North-Western European Journal of Mathematics

Simplicial Random Variables

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Abstract

We introduce a new 'geometric realization' of an (abstract) simplicial complex, inspired by probability theory. This space (and its completion) is a metric space, which has the right (weak) homotopy type, and which can be compared with the usual geometric realization through a natural map, which has probabilistic meaning : it associates to a random variable its probability mass function. This 'probability law' map is proved to be a Serre fibration and an homotopy equivalence.

Keywords: Geometric realization, Random variables, Simplicial complexes.

MSC: 55P10, 60A99.

1 Introduction and main results

In this paper we consider a new 'geometric realization' of an (abstract) simplicial complex, inspired by probability theory. This space is a metric space, which has the right (weak) homotopy type, and can be compared with the usual geometric realization through a map, which is very natural in probabilistic terms : it associates to a random variable its probability mass function. This 'probability law' function is proved to be a (Serre) fibration and a (weak) homotopy equivalence. This construction passes to the completion, and has nice functorial properties.

We specify the details now. Let S be a set, and $\mathscr{P}_{\mathrm{f}}(S)$ the set of its finite subsets. We set $\mathscr{P}_{\mathrm{f}}^*(S) = \mathscr{P}_{\mathrm{f}}(S) \setminus \{\emptyset\}$. Recall that an (abstract) simplicial complex is a collection of subsets $\mathscr{K} \subset \mathscr{P}_{\mathrm{f}}^*(S)$ with the property that, for all $X \in \mathscr{K}$ and $Y \in \mathscr{P}_{\mathrm{f}}^*(S)$, $Y \subset X \Rightarrow Y \in \mathscr{K}$. The elements of $\mathscr{K}$ are called its faces, and the vertices of $\mathscr{K}$ are the union of the elements of $\mathscr{K}$.

We endow S with the discrete metric of diameter 1, and with the Borel σ-algebra associated to this topology. We let Ω denote a nonatomic standard probability space with measure λ. Recall that all such probability spaces are isomorphic and can be identified in particular with any hypercube $[0,1]^n$, $n \geq 1$ endowed with the Lebesgue measure. We define $L(\Omega, S)$ as the set of random variables $\Omega \to S$, that is the set of measurable maps $\Omega \to S$ modulo the equivalence relation $f \equiv g$ if f and g agree

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almost everywhere, that is $\lambda(\{x; f(x) = g(x)\}) = 0$. We consider it as a metric space, endowed with the metric

$$d(f,g) = \int_\Omega d(f(t),g(t))\mathrm{d}t = \lambda(\{x \in \Omega; f(x) \neq g(x)\}).$$

We define $L(\Omega,\mathcal{K})$ as the subset of $L(\Omega,S)$ made of the (equivalence classes of) measurable maps $f : \Omega \to S$ such that $\{s \in S \mid \lambda(f^{-1}(\{s\})) > 0\} \in \mathcal{K}$.

Recall that the (usual) 'geometric' realization of $\mathcal{K}$ is defined as

$$|\mathcal{K}| = \{t : S \to [0,1] \mid \{s \in S; t_s > 0\} \in \mathcal{K} \ \& \ \sum_{s\in S} t_s = 1\}$$

and that its topology is given by the direct limit of the $[0,1]^A$ for $A \in \mathcal{P}_{\mathrm{f}}(S)$. There is a natural map $L(\Omega,\mathcal{K}) \to |\mathcal{K}|$ which associates to $f : \Omega \to \mathcal{K}$ the element $t : S \to [0,1]$ defined by $t_s = \lambda(f^{-1}(\{s\}))$. In probabilistic terms, it associates to the random variable f its probability law, or probability mass function. We denote $|\mathcal{K}|_1$ the same set as $|\mathcal{K}|$, but with the topology defined by the metric $|\alpha-\beta|_1 = \sum_{s\in S}|\alpha(s)-\beta(s)|$. We denote $\overline{|\mathcal{K}|_1}$ its completion as a metric space.

It is easily checked that, unless S is finite, $L(\Omega,\mathcal{K})$ is not in general closed in $L(\Omega,S)$, and therefore not complete. We denote $\bar{L}(\Omega,\mathcal{K})$ its closure inside $L(\Omega,S)$. The 'probability law' map $\Psi : L(\Omega,\mathcal{K}) \to |\mathcal{K}|_1$ is actually continuous, and can be extended to a map $\overline{\Psi} : \bar{L}(\Omega,\mathcal{K}) \to \overline{|\mathcal{K}|_1}$. Keane's Theorem about the contractibility of $\mathrm{Aut}(\Omega)$ (see Keane 1970) easily implies that these maps have contractible fibers. The goal of this note is to specify the homotopy-theoretic features of them. We get the following results.

Theorem 1 –

1. *The map $L(\Omega,\mathcal{K}) \to \bar{L}(\Omega,\mathcal{K})$ is a weak homotopy equivalence.*
2. *The 'probability law' map $L(\Omega,\mathcal{K}) \to |\mathcal{K}|_1$ is a Serre fibration and an homotopy equivalence. It admits a continuous global section.*
3. *The 'probability law' map $\bar{L}(\Omega,\mathcal{K}) \to \overline{|\mathcal{K}|_1}$ is a Serre fibration and an homotopy equivalence. It admits a continuous global section.*
4. *$L(\Omega,\mathcal{K})$ and $\bar{L}(\Omega,\mathcal{K})$ have the same weak homotopy type as the 'geometric realization' $|\mathcal{K}|$ of $\mathcal{K}$.*

In particular, in the commutative diagram below, the vertical maps are Serre fibrations, and all the maps involved are weak homotopy equivalences. When $\mathcal{K}$ is finite, $L(\Omega,\mathcal{K}) = \bar{L}(\Omega,\mathcal{K})$ and we prove in addition that the map $\Psi_{\mathcal{K}} = \overline{\Psi}_{\mathcal{K}}$ is a Hurewicz fibration (see Theorem 2).

$$\begin{array}{ccccc} & & L(\Omega,\mathcal{K}) & \hookrightarrow & \bar{L}(\Omega,\mathcal{K}) \\ & & \downarrow \Psi_{\mathcal{K}} & & \downarrow \bar{\Psi}_{\mathcal{K}} \\ |\mathcal{K}| & \longrightarrow & |\mathcal{K}|_1 & \hookrightarrow & \overline{|\mathcal{K}|_1} \end{array}$$

We now comment on the functorial properties of this construction. By definition, a morphism $\varphi : \mathcal{K}_1 \to \mathcal{K}_2$ between simplicial complexes is a map from the set $\bigcup \mathcal{K}_1$ of vertices of $\mathcal{K}_1$ to the set of vertices of $\mathcal{K}_2$ with the property that $\forall F \in \mathcal{K}_1 \ \ \varphi(F) \in \mathcal{K}_2$. We denote **Simp** the corresponding category of simplicial complexes. For such an abstract simplicial complex $\mathcal{K}$, our space $L(\Omega,\mathcal{K})$ has for ambient space $L(\Omega, S)$ with $S = \bigcup \mathcal{K}$ the set of vertices of $\mathcal{K}$.

Let **Set** denote the category of sets and $\mathbf{Met}_1$ denote the full subcategory of the category of metric spaces and contracting maps made of the spaces of diameter at most 1. Here a map $f : X \to Y$ between two metric spaces is called contracting if $\forall a, b \in X \ d(f(a), f(b)) \leq d(a, b)$. Let $\mathbf{CMet}_1$ be the full subcategory of $\mathbf{Met}_1$ made of complete metric spaces. There is a completion functor $Comp : \mathbf{Met}_1 \to \mathbf{CMet}_1$ which associates to each metric space its completion. Then $L(\Omega, \bullet) : X \rightsquigarrow L(\Omega, X)$ defines a functor $\mathbf{Set} \to \mathbf{CMet}_1$ (see Marin 2017). It can be decomposed as $L(\Omega, \bullet) = Comp \circ L_{\mathrm{f}}(\Omega, \bullet)$ where $L_{\mathrm{f}}(\Omega, S)$ is the subspace of $L(\Omega, S)$ made of the (equivalence classes of) functions $f : \Omega \to S$ of essentially finite image, that is such that there exists $S_0 \subset S$ finite such that $\sum_{s \in S_0} \lambda(f^{-1}(\{s\})) = 1$.

We prove in section 2.1 below that our simplicial constructions have similar functorial properties, which can be summed up as follows.

Proposition 1 – *$L(\Omega, \bullet)$ and $\bar{L}(\Omega, \bullet)$ define functors $\mathbf{Simp} \to \mathbf{Met}_1$ and $\mathbf{Simp} \to \mathbf{CMet}_1$, with the property that $\bar{L}(\Omega, \bullet) = Comp \circ L(\Omega, \bullet)$.*

2 Simplicial properties and completion

In this section we prove part (1) of Theorem 1. We start by proving the functorial properties stated in the introduction.

2.1 Functorial properties

We denote, as in the previous section, $\bar{L}(\Omega,\mathcal{K})$ the closure of $L(\Omega,\mathcal{K})$ inside $L(\Omega, S)$. As a closed subset of a complete metric space, it is a complete metric space. For any $f \in L(\Omega, S)$, we denote

$$f(\Omega) = \{s \in S \mid \lambda(f^{-1}(\{s\})) > 0\}$$

the essential image of an arbitrary measurable map $\Omega \to S$ representing f.

Lemma 1 – *Let $f \in L(\Omega, S)$. Then $f \in \overline{L}(\Omega, \mathcal{K})$ if and only if every nonempty finite subset of $f(\Omega)$ belongs to $\mathcal{K}$.*

Proof. Assume $f \in \bar{L}(\Omega, \mathcal{K})$ and let $F \subset f(\Omega)$ be a nonempty finite subset as in the statement. We set $m = \min\{\lambda(f^{-1}(\{s\})) \mid s \in F\}$. We have $m > 0$. Since $f \in \bar{L}(\Omega, \mathcal{K})$, there exists $f_0 \in L(\Omega, \mathcal{K})$ such that $d(f, f_0) < m$. We then have $F \subset f_0(\Omega)$. Indeed, there would otherwise exist $s \in F \setminus f_0(\Omega)$, and then $d(f, f_0) \geq \lambda(f^{-1}(\{s\})) \geq m$, a contradiction. From this we get $F \in \mathcal{K}$. Conversely, assume that every nonempty finite subset of $f(\Omega)$ belongs to $\mathcal{K}$. From Marin 2017 Proposition 3.3 we know that $f(\Omega) \subset S$ is countable. If $f(\Omega)$ is finite we have $f(\Omega) \in \mathcal{K}$ by assumption and $f \in L(\Omega, \mathcal{K})$. Otherwise, let us fix a bijection $\mathbf{N} \to f(\Omega)$, $n \mapsto x_n$ and define $f_n \in L(\Omega, S)$ by $f_n(t) = f(t)$ if $f(t) \in \{x_0, \dots, x_n\}$, and $f_n(t) = x_0$ otherwise. Clearly $f_n(\Omega) \subset f(\Omega)$ is nonempty finite hence belongs to $\mathcal{K}$, and $f_n \in L(\Omega, \mathcal{K})$. On the other hand, $d(f_n, f) \leq \sum_{k>n} \lambda(f^{-1}(\{x_k\})) \to 0$, hence $f \in \bar{L}(\Omega, \mathcal{K})$ and this proves the claim. □

We prove that, as announced in the introduction, $\bar{L}(\Omega, \bullet)$ provides a functor $\mathbf{Simp} \to \mathbf{CMet}_1$ that can be decomposed as $Comp \circ L(\Omega, \bullet)$, where $L(\Omega, \bullet)$ is itself a functor $\mathbf{Simp} \to \mathbf{Met}_1$.

Let $\varphi \in \mathrm{Hom}_{\mathbf{Simp}}(\mathcal{K}_1, \mathcal{K}_2)$ that is $\varphi : \bigcup \mathcal{K}_1 \to \bigcup \mathcal{K}_2$ such that $\varphi(F) \in \mathcal{K}_2$ for all $F \in \mathcal{K}_1$. If $f \in L(\Omega, \mathcal{K}_1)$, $g = L(\Omega, \varphi)(f) = \varphi \circ f$ is a measurable map and $g(\Omega) = \varphi(f(\Omega))$. Since $f(\Omega) \in \mathcal{K}_1$ and φ is simplicial we get that $\varphi(f(\Omega)) \in \mathcal{K}_2$ hence $g \in L(\Omega, \mathcal{K}_2)$. From this one gets immediately that $L(\Omega, \bullet)$ indeed defines a functor $\mathbf{Simp} \to \mathbf{Met_1}$.

Similarly, if $f \in \bar{L}(\Omega, \mathcal{K}_1)$ and $g = \varphi \circ f = L(\Omega, \varphi)(f) \in L(\Omega, S)$, then again $g(\Omega) = \varphi(f(\Omega))$. But, for any finite set $F \subset g(\Omega) = \varphi(f(\Omega))$ there exists $F' \subset f(\Omega)$ finite and with the property that $F = \varphi(F')$. Now $f \in \bar{L}(\Omega, \mathcal{K}_1) \Rightarrow F' \in \mathcal{K}_1$, by Lemma 1, hence $F \in \mathcal{K}_2$ because φ is a simplicial morphism. By Lemma 1 one gets $g \in \bar{L}(\Omega, \mathcal{K}_2)$, hence $\bar{L}(\Omega, \bullet)$ defines a functor $\mathbf{Simp} \to \mathbf{CMet_1}$. We checks immediately that $\bar{L}(\Omega, \bullet) = Comp \circ L(\Omega, \bullet)$, and this proves Proposition 1.

2.2 Technical preliminaries

We denote by 2 in the notation $L(\Omega, 2)$ a set with two elements. When needed, we will also assume that this set is pointed, that is contains a special point called 0, so that $f \in L(\Omega, 2)$ can be identified with $\{t \in \Omega; f(t) \neq 0\}$, up to a set of measure 0. Note that these conventions agree with the set-theoretic definition of $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$.

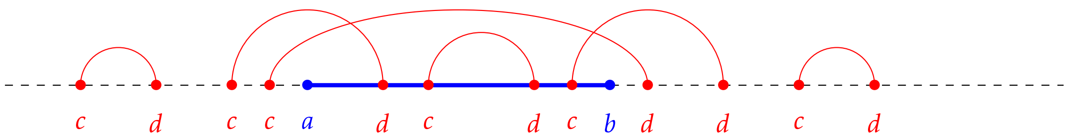
Lemma 2 – *Let F be a set. The map $f \mapsto \{t \in \Omega; f(t) \notin F\}$ is uniformly continuous $L(\Omega, S) \to L(\Omega, 2)$, and even contracting.*

Proof. Let $f_1, f_2 \in L(\Omega, S)$, and $\Psi : L(\Omega, S) \to L(\Omega, 2)$ the map defined by the statement. Then $\Psi(f_1)(t) \neq \Psi(f_2)(t) \Rightarrow f_1(t) \neq f_2(t)$, hence $d(\Psi(f_1)(t), \Psi(f_2)(t)) \leq d(f_1(t), f_2(t))$ for all $t \in \Omega$ and finally $d(\Psi(f_1), \Psi(f_2)) \leq d(f_1, f_2)$, whence Ψ is contracting and uniformly continuous. □

Lemma 3 – *Let $a,b,c,d \in \mathbf{R}$ with $a \le b$ and $c \le d$. Then*

$$\lambda([a,b]\setminus]c,d[) \leqslant |a-c| + |b-d|.$$

Proof. There are six possible relative positions of $c \le d$ with respect to $a \le b$ to consider, which are depicted as follows.



In three of them, namely $a \le b \le c \le d$, $c \le d \le a \le b$, and $c \le a \le b \le d$, we have $\lambda([a,b]\setminus]c,d[) = 0$. In case $c \le a \le d \le b$, we have $\lambda([a,b]\setminus]c,d[) = \lambda([d,b]) = |b-d| \le |a-c|+|b-d|$. In case $a \le c \le b \le d$, we have $\lambda([a,b]\setminus]c,d[) = \lambda([a,c]) = |a-c| \le |a-c|+|b-d|$. Finally, when $a \le c \le d \le b$, we have $\lambda([a,b]\setminus]c,d[) = \lambda([a,c]\sqcup[d,b]) = |a-c|+|b-d|$, and this proves the claim. □

Lemma 4 – *Let $\Delta^r = \{\underline{\alpha} = (\alpha_1,\dots,\alpha_r) \in \mathbf{R}_+^r \mid \alpha_1 + \dots + \alpha_r = 1\}$ denote the r-dimensional simplex. The map $\Delta^r \to L(\Omega,\{1,\dots,r\})$ defined by $\underline{\alpha} \mapsto f_{\underline{\alpha}}$ where $f_{\underline{\alpha}}(t) = i$ iff $t \in [\alpha_1 + \dots + \alpha_{i-1}, \alpha_1 + \dots + \alpha_i[$ is continuous. More precisely it is $2r$-Lipschitz if Δ^r is equipped with the metric $d(\underline{\alpha},\underline{\alpha'}) = \sum_i |\alpha_i - \alpha_i'|$.*

Proof. We fix an identification $\Omega \simeq [0,1]$. Let $\underline{\alpha},\underline{\alpha'} \in \Delta^r$. We denote $\beta_i = \alpha_1 + \dots + \alpha_i$, $\beta_0 = 0$, and we similarly define the β_i'. We have $\beta_i - \beta_{i-1} = \alpha_i$ hence $|\beta_i' - \beta_i| \le \sum_{k\le i} |\alpha_k' - \alpha_k|$ and finally $\sum_i |\beta_i' - \beta_i| \le r\sum_i |\alpha_i' - \alpha_i|$. Now, for $t \in [\beta_i, \beta_{i+1}[$ we have $f_{\underline{\alpha}}(t) = f_{\underline{\alpha'}}(t)$ unless $t \notin [\beta_i', \beta_{i+1}'[$. From this and Lemma 3 we get that $d(f_{\underline{\alpha}}, f_{\underline{\alpha'}})$ is no greater than

$$\sum_{i=1}^r \lambda\Big([\beta_i,\beta_{i+1}[\setminus[\beta_i',\beta_{i+1}'[\Big) \leqslant \sum_{i=1}^r |\beta_i - \beta_i'| + |\beta_{i+1} - \beta_{i+1}'| \leqslant 2\sum_{i=1}^r |\beta_i - \beta_i'| \leqslant 2r\sum_{i=1}^r |\alpha_i - \alpha_i'|$$

and this proves the claim. □

Lemma 5 – *Let $\mathcal{K}$ be a simplicial complex and X a topological space, and $A \subset X$. If $\gamma_0,\gamma_1 : X \to \bar{L}(\Omega,\mathcal{K})$ are two continuous maps such that $\forall x \in X \;\; \gamma_0(x)(\Omega) \subset \gamma_1(x)(\Omega)$, and $(\gamma_0)_{|A} = (\gamma_1)_{|A}$, then γ_0 and γ_1 are homotopic relative to A. Moreover, if γ_0 and γ_1 take value inside $L(\Omega,\mathcal{K})$, then the homotopy takes values inside $L(\Omega,\mathcal{K})$.*

Proof. We fix an identification $\Omega \simeq [0,1]$. We define $H : [0,1] \times X \to L(\Omega, S)$ by $H(u,x)(t) = \gamma_0(x)(t)$ if $t \geq u$ and $H(u,x)(t) = \gamma_1(x)(t)$ if $t < u$. We have $H(0,\bullet) = \gamma_0$ and $H(1,\bullet) = \gamma_1$.

We first check that H is indeed a (set-theoretic) map $[0,1] \times X \to \bar{L}(\Omega,\mathcal{K})$. For all $u \in [0,1]$ and $x \in X$ we have $H(u,x)(\Omega) \subset \gamma_0(x)(\Omega) \cup \gamma_1(x)(\Omega) = \gamma_1(x)(\Omega)$. Therefore $H(u,x)(\Omega) \in \mathcal{K}$ if $\gamma_1(x) \in L(\Omega,\mathcal{K})$, and all nonempty finite subsets of $H(u,x)(\Omega) \subset \gamma_1(x)(\Omega)$ belong to $\mathcal{K}$ if $\gamma_1(x) \in \bar{L}(\Omega,\mathcal{K})$. From this, by Lemma 1 we get that H takes values inside $\bar{L}(\Omega,\mathcal{K})$, and even inside $L(\Omega,\mathcal{K})$ if $\gamma_1 : X \to L(\Omega,\mathcal{K})$.

Now, we check that H is continuous over $[0,1] \times X$. We have $d(H(u,x),H(v,x)) \leq |u-v|$ for all $u,v \in [0,1]$ and, for all $x,y \in X$ and $u \in [0,1]$, we have

$$d(H(u,x),H(u,y)) = \int_0^u d(\gamma_1(x)(t),\gamma_1(y)(t))\mathrm{d}t + \int_u^1 d(\gamma_0(x)(t),\gamma_0(y)(t))\mathrm{d}t$$

$$\leqslant \int_0^1 d(\gamma_1(x)(t),\gamma_1(y)(t))\mathrm{d}t + \int_0^1 d(\gamma_0(x)(t),\gamma_0(y)(t))\mathrm{d}t = d(\gamma_1(x),\gamma_1(y)) + d(\gamma_0(x),\gamma_0(y))$$

from which we get $d(H(u,x),H(v,y)) \leq |u-v| + d(\gamma_1(x),\gamma_1(y)) + d(\gamma_0(x),\gamma_0(y))$ for all $x,y \in X$ and $u,v \in [0,1]$. For any given $(u,x) \in [0,1] \times X$ this proves that H is continuous at (u,x). Indeed, given $\varepsilon > 0$, from the continuity of γ_0,γ_1 we get that, for some open neighborhood V of x we have $d(\gamma_0(x),\gamma_0(y)) \leq \varepsilon/3$ and $d(\gamma_1(x),\gamma_1(y)) \leq \varepsilon/3$ for all $y \in V$. This proves that $d(H(u,x),H(v,y)) \leq \varepsilon$ for all $(v,y) \in]u-\varepsilon/3, u+\varepsilon/3[\times V$ and this proves the continuity of H.

Finally, it is clear that $\gamma_0(x) = \gamma_1(x)$ implies $H(u,x) = \gamma_0(x) = \gamma_1(x)$ for all $u \in [0,1]$, therefore the homotopy indeed fixes A. □

2.3 Weak homotopy equivalence

We now prove part (1) of the main theorem, through a series of propositions, which might be of independent interest.

Proposition 2 – *Let C be a compact subspace of $\bar{L}(\Omega,\mathcal{K})$ and $C_0 \subset C \cap L(\Omega,\mathcal{K})$ a (possibly empty) subset such that $\bigcup_{c \in C_0} c(\Omega)$ is finite. Then there exists a continuous map $p : C \to L(\Omega,\mathcal{K})$ such that $p(c) = c$ for all $c \in C_0$. Moreover, $p(c)(\Omega) \subset c(\Omega)$ for all $c \in C$ and $\bigcup_{c \in C} p(c)(\Omega)$ is finite.*

Proof. For any $s \in S$ and $n \in \mathbf{N}^* = \mathbf{N} \setminus \{0\}$ we denote $O_{s,n} = \{f \in L(\Omega,S) \mid \lambda(f^{-1}(\{s\})) > 1/n\}$. It is an open subset of $L(\Omega,S)$, hence $C_{s,n} = C \cap O_{s,n}$ is an open subset of C. Now, for every $c \in C$ there exists $s \in S$ such that $\lambda(c^{-1}(\{s\})) > 0$ hence $c \in C_{s,n}$ for some n. Then C is compact and covered by the $C_{s,n}$ hence there exists $s_1,\dots,s_r \in S$ and $n_1,\dots,n_r \in \mathbf{N}^*$ such that $C \subset \bigcup_{i=1}^r O_{s_i,n_i}$. Up to replacing the n_i's by their maximum, we may suppose $n_1 = \dots = n_r = n_0$. Let then $F' = \bigcup_{c \in C_0} c(\Omega) \subset S$. We set $F = \{s_1,\dots,s_r\} \cup F'$. For any $i \in \{1,\dots,r\}$ we set $O_i = O_{s_i,n_0}$.

For any $c \in C$, we set $\Omega_c = \{t \in \Omega; c(t) \notin F\}$, and

$$\alpha_i(c) = \frac{d(c, {}^c O_i)}{\sum_j d(c, {}^c O_j)}$$

and $\beta_i(c) = \sum_{k \leq i} \alpha_k(c)$, where ${}^c X$ denotes the complement of X. These define continuous maps $C \to \mathbf{R}_+$. We fix an identification $\Omega \simeq [0,1]$, so that intervals make sense inside Ω. We then set

$$\begin{array}{rcll} p(c)(t) & = & c(t) & \text{if } c(t) \in F, \text{ i.e. } t \notin \Omega_c \\ & = & s_i & \text{if } t \in \Omega_c \cap [\beta_{i-1}(c), \beta_i(c)[\end{array}$$

Let $c_1, c_2 \in C$ and $\underline{\alpha}^s$, $s = 1,2$ the corresponding r-tuples $\underline{\alpha}^s = (\alpha_1^s, \dots, \alpha_r^s) \in \Delta^r$ given by $\alpha_i^s = \alpha_i(c_s)$. When $t \notin \Omega_{c_1} \cup \Omega_{c_2}$ we have $p(c_s)(t) = c_s(t)$, hence

$$\int_{\Omega \setminus (\Omega_{c_1} \cup \Omega_{c_2})} d(p(c_1)(t), p(c_2)(t)) \mathrm{d}t \leqslant \int_{\Omega} d(c_1(t), c_2(t)) \mathrm{d}t = d(c_1, c_2)$$

and we have

$$\int_{\Omega_{c_1} \cup \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) \mathrm{d}t \leqslant \lambda(\Omega_{c_1} \Delta \Omega_{c_2}) + \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) \mathrm{d}t.$$

Since we know that $\lambda(\Omega_{c_1} \Delta \Omega_{c_2}) \leq d(c_1, c_2)$ by Lemma 2, we get

$$d(p(c_1), p(c_2)) \leq 2d(c_1, c_2) + \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) \mathrm{d}t$$

and there only remains to check that the term $\int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) \mathrm{d}t$ is continuous. But, by Lemma 4, we have

$$\int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) \mathrm{d}t = \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(f_{\underline{\alpha}^1}(t), f_{\underline{\alpha}^2}(t)) \mathrm{d}t \leq d(f_{\underline{\alpha}^1}, f_{\underline{\alpha}^2}) \leq 2r |\underline{\alpha}^1 - \underline{\alpha}^2|$$

whence the conclusion, by continuity of $c \mapsto \underline{\alpha}$.

We must now check that p takes values inside $L(\Omega, \mathcal{K})$. Let $c \in C$. We know that $p(c)(\Omega) \subset F$ is finite, and

$$p(c)(\Omega) \subset c(\Omega) \cup \{s_i; c \in O_i\}.$$

But $c \in O_i$ implies that $s_i \in c(\Omega)$ hence $p(c)(\Omega)$ is nonempty finite subset of $c(\Omega)$. Since $c \in \bar{L}(\Omega, \mathcal{K})$, by Lemma 1 this proves $p(c)(\Omega) \in \mathcal{K}$ and $p(c) \in L(\Omega, \mathcal{K})$.

Finally, we have $p(c) = c$ for all $c \in C_0$ since $F \supset F'$.

We immediately get the following corollary, by letting $C_0 = \{c_1^0, \dots, c_k^0\}$.

Corollary 1 – *Let C be a compact subset of $\bar{L}(\Omega, \mathcal{K})$ and $c_1^0, \dots, c_k^0 \in C \cap L(\Omega, \mathcal{K})$. Then there exists a continuous map $p : C \to L(\Omega, \mathcal{K})$ such that $p(c_i^0) = c_i^0$ for all $i \in \{1, \dots, k\}$. Moreover, $p(c)(\Omega) \subset c(\Omega)$ for all $c \in C$ and $\bigcup_{c \in C} p(c)(\Omega)$ is finite.*

Proposition 3 – *Let C be a compact space, and $x_0 \in C$. For any simplicial complex $\mathcal{K}$, and any continuous map $\gamma : C \to L(\Omega, \mathcal{K})$, there exists a continuous map $\hat{\gamma} : (C, x_0) \to (L(\Omega, \mathcal{K}), \gamma(x_0))$ which is homotopic to γ relative to $(\{x_0\}, \{\gamma(x_0)\})$, and such that $\bigcup_{x \in C} \hat{\gamma}(x)(\Omega)$ is finite.*

Proof. Let $C' = \gamma(C) \subset L(\Omega, \mathcal{K})$. It is compact, hence applying corollary 1 to it and to $\{c_1^0\} = \{\gamma(x_0)\}$ we get a continuous map $p : C' \to L(\Omega, \mathcal{K})$ such that $\bigcup_{c \in C'} p(c)(\Omega)$ is finite, and $p(c)(\Omega) \subset c(\Omega)$ for all $c \in C'$. Therefore, letting $\hat{\gamma} = p \circ \gamma : C \to L(\Omega, \mathcal{K})$, we get that $\bigcup_{x \in C} \hat{\gamma}(x)(\Omega)$ is finite. Since $\hat{\gamma}(x)(\Omega) \subset \gamma(x)(\Omega)$ for all $x \in C$, we get from Lemma 5 that γ and $\hat{\gamma}$ are homotopic, hence the conclusion.

Proposition 4 – *Let C be a compact space (and $x_0 \in C$), $\mathcal{K}$ a simplicial complex, and a pair of continuous maps $\gamma_0, \gamma_1 : C \to L(\Omega, \mathcal{K})$ (with $\gamma_0(x_0) = \gamma_1(x_0)$). If γ_0 and γ_1 are homotopic as maps in $\bar{L}(\Omega, \mathcal{K})$ (relative to $(\{x_0\}, \{\gamma_0(x_0)\})$), then they are homotopic inside $L(\Omega, \mathcal{K})$ (relative to $(\{x_0\}, \{\gamma_0(x_0)\})$).*

Proof. After Proposition 3, there exists $\hat{\gamma}_0, \hat{\gamma}_1 : C \to L(\Omega, \mathcal{K})$ such that $\hat{\gamma}_i$ is homotopic to γ_i with the property that $\bigcup_{x \in C} \hat{\gamma}_i(x)(\Omega)$ is finite, for all $i \in \{0, 1\}$. Without loss of generality, one can therefore assume that $\bigcup_{x \in C} \gamma_i(x)(\Omega)$ is finite, for all $i \in \{0, 1\}$. Let $H : C \times [0, 1] \to \bar{L}(\Omega, \mathcal{K})$ be an homotopy between γ_0 and γ_1. Let $C' = H(C \times [0, 1])$ and $C_0 = \gamma_0(C) \cup \gamma_1(C)$. These are two compact spaces which satisfy the assumptions of Proposition 2. If $p : C' \to L(\Omega, \mathcal{K})$ is the continuous map afforded by this proposition, then $\hat{H} = p \circ H$ provides a homotopy between γ_0 and γ_1 inside $L(\Omega, \mathcal{K})$. The 'relative' version of the statement is proved similarly. □

In particular, when C is equal to the n-sphere S^n, this proves that the natural map $[S^n, L(\Omega, \mathcal{K})]_* \to [S^n, \bar{L}(\Omega, \mathcal{K})]_*$ between sets of pointed homotopy classes is injective. In order to prove Theorem 1 (1), we need to prove that it is surjective. Let us consider a continuous map $\gamma : S^n \to \bar{L}(\Omega, \mathcal{K})$ and set $C = \gamma(S^n)$. It is a compact subspace of $\bar{L}(\Omega, \mathcal{K})$. Applying Proposition 2 with $C_0 = \emptyset$ we get $p : C \to L(\Omega, \mathcal{K})$ such that $p(c)(\Omega) \subset c(\Omega)$ for any $c \in C$. Let then $\hat{\gamma} = p \circ \gamma : S^n \to L(\Omega, \mathcal{K})$. From Lemma 5 we deduce that $\hat{\gamma}$ and γ are homotopic inside $\bar{L}(\Omega, \mathcal{K})$, and this concludes the proof of part (1) of Theorem 1.

3 Homotopies inside $L(\Omega,\{0,1\})$

In this section we denote $L(2) = L(\Omega,2) = L(\Omega,\{0,1\})$, with $d(0,1) = 1$. Since we are going to use Lipschitz properties of maps, we specify our conventions on metrics. When (X,d_X) and (Y,d_Y) are two metric spaces, we endow $X\times Y$ with the metric $d_X + d_Y$, and the space $C^0([0,1],X)$ of continuous maps $[0,1]\to X$ with the metric of uniform convergence $d(\alpha,\beta) = \|\alpha-\beta\|_\infty = \sup_{t\in I}|\alpha(t)-\beta(t)|$. Recall that the topology on $C^0([0,1],X)$ induced by this metric is the compact-open topology. For short we set $C^0(X) = C^0([0,1],X)$.

Identifying $L(2) = L(\Omega,2)$ with the space of measurable subsets of Ω (modulo subsets of measure 0) endowed with the metric $d(E,F) = \lambda(E\Delta F)$, where Δ is the symmetric difference operator, we have the following lemma. This lemma can be viewed as providing a continuous reparametrization by arc-length of natural geodesics inside the metric space $L(2)$.

Lemma 6 – *The exists a continuous map* $\mathbf{g} : L(2)\times[0,1]\to L(2)$ *such that* $\mathbf{g}(A,0) = A$, $\lambda(\mathbf{g}(A,u)) = \lambda(A)(1-u)$ *and* $\mathbf{g}(A,u)\supset\mathbf{g}(A,v)$ *for all* A *and* $u\le v$. *Moreover, it satisfies*

$$\lambda\left(\mathbf{g}(E,u)\Delta\mathbf{g}(F,v)\right) \le 4\lambda(E\Delta F) + |v-u|$$

for all $E,F\in L(2)$ *and* $u,v\in[0,1]$.

Proof. We fix an identification $\Omega\simeq[0,1]$. For $E\in L(2)\setminus\{\varnothing\}$ we define $\varphi_E(t) = \lambda(E\cap[t,1])/\lambda(E)$. The map φ_E is obviously (weakly) decreasing and continuous $[0,1]\to[0,1]$, with $\varphi_E(0) = 1$ and $\varphi_E(1) = 0$. It is therefore surjective, and we can define a (weakly) decreasing map $\psi_E : [0,1]\to[0,1]$ by $\psi_E(u) = \inf\varphi_E^{-1}(\{u\})$. Since φ_E is continuous, we have $\varphi_E(\psi_E(u)) = u$.

One defines $\mathbf{g}(E,u) = E\cap[\psi_E(1-u),1]$ if $\lambda(E)\neq 0$, and $\mathbf{g}(\varnothing,u) = \varnothing$. We have $\lambda(\mathbf{g}(E,u)) = \lambda(E\cap[\psi_E(1-u),1]) = \varphi_E(\psi_E(1-u))\lambda(E) = (1-u)\lambda(E)$ when $\lambda(E)\neq 0$, and $\lambda(\mathbf{g}(\varnothing,u)) = 0 = \lambda(E)(1-u)$ if $\lambda(E) = 0$. It is clear that $\mathbf{g}(E,u)\subset\mathbf{g}(E,v)$ for all $u\ge v$.

Moreover, clearly $\mathbf{g}(E,0) = E$ since $E\cap[\psi_E(1),1]\subset E$ and $\lambda(E\cap[\psi_E(1),1]) = \varphi_E(\psi_E(1))\lambda(E) = \lambda(E)$. It remains to prove that $\mathbf{g}$ is continuous.

Let $E,F\in L(2)$ and $u,v\in[0,1]$. We first assume $\lambda(E)\lambda(F) > 0$. Without loss of generality we can assume $\psi_E(1-u)\le\psi_F(1-v)$. Then $[\psi_E(1-u),1]\supset[\psi_F(1-v),1]$, and $\mathbf{g}(E,u)\Delta\mathbf{g}(F,v)$ can be decomposed as

$$((E\setminus F)\cap[\psi_E(1-u),1])\cup((F\setminus E)\cap[\psi_F(1-v),1])\cup((E\cap F)\cap[\psi_E(1-u),\psi_F(1-v)]).$$

Since the first two pieces are included inside $E\Delta F$, we get $\lambda(\mathbf{g}(E,u)\Delta\mathbf{g}(F,v))\le\lambda(E\Delta F)+\lambda((E\cap F)\cap[\psi_E(1-u),\psi_F(1-v)])$. Now $(E\cap F)\cap[\psi_E(1-u),\psi_F(1-v)] = (E\cap F\cap[\psi_E(1-u),1])\setminus(E\cap F\cap[\psi_F(1-v),1])$ hence

$$\begin{aligned}\lambda((E\cap F)\cap[\psi_E(1-u),\psi_F(1-v)]) &= \lambda(E\cap F\cap[\psi_E(1-u),1]) - \lambda(E\cap F\cap[\psi_F(1-v),1])\\ &\le \lambda(E\cap[\psi_E(1-u),1]) - \lambda(E\cap F\cap[\psi_F(1-v),1])\\ &\le (1-u)\lambda(E) - \lambda(E\cap F\cap[\psi_F(1-v),1])\end{aligned}$$

Now, since $F = (E \cap F) \sqcup (F \setminus E)$, we have $F \cap [\psi_F(1-v), 1] = ((E \cap F) \cap [\psi_F(1-v), 1]) \sqcup ((F \setminus E) \cap [\psi_F(1-v), 1])$ hence

$$\begin{array}{rcl} (1-v)\lambda(F) & = & \lambda\left((E \cap F) \cap [\psi_F(1-v), 1]\right) + \lambda\left((F \setminus E) \cap [\psi_F(1-v), 1]\right) \\ & \leq & \lambda\left((E \cap F) \cap [\psi_F(1-v), 1]\right) + \lambda(F \setminus E) \\ & \leq & \lambda\left((E \cap F) \cap [\psi_F(1-v), 1]\right) + \lambda(F \Delta E). \end{array}$$

It follows that $-\lambda\left((E \cap F) \cap [\psi_F(1-v), 1]\right) \leq \lambda(F \Delta E) - (1-v)\lambda(F)$ hence

$$\lambda\left((E \cap F) \cap [\psi_E(1-u), \psi_F(1-v)]\right) \leqslant (1-u)\lambda(E) + \lambda(F \Delta E) - (1-v)\lambda(F)$$

and finally

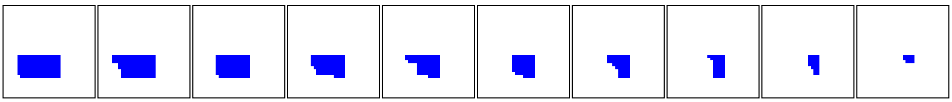
$$\begin{array}{rcl} \lambda(\mathbf{g}(E,u)\Delta \mathbf{g}(F,v)) & \leq & 2\lambda(E\Delta F) + (1-u)\lambda(E) - (1-v)\lambda(F) \\ & \leq & 2\lambda(E\Delta F) + (\lambda(E) - \lambda(F)) + (v-u)\lambda(E) + v(\lambda(F) - \lambda(E)) \\ & \leq & 2\lambda(E\Delta F) + |\lambda(E) - \lambda(F)| + |v-u|\lambda(E) + v|\lambda(F) - \lambda(E)| \\ & \leq & 2\lambda(E\Delta F) + 2|\lambda(E) - \lambda(F)| + |v-u| \\ & \leq & 4\lambda(E\Delta F) + |v-u|. \end{array}$$

Therefore we get the inequality $\lambda(\mathbf{g}(E,u)\Delta \mathbf{g}(F,v)) \leq 4\lambda(E\Delta F) + |v-u|$, that we readily check to hold also when $\lambda(E)\lambda(F) = 0$. This proves that $\mathbf{g}$ is continuous, whence the claim. □

We provide a 2-dimensional illustration, with $\Omega = [0,1]^2$. The map constructed in the proof depends on an identification $[0,1]^2 \simeq [0,1]$ (up to a set of measure 0). An explicit one is given by the binary-digit identification

$$0.\varepsilon_1\varepsilon_2\varepsilon_3 \cdots \mapsto (0.\varepsilon_1\varepsilon_3\varepsilon_5 \dots, 0.\varepsilon_2\varepsilon_4\varepsilon_6 \dots)$$

with the $\varepsilon_i \in \{0,1\}$. Then, when A is some (blue) rectangle, the map $u \mapsto \mathbf{g}(A,u)$ looks as follows.



$u \sim 0.0$ $u \sim 0.1$ $u \sim 0.2$ $u \sim 0.3$ $u \sim 0.4$ $u \sim 0.5$ $u \sim 0.6$ $u \sim 0.7$ $u \sim 0.8$ $u \sim 0.9$

The above lemma is actually all what is needed to prove Theorem 1 in the case of binary random variables, that is $S = \{0,1\}$, as we will illustrate later (see corollary 2). In the general case however, we shall need a more powerful homotopy, provided by Proposition 5 below. The next lemmas are preliminary technical steps in view of its proof.

Lemma 7 – *The map $C^0(L(2)) \times L(2) \to C^0([0,1])$ defined by $(E_\bullet, A) \to \alpha$ where $\alpha(u) = \lambda(E_u \cap A)$, is 1-Lipschitz.*

Proof. Let α, β denote the images of $(E_\bullet, A)$ and $(F_\bullet, B)$, respectively. Then, for all $u \in I$, we have

$$|\alpha(u) - \beta(u)| = |\lambda(E_u \cap A) - \lambda(F_u \cap B)| \leq \lambda\left((E_u \cap A)\Delta(F_u \cap B)\right)$$

From the general set-theoretic inequality $(X \cap A)\Delta(Y \cap B) \subset (X\Delta Y) \cup (A\Delta B)$ one gets

$$\lambda\left((E_u \cap A)\Delta(F_u \cap B)\right) \leq \lambda(E_u \Delta F_u) + \lambda(A\Delta B),$$

hence $\|\alpha - \beta\|_\infty \leq \sup_u \lambda(E_u \Delta F_u) + \lambda(A\Delta B)$ and this proves the claim. □

Lemma 8 – *A map $\Phi_- : C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))$ is defined as follows. To $(a, E_\bullet, A) \in C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))$ one associates the map*

$$\Phi_-(a, E_\bullet, A) : u \mapsto \mathbf{g}\left(E_u \cap A, 1 - \frac{\min(a(u)\lambda(E_u), \alpha(u))}{\alpha(u)}\right)$$

if $\alpha(u) \neq 0$, and otherwise $u \mapsto \emptyset$, where $\alpha(u) = \lambda(A \cap E_u)$. Then, the map Φ_- is continuous.

Proof. Let us fix $(a, E_\bullet, A) \in C^0([0,1]) \times C^0(L(2)) \times L(2)$, and let $\varepsilon > 0$. Consider $\hat{m} : [0,1] \times [\varepsilon/12, 1] \to [0,1]$ be defined by $\hat{m}(x,y) = \min(x,y)/y$. It is clearly continuous on the compact space $[0,1]\times[\varepsilon/12,1]$, hence unformly continuous, hence there exists $\eta > 0$ such that $\max(|x_1 - x_2|, |y_1 - y_2|) < \eta \Rightarrow |\hat{m}(x_1,y_1) - \hat{m}(x_2,y_2)| \leq \varepsilon/6$. Clearly one can assume $\eta \leq \varepsilon/6$ as well.

Let us then consider $(b, F_\bullet, B) \in C^0([0,1]) \times C^0(L(2)) \times L(2)$ such that $\|a - b\|_\infty + \sup_u \lambda(E_u \Delta F_u) + \lambda(A\Delta B) \leq \eta$. From Lemma 7, we get $\|\alpha - \beta\|_\infty \leq \eta$. Let us consider $I_0 = \{u \in [0,1] \mid \alpha(u) \leq \varepsilon/3\}$. We have by definition $\alpha([0,1] \setminus I_0) \subset]\varepsilon/3, 1] \subset [\varepsilon/12, 1]$ and, since $\|\alpha - \beta\|_\infty \leq \varepsilon/6$, we have $\beta([0,1] \setminus I_0) \subset]\varepsilon/6, 1] \subset [\varepsilon/12, 1]$. Moreover, since

$$\begin{aligned} |a(u)\lambda(E_u) - b(u)\lambda(F_u)| &\leq |a(u) - b(u)|\lambda(E_u) + b(u)|\lambda(E_u) - \lambda(F_u)| \\ &\leq |a(u) - b(u)| + \lambda(E_u \Delta F_u) \qquad \leq \eta \end{aligned}$$

we get that, for all $u \notin I_0$, we have $|\hat{m}(a(u)\lambda(E_u)), \alpha(u)) - \hat{m}(b(u)\lambda(F_u), \beta(u))| \leq \varepsilon/6$. Moreover, since in particular $\alpha(u)\beta(u) \neq 0$, we get from the general inequality $\lambda(\mathbf{g}(X,x)\Delta\mathbf{g}(Y,y)) \leq 4\lambda(X\Delta Y) + |x - y|$ of Lemma 6 that, for all $u \notin I_0$,

$$\begin{aligned} d\left(\Phi_-(a, E_\bullet, A)(u), \Phi_-(b, F_\bullet, B)(u)\right) &\leq 4\lambda((E_u \cap A)\Delta(F_u \cap B)) \\ &\quad + |\hat{m}(a(u)\lambda(E_u), \alpha(u)) - \hat{m}(b(u)\lambda(F_u), \beta(u))| \\ &\leq 4\left(\lambda(E_u \Delta F_u) + \lambda(A\Delta B)\right) + \varepsilon/6 \\ &\leq 4\varepsilon/6 + \varepsilon/6 \\ &< \varepsilon \end{aligned}$$

Now, if $u \in I_0$, then $\Phi_-(a, E_\bullet, A)(u) \subset E_u \cap A$ hence $\lambda(\Phi_-(a, E_\bullet, A)(u)) \leq \lambda(E_u \cap A) = \alpha(u) \leq \varepsilon/3$ and $\lambda(\Phi_-(b, F_\bullet, B)(u)) \leq \lambda(F_u \cap B) = \beta(u) \leq \varepsilon/3 + \varepsilon/6 = \varepsilon/2$, whence

$$d\left(\Phi_-(a, E_\bullet, A)(u), \Phi_-(b, F_\bullet, B)(u)\right) \leqslant \lambda(\Phi_-(a, E_\bullet, A)(u)) + \lambda(\Phi_-(b, F_\bullet, B)(u)) \leqslant 5\varepsilon/6 < \varepsilon.$$

It follows that $d(\Phi_-(a,E_\bullet,A),\Phi_-(b,F_\bullet,B)) \leq \varepsilon$ and Φ_- is continuous at $(a,E_\bullet,A)$, which proves the claim.

We use the convention $\mathbf{g}(X,t) = X$ for $t \leq 0$ and $\mathbf{g}(X,t) = \varnothing$ for $t > 1$, so that $\mathbf{g}$ is extended to a continuous map $L(2)\times\mathbf{R} \to L(2)$. The notation ${}^cA$ denotes the complement inside Ω of the set A, identified with an element of $L(\Omega,2)$.

Lemma 9 – *A map $\Phi_+ : C^0([0,1])\times C^0(L(2))\times L(2) \to C^0(L(2))$ is defined as follows. To $(a,E_\bullet,A) \in C^0([0,1])\times C^0(L(2))\times L(2) \to C^0(L(2))$ one associates the map*

$$\Phi_+(a,E_\bullet,A) : u \mapsto \mathbf{g}\left(E_u \cap ({}^cA), 1 - \frac{\max(0, a(u)\lambda(E_u) - \alpha(u))}{\lambda(E_u) - \alpha(u)}\right)$$

if $\alpha(u) \neq \lambda(E_u)$, and otherwise $u \mapsto \varnothing$, where $\alpha(u) = \lambda(A\cap E_u)$. Then, the map Φ_+ is continuous.

The proof is similar to the one of the previous lemma, and left to the reader.

Lemma 10 – *The map $(f,g) \mapsto (t \mapsto f(t)\cup g(t))$ is continuous $C^0(L(2))^2 \to C^0(L(2))$, and even 1-Lipschitz.*

Proof. The map $(X,Y) \mapsto X\cup Y$ is 1-Lipschitz because of the general set-theoretic fact $(X_1\cup Y_1)\Delta(X_2\cup Y_2) \subset (X_1\Delta X_2)\cup(Y_1\Delta Y_2)$ from which we deduce $\lambda((X_1\cup Y_1)\Delta(X_2\cup Y_2)) \leq \lambda(X_1\Delta X_2)+\lambda(Y_1\Delta Y_2)$, which proves that $(X,Y)\mapsto X\cup Y$ is 1-Lipschitz $L(2)^2 \to L(2)$. It follows that the induced map $C^0(L(2)^2) = C^0(L(2))^2 \to C^0(L(2))$ is 1-Lipschitz and thus continuous, too. □

The following proposition informally says that, when $E_\bullet \in C^0(L(2))$ is a path inside $L(2)$ with $A \subset E_0$, then we can find another path $\Phi_\bullet \in C^0(L(2))$ such that $\Phi_u \subset E_u$ for all u, and the ratio $\lambda(\Phi_\bullet)/\lambda(E_\bullet)$ follows any previously specified variation starting at $\lambda(A)/\lambda(E_0)$ – and, moreover, that this can be done continuously.

Proposition 5 – *There exists a continuous map $\Phi : C^0([0,1])\times C^0(L(2))\times L(2) \to C^0(L(2))$ having the following properties.*

- *for all $(a,E_\bullet,A) \in C^0([0,1])\times C^0(L(2))\times L(2)$ such that $A \subset E_0$ and $a(0)\lambda(E_0) = \lambda(A)$, we have $\Phi(a,E_\bullet,A)(0) = A$*
- *for all $u \in [0,1]$, $\Phi(a,E_\bullet,A)(u) \subset E_u$ and $\lambda(\Phi(a,E_\bullet,A)(u)) = a(u)\lambda(E_u)$*
- *if a and $E_\bullet$ are constant maps, then so is $\Phi(a,E_\bullet,A)$.*

Proof. We define $\Phi(a,E_\bullet,A)(u) = \Phi_-(a,E_\bullet,A)(u)\cup\Phi_+(a,E_\bullet,A)(u)$. By the definition of $\Phi_\pm$ in Lemmas 8 and 9, the last property is clear. By combining Lemmas 8, 9 and 10 we get that Φ is continuous. Moreover, $\Phi_-(a,E_\bullet,A)(u) \subset E_u\cap A$ and $\Phi_+(a,E_\bullet,A)(u) \subset E_u\cap({}^cA)$ hence $\Phi(a,E_\bullet,A)(u) = \Phi_-(a,E_\bullet,A)(u)\sqcup\Phi_+(a,E_\bullet,A)(u) \subset E_u$, with $\lambda(\Phi(a,E_\bullet,A)(u)) = \lambda(\Phi_-(a,E_\bullet,A)(u)) + \lambda(\Phi_+(a,E_\bullet,A)(u))$. Letting $\alpha(u) = \lambda(E_u \cap A)$, again by Lemmas 8 and 9 we get

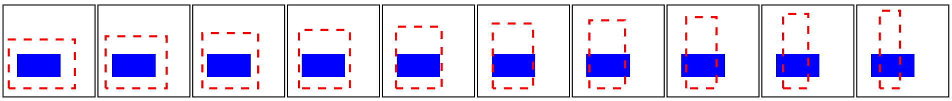
$$\lambda(\Phi_-(a,E_\bullet,A)(u)) = \lambda\left(\mathbf{g}\left(E_u \cap A, 1 - \frac{\min(a(u)\lambda(E_u),\alpha(u))}{\alpha(u)}\right)\right) = \min(a(u)\lambda(E_u),\alpha(u))$$

and, since $\lambda(E_u) - \alpha(u) = \lambda(E_u) - \lambda(A \cap E_u) = \lambda((^cA) \cap E_u)$, $\lambda(\Phi_+(a,E_\bullet,A)(u))$ is equal to

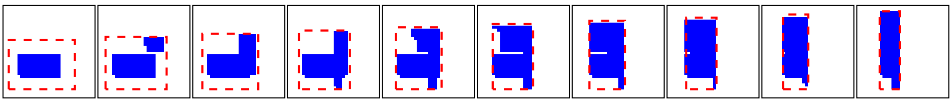
$$\lambda\left(\mathbf{g}\left(E_u \cap (^cA), 1 - \frac{\max(0, a(u)\lambda(E_u) - \alpha(u))}{\lambda((^cA) \cap E_u)}\right)\right) = \max(0, a(u)\lambda(E_u) - \alpha(u)).$$

Therefore we get $\lambda(\Phi(a,E_\bullet,A)(u)) = \max(0,a(u)\lambda(E_u)-\alpha(u)) + \min(a(u)\lambda(E_u),\alpha(u)) = a(u)\lambda(E_u)$ for all $u \in [0,1]$. Finally, since $A \subset E_0$ and $\alpha(0) = \lambda(E_0 \cap A) = \lambda(A) = \lambda(E_0)a(0)$, we get that $\Phi(a,E_\bullet,A)(0) = \mathbf{g}(E_0 \cap A, 0) \cup \mathbf{g}(E_0 \cap (^cA), 1) = A \cup \varnothing = A$, and this proves the claim. □

As before, we provide an illustration, when $A \subset \Omega$ is the same (blue) rectangle, and $E_\bullet$ associates continuously to any $u \in [0,1]$ some rectangle, whose boundary is dashed and in red. In this example, the map a is taken to be affine, from $\lambda(A)/\lambda(E_0)$ to 0. The first row depicts the map $u \mapsto E_u$, and the second row superposes it with the map $u \mapsto \Phi(a,E_\bullet,A)(u)$, depicted in blue.



$u \sim 0.0$ $u \sim 0.1$ $u \sim 0.2$ $u \sim 0.3$ $u \sim 0.4$ $u \sim 0.5$ $u \sim 0.6$ $u \sim 0.7$ $u \sim 0.8$ $u \sim 0.9$



$u \sim 0.0$ $u \sim 0.1$ $u \sim 0.2$ $u \sim 0.3$ $u \sim 0.4$ $u \sim 0.5$ $u \sim 0.6$ $u \sim 0.7$ $u \sim 0.8$ $u \sim 0.9$

4 Probability law

4.1 The law maps

Recall from Spanier 1966 that the weak (or coherent) topology on $|\mathcal{K}|$ is the topology such that U is open in $|\mathcal{K}|$ iff $U \cap |F|$ is open for every $F \in \mathcal{K}$, where $|F| = \{\alpha : F \to [0,1] \mid \sum_{s\in F} \alpha(s) = 1\}$ is given the topology induced from the product topology of $[0,1]^F$. For each $p \geq 1$, we can put a metric topology on the same set, in order to define a metric space $|\mathcal{K}|_{d_p}$ by the metric $d_p(\alpha,\beta) = \sqrt[p]{\sum_{s\in S} |\alpha(s) - \beta(s)|^p}$. The map

$|\mathcal{K}| \to |\mathcal{K}|_{d_p}$ is continuous, and it is an homeomorphism iff $|\mathcal{K}|$ is metrizable iff it is satisfies the first axiom of countability, iff $\mathcal{K}$ is locally finite (see Spanier 1966 p. 119 ch. 3 sec. 2 Theorem 8 for the case $p = 2$, but the proof works for $p \neq 2$ as well).

For $\alpha : S \to [0,1]$, we denote the *support* of α by $\mathrm{supp}(\alpha) = \{s \in S \mid \alpha(s) \neq 0\}$. We let $\Psi_0 : L(\Omega, \mathcal{K}) \to |\mathcal{K}|$ be defined by associating to a random variable $f \in L(\Omega, \mathcal{K})$ its probability law $s \mapsto \lambda(f^{-1}(\{s\}))$.

4.2 Non-continuity of Ψ_0

We first prove that Ψ_0 is *not* continous in general, by providing an example. Let us consider $S = \mathbf{N} = \mathbf{Z}_{\geq 0}$, and $\mathcal{K} = \mathcal{P}_f^*(\mathbf{N})$. We introduce

$$U = \left\{ \alpha \in |\mathcal{K}| \ ; \ \forall s \neq 0 \ \ \alpha(s) < \frac{1}{\#\mathrm{supp}(\alpha)} \right\}.$$

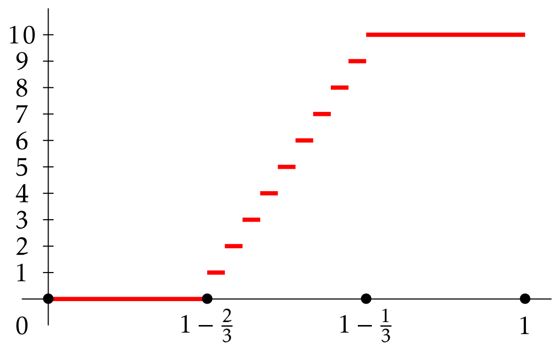
We note that U is open in $|\mathcal{K}|$. Indeed, if $F \in \mathcal{K}$ we have

$$U \cap |F| = \left\{ \alpha : F \to [0,1] \mid \sum_{s \in F} \alpha(s) = 1 \ \& \ \forall s \neq 0 \mid \alpha(s) < \frac{1}{\#\mathrm{supp}(\alpha)} \right\}$$

which is equal to

$$\bigcup_{G \subset F \setminus \{0\}} \left\{ \alpha : G \to [0,1] \mid \alpha(0) + \sum_{s \in G} \alpha(s) = 1 \ \& \ \forall s \in G \mid 0 < \alpha(s) < \frac{1}{\#G + 1} \right\}$$

and it is open as the union of a finite collection of open sets. Now consider $\Psi_0^{-1}(U)$, and let $f_0 \in L(\Omega, \mathcal{K})$ be the constant map $t \mapsto 0$. Clearly $\alpha_0 = \Psi_0(f_0)$ is the map $0 \mapsto 1$, $k \mapsto 0$ for $k \geq 1$, and $\alpha_0 \in U$. If $\Psi_0^{-1}(U)$ is open, there exists $\varepsilon > 0$ such that it contains the open ball centered at f_0 with radius ε. Let n be such that $1/n < \varepsilon/3$, and define $f \in L([0,1], \mathcal{K})$ by $f(t) = 0$ for $t \in [0, 1 - 2/n[$, $f(t) = k$ for $t \in [1 - \frac{2}{n} + \frac{k-1}{n^3}, 1 - \frac{2}{n} + \frac{k}{n^3}[$ and $1 \leq k \leq n^2$, and finally $f(t) = n^2 + 1$ for $t \in [1 - \frac{1}{n}, 1]$. The graph of f for $n = 3$ is depicted below.



We have $d(f,f_0) = 2/n < 2\varepsilon/3 < \varepsilon$ hence we should have $\alpha = \Psi_0(f) \in U$. But the support of α has cardinality n^2+2, and $\alpha(n^2+1) = 1/n > 1/(n^2+2)$, contradicting $\alpha \in U$. This proves that Ψ_0 is not continuous.

4.3 Continuity of Ψ and existence of global sections

For short, we now denote $|\mathscr{K}|_p = |\mathscr{K}|_{d_p}$. We consider the same 'law' map $\Psi : L(\Omega,\mathscr{K}) \to |\mathscr{K}|_1$. We prove that it is uniformly continuous (and actually 2-Lipschitz). Indeed, if $f,g \in L(\Omega,\mathscr{K})$, and $\alpha = \Psi(f)$, $\beta = \Psi(g)$, then

$$d_1(\alpha,\beta) = \sum_{s\in S} |\alpha(s)-\beta(s)| = \sum_{s\in S} |\lambda(f^{-1}(s)) - \lambda(g^{-1}(s))|$$

and $|\lambda(f^{-1}(s)) - \lambda(g^{-1}(s))| \leq \lambda(f^{-1}(s)\Delta g^{-1}(s))$. But $f^{-1}(s)\Delta g^{-1}(s) = \{t \in f^{-1}(s) \mid f(t) \neq g(t)\} \cup \{t \in g^{-1}(s) \mid f(t) \neq g(t)\}$ whence

$$d_1(\alpha,\beta) \leqslant \sum_{s\in S} \int_{f^{-1}(s)} d(f(t),g(t))\mathrm{d}t + \sum_{s\in S} \int_{g^{-1}(s)} d(f(t),g(t))\mathrm{d}t = 2\int_{\Omega} d(f(t),g(t))\mathrm{d}t$$

whence $d_1(\alpha,\beta) \leq 2d(f,g)$. It follows that it induces a continuous map $\bar{L}(\Omega,\mathscr{K}) \to \overline{|\mathscr{K}|_1}$, where

$$\overline{|\mathscr{K}|_1} = \{\alpha : S \to [0,1] \mid \mathscr{P}_{\mathrm{f}}^*(\mathrm{supp}(\alpha)) \subset \mathscr{K} \ \& \ \sum_{s\in S} \alpha(s) = 1\}$$

endowed with the metric $d(\alpha,\beta) = \sum_{s\in S} |\alpha(s)-\beta(s)|$ is the completion of $|\mathscr{K}|_1$. This map associates to $f \in \bar{L}(\Omega,\mathscr{K})$ the map $\alpha(s) = \lambda(f^{-1}(s))$. Notice that the condition $\sum_s \alpha(s) = 1 < \infty$ implies that the support $\mathrm{supp}(\alpha)$ of α is finite.

The fact that $|\mathscr{K}|_1$ has the same homotopy type than $|\mathscr{K}|$ has originally been proved by Dowker in Dowker 1952 in a more general context, and another proof was subsequently provided by Milnor in Milnor 1959.

It is clear that every mass distribution on the discrete set S is realizable by some random variable. We first show that it is possible to do this *continuously*. In topological terms, this proves the following statement.

Proposition 6 – *The maps Ψ and $\overline{\Psi}$ admit global (continuous) sections.*

Proof. We fix some (total) ordering $\leq$ on S and some identification $\Omega \simeq [0,1]$. We define $\sigma : \overline{|\mathscr{K}|_1} \to \bar{L}(\Omega,\mathscr{K})$ as follows. For any $\alpha \in \overline{|\mathscr{K}|_1}$, $S_\alpha = \mathrm{supp}(\alpha) \subset S$ is countable. Let $A_\pm : S \to \mathbf{R}_+$ denote the associated cumulative mass functions $A_+(s) = \sum_{u\leq s} \alpha(u)$ and $A_-(s) = \sum_{u<s} \alpha(u)$. They induce increasing injections $(S_\alpha,\leq) \to [0,1]$. The map $\sigma(\alpha)$ is defined by $\sigma(\alpha)(t) = a$ if $A_-(a) \leq t < A_+(a)$. We have $\sigma(\alpha)(\Omega) = S_\alpha$. Since $\alpha \in \overline{|\mathscr{K}|_1}$ every non-empty finite subset of S_α belongs to $\mathscr{K}$ hence $\sigma(\alpha) \in \bar{L}(\Omega,\mathscr{K})$, and $\sigma(\alpha) \in L(\Omega,\mathscr{K})$ as soon as $\alpha \in |\mathscr{K}|_1$.

Clearly $\bar{\Psi} \circ \sigma$ is the identity. We prove now that σ is continuous at any $\alpha \in \overline{|\mathcal{K}|_1}$. Let $\varepsilon > 0$. There exists $S_\alpha^0 \subset S_\alpha$ finite (and non-empty) such that $\sum_{s \in S_\alpha \setminus S_\alpha^0} \alpha(s) \leq \varepsilon/3$. Let $n = |S_\alpha^0| > 0$. We set $\eta = \varepsilon/3n$. Let $\beta \in \overline{|\mathcal{K}|_1}$ with $|\alpha - \beta|_1 \leq \eta$, and set $B_+(s) = \sum_{u \leq s} \beta(u)$ and $B_-(s) = \sum_{u<s} \beta(u)$. We have

$$d(\sigma(\alpha), \sigma(\beta)) \leq \varepsilon/3 + \sum_{a \in S_\alpha^0} \int_{A_-(a)}^{A^+(a)} d(\sigma(\alpha)(t), \sigma(\beta)(t)) \mathrm{d}t$$

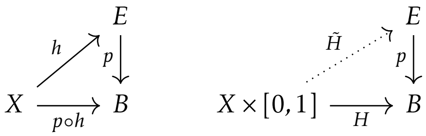
Now note that $|A_\pm(a) - B_\pm(a)| \leq |\alpha - \beta|_1 \leq \varepsilon/3n$ for each $a \in S_\alpha^0$ hence

$$\int_{A_-(a)}^{A^+(a)} d(\sigma(\alpha)(t), \sigma(\beta)(t)) \mathrm{d}t \leq \frac{2\varepsilon}{3n} + \int_{\max(A_-(a), B_-(a))}^{\min(A_+(a), B_+(a))} d(\sigma(\alpha)(t), \sigma(\beta)(t)) \mathrm{d}t = \frac{2\varepsilon}{3n}$$

since $\sigma(\alpha)(t) = \sigma(\beta)(t)$ for each $t \in [\max(A_-(a), B_-(a)), \min(A_+(a), B_+(a))]$, and this yields $d(\sigma(\alpha), \sigma(\beta)) \leq \varepsilon$. This proves that σ is continuous at any $\alpha \in \bar{L}(\Omega, \mathcal{K})$. Therefore σ provides a continuous global section of $\bar{\Psi}$, which obviously restricts to a continuous global section of Ψ. □

4.4 Homotopy lifting properties

Let $\Psi_{\mathcal{K}} : L(\Omega, \mathcal{K}) \to |\mathcal{K}|_1$ and $\bar{\Psi}_{\mathcal{K}} : \bar{L}(\Omega, \mathcal{K}) \to \overline{|\mathcal{K}|_1}$ denote the law maps. If α is a cardinal, we let Ψ_α (resp. $\bar{\Psi}_\alpha$) denote the map associated to the simplicial complex $\mathcal{P}_{\mathrm{f}}^*(\alpha)$. Recall that a continuous map $p : E \to B$ is said to have the homotopy lifting property (HLP) with respect to some topological space X if, for any (continuous) maps $H : X \times [0,1] \to B$ and $h : X \to E$ such that $p \circ h = H(\bullet, 0)$, there exists a map $\tilde{H} = X \times [0,1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H}(\bullet, 0) = h$.



A Hurewicz fibration is a map having the HLP w.r.t. arbitrary topological spaces. A Serre fibration is a map having the HLP w.r.t. all n-spheres, and this is equivalent to having the HLP w.r.t. any CW-complex.

Lemma 11 – *If Ψ_α (resp. $\bar{\Psi}_\alpha$) has the HLP w.r.t. the space X, then the map $\Psi_{\mathcal{K}}$ (resp. $\bar{\Psi}_{\mathcal{K}}$) has the HLP w.r.t. the space X for every simplicial complex whose vertex set has cardinality α.*

Proof. This is a straightforward consequence of the fact that, by definition, the following natural square diagrams are cartesian, where $S = \bigcup \mathcal{K}$ is the vertex set of $\mathcal{K}$.

$$\begin{array}{ccc} L(\Omega,\mathcal{K}) & \hookrightarrow & L_{\mathrm{f}}(\Omega,S) \\ \downarrow & & \downarrow \\ |\mathcal{K}|_1 & \hookrightarrow & |\mathcal{P}_{\mathrm{f}}^*(S)|_1 \end{array} \qquad \begin{array}{ccc} \bar{L}(\Omega,\mathcal{K}) & \hookrightarrow & L(\Omega,S) \\ \downarrow & & \downarrow \\ \overline{|\mathcal{K}|_1} & \hookrightarrow & \overline{|\mathcal{P}_{\mathrm{f}}^*(S)|_1} \end{array}$$

Notice that the following lemma applies in particular to every compact metrizable space (e.g. the n-spheres). Recall that $\aleph_0$ denotes the cardinality of $\mathbf{N}$.

Lemma 12 – *Let X be a separable space. If $\Psi_{\aleph_0}$ (resp. $\bar{\Psi}_{\aleph_0}$) has the HLP w.r.t. the space X then, for every infinite cardinal γ, the map Ψ_γ (resp. $\bar{\Psi}_\gamma$) has the HLP w.r.t. the space X.*

Proof. Let S be a set of cardinality γ, $H : X \times [0,1] \to |\mathcal{P}_{\mathrm{f}}^*(S)|_1$ (resp. $\bar{H} : X \times [0,1] \to \overline{|\mathcal{P}_{\mathrm{f}}^*(S)|_1}$) and $h : X \to L_{\mathrm{f}}(\Omega,S)$ (resp. $\bar{h} : X \to L(\Omega,S)$) be continuous maps such that $\Psi_S \circ h = H(\bullet,0)$ (resp. $\bar{\Psi}_S \circ \bar{h} = \bar{H}(\bullet,0)$). Since X is separable, $X \times [0,1]$ is also separable and so are $H(X \times [0,1])$ and $\bar{H}(X \times [0,1])$. Let $(x_n)_{n \in \mathbf{N}}$ be a dense sequence of elements of $H(X \times [0,1])$ (resp. $\bar{H}(X \times [0,1])$). Each $\mathrm{supp}(x_n) \subset S$ is countable, and therefore so is $D = \bigcup_n \mathrm{supp}(x_n)$.

We first claim that, for any $\alpha \in H(X \times [0,1])$ (resp. $\alpha \in \bar{H}(X \times [0,1])$) we have $\mathrm{supp}(\alpha) \subset D$. Indeed, if $\alpha(s_0) \neq 0$ for some $s_0 \notin D$, then there exists x_n such that $d(x_n,\alpha) < \alpha(s_0)$. But since $d(x_n,\alpha) = \sum_{s \in S} |\alpha(s) - x_n(s)|$, this condition implies $x_n(s_0) \neq 0$, contradicting $\mathrm{supp}(x_n) \subset D$. Therefore $\mathrm{supp}(\alpha) \subset D$ for all $\alpha \in H(X \times [0,1])$ (resp. $\alpha \in \bar{H}(X \times [0,1])$), and H (resp. $\bar{H}$) factorizes through a map $H_D : X \times [0,1] \to |\mathcal{P}_{\mathrm{f}}^*(D)|_1$ (resp. $\bar{H}_D : X \times [0,1] \to \overline{|\mathcal{P}_{\mathrm{f}}^*(D)|_1}$) and the natural inclusion $|\mathcal{P}_{\mathrm{f}}^*(D)|_1 \subset |\mathcal{P}_{\mathrm{f}}^*(S)|_1$ (resp. $\overline{|\mathcal{P}_{\mathrm{f}}^*(D)|_1} \subset \overline{|\mathcal{P}_{\mathrm{f}}^*(S)|_1}$).

Notice that this implies that h (resp. $\bar{h}$) takes values in $L_{\mathrm{f}}(\Omega,D)$ (resp. $L(\Omega,D)$), too. By assumption, there exists $\tilde{H}_D : X \times [0,1] \to L_{\mathrm{f}}(\Omega,D)$ (resp. $\tilde{\bar{H}}_D : X \times [0,1] \to L(\Omega,D)$) such that $\Psi_D \circ \tilde{H}_D = H_D$ and with $\tilde{H}_D(\bullet,0) = h$ (respectively, $\tilde{\bar{H}}_D(\bullet,0) = \bar{h}$). Composing $\tilde{H}_D$ (resp. $\tilde{\bar{H}}_D$) with the natural injection $L_{\mathrm{f}}(\Omega,D) \hookrightarrow L_{\mathrm{f}}(\Omega,S)$ (resp. $L(\Omega,D) \hookrightarrow L(\Omega,S)$) we get the lifting $\tilde{H}$ (resp. $\tilde{\bar{H}}$) we want, and this proves the claim.

$$\begin{array}{ccccc} & & L_{\mathrm{f}}(\Omega,D) & \hookrightarrow & L_{\mathrm{f}}(\Omega,S) \\ & \overset{\tilde{H}_D}{\nearrow} & \downarrow{\scriptstyle \Psi_D} & & \downarrow{\scriptstyle \Psi_S} \\ X \times [0,1] & & \xrightarrow{\qquad H \qquad} & & |\mathcal{P}_{\mathrm{f}}^*(S)|_1 \\ & \underset{H_D}{\searrow} & \downarrow & \nearrow & \\ & & |\mathcal{P}_{\mathrm{f}}^*(D)|_1 & & \end{array}$$

Proposition 7 – *Let X be a topological space and γ a countable cardinal. Then Ψ_γ has the HLP property w.r.t. X as soon as γ is finite or X is compact. Moreover $\bar{\Psi}_\gamma$ has the HLP w.r.t. X as soon as X is compact.*

Proof. Let X be an arbitrary topological space. Our cardinal γ is the cardinal of some initial segment $S \subset \mathbf{N} = \mathbf{Z}_{\geq 0}$ that is, either $S = [0,m]$ for some m, or $S = \mathbf{N}$. Let $H : X \times [0,1] \to |\mathscr{P}_f^*(S)|_1$ and $h : X \to L_f(\Omega, S)$ such that $H(\bullet, 0) = \Psi_S \circ h$. For $(x,u) \in X \times [0,1]$, the element $H(x,u) \in |\mathscr{P}_f^*(S)|_1$ is of the form $(H(x,u)_s)_{s\in S}$, with $\sum_{s\in S} H(x,u)_s = 1$. Since, for each $s \in S$, the map $|\mathscr{P}_f^*(S)|_1 \to [0,1]$ given by $\alpha \mapsto \alpha(s)$ is 1-Lipschitz, the composite map $(x,u) \mapsto H(x,u)_s$ defines a continuous map $X \times [0,1] \to [0,1]$.

Let us choose $x \in X$. We set, with the convention $0/0 = 0$,

$$a_n(x,u) = \frac{H(x,u)_n}{1 - \sum_{k<n} H(x,u)_k} \in [0,1], \quad A_n(x) = h(x)^{-1}(\{n\}) \in L(2)$$

and we construct recursively, for each $n \in \mathbf{N}$,

- maps $\Omega_n(x,\bullet) : [0,1] \to L(2)$
- maps $E^{(n)}_{x,\bullet} : [0,1] \to L(2)$

by letting

$$E^{(n)}_{x,u} = \Omega \setminus \bigcup_{k<n} \Omega_k(x,u), \quad \Omega_n(x,u) = \Phi(a_n(x,\bullet), E^{(n)}_{x,\bullet}, A_n(x))(u)$$

where Φ is the map afforded by Proposition 5.

In order for this to be defined at any given n, one needs to check that $A_n(x) \subset E^{(n)}_{x,0}$ and $a_n(x,0)\lambda(E^{(n)}_{x,0}) = \lambda(A_n(x))$. This is easily checked by induction because, if $\Omega_k, E^{(k)}$ are defined for $k < n$, then

$$\Omega_k(x,0) = \Phi\Big(a_n(x,\bullet), E^{(n)}_{x,\bullet}, A_n(x)\Big)(0) = A_n(x) = h(x)^{-1}(\{n\})$$

hence

$$E^{(n)}_{x,0} = \Omega \setminus \bigcup_{k<n} A_k(x) = h(x)^{-1}(S \setminus [0,n[) \supset h(x)^{-1}(\{n\}) = A_n(x)$$

and moreover $\lambda(A_n(x)) = \lambda(h(x)^{-1}(\{n\})) = H(x,0)_n = a(x,0)\lambda(E^{(n)}_{x,0})$. Therefore these maps are well-defined.

From their definitions and the properties of Φ one gets immediately by induction that

$$a_n(x,u)\lambda(E^{(n)}_{x,u}) = H(x,u)_n = \lambda(\Omega_n(x,u))$$

for all $(x,u) \in X \times [0,1]$.

For a given (x,u), the sets $\Omega_n(x,u)$ are essentially disjoint, since $\Omega_n(x,u) \subset E^{(n)}_{x,u} = \Omega \setminus \bigcup_{k<n} \Omega_k(x,u)$, and moreover $\bigcup_n \Omega_n(x,u) = \Omega$ since $\sum_n \lambda(\Omega_n(x,u)) = \sum_n H(x,u)_n = 1$. Therefore, we can define a map $\tilde{H} : X \times [0,1] \to L_{\mathrm{f}}(S)$ by setting $\tilde{H}(x,u)(t) = n$ if $t \in \Omega_n(x,u)$. Clearly $(\Psi_S \circ \tilde{H}(x,u))_n = \lambda(\Omega_n(x,u)) = H(x,u)_n$ for all n, hence $\Psi_S \circ \tilde{H} = H$. Moreover $\tilde{H}(x,0)_n = \Omega_n(x,0) = A_n(x) = h(x)^{-1}(\{n\}$ hence $\tilde{H}(x,0) = h(x)$ for all $x \in X$.

Therefore it only remains to prove that $\tilde{H} : X \times [0,1] \to L_{\mathrm{f}}(\Omega, S)$ is continuous.

Let us define the auxiliary maps $\tilde{H}_n : X \times [0,1] \to L(\Omega, \{0,\dots,n\})$ by $\tilde{H}_n(x,u)(t) = \tilde{H}(x,u)(t)$ if $\tilde{H}(x,u)(t) < n$, and $\tilde{H}_n(x,u)(t) = n$ if $\tilde{H}(x,u)(t) \geq n$ – that is, $\tilde{H}_n(x,u)(t) = \min(n, \tilde{H}(x,u)(t))$.

We first prove that each $\tilde{H}_n$ is continuous. Let $(x_0,u_0),(x,u) \in X \times [0,1]$. We have

$$d(\tilde{H}_n(x,u), \tilde{H}_n(x_0,u_0)) = \sum_{k=0}^{n} \int_{\Omega_k(x_0,u_0)} d((\tilde{H}_n(x,u)(t), \tilde{H}_n(x_0,u_0)(t))\mathrm{d}t$$

hence

$$d(\tilde{H}_n(x,u), \tilde{H}_n(x_0,u_0)) \leqslant \sum_{k=0}^{n} \lambda\left(\Omega_k(x_0,u_0) \setminus \Omega_k(x,u)\right) \leqslant \sum_{k=0}^{n} \lambda(\Omega_k(x_0,u_0) \Delta \Omega_k(x,u))$$

and therefore it remains to prove that the maps $(x,u) \mapsto \Omega_n(x,u)$ are continuous for each $n \in \mathbf{N}$.

We thus want to prove that $\Omega_n(\bullet,\bullet) \in C^0(X \times [0,1], L(2))$, which we identify with the space $C^0(X, C^0([0,1], L(2))) = C^0(X, C^0(L(2)))$ since $[0,1]$ is (locally) compact. Recall that Φ is continuous $C^0([0,1]) \times C^0(L(2)) \times L(2) \to C^0(L(2))$. Moreover, for arbitrary spaces Y, Z and a map $g \in C^0(Y,Z)$, the induced map $C^0(X,Y) \to C^0(X,Z)$ given by $f \mapsto g \circ f$ is continuous. Letting $Y = C^0([0,1]) \times C^0(L(2)) \times L(2)$ and $Z = C^0(L(2))$, we deduce from $\Phi : Y \to Z$ a continuous map $\hat{\Phi} : C^0(X,Y) \to C^0(X,Z)$, that is

$$\begin{array}{ccc}
C^0(X, C^0([0,1]) \times C^0(L(2))) \times L(2)) & \xrightarrow{\quad\hat{\Phi}\quad} & C^0(X, C^0(L(2)) \\
\| & & \| \\
C^0(X \times [0,1], [0,1]) \times C^0(X \times [0,1], L(2)) \times C^0(X, L(2)) & & C^0(X \times [0,1], L(2))
\end{array}$$

By induction and because the maps a_n, A_n are clearly continuous for any n, we get that all the maps involved are continuous, through the recursive identities

- $\Omega_n = \hat{\Phi}(a_n, E^{(n)}_{\bullet,\bullet}, A_n(\bullet))$
- $E^{(n)}_{x,u} = \Omega \setminus \bigcup_{k<n} \Omega_k(x,u)$ □

and this proves the continuity of $\tilde{H}_n$.

If S is finite this proves that $\tilde{H}$ is continuous, because $\tilde{H} = \tilde{H}_n$ for n large enough in this case. Let us now assume that $S = \mathbf{N}$ and X is compact. We want to prove that the sequence $\tilde{H}_n$ converges uniformly to $\tilde{H}$. Since each $\tilde{H}_n$ is continuous this will prove that $\tilde{H}$ is continuous. Let $\varepsilon > 0$. Let $U_n = \{(x,u) \in X \times [0,1] \mid \sum_{k \leq n} H(x,u)_k > 1-\varepsilon\}$. Since H is continuous this defines a collection of open subsets in the compact space $X \times [0,1]$, and since $\sum_{k \leq n} H(x,u) \to 1$ when $n \to \infty$ for any $(x,u) \in X \times [0,1]$, this collection is an open covering of $X \times [0,1]$. By compactness, and because this collection is a filtration, we have $X \times [0,1] = U_{n_0}$ for some $n_0 \in \mathbf{N}$. But then, for any $(x,u) \in X \times [0,1]$ and $n \geq n_0$ we have

$$d(\tilde{H}_n(x,u), \tilde{H}(x,u)) = \lambda\left(\bigcup_{k>n} \Omega_k(x,u)\right) = \sum_{k>n} H(x,u)_k \leqslant \varepsilon$$

and this proves the claim.

Remark 1 – We notice that the liftings constructed in the above proof have the following additional property that, whenever $H(x,\bullet)$ is a constant map for some $x \in X$, then so is the map $\tilde{H}(x,\bullet)$. This follows from the fact that the maps $a_r(x,\bullet)$ are constant as soon as $H(x,\bullet)$ is constant, and then one gets by induction on n that $\Omega_n(x,u) = \Phi(a_n(x,\bullet), E^{(n)}_{x,\bullet}, A_n(x))$ is constant in u by the last item of Proposition 5, and thus so is $E^{(n)}_{x,u}$.

Since it is far simpler in this case, we provide an alternative proof for the case of binary random variables.

Corollary 2 – *The map $\Psi_2 = \Psi_{\{0,1\}}$ is a Hurewicz fibration.*

Proof. (alternative proof) Let X be a space, and $H : X \times [0,1] \to |\mathscr{P}^*_{\mathrm{f}}(2)|_1$ and $h : X \to L(\Omega, 2)$ such that $H(\bullet, 0) = \Psi_2 \circ h$. Note that $|\mathscr{P}^*_{\mathrm{f}}(2)|_1 = \{\alpha : \{0,1\} \to \mathbf{R}_+ \mid \alpha(0)+\alpha(1) = 1\}$ is isometric to $[0,1]$ through the isometry $j : \alpha \mapsto \alpha(1)$, where the metric on $[0,1]$ is the Euclidean one. If $\alpha = \Psi_2 \circ h(x)$, we have $\alpha(0) = 1 - \lambda(h(x))$, $j(\Psi_2(h(x))) = \alpha(1) = \lambda(h(x))$.

Using the map $\mathbf{g}$ of Lemma 6 we note that $\lambda({}^c\mathbf{g}({}^cA, u)) = u + (1-u)\lambda(A) = u\lambda(\Omega) + (1-u)\lambda(A)$ and we define, for $A \in L(2)$ and $a \in [0,1]$,

- $\tilde{\mathbf{g}}(A,a) = \mathbf{g}(A, 1 - a/\lambda(A))$ if $a < \lambda(A)$,
- $\tilde{\mathbf{g}}(A, \lambda(A)) = A$,
- $\tilde{\mathbf{g}}(A,a) = {}^c\,\mathbf{g}({}^cA, (a - \lambda(A))/(1-\lambda(A)))$ if $a > \lambda(A)$.

We prove that $\tilde{\mathbf{g}} : L(2) \times [0,1] \to L(2)$ is continuous at each $(A_0, a_0) \in L(2)$. The case $a_0 \neq \lambda(A_0)$ is clear from the continuity of $\mathbf{g}$, as there is an open neighborhood of (A_0, a_0) on which $a - \lambda(A)$ has constant sign. Thus we can assume $a_0 = \lambda(A_0)$. Then

$$d(\tilde{\mathbf{g}}(A,a), \tilde{\mathbf{g}}(A_0,a_0)) = d(\tilde{\mathbf{g}}(A,a), A_0) \leqslant d(\tilde{\mathbf{g}}(A,a), A) + d(A, A_0)$$

But, if $a < \lambda(A)$ we have by the inequality of Lemma 6

$$d(\tilde{\mathbf{g}}(A,a),A) = d\left(\mathbf{g}\left(A, 1-\frac{a}{\lambda(A)}\right), \mathbf{g}(A,0)\right) \leqslant \left|1-\frac{a}{\lambda(A)}\right|$$

and, if $a > \lambda(A)$, we have, noticing that $A \mapsto^c A$ is an isometry of $L(2)$ (as $A\Delta B = ({}^cA)\Delta({}^cB)$),

$$d(\tilde{\mathbf{g}}(A,a),A) = d\left({}^c\mathbf{g}\left({}^cA, \frac{a-\lambda(A)}{1-\lambda(A)}\right), A\right) = d\left(\mathbf{g}\left({}^cA, \frac{a-\lambda(A)}{1-\lambda(A)}\right), ({}^cA)\right) \leqslant \left|\frac{a-\lambda(A)}{1-\lambda(A)}\right|$$

which altogether imply

$$d(\tilde{\mathbf{g}}(A,a), \tilde{\mathbf{g}}(A_0,a_0)) \leqslant d(A,A_0) + \max\left(\left|1-\frac{a}{\lambda(A)}\right|, \left|\frac{a-\lambda(A)}{1-\lambda(A)}\right|\right)$$

Since the RHS is continuous with value 0 at (A_0, a_0) with $a_0 = \lambda(A_0)$, this proves the continuity of $\tilde{\mathbf{g}}$.

It is readily checked that $\lambda(\tilde{\mathbf{g}}(A,a)) = a$ for all A, a. We then define $\tilde{H} : X \times [0,1] \to L(\Omega,2)$ by $\tilde{H}(x,u) = \tilde{\mathbf{g}}(h(x), j(H(x,u)))$. We have $\lambda(\tilde{H}(x,u)) = j(H(x,u))$ hence $\Psi_2 \circ \tilde{H} = H$, and $\tilde{H}(x,0) = h(x)$ for all $x \in X$, therefore $\tilde{H}$ provides the lifting we want. □

Altogether, these statements imply the following result, which completes the proof of Theorem 1.

Theorem 2 – *For an arbitrary simplicial complex $\mathcal{K}$, the maps $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Serre fibrations and (strong) homotopy equivalences. If $\mathcal{K}$ is finite, then $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Hurewicz fibrations.*

Proof. Let $\mathcal{K}$ be an arbitrary simplicial complex. We first prove that $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Serre fibrations. By Lemmas 11 and 12, and since the n-spheres are separable spaces, we can restrict ourselves to proving the same statement for Ψ_{γ} and $\overline{\Psi}_{\gamma}$ when $\gamma \leq \aleph_0$, and this is true in this case because the n-spheres are compact, by Proposition 7. If $\mathcal{K}$ is a finite simplicial complex, by Lemma 11 and Proposition 7 we get that $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Hurewicz fibrations.

Now, by Proposition 6 we know that $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ admit global sections. We denote them $\sigma_{\mathcal{K}}$ and $\overline{\sigma}_{\mathcal{K}}$, respectively. In order to prove that these are homotopy inverses for $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$, we need to check that $\sigma_{\mathcal{K}} \circ \Psi_{\mathcal{K}}$ and $\overline{\sigma}_{\mathcal{K}} \circ \overline{\Psi}_{\mathcal{K}}$ are homotopic to the identity map. Taking $\gamma_1 : X = L(\Omega,\mathcal{K}) \to L(\Omega,\mathcal{K})$ to be $\sigma_{\mathcal{K}} \circ \Psi_{\mathcal{K}}$ and $\gamma_0 = \mathrm{Id}_X$, one checks easily that, for all $f \in L(\Omega,\mathcal{K})$, $\gamma_1(f)(\Omega)$ is equal to

$$\sigma_{\mathcal{K}} \circ \Psi_{\mathcal{K}}(f)(\Omega) = \mathrm{supp}\,(\Psi_{\mathcal{K}} \circ \sigma_{\mathcal{K}} \circ \Psi_{\mathcal{K}}(f)) = \mathrm{supp}\,(\Psi_{\mathcal{K}}(f)) = f(\Omega)$$

which is equal to $\gamma_0(f)(\Omega)$. Therefore we can apply Lemma 5 (with $A = \varnothing$) and get that γ_0, γ_1 are homotopic. The proof for $\overline{\sigma}_{\mathcal{K}} \circ \overline{\Psi}_{\mathcal{K}}$ is similar.

Acknowledgments

I thank D. Chataur and A. Rivière for discussions and references. I foremost thank an anonymous referee for a careful check.

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