|$ if W is finite. The algebra morphism $\mathfrak{p} : \mathcal{C}_W(\underline{u}) \twoheadrightarrow H_W(\underline{u})$ factorizes through the natural projection $\mathcal{C}_W(\underline{u}) \twoheadrightarrow \mathcal{C}^{\mathcal{F}}_W(\underline{u})$.*

Proof. Let I'_p be the $\mathbf{k}$-module spanned by the $(e_{\langle J\rangle} - e_{\langle K\rangle})g_w$ for $w \in W$ and $p(\langle J\rangle) = p(\langle K\rangle)$. Since $p(\langle J\rangle) = p(\langle K\rangle)$ implies $p(\langle J, s\rangle) = p(\langle K, s\rangle)$ we know that $e_s I'_p \subset I'_p$ for all $s \in S$; since p is equivariant we have $g_w I'_p g_w^{-1} \subset I'_p$ for all $w \in W$ and therefore $I'_p e_s \subset I'_p$. From this and the defining relation (6) of $\mathcal{C}_W(\underline{u})$ we get $I'_p g_s \subset I'_p$ for all $s \in S$, and $g_s I'_p = g_s I'_p g_s^{-1}.g_s \subset I'_p$. Therefore I'_p is an ideal. Since $I'_p \subset I_p$ we get $I_p = I'_p$ hence I_p is spanned by the $(e_{\langle J\rangle} - e_{\langle K\rangle})g_w$ for $w \in W$ and $p(\langle J\rangle) = p(\langle K\rangle)$. The assertion on the structure as a module and the rank then follows from the previous theorem. The factorization assertion is clear from the definition of I_p and proposition 3.3. □

Important examples of such p are the following ones :

1. $\mathcal{F} = \mathcal{F}_{parab}$ is the collection of parabolic subgroups, and the map p associates to $G \in \mathcal{W}$ the fixer of the fixed-point set $\{x \in \mathbf{R}^n; \forall g \in G \;\; g.x = x\}$
2. If W is the Weyl group of a reduced root system R, then $\mathcal{W}$ can be identified with the collection of root subsystems of R. Then, one can take for $\mathcal{F} = \mathcal{F}_{closed}(R)$ the collection of closed symmetric subsystems, and for p the map which associate to a root subsystem its closure.

The first example arises for arbitrary groups, and is the smaller of the two types, when both can be compared : there is a natural surjective map $\mathcal{F}_{closed}(R) \to \mathcal{F}_{parab}$ which is not bijective in general (e.g. see A_2 as the set of long roots inside G_2). The second one is the one which is the most relevant to the original Yokonuma-Hecke algebra $\mathcal{H}(G, U)$, as $\mathcal{C}_W^{\mathcal{F}_{closed}(R)}$ 'generically' embeds into $\mathcal{H}(G, U)$ (see section 2.1). For short, we let $\mathcal{C}_W^R(\underline{u}) = \mathcal{C}_W^{\mathcal{F}_{closed}(R)}(\underline{u})$ and $\mathcal{C}_W^p(\underline{u}) = \mathcal{C}_W^{\mathcal{F}_{parab}}(\underline{u})$.

Note that, when W has type A_n, and R is the root system of type A_n, then $\mathcal{C}_W(\underline{u}) = \mathcal{C}_W^R(\underline{u}) = \mathcal{C}_W^p(\underline{u})$. Moreover, in general the morphism onto $H_W(\underline{u})$ factorizes as follows

$$\mathcal{C}_W(\underline{u}) \twoheadrightarrow \mathcal{C}_W^R(\underline{u}) \twoheadrightarrow \mathcal{C}_W^p(\underline{u}) \twoheadrightarrow H_W(\underline{u}).$$

3.5. **Lusztig's involution and Kazhdan-Lusztig bases.** Our reference on Kazhdan-Lusztig bases is [32]. We choose $\mathbf{k} = \mathbf{Z}[v_s, v_s^{-1}; s \in S]$, where there is one formal parameter v_s for each $s \in S$, with the condition $v_s = v_t$ is $s \sim t$. For short, we denote it $\mathbf{k} = \mathbf{Z}[v, v^{-1}]$. We set $u_s = v_s^2$ for each $s \in S$. We set $H_s = (-v_s^{-1})g_s$ for $s \in S$ and $H_w = H_{s_1} \dots H_{s_m}$ for $w \in W$ and $w = s_1 \dots s_m$ a minimal decomposition. The following equalities are easily checked

$$\begin{array}{rcl} H_s^{-1} & = & v^2 H_s + (v - v^{-1})e_s(1 - vH_s) \\ H_s^2 & = & v^{-2} + (v - v^{-1})e_s(v^{-1} - H_s) \\ H_s^{-2} & = & v^2 + (v^{-1} - v)e_s(v - H_s^{-1}) \end{array}$$

Moreover, we have $H_s e_{W_0} H_s^{-1} = e_{sW_0s^{-1}} = H_s^{-1} e_{W_0} H_s$. From this the following proposition readily follows.

Proposition 3.7. *There exists an involutive ring automorphism of $\mathcal{C}_W$ over $\mathbf{k} = \mathbf{Z}[v, v^{-1}]$ which maps $v \mapsto v^{-1}$, $H_w \mapsto H_{w^{-1}}^{-1}$, $e_{W_0} \mapsto e_{W_0}$ for each $w \in W$ and $W_0 \in \mathcal{W}$. It induces similar automorphisms of $\mathcal{C}_W^p$ and $\mathcal{C}_W^R$ (when defined). It is compatible with the ring automorphism of $\mathbf{Z}[v, v^{-1}]B$ which maps $v \mapsto v^{-1}$, $s \mapsto s^{-1}$ for $s \in B$, and with Lusztig's involution of H_W (as in* [32]*), that is the following diagram commutes, where the vertical maps are these involutive automorphisms and the horizontal ones are the natural maps.*

$$\begin{array}{ccccc} \mathbf{Z}[v, v^{-1}]B & \longrightarrow & \mathcal{C}_W & \longrightarrow & H_W \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z}[v, v^{-1}]B & \longrightarrow & \mathcal{C}_W & \longrightarrow & H_W \end{array}$$

Recall that the Kazhdan-Lusztig basis $(\mathbf{H}_w^0)_{w \in W}$ of H_W is characterised by the properties $\overline{\mathbf{H}^0}_w = \mathbf{H}_w^0$ and $\mathbf{H}_w^0 \in H_w^0 + \sum_{y \in w} v\mathbf{Z}[v] H_y^0$. One readily checks that, for $\mathbf{H}_s \in C_W$ and $W = A_1$, the conditions $\overline{\mathbf{H}_s} = \mathbf{H}_s$ and $\mathbf{H}_s \in H_s + \sum_{y \in W, W_0 < W} v\mathbf{Z}[v] H_y e_{W_0}$ are equivalent to saying that $\mathbf{H}_s = H_s + (xv + v^2)(1 - e_s)H_s + ve_s$ for *some* $x \in \mathbf{Z}$. Note that such a $\mathbf{H}_s$ clearly maps onto $\mathbf{H}_s^0 = H_s + v$.

W	$\|W\|$	$\mathrm{Bell}^p(W)$	$\mathrm{Bell}^R(W)$	$\mathrm{Bell}(W)$	$\mathrm{rk}\,\mathcal{C}_W(\underline{u})$
G_2	12	8	12	13	156
H_3	120	48		53	6360
H_4	14400	2104		2760	39744000
F_4	1152	268	447	637	733824
E_6	51840	4598	5079	5079	263295360
E_7	2903040	90408	107911	107911	313269949440
E_8	696729600	5506504	7591975	7591975	5289553704960000

TABLE 1. Bell numbers in exceptional types.

3.6. **Combinatorics and Bell numbers.** In type A_{n-1}, reflections have the form $(i,j), 1 \leq i < j \leq n$, and therefore a subset of $\mathcal{R}$ can be identified with a graph on n vertices. If $J \subset \mathcal{R}$, then $\bar{J}$ is the graph of the transitive closure of the graph given by J, and the set of all graphs of this form is the set of disjoint unions of complete graphs on $\{1,\dots,n\}$. This set is in natural 1-1 correspondence with partitions of the set $\{1,\dots,n\}$, and therefore has for cardinality the n-th Bell number $Bell_n$: $1,1,2,5,15,52,203,877,\dots$. Because of this, we will call in general the Bell number of type W the number of reflection subgroups of W, and we will call W-partitions the elements $\bar{J}$, $J \subset \mathcal{R}$.

In type D_n, it can be interpreted as the number of symmetric partitions of $\{-n,\dots,n\} \setminus \{0\}$ such that none of the subsets is of the form $\{j,-j\}$, see sequence A086365 in Sloane's Online Encyclopaedia of Integer Sequences. Here symmetric means that, for every part X of the partition, its opposite $-X$ is a part of the partition.

Indeed, the reflections have the form s_{ij} or s'_{ij}, where

$$\begin{array}{rcl} s_{ij}.(z_1,\dots,z_i,\dots,z_j,\dots,z_n) & = & (z_1,\dots,z_j,\dots,z_i,\dots,z_n) \\ s'_{ij}.(z_1,\dots,z_i,\dots,z_j,\dots,z_n) & = & (z_1,\dots,-z_j,\dots,z_i,\dots,z_n); \end{array}$$

then, to a stable subset $\mathcal{R}_0$ of $\mathcal{R}$ we associate the partition of $\{-n,\dots,n\} \setminus \{0\}$ made of the equivalence classes under the relation $i \sim j$ for $ij > 0$ if $s_{ij} \in \mathcal{R}_0$, for $ij < 0$ if $s'_{ij} \in \mathcal{R}_0$. Conversely, we associate to a partition $\mathcal{P}$ the collection of reflections made of the s_{ij} for $i,j > 0$ in the same part of $\mathcal{P}$, and of the s'_{ij} for $i,j > 0$ when $-i,j$ belong to the same part of $\mathcal{P}$. These two maps provide a bijective correspondence. An exponential generating function for this sequence is

$$-1 + \exp\left(-x + e^x - 1 + \frac{e^{2x}-1}{2}\right)$$

and the first terms are $1,4,15,75,428$. In type $B_n, n \geq 2$, we get the numbers $8,38,218,1430,10514,\dots$, which we could not relate to other mathematical objects. In type $I_2(m)$, we get $1+\sigma(m)$, where $\sigma(m)$ is the sum of divisors of m. Indeed, the non-trivial reflection subgroups are the stabilizer of the d-gons with vertices $\exp(2\pi\mathrm{i}(\frac{k_0}{m}+\frac{k}{d}))$ for some $k_0 \in [0, m/d[$ and k running from 0 to $d-1$, and d is a divisor of m. Since there are m/d such d-gons for d dividing m, there are exactly $\sigma(m)$ non-trivial reflection subgroups.

Among the exceptional groups, we computed the number of reflection subgroups by using elementary methods in the computer system GAP3 together with its CHEVIE package, except for the largest ones E_7 and E_8, for which this was not sufficient. Therefore, we used the classification of their reflection subgroups provided in [11] in this case : the total number is then the sum of the number of conjugacy classes provided in the third columns of tables 4 and 5 of [11]. The result can be found in table 1.

In order to find the dimension of $\mathcal{C}^p(W)$, we need to know the number of parabolic subgroups. These are in 1-1 correspondence with the elements of the lattice of the corresponding hyperplane arrangements, and with this interpretation they are described in [30]. We call parabolic Bell number of type W and denote $Bell^p(W)$ this number. Finally, when R is (one of) the classical root systems attached to W, we call Bell number of type R and denote $Bell^R(W)$ the number of closed root subsystems. If W is of simply laced (ADE) type, then $Bell^R(W) = Bell(W)$. For

n	2	3	4	5	6	7
$Bell(B_n)$	8	38	218	1430	10514	85202
$Bell^R(B_n)$	7	31	164	999	6841	51790
$Bell^p(B_n)$	6	24	116	648	4088	28640
$Bell^{(R)}(D_n)$	4	15	75	428	2781	20093
$Bell^p(D_n)$	4	15	72	403	2546	17867

TABLE 2. Bell numbers in classical types.

exceptional groups, both numbers are also listed in table 1. For the infinite series B_n and D_n, the first values are listed in table 2.

The series $Bell^p(D_n)$ and $Bell^p(B_n)$ are investigated and presented as analogues of Bell numbers in [36]. J. East communicated to us that he too generalized Bell numbers to series B, D and $I_2(m)$ (unpublished). In his approach, the 'right analogues' are $Bell^R(B_n)$, $Bell(D_n)$ and $Bell(I_2(m))$, respectively, which correspond to the sequences A002872, A086365 and A088580 in Sloane's encyclopaedia of integer sequences. To the best of our knowledge, the sequence $Bell(B_n)$ has not yet been investigated.

3.7. **Specialization at $u = 1$ and semisimplicity.** The algebra $\mathcal{C}_W(1)$ is obviously a semidirect product $\mathbf{k}W \ltimes A$, where A is the subalgebra generated by the idempotents e_J.

Let L be a join semilattice. That is, we have a finite partially ordered set L for which there exists a least upper bound $x \vee y$ for every two $x, y \in L$. Let M be the semigroup with elements $e_\lambda, \lambda \in L$ and product law $e_\lambda e_\mu = e_{\lambda \vee \mu}$. Such a semigroup is sometimes called a band.

If L is acted upon by some group G in an order-preserving way (that is $x \leq y \Rightarrow g.x \leq g.y$ for all $x, y \in L$ and $g \in G$) then M is acted upon by G, so that we can form the algebra $\mathbf{k}M \rtimes \mathbf{k}G$. Up to exchanging meet and joint, the algebra $\mathbf{k}M$ is the Möbius algebra of [35], definition 3.9.1 (this reference was communicated to us by V. Reiner). We will need the following proposition, which is in part a G-equivariant version of [35], theorem 3.9.2.

Proposition 3.8. *Let M be the band associated to a finite join semilattice L. For every commutative ring $\mathbf{k}$, the semigroup algebra $\mathbf{k}M$ is isomorphic to $\mathbf{k}^L$. If L is acted upon by some group G as above, then $\mathbf{k}M \rtimes \mathbf{k}G \simeq \mathbf{k}^L \rtimes \mathbf{k}G$. If G is finite and $\mathbf{k}$ is a field whose characteristic does not divide $|G|$, then the algebra $\mathbf{k}M \rtimes \mathbf{k}G$ is semisimple. If $\mathbf{k}G_\lambda$ is split semisimple for all $\lambda \in L$, where $G_\lambda < G$ is the stabilizer of λ, then so is $\mathbf{k}M \rtimes \mathbf{k}G$.*

Proof. To each $\lambda \in L$ we associate $\varphi_\lambda : L \to \mathbf{k}$ defined by $\varphi_\lambda(\mu) = 1$ if $\lambda \leq \mu$ and $\varphi_\lambda(\mu) = 0$ otherwise. We define a $\mathbf{k}$-linear map $c : M \to \mathbf{k}^L$ by $e_\lambda \mapsto \varphi_\lambda$. We prove that c is an algebra homomorphism. We have that $\varphi_{\lambda_1}\varphi_{\lambda_2}$ maps $\mu \in L$ to 1 iff $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$, and to 0 otherwise ; $\varphi_{\lambda_1 \vee \lambda_2}$ maps $\mu \in L$ to 1 iff $\lambda_1 \vee \lambda_2 \leq \mu$, and to 0 otherwise. These two conditions being equivalent, this proves $c(e_{\lambda_1} e_{\lambda_2}) = c(e_{\lambda_1})c(e_{\lambda_2})$, hence c is a $\mathbf{k}$-algebra homomorphism. We now prove that c is injective. We assume $\sum_{\lambda \in L} a_\lambda \varphi_\lambda = 0$ for a collection of $a_\lambda \in \mathbf{k}$, and we want to prove that all a_λ's are zero. If not, let λ_0 be a minimal element (w.r.t. $\leq$) among the elements of L such that $a_\lambda \neq 0$. Then $0 = \sum_{\lambda \in L} a_\lambda \varphi_\lambda(\lambda_0) = a_{\lambda_0}$ provides a contradiction. Therefore, c is injective. We now prove that c is surjective. Let $f_\lambda \in \mathbf{k}^L$ being defined by $f_\lambda(\mu) = \delta_{\lambda,\mu}$ (Kronecker symbol). The f_λ's obviously form a basis of $\mathbf{k}^L$ and we need to prove that they belong to the image of c, that is to the submodule V spanned by the φ_λ's. Let $\lambda_0 \in L$. We prove that f_{λ_0} belongs to V by induction with respect to $\leq$. If λ_0 is minimal in L, then $\varphi_{\lambda_0} = f_{\lambda_0}$ and this holds true. Now assume $f_\lambda \in V$ for all $\lambda < \lambda_0$. Let $g = f_{\lambda_0} - \varphi_{\lambda_0}$. We have $g(\mu) = 0$ unless $\mu < \lambda_0$. Therefore g is a linear combination of the f_μ's for $\mu < \lambda_0$ hence $g \in V$ and this implies $f_{\lambda_0} \in \varphi_{\lambda_0} + V \subset V$. By induction we conclude that c is surjective, and therefore is an isomorphism.

Now assume that L is acted upon by G. Then $\mathbf{k}M$ and $\mathbf{k}^L$ are both natural $\mathbf{k}G$-modules : if $g \in G$, then $g.e_\lambda = e_{g.\lambda}$ and, if $f : L \to \mathbf{k}$, then $g.f : \lambda \mapsto f(g^{-1}.\lambda)$. For these actions, c is an isomorphism of $\mathbf{k}G$-modules. Indeed, $g.\varphi_\lambda(\mu) = \varphi_\lambda(g^{-1}.\mu)$ is 1 is $\lambda \leq g^{-1}.\mu$ and 0 otherwise, while $\varphi_{g.\lambda}(\mu)$ is 1 if $g.\lambda \leq \mu$ and 0 otherwise. Since the action of G is order-preserving, the two conditions are equivalent and this proves the claim. Therefore c induces an isomorphism $\mathbf{k}M \rtimes \mathbf{k}G \simeq \mathbf{k}^L \rtimes \mathbf{k}G$.

When G is a finite and $\mathbf{k}$ is a field, we have $\mathbf{k}M \rtimes \mathbf{k}G \simeq \mathbf{k}^L \rtimes \mathbf{k}G \simeq \bigoplus_{X \in E} \left(\mathbf{k}^X \rtimes \mathbf{k}G\right)$ where E is the set of orbits of the action of G on L. Each X is a finite, transitive G-set, and therefore $\mathbf{k}^X \rtimes \mathbf{k}G \simeq Mat_X(\mathbf{k}G_0)$, where $Mat_X(R)$ denotes the $|X| \times |X|$ matrix ring over the ring R, and $G_0 < G$ is the stabilizer of an element of X (see e.g. [6], proposition 3.4). Therefore $\mathbf{k}M \rtimes \mathbf{k}G$ is isomorphic to a direct sum of matrix algebras over group algebras of finite groups. It is thus semisimple if and only if all these group algebras are semisimple. This is the case as soon as the characteristic of $\mathbf{k}$ does not divide $|G|$. Similarly, it is split semisimple if all these group algebras are split semisimple, and this concludes the proof of the proposition.

□

We use this proposition to prove the following.

Theorem 3.9. *Let W be a finite Coxeter group. The algebra $\mathcal{C}_W(1)$ is isomorphic to $\mathbf{k}^{\mathcal{W}} \rtimes \mathbf{k}W$. Moreover, if $\mathbf{k}$ is a field then the following holds.*

1. *If the characteristic of $\mathbf{k}$ does not divide the order of $|W|$, then $\mathcal{C}_W(1)$ is semisimple. If $\mathbf{k}$ has characteristic 0, then the algebra $\mathcal{C}_W(\underline{u})$ is generically semisimple, and $\mathcal{C}_W(\underline{u}) \simeq \mathbf{k}^{\mathcal{W}} \rtimes \mathbf{k}W$ for generic $\underline{u}$, up to a finite extension of $\mathbf{k}$.*
2. *If moreover the group algebra $\mathbf{k}N_W(W_0)$ of the normalizer of W_0 inside W is split semisimple for every reflection subgroup W_0 of W, then $\mathcal{C}_W(1)$ is split semisimple.*

Proof. We apply the above proposition with L the semilattice made of all the reflection subgroups $\mathcal{W}$, with $\leq$ denoting the inclusion of reflection subgroups, and the action of W is by conjugation. This proves one part of (1), and the remaining part is a consequence of Tits' deformation theorem (see e.g. [19], §7.4) and of the fact that $\mathcal{C}_W(\underline{u})$ is a free module of finite rank over $\mathbf{k}[\underline{u}]$, by theorem 3.4. Part (2) is the consequence of the proposition above together with the fact that the stabilizers of the action of W on $\mathcal{W}$ are exactly the normalizers of reflection subgroups.

□

In particular, for $W = \mathfrak{S}_n$, this has the following consequence.

Corollary 3.10. *If $W = \mathfrak{S}_n$ and $\mathbf{k}$ has characteristic not dividing $n!$, then $\mathcal{C}_W(1)$ is split semisimple over $\mathbf{k}$.*

Proof. From the theorem above, we need to prove that, for every reflection subgroup W_0 of $\mathfrak{S}_n$, its normalizer N_0 has a split semisimple group algebra over $\mathbf{k}$. Recall that a reflection subgroup W_0 of $\mathfrak{S}_n$ naturally corresponds to a partition $\mathcal{P}$ of $\{1, \dots, n\}$. The normalizer of W_0 is easily seen to be the subgroup of $\mathfrak{S}_n$ stabilizing the partition, and is therefore a direct product of wreath products of the form $\mathfrak{S}_m \wr \mathfrak{S}_d = (\mathfrak{S}_m)^d \rtimes \mathfrak{S}_d$ for $md \leq n$. The group algebras of these groups are split semisimple as soon as they are semisimple (see [15], cor. 4.4.9). By Maschke's theorem this holds true as soon as the characteristic of p does not divides $n!$, and this proves the claim. □

We do not know the class of groups, for which the above corollary holds (in characteristic 0). When W is not a Weyl group, the field $\mathbf{Q}$ should of course be replaced by the field of definition $K = \langle tr(w); w \in W\rangle$. Also, we might want to generalize this statement either to $\mathcal{C}_W(1)$ or, more cautiously, to $\mathcal{C}_W^p(1)$ or some $\mathcal{C}_W^R(1)$. The question therefore is : for which W and which class of reflection subgroups G of W can we expect that the group algebra $KN_W(G)$ of the normalizer is split semisimple ?

One may wonder whether this is actually true for an arbitrary *reflection subgroup*. That this is probably too much can be already hinted from the fact that the normalizer of a 2-Sylow subgroup $S \simeq (\mathbf{Z}/2\mathbf{Z})^3$ of the symmetric group $W = H_3$ of the icosahedron is a semi-direct product $S \rtimes C_3$, and S is a reflection subgroup – generated by the reflections around three orthogonal golden rectangles, see figure 1, and the element of order 3 is a rotation whose axis goes through the two opposite faces painted in blue. Therefore this normalizer has (1-dimensional) representations that can be realized only over $\mathbf{Q}(\zeta_3)$, while the group algebra of W splits only over $\mathbf{Q}(\sqrt{5})$. But an actual counterexample for $K = \mathbf{Q}$ (that is, for W a Weyl group), is not so easy to find, because it can be

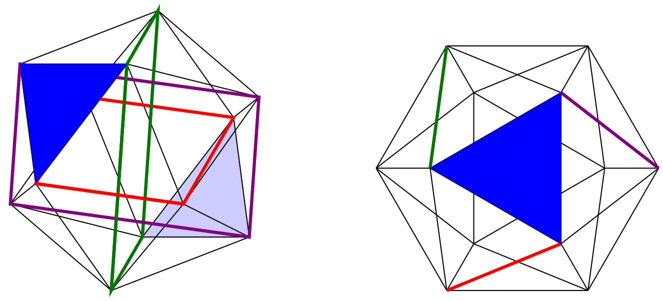

FIGURE 1. Normalizer of a maximal reflection 2-group in type H_3.

checked by computer that the characters of the normalizers of the reflection subgroups of all Weyl groups of rank ≤ 6 are rationnally-valued. In type E_7 however, there is a 2-reflection subgroup W_0 isomorphic to $\mathbf{Z}_2^7$, whose normalizer N_0 has for quotient $N_0/W_0 \simeq \mathrm{PSL}_2(\mathbf{F}_7) \simeq \mathrm{SL}_3(\mathbf{F}_2)$. From the character table of $\mathrm{SL}_3(\mathbf{F}_2)$ (that can be found e.g. in the ATLAS [10]) one gets that it admits (for example, 3-dimensional) irreducible characters whose values generate $\mathbf{Q}(\sqrt{-7})$, and therefore the irreducible characters N_0 are not all rationally-valued. Interestingly enough, the reflection subgroups appearing as counter-examples here (for H_3 and E_7) both arise from the decomposition of $-1 \in W$ as a product of orthogonal reflections, established in [33]. For the interested reader, one can check that, in type E_7, we have $N_0 = \mathrm{SL}_3(\mathbf{F}_2) \ltimes \mathbf{F}_2^7$, and the action of $\mathrm{SL}_3(\mathbf{F}_2)$ on $\mathbf{F}_2^7$ is the permutation representation over $\mathbf{F}_2$ associated to a transitive action of $\mathrm{SL}_3(\mathbf{F}_2)$ on 7 elements. Up to automorphism, there is only one transitive action of $\mathrm{SL}_3(\mathbf{F}_2)$, and this is its natural action on the seven non-zero elements of $\mathbf{F}_2^3$. I thank R. Stancu for discussions on this last topic.

The question however remains open for the class of parabolic subgroups, our counterexamples not being parabolic since they have the same rank as the whole group. In that case, the conclusion does not appear to be completely obvious from Howlett's general description of their normalizer (see [21]). Also, one may be interested to determine a minimal splitting field for the whole algebra $\mathcal{C}_W(1)$, for each finite Coxeter group W.

4. BRAID IMAGE

In this section we study the image of the (generalized) braid group B inside the algebra $\mathcal{C}_W(\underline{u})$. We let B^+ denote the positive braid monoid (or Artin monoid) associated to W.

4.1. Braid morphisms.

Proposition 4.1. *For every collection $(\lambda_s)_{s\in S} \in (\mathbf{k} \setminus \{0\})^S$ such that $s \sim t \Rightarrow \lambda_s = \lambda_t$, there exists a morphism $B^+ \to \mathcal{C}_W(\underline{u})$ defined by $G_s \mapsto g_s + \lambda_s g_s e_s$. When $\mathbf{k}$ is a field, it can be extended to a morphism $\Phi_\lambda : \mathbf{k}B \to \mathcal{C}_W(\underline{u})$ if and only if $\forall s \in S\ \lambda_s \neq \{-1\}$.*

Proof. Let $s, t \in S$, and m_{st} denote the order of $st \in W$. We have

$$\begin{array}{rl} & g_s(1+\lambda_s e_s) g_t(1+\lambda_t e_t) g_s(1+\lambda_s e_s)\dots \\ = & \sum g_s(\lambda_s e_s)^{\varepsilon_1} g_t(\lambda_t e_t)^{\varepsilon_2} g_s(\lambda_s e_s)^{\varepsilon_3}\dots \\ = & \sum g_s(\lambda_s e_s)^{\varepsilon_1} g_t(\lambda_t e_t)^{\varepsilon_2} g_s(\lambda_s e_s)^{\varepsilon_3}\dots \\ = & \sum (\lambda_s e_s)^{\varepsilon_1} (\lambda_{sts} e_{sts})^{\varepsilon_2} g_s g_t g_s (\lambda_s e_s)^{\varepsilon_3}\dots \\ = & (\sum \underbrace{(\lambda_s e_s)^{\varepsilon_1} (\lambda_{sts} e_{sts})^{\varepsilon_2} (\lambda_{ststs} e_{ststs})^{\varepsilon_3}\dots}_{m_{st}}) \underbrace{g_s g_t g_s\dots}_{m_{st}} \end{array}$$

where the sums are over all the $(\varepsilon_1, \dots, \varepsilon_{m_{st}}) \in \{0,1\}^{m_{st}}$. By the braid relations inside $\mathcal{C}_W(\underline{u})$ we have $\underbrace{g_s g_t g_s\dots}_{m_{st}} = \underbrace{g_t g_s g_t\dots}_{m_{st}}$. Finally, inside the dihedral group $\langle s, t\rangle$, the set of cardinality m_{st} given by $\{s, sts, ststs, stststs, \dots\}$ is exactly the union of all the reflections (this is for instance a consequence of the fact that $\ell(\underbrace{sts\dots}_{m_{st}}) = m_{st}$, see e.g. [4] ch. 4 §1 no. 4, Lemme 2). From this we get

$$\underbrace{g_s(1+\lambda_s e_s)g_t(1+\lambda_t e_t)g_s(1+\lambda_s e_s)\dots}_{m_{st}} = \underbrace{g_t(1+\lambda_t e_t)g_s(1+\lambda_s e_s)g_t(1+\lambda_t e_t)\dots}_{m_{st}}$$

and this proves the first part. In order to extend this morphism to B it is necessary and sufficient to have $g_s(1+\lambda_s e_s)$ invertible for all $s \in S$. Since g_s is invertible, this means $(1+\lambda_s e_s)$ invertible. Since $e_s^2 = e_s$ and $e_s \notin \{0,1\}$ this means $\lambda_s + 1 \neq 0$. Indeed, we have $(1+\lambda_s e_s)(1+\lambda_s e_s - \lambda_s - 2) = -(\lambda_s+1)$ hence $(1+\lambda_s e_s)$ is invertible as soon as $\lambda_s + 1 \neq 0$, and conversely $1 - e_s$ is not invertible since $(1-e_s)e_s = 0$. □

4.2. **Description in type A_1, and beyond for generic λ.** If W has type A_1, the algebra $\mathcal{C}_W(\underline{u})$ can be described by two generators g, e and relations $e^2 = e$, $ge = eg$, $g^2 = 1 + (u-1)e(1+g)$. We know that it is a free module with basis $1, e, g, eg$. We let $a_0 = (1+g)(1-e)$, $a_1 = e(1+g)$, $a_2 = (g-1)(1-e)$, $a_3 = (g-u)e$. If $2(u+1)$ is invertible in $\mathbf{k}$, then a_0, a_1, a_2, a_3 is again a basis over $\mathbf{k}$. It is made of eigenvectors for g and e. The eigenvalues are

	a_0	a_1	a_2	a_3
e	0	1	0	1
g	1	u	-1	-1
eg	0	u	0	-1

It follows that $g + \lambda ge$ has eigenvalues $1, u(1+\lambda), -1, -1-\lambda$. The discriminant of its characteristic polynomial $(X-1-\lambda)(X-u(1+\lambda))(X+1)(X+1+\lambda)$ is

$$Q(\lambda, u) = 4(\lambda+2)^2(\lambda u + u - 1)^2(1+u)^2(1+\lambda)^2(\lambda u + 1 + u)^2\lambda^2.$$

When this discriminant vanishes, and over a domain, $g + \lambda ge$ satisfies a cubic relation, because 2 of the 4 eigenvalues are equal. When it is invertible, $g + \lambda ge$ generates the whole algebra. As a consequence, we get for an arbitrary Coxeter group W the following.

Proposition 4.2. *If the λ_s, u_s as in the previous proposition are such that $Q(\lambda_s, u_s)$ is invertible for all $s \in S$, then $\mathcal{C}_W(\underline{u})$ is generated by the $g_s, s \in S$.*

4.3. **Description in type A_2 and beyond for $\lambda = 0$.**

4.3.1. *The cubic Hecke algebra.* For $a, b, c \in \mathbf{k}$, the $\mathbf{k}$-algebra $H_3(a,b,c)$ presented by generators s, t and relations $sts = tst$, $(s-a)(s-b)(s-c) = (t-a)(t-b)(t-c) = 0$ is known to be a free deformation of the group algebra of the group $\Gamma_3 = Q_8 \rtimes (\mathbf{Z}/3\mathbf{Z})$, where Q_8 is the quaternion group of order 8 (see [25]). Moreover it is known to be a symmetric algebra, with explicitely determined Schur elements. Specializing a, b, c to $1, -1, u$ we get from [28] that, when $\Delta(u) = 6u(1-u)(u+1)(u^2-u+1)(u^2+u+1)$ is invertible in $\mathbf{k}$, then $H_3(1,-1,u)$ is a semisimple algebra, isomorphic to $\mathbf{k}\Gamma_3$, possibly after some extension of scalars. We know that $H_3(1,-1,u)$ is a free module of rank $|\Gamma_3| = 24$ and that $\mathcal{C}_{A_2}(\underline{u})$ as rank 30. Over the field $\mathbf{k} = \mathbf{Q}(u)$, the image of the natural map $H_3(1,-1,u) \to \mathcal{C}_{A_2}(\underline{u})$ can be easily computed, starting from a basis of $H_3(1,-1,u)$. We get a vector space of dimension 20. Therefore, this image is the quotient of $H_3(1,-1,u)$ by one of its three 2-sided ideals corresponding to its simple modules of dimension 2. This quotient also appears in the study of the Links-Gould invariant, see [27]. This incites to look at skein relation of braid type satisfied by the Links-Gould invariant on 3 strands. Ishii has established ([14] and also private communication, 2012) that, besides a cubic relation of the form $(\sigma_i - t_0)(\sigma_i - t_1)(\sigma_i + 1) = 0$, the Links-Gould invariant vanishes on the following relation

$$\begin{array}{rl} & s_1 s_2 s_1^{-1} + s_1^{-1} s_2^{-1} s_1 + s_1 s_2 + s_1^{-1} s_2^{-1} + s_2 s_1^{-1} + s_2^{-1} s_1 \\ = & s_1 s_2^{-1} s_1^{-1} + s_1^{-1} s_2 s_1 + s_1 s_2^{-1} + s_1^{-1} s_2 + s_2^{-1} s_1^{-1} + s_2 s_1 \end{array}$$

From explicit calculations inside $H_3(1, t_0, t_1)$ one checks that this relation is non-trivial in this algebra. Therefore it is a generator of the simple ideal defining the Links-Gould quotient LG_3 in the notations of [27].

Another relation communicated by Ishii is the following one.

$$\begin{array}{rl} & s_1 s_2^{-1} s_1 - s_2 s_1^{-1} s_2 + t_0 t_1 s_2^{-1} s_1 s_2^{-1} - t_0 t_1 s_1^{-1} s_2 s_1^{-1} \\ = & -(t_0-1)(t_1-1)\left(s_2^{-1}s_1 - s_1^{-1}s_2 + s_1 s_2^{-1} - s_2 s_1^{-1} + s_1 - s_2 + s_2^{-1} - s_1^{-1}\right) \end{array}$$

One checks similarly that it is nontrivial in $H_3(1,t_0,t_1)$. By explicit computations inside $\mathcal{C}_{A_2}(u)$, one checks that both relations are valid there. For the second one one neeeds to specialize at $\{t_0,t_1\} = \{1,u\}$. This proves

Proposition 4.3. *The two relations above are satisfied inside* $\mathcal{C}_{A_k}(u)$ *(and therefore inside* $Y_{d,k+1}(u)$*), for all* $k \geq 2$*. Moreover, if* $\mathbf{k}$ *is a field and* $\Delta(u) \neq 0$*, then the image of* $\mathbf{k}B_3$ *inside the algebra* $\mathcal{C}_{A_2}(u)$ *is semisimple, has dimension 20, and can be presented by generators* s_1, s_2*, and the braid relations together with the cubic relation* $(s_1-1)(s_1+1)(s_1-u)=0$ *and* one *of the two relations above.*

The study of the algebra for a higher number of strands cannot be continued using the same methods as in [27], because the cubic quotient $H_4(1,-1,u)$, though still being finite dimensional, is conjecturally not semisimple. Indeed, the Schur elements of a conjectural symmetric trace for $H_4(a,b,c)$ were computed and included in the development version of the CHEVIE package for GAP3 (see [29]) , and some of them vanish when $(a+b)(a+c)(b+c)=0$.

We computed the dimension of the algebra generated by the braid generators inside $\mathcal{C}_{A_k}(u)$, $k \in \{3,4\}$ for a few rational values of u (including, for $k=4$, $u \in \{17,127,217\}$). We obtained 217 for $k=3$ and 3364 for $k=4$. This sequence $3,20,217,3364$ of dimensions does not appear for now in Sloane's encyclopaedia of integer sequences, so we could not extrapolate a general formula from this.

4.4. **Positive representation of the braid monoid for** $\lambda = -1$**.** When $\lambda=-1$, the images of the Artin generators still satisfy the braid relations, but they are not invertible anymore. Therefore, they define a representation of the positive braid monoid, or Artin monoid, that we denote B^+. We denote $b_s = g_s - g_s e_s$ the action of $s \in S$. We have $b_s^3 = b_s$, and a straightforward computation shows that, for all $J \in \mathcal{P}_f(\mathcal{W})$ and $w \in W$, we have

$$b_s.v_{J,w} = v_{sJs,sw} - v_{sJs\cup\{s\},sw}.$$

It is remarkable that this action does not depend on the parameters u_s anymore. Moreover, when W is finite, we can convert it to a linear action with *positive* coefficients, as follows. Composing through the natural projection $\mathcal{C}_W(\underline{u}) \to \mathcal{C}_W^{\mathcal{F}_{parab}}(\underline{u})$ we get a linear action on a vector space with basis the $x_{W_0,w}$ with W_0 a parabolic subgroup of W and $w \in W$. Letting $[J]$ denote the parabolic closure of $\langle J \rangle$, and $x_{J,w} = x_{[J],w}$ we get that $b_s.x_{[J],w} = x_{[sJs],sw} - x_{[sJs\cup\{s\}],sw}$ (that is the fixer of the fixed point subspace of $\langle J\rangle$). The rank of $[J]$ is equal to the codimension of the fixed point space of $\langle J \rangle$.

Note that $b_s.x_{[J],w} = 0$ iff $s \in [sJs]$ iff $s \in [J]$ iff $\mathrm{rk}([J\cup\{s\}]) = \mathrm{rk}([J])$. Otherwise, $\mathrm{rk}([J\cup\{s\}]) = \mathrm{rk}([J])+1$. Because of this, letting $y_{[J],w} = (-1)^{\mathrm{rk}([J])}x_{[J],w}$ we get the formula

$$\begin{array}{rcl} b_s.y_{[J],w} & = & y_{[sJs],sw} + y_{[sJs\cup\{s\}],sw} \text{ if } s \in [J] \\ b_s.y_{[J],w} & = & 0 \text{ otherwise.} \end{array}$$

In particular, if $g \in B^+$ is divisible by $s \in S$, then $g.y_{[J],w} = 0$ for all $s \in [J]$. Therefore, one could hope that this representation $g \mapsto b_g$ of B^+ is initially injective in the sense given by Hée in his analysis of Krammer's faithfulness criterium (see [20]), meaning that b_g determines the leftmost (or rightmost) simple factor of g. This would imply that the representation $s \mapsto g_s + \lambda g_s e_s$ of B is faithful, for generic λ. However, this is not the case : in type A_2, with generators s,t, a straightforward computation shows that $b_s^2 b_t^3 b_s^2 = b_s b_t b_s b_t b_s$ while $s^2t^3s^2$ is divisible by s and not by t (on both sides), while $ststs = tstts = sttst$ is divisible by s and t on both sides.

Finally, we remark that this representation with positive coefficients cannot be readily transposed to infinite Coxeter groups. Indeed, although the intersection of all parabolic subgroups containing a *finitely generated* reflection subgroup of W *is* a parabolic subgroup, and therefore the notion of parabolic closure remains well-defined, the relation $\mathrm{rk}([J\cup\{r\}]) = \mathrm{rk}([J])+1$ whenever $r \notin [J]$ fails. The following easy example was communicated to me by T. Gobet. Let (W,S)

be an affine Coxeter group of type $\tilde{A}_2$, and $S = \{s, t, u\}$. Let $J = I = \{s\}$ and $r = tut = utu$. Then $\langle s, t\rangle$ is an infinite dihedral group, whose parabolic closure is W, because every proper parabolic subgroup of W is finite. Therefore $\mathrm{rk}[J \cup \{r\}] = 2 + \mathrm{rk}[J]$ in this case.

5. Generalization to complex reflection groups

Let $W < \mathrm{GL}(V)$ be a finite complex reflection group, $\mathcal{R}$ its set of pseudo-reflections, $\mathcal{W}_{parab}$ the collection of its parabolic subgroups, defined as the fixers of some linear subspace of V. We let $\mathcal{A} = \{\mathrm{Ker}\,(s-1), s \in \mathcal{R}\}$ denote the associated hyperplane arrangement, $X = V \setminus \bigcup \mathcal{A}$ the hyperplane complement and $B = \pi_1(X/W)$ its braid group. Without loss of generality we may assume that $\mathcal{A}$ is essential, meaning $\bigcap \mathcal{A} = \{0\}$. We let $\mathcal{L}$ denote the lattice of the arrangement, formed by the intersections of reflecting hyperplanes. There is a 1-1 correspondence $\mathcal{L} \to \mathcal{W}$ given by $L \mapsto W_L$ where $W_L = \{w \in W; w_{|L} = \mathrm{Id}_{|L}\}$. This bijection is an isomorphism of lattices, and it is equivariant under the natural actions of W.

5.1. Generalization of $\mathcal{C}_W^p(1)$, and a monodromy representation.

For $\mathbf{k}$ an arbitrary unital commutative ring, we let $\mathbf{k}\mathcal{W}_{parab} = \mathbf{k}\mathcal{L}$ denote the commutative algebra spanned by a basis of idempotents $e_G, G \in \mathcal{W}$ with relations $e_{G_1}e_{G_2} = e_{[G_1,G_2]}$, where $[A]$ denotes the parabolic closure of A, that is the fixer of the fixed point set of $A \subset W$. Equivalently, it is spanned by idempotents $e_L, L \in \mathcal{L}$ with relations $e_{L_1}e_{L_2} = e_{L_1 \vee L_2}$, where $e_L = e_{W_L}$. In particular, $e_s = e_{Ker(s-1)}$ for all $s \in \mathcal{R}$. This algebra is naturally acted upon by W, through $w.e_G = e_{wGw^{-1}}$, or equivalently $w.e_L = e_{w(L)}$. We define $\mathcal{C}_W^p(1)$ as the semidirect product $W \ltimes \mathbf{k}\mathcal{W}_{parab} \simeq W \ltimes \mathbf{k}\mathcal{L}$. It is again acted upon by W through $w.(w_1.e_G) = (ww_1w^{-1})e_{wGw^{-1}}$. Applying proposition 3.7, we have the following analogue of theorem 3.8.

Proposition 5.1. *Let W be a finite complex reflection group, and $\mathbf{k}$ be a field. The algebra $\mathcal{C}_W^p(1)$ is isomorphic to $\mathbf{k}^{\mathcal{L}} \rtimes \mathbf{k}W$. Moreover, if char. $\mathbf{k}$ does not divide $|W|$, then $\mathcal{C}_W^p(1)$ is semisimple. It is split semisimple as soon as the group algebra $\mathbf{k}N_W(W_0)$ is split semisimple for all $W_0 \in \mathcal{W}_{parab}$, where $N_W(W_0)$ denotes the normalizer of W_0 inside W.*

We let $\mathcal{T}$ denote the holonomy Lie algebra of the hyperplane complement $V \setminus \bigcup \mathcal{A}$. Recall from [22] that it is presented by generators $t_H, H \in \mathcal{A}$ and relations $[t_{H_0}, t_E] = 0$ for all $H_0 \in \mathcal{A}$ and E a codimension 2 subspace contained in H_0 inside the hyperplane lattice (such a subspace is called a flat), where $t_E = \sum_{H \supset E} t_H$. It is acted upon by W through $w.t_H = t_{w(H)}$. For $H \in \mathcal{A}$ we let $W_H = \{w \in W; w_{|H} = \mathrm{Id}_H\} \in \mathcal{W}_{parab}$. It is a cyclic group of order $m_H \in \mathbf{Z}_{\geq 2}$. It contains a unique generator s_H with eigenvalue $\exp(2i\pi/m_H)$, that we call the distinguished reflection associated to $H \in \mathcal{A}$. We remark that, if $H_2 = w(H_1)$ for some $w \in W$, then $e_{H_2} = e_{H_1}$ and $ws_{H_1}w^{-1} = s_{H_2}$. The following simple fact will be crucial for us. We state it as a lemma.

Lemma 5.2. *Let $H, H_0 \in \mathcal{A}$ and $s \in \mathcal{R}$ such that* $\mathrm{Ker}\,(s-1) = H$. *Then* $[se_H, e_{H_0}] = 0$.

Proof. We have $se_H.e_{H_0} = se_{H \cap H_0}$ and $e_{H_0}.se_H = se_{s^{-1}(H_0)}e_H = se_{s^{-1}(H_0)\cap H}$. Since s^{-1} acts by the identity on H, and $H_0 \cap H \subset H$, we get $s^{-1}(H_0) \cap H = s^{-1}(H_0) \cap s^{-1}(H) = s^{-1}(H_0 \cap H) = H_0 \cap H$ hence $e_{H_0}.se_H = se_H.e_{H_0}$, which proves the claim. □

Let us choose for each $H \in \mathcal{A}$ a collection of scalar parameters $\lambda_H^{(i)}, 0 \leq i < m_H$, such that the condition $H_2 = w(H_1)$ for some $w \in W$ implies $\lambda_{H_1}^{(i)} = \lambda_{H_2}^{(i)}$ for all $0 \leq i < m_{H_1} = m_{H_2}$.

Proposition 5.3. *There exists a (necessarily unique) morphism of Lie algebras $\varphi : \mathcal{T} \to \mathcal{C}_W(1)$ satisfying*

$$t_H \mapsto \left(\sum_{0 \leq i < m_H} \lambda_H^{(i)} s_H^i \right) e_H$$

This morphism is W-equivariant.

Proof. Let E be a codimension 2 flat, and $H_0 \in \mathcal{A}$ such that $E \subset H_0$. Let $s \in \mathcal{R}$ such that $\mathrm{Ker}\,(s-1) = H_0$. It is enough to prove that se_{H_0} commutes with the image of t_E for all such s. We have

$$se_{H_0}\left(\sum_{H\supset E}\left(\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i\right)e_H\right) = \sum_{H\supset E}\sum_{0\le i<m_H}\lambda_H^{(i)}ss_H^i e_{s_H^i(H_0)}e_H$$
$$= \sum_{0\le i<m_{H_0}}\lambda_{H_0}^{(i)}ss_{H_0}^i e_{H_0} + \sum_{\substack{H\supset E\\ H\neq H_0}}\sum_{0\le i<m_H}\lambda_H^{(i)}ss_H^i e_{s_H^i(H_0)}e_H$$

and

$$\left(\sum_{H\supset E}\left(\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i\right)e_H\right)se_{H_0} = \sum_{H\supset E}\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i se_{s^{-1}(H)}e_{H_0}$$
$$= \sum_{0\le i<m_{H_0}}\lambda_{H_0}^{(i)}s_{H_0}^i se_{H_0} + \sum_{\substack{H\supset E\\ H\neq H_0}}\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i se_{s^{-1}(H)}e_{H_0}.$$

We notice that $ss_{H_0}^i = s_{H_0}^i s$ for all i, since $s \in W_{H_0} = \langle s_{H_0}\rangle$. Moreover, if $H \neq H_0$, then $s^{-1}(H) \neq H_0$ and $s_H^i(H_0) \neq H$, and therefore $e_{s^{-1}(H)}e_{H_0} = e_{s_H^i(H_0)}e_H = e_E$. Therefore, it is sufficient to prove that

$$\sum_{\substack{H\supset E\\ H\neq H_0}}\sum_{0\le i<m_H}\lambda_H^{(i)}ss_H^i = s\left(\sum_{\substack{H\supset E\\ H\neq H_0}}\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i s\right)s^{-1}$$
$$= \sum_{\substack{s(H)\supset E\\ s(H)\neq H_0}}\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i s = \sum_{\substack{H\supset E\\ H\neq H_0}}\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i s$$

which holds true. The W-equivariance is clear. □

Now assume that the $\lambda_{H,i}$ are complex numbers, and that $\mathbf{k}$ contains $\mathbf{C}[[h]]$ as a subring. We consider the $\mathcal{C}_W(1)$-valued 1-form

$$\omega = \frac{1}{\mathrm{i}\pi}\sum_{H\in\mathcal{A}}h\varphi(t_H)\omega_H$$

with ω_H is the canonical logarithmic form around H defined as $\mathrm{d}\alpha_H/\alpha_H$ for some linear form $\alpha_H \in V^*$ with kernel H. Proposition 5.3 states that $\omega \in \Omega^1(X)\otimes\mathcal{C}_W(1)$ is integrable and W-invariant. By monodromy we get a representation $B \to \mathcal{C}_W(1)$, where the image of a braided reflection σ associated to $s = s_H^k \in \mathcal{R}$ is conjugated to $s_H^k\exp(h\varphi(t_H))$, that is

$$s_H^k\exp\left(h\left(\sum_{0\le i<m_H}\lambda_H^{(i)}s_H^i\right)e_H\right).$$

The algebra $\mathcal{C}_{W_H}(1)$ can be decomposed as $(\mathbf{k}C)(e_H-1)\oplus(\mathbf{k}C)e_H$ where $C = \langle s_H\rangle$, since e_H is a central idempotent inside $\mathcal{C}_{W_H}(1)$. This proves that σ is semisimple with eigenvalues $1, \zeta^k$ together with the $\zeta^{kr}\exp(h\sum_{0\le i<m_H}\lambda_H^{(i)}(\zeta)^{ri})$ for $0\le r<m_r$ for $\zeta = \exp(2\mathrm{i}\pi/m_H)$.

5.2. **Generalization of $\mathcal{C}_W^p$.** Let $a_{H,i}, 0 \le i < m_H$ be a collection of parameters in $\mathbf{k}$ such that $a_{w(H),i} = a_{H,i}$ for all $H \in \mathcal{A}$, $w \in W$, and $a_{H,0} \in \mathbf{k}^\times$. The natural action of W on $\mathcal{L}$ can be considered as an action of B, through the natural morphism $\pi : B \twoheadrightarrow W$, so that the relation $ge_L = e_{\pi(g)(L)}g$ for all $g \in B$, $L \in \mathcal{L}$ holds inside $B \ltimes \mathbf{k}\mathcal{L}$.

Definition 5.4. *The algebra $\mathcal{C}_W^p(\underline{a})$ is defined as the quotient of the group algebra $B \ltimes \mathbf{k}\mathcal{L}$ by the relations*

$$\sigma^{m_H} = 1 + e_H(\sum_{k=0}^{m_H-1}a_{H,k}\sigma^k - 1)$$

for any braided reflection σ associated to $s_H \in \mathcal{R}$ with $H \in \mathcal{A}$.

We remark that the relation of the semidirect product is the consequence of the more elementary relations $ge_H = e_{\pi(g)(H)}g$ for $H \in \mathcal{A}$ and g running among a generating system of B. Similarly, since all braided reflections corresponding to some $H \in \mathcal{A}$ are conjugated one to the other by an element of B, using the relations of the semidirect product, the defining relations of $\mathcal{C}^p_W(\underline{a})$ can be replaced by imposing the same relations but only for one braided reflection per conjugacy class of hyperplanes. Since B is known to be finitely generated (and even finitely presentable), these remarks prove the following

Proposition 5.5.

(1) The algebra $\mathcal{C}^p_W(\underline{a})$ is finitely generated (and even finitely presentable) as an algebra.
(2) If W is a finite Coxeter group with generating system S, and if $a_{\mathrm{Ker}\,(s-1),0} = u_s$, $a_{\mathrm{Ker}\,(s-1),1} = u_s - 1$, then $\mathcal{C}^p_W(\underline{a}) = \mathcal{C}^p_W(\underline{u})$.

The Hecke algebra $H_W(\underline{a})$ of W is defined as the quotient of $\mathbf{k}B$ by the relations $\sigma^{m_H} = \sum_{k=0}^{m_H-1} a_{H,k}\sigma^k$ for any braided reflection σ associated to $s_H \in \mathcal{R}$ with $H \in \mathcal{A}$. We remark that the algebra $\mathbf{k}\mathcal{L}$ admits an augmentation map $\mathbf{k}\mathcal{L} \to \mathbf{k}$ defined by $e_L \mapsto 1$, which is split through $1 \mapsto e_W = e_{\{0\}}$. From this the following is immediate.

Proposition 5.6. *The maps $\mathbf{k}B \to H_W(\underline{a})$ and $\mathbf{k}\mathcal{L} \mapsto \mathbf{k}$ together induce an algebra morphism $B \ltimes \mathbf{k}\mathcal{L} \to H_W(\underline{a}) \otimes_{\mathbf{k}} \mathbf{k} = H_W(\underline{a})$. It factorizes through a surjective algebra morphism $\mathfrak{p} : \mathcal{C}^p_W(\underline{a}) \twoheadrightarrow H_W(\underline{a})$. The morphism $\mathbf{k}B \to \mathcal{C}^p_W(\underline{a})$ induced by $b \mapsto be_W$ for $b \in B$ factors through a splitting $\mathfrak{q} : H_W(\underline{a}) \to \mathcal{C}^p_W(\underline{a})$ of $\mathfrak{p}$.*

Proposition 5.7. *Assume that $\mathbf{k} = \mathbf{C}[[h]]$, and $v_{H,i} = \zeta^i_{m_H}\exp(h\tau_{H,i})$ for some $\tau_{H,i} \in \mathbf{C}$. Then there exists a surjective morphism $\mathcal{C}^p_W(\underline{a}) \to \mathcal{C}^p_W(1)$, where*

$$X^{m_H} - \sum_{0 \le i < m_H} a_{H,i}X^i = \prod_{0 \le i < m_H} (X - v_{H,i}).$$

Proof. For instance by invertibility of the Vandermonde determinant, one can find complex scalars $\lambda_{H,i}$ such that $\sum_{0 \le i < m_H} \lambda^{(i)}_H (\zeta_H)^{ri}) = \tau_{H,r}$ for $0 \le r < m_H$, with $\zeta_H = \exp(2i\pi/m_H)$. We consider the monodromy morphism $R : \mathbf{k}B \to \mathcal{C}^p_W(1)$ constructed above. The image of a braided reflection σ associated so s_H has eigenvalues $1, \zeta$ and $\zeta^r \exp(h\tau_{H,r}) = v_{H,r}$. For instance by using Chen's iterated integrals, we notice that, for $b \in B$, $R(b)$ has the form $\beta M(h)$, where $\beta \in W \subset \mathcal{C}^p_W(1)$ is the image of $b \in B$ under the natural map $B \twoheadrightarrow W$, and $M(h) \in A_0[[h]]$, where A_0 is the subalgebra of $\mathcal{C}^p_W(1)$ generated by the se_H, for $s \in \mathcal{R}$ and $H = \mathrm{Ker}\,(s-1)$. Lemma 5.2 implies that A_0 commutes with all $e_L, L \in \mathcal{L}$. Therefore, we have $R(b)e_L R(b)^{-1} = \beta M(h) e_L M(h)^{-1}\beta^{-1} = \beta e_L \beta^{-1} = e_{\beta(L)} = e_{b(L)}$. This proves that R can be extended to a morphism $B \ltimes \mathbf{k}\mathcal{L} \to \mathcal{C}^p_W(1)$ through $b \otimes e_L \mapsto R(b) \otimes e_L$.

It remains to prove that the defining relations of $\mathcal{C}^p_W(\underline{a})$ are satisfied. Let $H \in \mathcal{A}$, $s = s_H$ and σ a braided reflection associated to them. For short, let $S = R(\sigma)$ and $S_0 = s\exp(h\varphi(t_H))$. We have $S = Ps\exp(h\varphi(t_H))P^{-1}$ for some $P \in A_0[[h]]$. Since $\varphi(t_H)$ commutes with A_0, we get $S^m = P\exp(mh\varphi(t_H))P^{-1} = 1 + P(\exp(mh\varphi(t_H)) - 1)P^{-1}$. We have $\exp(mh\tau_{H,r}) = v^{m_H}_{H,r}$ and $v^{m_H}_{H,r} - \sum_i a_{H,i}v^i_{H,r} = \prod_i (v_{H,r} - v_{H,i}) = 0$. Now, the compared spectrum of the elements in play is as follows

$S_0 = s\exp(h\varphi(t_H))$	1	ζ	$v_{H,r}$
e_H	0	0	1
$S_0 e_H$	0	0	$v_{H,r}$
S_0^m	1	1	$v^m_{H,r}$
$S_0^m - 1$	0	0	$v^m_{H,r} - 1$
$e_H S_0^i$	0	0	$v^i_{H,r}$

Therefore, we have $S_0^m - 1 = e_H((\sum_{0 \le i < m_H} a_{H,i}S_0^i) - 1)$ hence $S^m = 1 + Pe_H((\sum_{0 \le i < m_H} a_{H,i}S_0^i) - 1)P^{-1} = 1 + e_H((\sum_{0 \le i < m_H} a_{H,i}(PS_0P^{-1})^i) - 1) = 1 + e_H((\sum_{0 \le i < m_H} a_{H,i}S^i) - 1)$ and this proves the claim.

□

We remark that proposition 4.1 admits no direct generalization to the complex reflection groups setting, namely there is not in general a 1-parameter family of morphisms $B \to \mathcal{C}_W^p(\underline{a})$ of a similar form. Indeed, let us consider for W the group generated by 2-reflections called G_{12} in the Shephard-Todd classification. Its braid group has the presentation $\langle s,t,u \mid stus = tust = ustu \rangle$ and $W = B/\langle s^2, t^2, u^2\rangle$. Letting $e_x \in \mathcal{C}_W^p(\underline{a})$ denote the idempotent associated to the hyperplane $\mathrm{Ker}\,(x-1)$, for $x \in W$ a reflection, one can check that there can be a morphism $B \to \mathcal{C}_W^p(\underline{a})$ satisfying $y \mapsto y + \lambda e_y y$, for $y \in \{s,t,u\}$ only if the 4 reflecting hyperplanes associated to the reflections $\{s, sts, stuts, stusuts\}$ are the same as the ones associated to the reflections $\{t, tut, tusut, tustsut\}$ (equivalently, that these two sets of 2-reflections are equal). One readily checks that this does *not* hold.

5.3. **An extended freeness conjecture.** For a W-orbit of hyperplanes c, the order m_H of W_H for $H \in c$ depends only on c. Therefore, we can denote it m_c, and define a generic ring $R_W = \mathbf{Z}[(a_{c,i}, a_{c,0}^{-1}]$ for $c \in \mathcal{A}/W$ and $0 \le i < m_c$. The *generic algebra* $\mathcal{C}_W^p$ is defined over the ring $\mathbf{k} = \mathbf{Z}[(a_{c,i}, a_{c,0}^{-1}]$ as in definition 5.4 by letting $a_{H,i} = a_{c,i}$ if $H \in c$.

Proposition 5.8. *If the algebra $\mathcal{C}_W^p$ is spanned by $|W|.|\mathcal{L}|$ elements as a R_W-module, then it is a free R_W-module of rank $|W|.|\mathcal{L}|$.*

Proof. The proof follows exactly the same lines as in [5] (proof of theorem 4.24), see also [26] proposition 2.4, the 'monodromic' ingredient being given by proposition 5.7 above. It is left to the reader. □

We consider $\mathcal{C}_W^p(\underline{a})$ as a $\mathbf{k}B$-module. As a $\mathbf{k}B$-module, it is generated by the $e_L, L \in \mathcal{L}$. Let $\mathcal{E}_L = \sum_{F\subset L} (\mathbf{k}B).e_F$, $\mathcal{E}'_L = \sum_{\substack{F\neq L\\ F\subset L}} (\mathbf{k}B).e_F$ and $\overline{\mathcal{E}}_L = \mathcal{E}_L/\mathcal{E}'_L$.

Lemma 5.9. *If each $\overline{\mathcal{E}}_L$ is spanned as a $\mathbf{k}$-module by $|W|$ elements of the form $b.e_L, b \in B$, then $\mathcal{C}_W^p(\underline{a})$ is spanned by $|W|.|\mathcal{L}|$ elements, and therefore it is a free $\mathbf{k}$-module of rank $|W|.|\mathcal{L}|$.*

Proof. Assume that, for each L, we have elements $b_{L,w}, w \in W$ such that $\overline{\mathcal{E}}_L$ is spanned by the $b_{L,w}.e_L$. We shall prove that $\mathcal{C}_W^p(\underline{a})$ is spanned by the $b_{L,w}.e_L$ for $L \in \mathcal{L}, w \in W$. Since $\mathcal{C}_W^p(\underline{a})$ is generated as a $\mathbf{k}B$-module by the $e_L, L \in \mathcal{L}$, it is spanned as a $\mathbf{k}$-module by the $be_L, L \in \mathcal{L}$. Therefore, it is sufficent to prove that such a be_{L_0} is a linear combination of the $b_{L,w}.e_L, L \in \mathcal{L}$. We prove this by induction on L_0 with respect to the well-ordering provided by the lattice $\mathcal{L}$. If $L_0 = \{0\}$, then $b.e_L = b.e_W \in \mathcal{E}_L = \overline{\mathcal{E}}$ and we have the conclusion by assumption. If not, we know that there exists scalars $\alpha_{L_0,w}, w \in W$ such that $x = b.e_{L_0} - \sum_{w\in W} \alpha_{L_0,w} b_{L_0,w}.e_{L_0} \in \mathcal{E}'_{L_0}$. By the induction assumption we can write x as a linear combination of the $b_{L,w}e_L$ for $L \subsetneq L_0$, and therefore $b.e_{L_0}$ as a linear combination of the $b_{L,w}e_L$ for $L \subset L_0$, and this proves the claim. □

We notice that the action of $\mathbf{k}B$ on $\overline{\mathcal{E}}_{\{0\}} = \mathcal{E}_{\{0\}}$ factorizes through $H_W(\underline{a})$, and therefore $\overline{\mathcal{E}}_{\{0\}}$ is spanned by $|W|$ elements if and only if the BMR freeness conjecture is true for W. We also notice that the action of $\mathbf{k}B$ on $\overline{\mathcal{E}}_V$ factorizes through the regular representation of $\mathbf{k}W$, hence $\overline{\mathcal{E}}_V$ is clearly spanned by $|W|$ elements.

In this way, the presumed fact that each $\overline{\mathcal{E}}_L$ is spanned by $|W|$ elements appears as an intermediate between the trivial fact that $\mathbf{k}W$ has this property and the BMR freeness conjecture that H_W is spanned by $|W|$ elements. For a given $L \neq \{0\}$, and if true, it should be easier to prove than the freeness conjecture for H_W, since, at each stage, the relation $g_s^m = \dots$ to be used can be either the complicated (Hecke) one or the trivial one ($g_s^m = 1$). However, it does not seem to readily follow from it, and therefore we propose it as a (a priori stronger) conjecture.

Conjecture 5.10. *(extended freeness conjecture) The algebra $\mathcal{C}_W^p(\underline{a})$ is a free $\mathbf{k}$-module of rank $|W|.|\mathcal{L}|$. Moreover, each module $\overline{\mathcal{E}_L}, L \in \mathcal{L}$, is spanned by $|W|$ elements of the form $b_{L,w}.e_L, w \in W$, with $b_{L,w} \in B$ mapping to $w \in W$ under the natural map $B \twoheadrightarrow W$.*

If $\mathcal{C}_W^p$ is a R_W-module of rank $|W|.|\mathcal{L}|$, then it is a free deformation of the algebra $\mathcal{C}_W(1)$, which is semisimple for $\mathbf{k} = \mathbf{Q}$ by proposition 5.1. Therefore, Tits' deformation theorem (see e.g. [19], §7.4) and proposition 5.1 imply the following, where K_W denotes a field containing R_W.

Proposition 5.11. *If the extended conjecture is true, then $\mathcal{C}_W^p \otimes_{R_W} K_W$ is semisimple. If moreover K_W is algebraically closed, then $\mathcal{C}_W^p \otimes_{R_W} K_W \simeq \mathcal{C}_W^p(1) \otimes K_W \simeq K_W^{\mathcal{L}} \rtimes K_W W$.*

If W has rank 2 and the BMR freeness conjecture is true for W, the proof is reduced to the consideration of the $\overline{\mathcal{E}}_H$ for $H \in \mathcal{A}$. Since $gbe_L g^{-1} = gbg^{-1}e_{\pi(g)(L)}$ for all $g \in B$, we moreover need to consider only one hyperplane per W-orbit.

5.4. **The case of G_4.** The smallest non-trivial example of an irreducible non-real complex reflection group is the group $Q_8 \rtimes \mathbf{Z}_3$ denoted G_4 in Shephard-Todd notation. It is also the group for which the original BMR freeness conjecture has had, so far, the more topological applications (see e.g. [27, 28]). In this case $B = \langle s_1, s_2 \mid s_1s_2s_1 = s_2s_1s_2 \rangle$ is the Artin group of type A_2 (a.k.a. the braid group on 3 strands) and W is the quotient of B by the realtions $s_1^3 = s_2^3 = 1$. A proof of the original BMR freeness conjecture for this case can be found for instance in [25].

We let $\mathcal{B} = \{1, s_1^{\varepsilon}, s_1^{\alpha}s_2^{\varepsilon}s_1^{\beta}, s_2^{-1}s_1s_2^{-1}s_1^{\alpha}\}$ where $\alpha, \beta \in \{-1,0,1\}$, $\varepsilon \in \{-1,1\}$. We have $\#\mathcal{B} = 24 = |W|$, and we want to prove that $\mathcal{B}.e_1 \subset E = \overline{\mathcal{E}_1}$ is a $\mathbf{k}B$-submodule. This will prove $\mathcal{B}.e_1 = E$ since $1 \in \mathcal{B}$. Since B is generated as a group by s_1, s_2 it is sufficient to prove that $s_i b e_1 \subset \mathcal{B}.e_1$ for $i \in \{1,2\}$ and $b \in \mathcal{B}$. We let $e_i = e_{\mathrm{Ker}\,(s_i - 1)}$ pour $i \in \{1,2,3,4\}$ by letting $s_3 = s_1s_2s_1^{-1}$, $s_4 = s_2s_1s_2^{-1}$. Inside E we have $e_ie_j = \delta_{ij}e_i = \delta_{ij}e_j$. We have $s_ie_j = e_{\sigma_i(j)}s_i$ where $\sigma_1 = (2,3,4)$ and $\sigma_2 = (1,4,3)$. By definition we have $s_i^3 = 1 + e_iP_i = 1 + P_ie_i$ for $P_i = as_i^2 + bs_i + c - 1$, and this implies $s_i^2 = s_i^{-1} + e_iQ_i = 1 + Q_ie_i$ with $Q_i = as_i + b + (c-1)s_i^{-1}$.

Lemma 5.12.

(1) For all $m \in \mathbf{Z}$, $b \in B$, $i \in \{1,2\}$, we have $s_i^m be_1 \in \langle s_i^{\alpha}be_1, \alpha \in \{-1,0,1\}\rangle$.
(2) For all $\alpha, \beta \in \mathbf{Z}$ we have $s_2^{\alpha}s_1s_2^{-1}s_1^{\beta}e_1 \in \langle \mathcal{B}e_1\rangle$.
(3) For all $x \in \langle s_1^m, m \in \mathbf{Z}\rangle$ we have $(s_2s_1^{-1}s_2)s_1xe_1 = s_2^{-1}s_1s_2^{-1}xe_1$
(4) For all $x \in \langle s_1^m, m \in \mathbf{Z}\rangle$ we have $(s_2^{-1}s_1s_2^{-1})xe_1 = x(s_2^{-1}s_1s_2^{-1})e_1$.

Proof. Since $s_i^2 = s_i^{-1} + e_iQ_i$ we have $s_i^2be_1 = s_i^{-1}be_1 + Q_ie_ibe_1 = s_i^{-1}be_1 + Q_ie_{\beta(i)}e_1 = s_i^{-1}be_1 + Q_i\delta_{\beta(i),1}e_1$ for some $\beta \in \mathfrak{S}_4$. This proves $s_i^2be_1 \in \langle s_i^{\alpha}be_1, \alpha \in \{-1,0,1\}\rangle$ hence (1).

We now prove (2). By item (1) we can assume $\alpha, \beta \in \{-1,0,1\}$. If $\alpha \in \{0,-1\}$, then $s_2^{\alpha}s_1s_2^{-1}s_1^{\beta}e_1 \in \mathcal{B}e_1$ by definition. If $\alpha = 1$, then $s_2^{\alpha}s_1s_2^{-1}s_1^{\beta}e_1 = (s_2s_1s_2^{-1})s_1^{\beta}e_1 = s_1^{-1}s_2s_1s_1^{\beta}e_1 = s_1^{-1}s_2s_1^{\beta+1}e_1$ by the braid relation. This element belongs to $\langle \mathcal{B}e_1\rangle$ by (1), and this proves (2).

We now prove (3). We have $s_2(s_1^{-1}s_2s_1)xe_1 = s_2^2s_1s_2^{-1}xe_1 = (s_2^{-1} - Q_2e_2)s_1s_2^{-1}xe_1 = s_2^{-1}s_1s_2^{-1}xe_1 - Q_2e_2s_1s_2^{-1}xe_1$. Now $e_2s_1s_2^{-1} = s_1s_2^{-1}e_3$ and $e_3s_1^{\alpha}e_1 = 0$ for all α, hence $Q_2e_2s_1s_2^{-1}xe_1 = Q_2s_1s_2^{-1}xe_3e_1$.

We now prove (4). We first assume $x = s_1^{-1}$. We have $s_1s_2^{-1}s_1^{-1}e_1 = (s_1^{-2} - Ae_1)s_2^{-1}s_1^{-1}e_1$for some $A \in \mathbf{k}B$. Since $e_1s_2^{-1}s_1^{-1}e_1 = 0$ we get $s_1s_2^{-1}s_1^{-1}e_1 = s_1^{-2}s_2^{-1}s_1^{-1}e_1 = s_1^{-1}(s_1^{-1}s_2^{-1}s_1^{-1})e_1 = s_1^{-1}s_2^{-1}s_1^{-1}s_2^{-1}e_1$. Thus $s_2^{-1}.s_1s_2^{-1}s_1^{-1}e_1 = s_2^{-1}.s_1^{-1}s_2^{-1}s_1^{-1}s_2^{-1}e_1 = s_1^{-1}s_2^{-1}s_1^{-1}s_1^{-1}s_2^{-1}e_1$ by the braid relation Now, $s_1^{-2}s_2^{-1}e_1 = s_1s_2^{-1}e_1$ since $e_1s_2^{-1}e_1 = 0$ hence $s_2^{-1}.s_1s_2^{-1}s_1^{-1}e_1 = s_1^{-1}s_2^{-1}s_1s_2^{-1}e_1$ and we have proven (4) for $x = s_1^{-1}$. This implies that (4) holds true for every power of s_1^{-1}, whence the claim.

∎

We now prove that $s_ibe_1 \in \langle \mathcal{B}e_1\rangle$ for all $b \in \mathcal{B}$, $i \in \{1,2\}$. For $b \in \{1, s_1^{\varepsilon}\}$ this is clearly true. Now assume $b = s_1^{\alpha}s_2^{\varepsilon}s_1^{\beta}$. If $i = 1$ this is an immediate consequence of the definition of $\mathcal{B}$ and of lemma 5.12 (1). We consider s_2be_1. If $\varepsilon = -1$, then $s_2be_1 = s_2s_1^{\alpha}s_2^{-1}s_1^{\beta}e_1 = s_1^{-1}s_2^{\alpha}s_1s_1^{\beta}e_1$ by the braid relation, hence $s_2be_1 \in \langle \mathcal{B}e_1\rangle$. Therefore we can assume $\varepsilon = 1$. If $\alpha = 0$, then $s_2be_1 = s_2^2s_1^{\beta}e_1 \in \langle \mathcal{B}e_1\rangle$ by lemma 5.12 (1). If $\alpha = 1$, we have $s_2be_1 = (s_2s_1s_2)s_1^{\beta}e_1 = (s_1s_2s_1)s_1^{\beta}e_1 \in \langle \mathcal{B}e_1\rangle$ by (1). If $\alpha = -1$, then $s_2be_1 = s_2s_1^{-1}s_2s_1^{\beta}e_1 = s_2s_1^{-1}s_2s_1s_1^{\beta-1}e_1 = s_2^{-1}s_1s_2^{-1}s_1^{\beta-1}e_1$ by (3), hence $s_2be_1 \in \langle \mathcal{B}e_1\rangle$. This concludes the case $b = s_1^{\alpha}s_2^{\varepsilon}s_1^{\beta}$. We now assume $b = s_2^{-1}s_1s_2^{-1}s_1^{\alpha}$. We have $s_2be_1 = s_1s_2^{-1}s_1^{\alpha}e_1 \in \mathcal{B}$, and $s_1be_1 = s_1s_2^{-1}s_1s_2^{-1}s_1^{\alpha}e_1 = s_2^{-1}s_1s_2^{-1}s_1^{\alpha+1}e_1$ by (4), which belongs to $\langle \mathcal{B}e_1\rangle$ by (1). This proves conjecture 5.10 for $W = G_4$.

5.5. **An extended Ariki-Koike algebra.** Let $W = G(d,1,n)$ be the group of $n \times n$ monomial matrices with entries in $\mu_d(\mathbf{C}) \cup \{0\}$. In this case H_W is known as the Ariki-Koike algebra, and B is the Artin group of type B_n/C_n, with generators $t = a_1, a_2, \dots, a_n$. The image of $a_i a_j$ inside W has order 4 if $\{i,j\} = \{1,2\}$, 2 if $|i-j| \geq 2$, and 3 otherwise. If we abuse notations by letting $e_b, b \in B$ be equal to $e_{\mathrm{Ker}\,(\beta - 1)}$ for $\beta \in W$ the image of b, we have inside $\mathcal{C}_W^p$ the relations $t^d = 1 + (q-1)e_t P(t)$ for some polynomial P of degree at most $d-1$, and $a_i^2 = 1 + (q-1)(a_i+1)e_{a_i}$. We adapt the arguments of [2] to prove conjecture 5.10.

First of all, we let $t_1 = t$, $t_i = a_i t_{i-1} a_i$. There is a classical injective morphism $B \to B_{n+1}$, where B_{n+1} is the usual braid group on $n+1$ strands, given by $t \mapsto \sigma_1^2$, $a_i \mapsto \sigma_{i+1}$ for $i \geq 2$, where $\sigma_1, \dots, \sigma_{n+1}$ denote the classical Artin generators. Under this map, each t_i is mapped to $\delta_{i+1} = \sigma_{i+1}\sigma_i \dots \sigma_2 \sigma_1^2 \sigma_2 \dots \sigma_{i+1}$, and $\delta_i = z_{i+1}/z_i$ where $z_i = (\sigma_1 \dots \sigma_{i-1})^i$ is the canonical generator of $Z(B_i)$. From this, we have the following relations inside B, and therefore inside $\mathcal{C}_W^p$:

(1) For all i,j with $j \notin \{i, i+1\}$ we have $a_j t_i = t_i a_j$
(2) For all i,j we have $t_i t_j = t_j t_i$
(3) For all i, we have $a_i t_{i-1} t_i = t_{i-1} t_i a_i$.

Let E denote the (commutative) subalgebra of $\mathcal{C}_W^p$ generated by the $e_H, H \in \mathcal{A}$. Note that $Eb \subset bE$ for all $b \in B$. The above equalities moreover imply that the $t_i, i \geq 1$ generate a commutative subalgebra of $\mathcal{C}_W^p$. We prove the following lemma.

Lemma 5.13. *For all $k \geq 1$,*

(1) For all $i \geq 2$, $a_i t_i^k \in t_{i-1}^k a_i E + \sum_{j=1}^k t_{i-1}^{j-1} t_i^{k+1-j} E$
(2) For all $i \geq 1$, $a_{i+1} t_i^k \in t_{i+1}^k a_{i+1} E + \sum_{j=1}^k t_i^{k-j} t_{i+1}^j E$

Proof. We prove (1) by induction on $k \geq 1$. We first assume $k=1$. We have $a_i t_i = a_i^2 t_{i-1} a_i = t_{i-1}a_i + (q-1)(a_i+1)e_{a_i} t_{i-1} a_i = t_{i-1}a_i + (q-1)e_{a_i} t_{i-1} a_i + (q-1) a_i e_{a_i} t_{i-1} a_i = t_{i-1}a_i + (q-1) t_{i-1} a_i e_{a_i^{-1} t_{i-1}^{-1} a_i t_{i-1} a_i} + (q-1) a_i t_{i-1} a_i e_{t_{i-1}^{-1} a_i t_{i-1}} = t_{i-1} a_i + (q-1) t_{i-1} a_i e_{a_i^{-1} t_{i-1}^{-1} a_i t_{i-1} a_i} + (q-1) t_i e_{t_{i-1}^{-1} a_i t_{i-1}} \in t_{i-1} a_i E + t_i E$. Now, if $k \geq 2$, then $a_i t_i^k = a_i t_i^{k-1} t_i \in t_{i-1}^{k-1} a_i E t_i + \sum_{j=1}^{k-1} t_{i-1}^{j-1} t_i^{k-j} E t_i \subset t_{i-1}^{k-1}(a_i t_i) E + \sum_{j=1}^{k-1} t_{i-1}^{j-1} t_i^{k-j} t_i E \subset t_{i-1}^{k-1}(t_{i-1} a_i E + t_i E) E + \sum_{j=1}^{k-1} t_{i-1}^{j-1} t_i^{k+1-j} E \subset t_{i-1}^k a_i E + t_{i-1}^{k-1} t_i E + \sum_{j=1}^{k-1} t_{i-1}^{j-1} j t_i^{k+1-j} E \subset t_{i-1}^k a_i E + \sum_{j=1}^k t_{i-1}^{j-1} t_i^{k+1-j} E$ and this proves (1).

We now prove (2) by induction on $k \geq 1$. If $k=1$, then $a_{i+1} t_i a_{i+1} = t_{i+1}$ implies $a_{i+1} t_i = t_{i+1} a_{i+1}^{-1} \in t_{i+1} a_{i+1} E + t_{i+1} E$. If $k \geq 2$, then $a_{i+1} t_i^k = a_{i+1} t_i^{k-1} t_i \in t_{i+1}^{k-1} a_{i+1} E t_i + \sum_{j=1}^{k-1} t_i^{k-1-j} t_{i+1}^j E t_i \subset t_{i+1}^{k-1}(a_{i+1} t_i) E + \sum_{j=1}^{k-1} t_i^{k-1-j} t_{i+1}^j t_i E \subset t_{i+1}^{k-1}(t_{i+1} a_{i+1} E + t_{i+1} E) E + \sum_{j=1}^{k-1} t_i^{k-j} t_{i+1}^j E \subset t_{i+1}^{k-1} t_{i+1} a_{i+1} E + t_{i+1}^{k-1} t_{i+1} E + \sum_{j=1}^{k-1} t_i^{k-j} t_{i+1}^j E \subset t_{i+1}^k a_{i+1} E + \sum_{j=1}^k t_i^{k-j} t_{i+1}^j E$ and this proves (2). ☐

Since $a_2, \dots, a_n$ satisfy the braid relations in type A_{n-1}, by Iwahori-Matsumoto theorem we know that, for each $g \in \mathfrak{S}_n$ there is a well-defined $a_g \in B$ such that $a_g = a_{i_1} \dots a_{i_r}$ for every reduced decomposition $g = s_{i_1} \dots s_{i_r}$ with $s_{i_m} = (m, m-1)$. We note that, for each $i \geq 2$, $a_i a_g \in \sum_{h \in \mathfrak{S}_n} a_h E$, as a consequence of the corresponding inequality inside $\mathcal{C}_{\mathfrak{S}_n}(u)$. From this we prove that

$$\mathcal{C}_W^p = \sum_{g \in \mathfrak{S}_n} \sum_{0 \leq k_1, \dots, k_n \leq d} t_1^{k_1} \dots t_n^{k_n} a_g E$$

Indeed, the RHS contains 1 and is clearly stable by left multiplication under

- $a_1 = t = t_1$, by the order relation $t^d = 1 + (q-1)P(t)e_t$
- $a_2, \dots, a_n$ by lemma 5.13 and the fact that $a_i a_g E \subset \sum_{h \in \mathfrak{S}_n} a_h E$ for all $i \geq 2$.

Since E is spanned by $|\mathcal{W}_p|$ elements, and $|W| = d^n n!$, this proves that the assumption of proposition 5.8 is satisfied, and this proves conjecture 5.10 for $W = G(d,1,n)$.

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