A CUBIC DEFINING ALGEBRA FOR THE LINKS-GOULD POLYNOMIAL

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1. Introduction

In 1992, Links and Gould introduced a polynomial invariant of knots and links out of a family of 4-dimensional representations of the quantum Lie superalgebra $\mathsf{U}_q\mathfrak{sl}(2|1)$. This invariant satisfies a *cubic* skein relation, that is the simplest skein relation on simple crossings that can be asked for, the quadratic one being characteristic of the HOMFLY-PT polynomial. It shares this property with the Kauffman polynomial (which corresponds to the quantum orthosymplectic Lie algebras and their standard representations), but behaves quite differently, notably with respect to disjoint union of links : the Kauffman polynomial is multiplicative

with respect to the disjoint union of two links, whereas the Links-Gould polynomial vanishes on such disjoint unions.

The additional skein relations satisfied by the Kauffman polynomial are relations of the so-called Birman-Wenzl-Murakami (BMW) algebra (see [BW], [Mu]). This BMW_n algebra is a quotient of the group algebra KB_n of the braid group over some field K of characteristic 0 by a generic cubic relations and by some other relation in KB_3, and there exists a single Markov trace on the tower of algebras $(BMW)_n$, whose value on closed braids provides the Kauffman invariant. This algebra is a deformation of the classical algebra of Brauer diagrams, and as such admits a basis with a nice combinatorial description. It describes the centralizer algebra of the action of $U_q\mathfrak{osp}(V)$ inside $V^{\otimes n}$. In addition, since the faithful Krammer representation factorizes through this BMW algebra, B_n embeds into the group $BMW_n^\times$ of invertible elements of the BMW algebra ; its Zariski closure is described in [M4].

The goal of this paper is to define a similar algebra for the Links-Gould polynomial. We first consider the corresponding centralizer algebra LG_n and prove the following statement, analogous to the well-known fact that the BMW_n algebra, defined as a (quantum) centralizer algebra, is a quotient of KB_n (see corollary 4.4).

Theorem 1.1. *The natural morphism $KB_n \to LG_n$ is surjective.*

As a consequence, LG_n is a natural candidate for being an analogue of the BMW_n algebra for the Links-Gould polynomial. As it is a centralizer algebra, we have a natural (combinatorial) description of its simple modules, but we do not have yet a satisfactory description of its elements. We have the following conjecture about its dimension.

Conjecture 1.2. *For all n,*

$$\dim LG_{n+1} = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$$

We checked this formula for $n \le 50$. It gives $\dim LG_1 = 1$, $\dim LG_2 = 3$, $\dim LG_3 = 20$, $\dim LG_4 = 175$, $\dim LG_5 = 1764$, $\dim LG_6 = 19404$. As communicated to us by F. Chapoton, this formula appears in the following setting. It is the number of pairs of paths, inside a square whose side has length $n + 1$, which go from the top left corner to the down right by down and right moves, which do not cross each other (see figure 1 for the corresponding 20 diagrams for LG_3). It suggests the possibility of a natural basis of this algebra indexed by such combinatorial objects. Sloane's encyclopedia of integer sequences suggests another (related) combinatorial interpretation, as the number of non-crossing partitions of $2n + 1$ inside $n + 1$ blocs. R. Blacher suggested to us to consider it as $(C_n)^2(2n + 1)$, where C_n is the Catalan number, and to interpret it as couples of binary trees together with a suitable marking of the leaves. In any case, it is natural to ask for a pictorial description of this algebra, and also if it admits a natural 'cellular structure' in the sense of Graham and Lehrer.

Our next goal is to get a presentation of LG_n by generators and relations. We know that it is a quotient of the generic cubic Hecke algebra, namely the quotient H_n of the group algebra of the braid group by a generic cubic relation. The latter is an infinite dimensional algebra for $n \ge 6$ (see section 5). In [I2], Ishii introduced a relation $r_2 \in H_3$ satisfied inside the skein module of the Links-Gould polynomial, and proved that the quotient of H_3 by (the image of) r_2 has dimension 20. We first prove that the natural morphism $H_4/(r_2) \twoheadrightarrow LG_4$ is *not* an isomorphism, and introduce a new relation (r_3) such that $H_4/(r_3) = H_4/(r_2, r_3) = LG_4$. We let $A_n = H_n/(r_2, r_3)$ for $n \ge 4$, $A_3 = H_3/(r_2)$, $A_2 = H_2$. By definition, $A_n \simeq LG_n$ for $n \le 4$. We prove (see section 7) the following.

Theorem 1.3.
  (i) $A_5 \simeq LG_5$.
  (ii) A_n is finite dimensional for all n.
  (iii) There exists only one Markov trace on $(A_n)_{n \geq 1}$.

We also get that there is only one Markov trace on LG_n. These two algebras A_n and LG_n are related by $A_n \twoheadrightarrow LG_n$ and the above theorem provides some evidence in favor of the following conjecture.

Conjecture 1.4. For all n, $A_n \simeq LG_n$.

The conjunction of these two conjectures clearly implies the next one :

Conjecture 1.5. For all n,
$$\dim A_{n+1} = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}.$$

While this conjecture is open, we have two (conjecturally equal) algebras, and one may use either of them, depending on our needs, to deal with the Links-Gould polynomial.

A computational use of the new relation r_3 (that is, of the algebra A_n) is theoretically possible, as it provides a new algorithm, which should use less memory space than the brutal use of the R-matrix. Unfortunately, the relation r_3 has too many terms to be printed here, not to mention to be used by a human. A computer implementation is in progress, though.

We also use these algebras to get new proofs of known results. For instance, using the classical notations for the parameters of the polynomial, the classical form of the R-matrix (recalled in section 3) implies that the Links-Gould polynomial takes values in $\mathbb{Z}[t_0^{\pm \frac{1}{2}}, t_1^{\pm \frac{1}{2}}, \sqrt{(t_0 - 1)(1 - t_1)}]$. Using our results we provide another proof of the following theorem of Ishii.

Theorem 1.6. (Ishii, [I3] theorem 1) The Links-Gould polynomial takes values in $\mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$.

Proof. The algebras A_3 and A_4 are defined and are split semisimple over $\mathbb{Q}(t_0, t_1)$. Moreover, the decomposition of A_{n+1} as a A_n-module described in section 7 is valid over the field $\mathbb{Q}(t_0, t_1)$, and the unicity of the Markov trace proves that it is also defined over $\mathbb{Q}(t_0, t_1)$. It follows that the Links-Gould invariant takes value in $\mathbb{Q}(t_0, t_1)$. The conclusion follows from the elementary fact that $\mathbb{Z}[t_0^{\pm \frac{1}{2}}, t_1^{\pm \frac{1}{2}}, \sqrt{(t_0 - 1)(1 - t_1)}] \cap \mathbb{Q}(t_0, t_1) = \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$. □

Finally, we investigate the image and kernel of the morphism $B_n \to LG_n^\times$, and get the following (see section 8).

Theorem 1.7. The morphism $B_n \to LG_n^\times$ is injective. If $LG_n = \bigoplus_{1 \leq k \leq N} Mat_{c(k)}(K)$ is the decomposition of LG_n in a sum of matrix algebras, the Zariski closure of B_n inside $LG_n^\times$ contains $\bigoplus_{1 \leq k \leq N} SL_{c(k)}(K)$.

Finally, the question of whether these conjecturally equal algebras are *the* 'right' algebras, in the sense of being the minimal ones, for the Links-Gould polynomial, is equivalent to the following one, that we leave open for the time being.

Question 1.8. Is this Markov trace non-degenerate on A_n ? on LG_n ?

We prove that it is non-degenerate on $A_4 = LG_4$, as it is a linear combination with non-zero coefficients of matrix traces (see table 10), thus providing some evidence to a 'yes' answer to
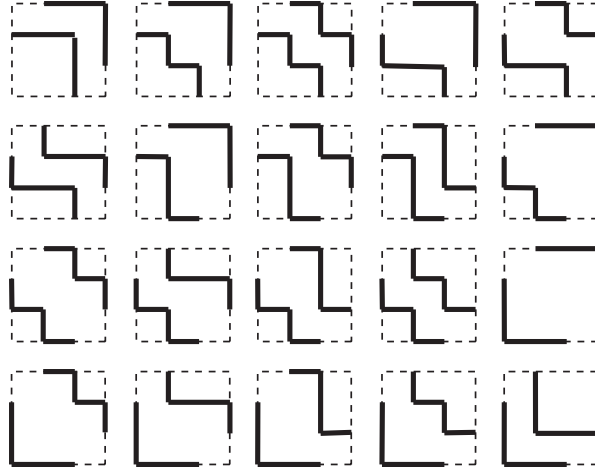
FIGURE 1. Non-crossing paths as a potential combinatorial model for LG_3.

this question. A complete answer of this question for LG_n should be given by the complete determination of the similar coefficients for arbitrary n, which is an interesting task in itself.

2. Quantum definitions and properties

2.1. **Lie superalgebras.** We recall the basic notions about Lie superalgebras. From now on the vector spaces under consideration are defined over a field $\Bbbk$ of characteristic 0. For simplicity we moreover assume $\Bbbk \subset \mathbb{C}$. A *superspace* V is a $\mathbb{Z}/2\mathbb{Z}$-graded space $V = V_0 \oplus V_1$. An homogeneous element in V is called even if it belongs to V_0, odd if it belongs to V_1. We denote $|a| \in \{0, 1\}$ the degree of an homogeneous element. A *superalgebra* A is an associative unital $\mathbb{Z}/2\mathbb{Z}$-graded algebra (that is $A = A_0 \oplus A_1$ with $A_i A_j \subset A_{i+j}$). The tensor superproduct $A \otimes_s B$ of two superalgebras A, B is defined as the vector space $A \otimes B$ endowed with the natural ($\mathbb{Z}/2\mathbb{Z}$)-graduation and by the multiplication $(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$ for homogeneous b, c. It is straightforward to check that $((A \otimes_s B) \otimes_s C)$ is naturally isomorphic to $(A \otimes_s (B \otimes_s C))$, hence the superproduct $A^{(1)} \otimes_s A^{(2)} \otimes_s \cdots \otimes_s A^{(n)}$ of an ordered (finite) family of superalgebras $A^{(1)}, \ldots, A^{(n)}$ is well-defined.

For a superspace $V = V_0 \oplus V_1$, the algebra $\mathrm{End}(V)$ has a natural superalgebra structure, with $\mathrm{End}(V)_0 = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1)$ and $\mathrm{End}(V)_1 = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0)$. For V, W two superspaces, the tensor product $V \otimes W$ has a natural superspace structure $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$, $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$. Assuming that V and W are finite dimensional, there is an isomorphism $\mathrm{End}(V) \otimes_s \mathrm{End}(W) \simeq \mathrm{End}(V \otimes W)$. It associates to $a \otimes b \in \mathrm{End}(V) \otimes_s \mathrm{End}(W)$, with homogeneous a, b, the endomorphism of $V \otimes W$ that maps $v \otimes w$ to $(-1)^{|b||v|}(av) \otimes (bw)$ for homogeneous v, w. It can be extended

to an isomorphism of superalgras $\mathrm{End}(V)^{\otimes_s n} = \mathrm{End}(V) \otimes_s \mathrm{End}(V) \otimes_s \cdots \otimes_s \mathrm{End}(V)$ with $\mathrm{End}(V \otimes \cdots \otimes V) = \mathrm{End}(V^{\otimes n})$.

For the axiomatic definition of Lie superalgebras we refer to [K1], and recall the lazy definition as a graded subspace of some associative superalgebra stable under the superbracket defined by $[a, b] = ab - (-1)^{|a||b|} ba$ for homogeneous elements a, b. To each Lie superalgebra $\mathfrak{g}$ is associated its universal envelopping algebra $\mathsf{U}\mathfrak{g}$. It is a superalgebra with a (super)coproduct $\Delta : \mathsf{U}\mathfrak{g} \to \mathsf{U}\mathfrak{g} \otimes_s \mathsf{U}\mathfrak{g}$ defined by $\Delta(a) = a \otimes 1 + 1 \otimes a$ for $a \in \mathfrak{g}$ (the definition of the 'diagonal homomorphism' of [K1] is well-known to be flawed). We let $\Delta_n : \mathsf{U}\mathfrak{g} \to (\mathsf{U}\mathfrak{g})^{\otimes_s n}$ denote the iterated coproduct.

2.2. **Casimir operators.** Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra, with a non-degenerate bilinear form $< \, , \, >$ which is invariant, that is $< [a, b], c > = < a, [b, c] >)$, supersymmetric, that is $< b, a > = (-1)^{|a||b|} < a, b >$ for homogeneous a, b, and consistent, that is $< \mathfrak{g}_0, \mathfrak{g}_1 > = 0$. Let $e_1, \ldots, e_n$ be an homogeneous basis of $\mathfrak{g}$, and denote $|i| = |e_i|$. The dual basis $(\tilde{e}_i)$ is defined by $< \tilde{e}_i, e_j > = \delta_{ij}$ (Kronecker symbol), and $|\tilde{e}_i| = |e_i| = |i|$ by consistency. By supersymmetry, $\tilde{\tilde{e}}_i = (-1)^{|i|} e_i$. Let $\Omega = \sum_i e_i \otimes \tilde{e}_i \in \mathfrak{g} \otimes \mathfrak{g}$. It is independent of the choice of the basis, as it is the image of $< \, , \, >$ under the isomorphism of vector spaces $(\mathfrak{g} \otimes \mathfrak{g})^* \simeq \mathfrak{g}^* \otimes \mathfrak{g}^* \simeq \mathfrak{g} \otimes \mathfrak{g}$ induced by $< \, , \, >$. In particular, $\Omega = \sum_i \tilde{e}_i \otimes \tilde{\tilde{e}}_i = \sum_i (-1)^{|i|} \tilde{e}_i \otimes e_i$.

The Casimir operator is the element $C = \sum_i e_i \tilde{e}_i = \sum_i (-1)^{|i|} \tilde{e}_i e_i \in \mathsf{U}\mathfrak{g}$. It does not depend on the choice of the basis either. Note that $C \in \mathsf{U}\mathfrak{g}$ is even.

Let $n \geq 2$. For $i < j \leq n$, we let

$$\Omega_{ij} = \sum_r 1 \otimes \cdots \otimes e_r \otimes 1 \otimes \cdots \otimes 1 \otimes \tilde{e}_r \otimes 1 \otimes \cdots \otimes 1 \in \mathsf{U}\mathfrak{g}^{\otimes_s n}$$

where the e_r's are in position i and the $\tilde{e}_r$'s are in position j. The following lemma is standard in the classical (that is, Lie algebra) case. Its extension to the 'super' case is straightforward.

Lemma 2.1.

$$2(\Omega_{1,n} + \Omega_{2,n} + \cdots + \Omega_{n-1,n}) = \Delta_n(C) - \Delta_{n-1}(C) \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes C$$

Proof. It is easily checked that $\sum_{i<n} \Omega_{i,n} = \sum_r \Delta_{n-1}(e_r) \otimes \tilde{e}_r$. On the other hand, $\Delta_n(C) = \sum_r \Delta_n(e_r) \Delta_n(\tilde{e}_r)$ is

$$\sum_r \left(\sum_i 1 \otimes \cdots \otimes \underbrace{e_r}_{(i)} \otimes \cdots \otimes 1 \right) \left(\sum_j 1 \otimes \cdots \otimes \underbrace{\tilde{e}_r}_{(j)} \otimes \cdots \otimes 1 \right)$$

that is the sum over r of

$$\sum_{i<j} 1 \otimes \cdots \otimes \underbrace{e_r}_{(i)} \otimes \cdots \otimes \underbrace{\tilde{e}_r}_{(j)} \otimes \cdots \otimes 1 + \sum_{j<i} (-1)^{|e_r||\tilde{e}_r|} 1 \otimes \cdots \otimes \underbrace{\tilde{e}_r}_{(j)} \otimes \cdots \otimes \underbrace{e_r}_{(i)} \otimes \cdots \otimes 1$$

and of

$$\sum_i 1 \otimes \cdots \otimes \underbrace{e_r \tilde{e}_r}_{(i)} \otimes \cdots \otimes 1$$

It follows that

$$\Delta_n(C) - \Delta_{n-1}(C) \otimes 1 = 1^{\otimes(n-1)} \otimes C + \sum_r \Delta_{n-1}(e_r) \otimes \tilde{e}_r + \sum_r \Delta_{n-1}(\tilde{e}_r) \otimes \tilde{\tilde{e}}_r$$

since $|e_r||\tilde{e}_r| = |r|^2 = |r|$ and $\tilde{\tilde{e}}_r = (-1)^{|r|}e_r$, this proves the lemma. $\square$

We recall that the Ω_{ij} satisfy the infinitesimal braids or 4T-relations, namely the 1-form

$$\sum_{i<j} \Omega_{i,j} \mathrm{dlog}(z_i - z_j)$$

is integrable on $\mathbb{C}_*^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$. There is a Kohno-Drinfeld type theorem due to N. Geer (see [Ge]) saying that the representations of the braid group obtained in this way are isomorphic to the ones originating from the Yamane quantization of the corresponding Lie superalgebra.

2.3. **The Lie superalgebras** $\mathfrak{sl}(2|1)$. When V is a superspace with $V_0 = \Bbbk^m$ and $V_1 = \Bbbk^n$ the Lie superalgebra associated to $\mathrm{End}(V)$ is traditionnaly denoted $\mathfrak{gl}(m|n)$. It admits a linear form called the supertrace $str : \mathfrak{gl}(m|n) \to \Bbbk$ defined by $str(\mathrm{End}(V)_1) = 0$ and $str_{|\mathrm{End}(V)_0} = \mathrm{Id}_{\mathrm{End}(V_0)} - \mathrm{Id}_{\mathrm{End}(V_1)}$. Choosing the natural homogeneous basis of $V = \Bbbk^m \oplus \Bbbk^n$ the elements of $\mathrm{End}(V)$ are represented by matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, with supertrace $\mathrm{tr}\alpha - \mathrm{tr}\delta$. The Lie superalgebra structure of $\mathfrak{gl}(m|n)$ restricts to a Lie superalgebra structure on $\mathrm{Ker}\,str$, denoted $\mathfrak{sl}(m|n)$. We assume $m \neq n$. Then $< a, b >= str(ab)$ defines on $\mathfrak{sl}(m|n)$ a bilinear form fulfilling the requirements of section 2.2.

We now specialize to the case $m = 2$, $n = 1$, and choose for basis of $\mathfrak{sl}(2|1)$ the following elements

$$h_1 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) h_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) e_1 = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) f_1 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$e_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right) f_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) [e_1, e_2] = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) [f_1, f_2] = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline -1 & 0 & 0 \end{array}\right)$$

The elements h_1, h_2, f_1, f_2 are even, the other four are odd. We have $[h_i, h_j] = 0$, $[h_i, e_j] = a_{ij}e_j$, $[h_i, f_j] = -a_{ij}f_j$, $[e_i, f_j] = \delta_{ij}h_i$, $[e_2, e_2] = [f_2, f_2] = 0$, $[e_1, [e_1, e_2]] = [f_1, [f_1, f_2]] = 0$ with $(a_{ij})_{i,j}$ the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$, so our notations are compatible with [GP].

These 8 elements are weight vectors under the adjoint representation for the Cartan subalgebra $\mathfrak{h}$ spanned by h_1, h_2. Clearly $\mathfrak{h}$ is a subspace of the space $\mathfrak{d}$ of the diagonal matrices in $\mathfrak{gl}(2|1)$. Following [K2] we let $\varepsilon_1^K, \varepsilon_2^K, \delta_1^K \in \mathfrak{d}^*$ denote the dual basis of the diagonal vectors E_{11}, E_{22}, E_{33}. Under the natural restriction map $\mathfrak{d}^* \to \mathfrak{h}^*$ one has $\varepsilon_1^K \mapsto h_1^*$, $\varepsilon_2^K \mapsto h_2^* - h_1^*$, $\delta_1^K \mapsto h_2^*$, where (h_1^*, h_2^*) denotes the dual basis of (h_1, h_2). Recall that a root $\alpha \in \mathfrak{h}^*$ is called even (odd) if the corresponding eigenspace is even (odd). The only positive even root is thus $\alpha_1 = \varepsilon_1^K - \varepsilon_2^K = 2h_1^* - h_2^*$, the odd ones are $\alpha_2 = \varepsilon_2^K - \delta_1^K = -h_1^*$, $\alpha_1 + \alpha_2 = \varepsilon_1^K - \delta_1^K = h_1^* - h_2^*$. It follows that the 'super half-sum' of the positive roots is $\rho = \frac{1}{2}(\alpha_1 - (\alpha_2 + \alpha_1 + \alpha_2)) = -\alpha_2 = h_1^*$.

2.4. **Highest weight modules for** $\mathfrak{sl}(2|1)$. To each $\lambda \in \mathfrak{h}^*$ one can associate a Kac module $V(\lambda)$ for $\mathfrak{sl}(2|1)$, which is an irreducible heighest weight module, if $< \lambda + \rho, \alpha > \neq 0$ for odd root α (see [K2]). Such a weight is called *typical*. The Cartan matrix is the matrix in the basis (h_1, h_2) of the Killing form $a, b \mapsto < a, b >= str(ab)$ to the Cartan subalgebra $\mathfrak{h}$. We have $h_1^* =< -h_2, \cdot >$ and $h_2^* =< -h_1 - 2h_2, \cdot >$. It follows that the induced bilinear form $< , >$ on $\mathfrak{h}^*$ satisfies $< h_1^*, h_1^* >= 0$, $< h_1^*, h_2^* >= -1$, $< h_2^*, h_2^* >= -2$. In particular, for

$\lambda = a_1 h_1^* + a_2 h_2^*$ one computes $< \lambda + \rho, -h_1^* > = a_2$ and $< \lambda + \rho, h_1^* - h_2^* > = a_1 + a_2 + 1$. Thus λ is typical if $a_2 \neq 0$ and $a_1 + a_2 + 1 \neq 0$.

In that case, the (usual) dimension of $V(\lambda)$ is $4 < \lambda - \alpha_2, \alpha_1 > = 4(a_1 + 1)$, and its odd and even parts have the same dimension $2(a_1 + 1)$.

Finally, the value of C on $V(\lambda)$ for a typical λ is $< \lambda, \lambda + 2\rho >$. We find $C_{|V(\lambda)} = -2a_2(a_1 + a_2 + 1)$.

Letting $V(a_1, a_2)$ denote the highest weight module with weight $a_1 h_1^* + a_2 h_2^*$, which is typical for $a_2 \neq 0$ and $a_1 + a_2 + 1 \neq 0$. If $a_1 \neq 0$, then $V(a_1, a_2) \otimes V(0, b_2)$ is isomorphic to

$$V(a_1, a_2 + b_2) \oplus V(a_1 + 1, a_2 + b_2) \oplus V(a_1 - 1, a_2 + b_2 + 1) \oplus V(a_1, a_2 + b_2 + 1)$$

whenever all the weights involved are typical. When $a_1 = 0$, and under the same conditions, it is isomorphic to $V(a_1, a_2 + b_2) \oplus V(a_1 + 1, a_2 + b_2) \oplus V(a_1, a_2 + b_2 + 1)$ (see [GP] lemma 1.3 or [Gr] proposition 4.1).

2.5. **The bimodules** $V(0, \alpha)^{\otimes 2}$. According to the Clebsh-Gordan decomposition above, for generic values of α we have $V(0, \alpha) \otimes V(0, \alpha) = V(0, 2\alpha) \oplus V(0, 2\alpha + 1) \oplus V(1, 2\alpha)$, with C taking values $-4\alpha(2\alpha + 1)$, $-4(\alpha + 1)(2\alpha + 1)$, $-8\alpha(\alpha + 1)$ and the supersymmetrizer τ the values $1, 1, -1$ (see [LG]). Since, on $V(0, \alpha)$, C takes value $-2\alpha(\alpha + 1)$, then Ω_{12} takes the values $-2\alpha^2$, $-2(\alpha + 1)^2$, $-2\alpha(\alpha + 1)$.

3. The Links-Gould invariant of links

In this section, we give a definition of the Links-Gould invariant [LG] of links. It is an 2-variable invariant constructed by using the Reshetikhin-Turaev recipe [RT] applied to a one-parameter family of representations of the quantum supergroup $\mathsf{U}_q(\mathfrak{sl}(2,1))$ – or $\mathsf{U}q\mathfrak{l}(2|1)$, which does not make any difference, see e.g. the introduction of [DIL]. It is naturally defined to be an invariant of (1-1)-tangle [DWKL] and turn's out to be in fact an invariant of links (see Remark [GP]). For convenience we define following closely Ishii [I1] the Links-Gould invariant as a partial trace on the braid group.

Recall that it follows from Alexander theorem [AL] that any link can be presented as the closure of a braid. In addition Markov's theorem [MA] precises exactly when two braids represent the same link. Hence one can always define a link invariant on a braid closure representative, as long as it is invariant under the Markov moves (see also Section 7).

Let K be a field of characteristic 0 containing two algebraically independent elements $t_0, t_1 \in K^\times$, as well as the square roots $\sqrt{t_0}$, $\sqrt{t_1}$ and $\sqrt{(t_0 - 1)(1 - t_1)}$. Consider a four-dimensional vector space V over K with basis (e_1, e_2, e_3, e_4). Recall that a *R-matrix* R is an invertible element of $\mathrm{End}(V \otimes V)$ that satisfies the Yang-Baxter equation:

$$(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R) = (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V).$$

An endomorphism A of $V \otimes V$ will be represented by a matrix $(A_{i,j})_{i,j \in [|1,16|]}$ where

$$A_{4(i-1)+j, 4(k-1)+l} = A_{k,l}^{i,j} \text{ with } A(e_k \otimes e_l) = \sum_{i,j=1}^{4} A_{k,l}^{i,j} e_i \otimes e_j \text{ for all } k, l = 1, \ldots, 4.$$

The endomorphism $R \in \mathrm{End}(V \otimes V)$ given by the following matrix $(R_{i,j})_{i,j \in [|1,16|]}$ [DWKL] is a R-matrix :

$$
\begin{pmatrix}
t_0 & \cdot & \cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & t_0^{1/2} & \cdot & \cdot & t_0-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0t_1-1 & \cdot & \cdot & -t_0^{1/2}t_1^{1/2} & \cdot & \cdot & -t_0^{1/2}t_1^{1/2}Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot & \cdot \\
\cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & t_0-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{1/2}t_1^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & -t_0^{1/2}t_1^{1/2}Y & \cdot & \cdot & Y & \cdot & \cdot & Y^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & t_1-1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot & \cdot & t_1-1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1
\end{pmatrix},
$$

where $Y = ((t_0-1)(1-t_1))^{\frac{1}{2}}$.

Let B_n be the braid group on n strands with Artin generators $s_1,\ldots,s_{n-1}$. A R-matrix $R \in \mathrm{End}(V \otimes V)$ defines a representation ρ_R^n of the braid group in $\mathrm{Aut}(V^{\otimes n})$ by sending s_i to $\mathrm{Id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{Id}_V^{\otimes(n-1-i)} \in \mathrm{Aut}(V^{\otimes n})$, for $i=1,\ldots,n-1$. In addition denote μ the automorphism of V defined by $\mu(e_i) = \mu_i e_i$ for $i=1,\ldots 4$ where $(\mu_1,\mu_2,\mu_3,\mu_4) = (t_0^{-1},-t_1,-t_0^{-1},t_1)$.

Given $A \in \mathrm{End}(V^{\otimes n})$ such that

$$A(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \sum_{1 \le j_1,\ldots,j_n \le 4} A_{i_1,\ldots,i_n}^{j_1,\ldots,j_n} e_{j_1} \otimes \cdots \otimes e_{j_n},$$

define for $m=1,\ldots,n$ the partial trace operator $tr_n^m(A) \in \mathrm{End}(V^{\otimes(n-m)})$ by

$$tr_n^m(A)(e_{i_1} \otimes \cdots \otimes e_{i_{n-m}}) = \sum_{1 \le j_1,\ldots,j_n \le 4} A_{i_1,\ldots,i_{n-m},j_{n-m+1}\ldots,j_n}^{j_1,\ldots,j_{n-m},j_{n-m+1}\ldots,j_n} e_{j_1} \otimes \cdots \otimes e_{j_{n-m}}.$$

Given a braid $\beta \in B_n$ we denote by $\widehat{\beta}$ the closure of β (see Figure (2)) we define the Links-Gould invariant $\mathfrak{LG}(\widehat{\beta}; t_0, t_1)$ of the link $\widehat{\beta}$ by the following formula:

$$tr_n^{n-1}((\mathrm{Id}_V \otimes \mu^{\otimes(n-1)})(\rho_R^n(\beta)) = \mathfrak{LG}(\widehat{\beta}; t_0, t_1)\mathrm{Id}_V.$$
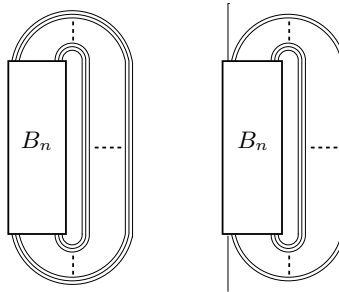

FIGURE 2. Closure of a braid and partial closure of a braid.

The Links-Gould invariant can also be defined by considering the right picture of Figure (2) which describes a topological partial trace and use the tangle invariant construction described in [DWKL] (notice that one just needs to assign an endomorphism to each cup or cap). These two constructions coincide up to a change of variables.

The R-matrix that we introduced here has for eigenvalues $-1, t_0, t_1$. Rescaling the R-matrix construction by $s_i \mapsto (-a)s_i$ with $a \in K^\times$, we get a representation $B_n \to \mathrm{GL}(V^{\otimes n})$ in which the image of s_i is annihilated by the cubic polynomial $(X - a)(X - b)(X - c)$, with $b = -t_0 a$, $c = -t_1 a$. We use this renormalization from now on.

4. The morphism $KB_n \twoheadrightarrow LG_n(\alpha)$

We take $\mathfrak{g} = \mathfrak{sl}(2|1)$ and we let $LG_n(\alpha)$ denote the commutant algebra of the image of $\mathsf{U}\mathfrak{g}$ inside $\mathrm{End}(V(0, \alpha)^{\otimes n})$. Note that $LG_n(\alpha) \subset LG_{n+1}(\alpha)$ under $m \mapsto m \otimes 1$.

It contains the group algebra $\Bbbk \mathfrak{S}_n$ of the symmetric group, as well as the image $\mathcal{LG}_n(\alpha)$ of the Lie algebra $\mathcal{T}_n$ of infinitesimal braids (or horizontal chord diagrams), also known as the holonomy Lie algebra of the space $\mathbb{C}_*^n$. Recall that this algebra $\mathcal{T}_n$ is generated by elements t_{ij} which are dual to the 1-forms $\mathrm{dlog}(z_i - z_j)$, and which satisfy the relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{kj}] = 0$ and $[t_{ij}, t_{kl}] = 0$ whenever $\#\{i, j, k, l\} = 4$. The morphism $\mathcal{T}_n \to \mathcal{LG}_n(\alpha)$ is given by $t_{ij} \mapsto \Omega_{ij}$.

Let T_n be the image of $\sum_{1 \le i, j \le n} t_{ij}$ in $\mathcal{LG}_n(\alpha)$.

We let $[a, k]_r = ah_1^* + (r\alpha + k)h_2^*$ for $0 \le a \le r-1$, $0 \le k \le r-a-1$, that we identify with the corresponding module $V(a, k + r\alpha)$. We also replace $\oplus$ by $+$ when it lightens notations. Notice that $r\alpha + k \ne 0$ and $a + r\alpha + k + 1 \ne 0$ for $\alpha \notin \mathbb{Q}_{<0}$. The tensor product decomposition above is thus translated as $[a, k]_r \otimes V(0, \alpha) = [a-1, k+1]_{r+1} + [a, k]_{r+1} + [a, k+1]_{r+1} + [a+1, k]_{r+1}$, under the convention that $[a, l]_{r+1}$ is omitted if $a < 0$. From this we get that, if $a \ge 2$,

$$\begin{aligned}
[a, k]_r \otimes V(0, \alpha)^{\otimes 2} = \ & [a-2, k+2]_{r+2} + 2[a-1, k+1]_{r+2} + 2[a-1, k+2]_{r+2} \\
& + 4[a, k+1]_{r+2} + [a, k]_{r+2} + 2[a+1, k]_{r+2} \\
& + [a, k+2]_{r+2} + 2[a+1, k+1]_{r+2} + [a+2, k]_{r+2}
\end{aligned}$$

and in particular $[0, 0]_1 \otimes V(0, \alpha) = [0, 0]_2 \oplus [0, 1]_2 \oplus [1, 0]_2$.

We note the following decompositions

$$\begin{aligned}
[a, k]_r \otimes V(0, 2\alpha) &= [a, k]_{r+2} + [a+1, k]_{r+2} + [a-1, k+1]_{r+2} + [a, k+1]_{r+2} && \text{if } a \ne 0 \\
[0, k]_r \otimes V(0, 2\alpha) &= [0, k]_{r+2} + [1, k]_{r+2} + [0, k+1]_{r+2} \\
[a, k]_r \otimes V(0, 2\alpha + 1) &= [a, k+1]_{r+2} + [a+1, k+1]_{r+2} + [a-1, k+2]_{r+2} + [a, k+2]_{r+2} && \text{if } a \ne 0 \\
[0, k]_r \otimes V(0, 2\alpha + 1) &= [0, k+1]_{r+2} + [1, k+1]_{r+2} + [0, k+2]_{r+2}
\end{aligned}$$

Since $V(0, \alpha) \otimes V(0, \alpha) = V(0, 2\alpha) + V(0, 2\alpha + 1) + V(1, 2\alpha)$ it follows that

$$\begin{aligned}
[a, k]_r \otimes V(1, 2\alpha) = \ & [a-2, k+2]_{r+2} + [a-1, k+1]_{r+2} + [a-1, k+2]_{r+2} + 2[a, k+1]_{r+2} \\
& + [a+1, k]_{r+2} + [a+1, k+1]_{r+2} + [a+2, k]_{r+2}
\end{aligned}$$

Theorem 4.1. *For generic values of α, the image of $\mathsf{U}\mathcal{T}_n$ in $\mathrm{End}(V(0, \alpha)^{\otimes n})$ is $LG_n(\alpha)$.*

Proof. The techniques used to prove the result are inherited from [M1], to which the reader can report for a detailed description of the general setting. The value of C on $[a, k]_r$ is $-2(r\alpha + k)(r\alpha + k + a + 1)$. It is a polynomial of degree 2 in α with dominant term $-2r^2$, hence it is uniquely defined by its roots $-k/r$ and $-(k + a + 1)/r$. For a given r, no two

such polynomial can be equal, as $\{-k/r, -(k+a+1)/r\} = \{-k'/r, -(k'+a'+1)/r\}$ iff $\{k, k+a+1\} = \{k', k'+a'+1\}$, and k is determined by $k = \min\{k, k+a+1\}$.

We now consider a given r, and the $\mathsf{U}\mathcal{T}_r$ module $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, V(0,\alpha)^{\otimes n})$, with $\mathfrak{g} = \mathfrak{sl}(2|1)$. We choose a basis compatible with the Bratteli diagram (that is, a basis of highest weight vectors for all $\Delta_s(\mathfrak{g}) \otimes 1^{\otimes(r-s)}$ for all $s \le r$). We call such a basis a standard basis. See table 1 for the beginning of the Bratteli diagram.

The above remark on the Casimir together with lemma 2.1 imply that, when restricted to the commutative Lie subalgebra $\mathcal{D}_r$ of $\mathcal{T}_r$ generated by the $t_{1,r} + \cdots + t_{r-1,r}$, the restriction of this $\mathcal{T}_r$-module is the multiplicity-free sum of 1-dimensional representations, this decomposition being necessarily given by a standard basis. In particular, the image of $\mathsf{U}\mathcal{D}_r$ in such a basis is the whole algebra of diagonal matrices. In order to prove the theorem, we moreover only need to show that such a $\mathsf{U}\mathcal{T}_r$-module is irreducible, as the eigenvalues of $\sum \Omega_{ij}$ determine the value of $\Delta_r(C)$ hence the module $[a,k]_r$.

In order to prove the irreducibility, we make the following remark. Let $t_r = t_{1,r} + \cdots + t_{r-1,r}$ and $Y = t_r - t_{r-1}$, s the transposition $(r-1, r)$, and $u = t_{r-1,r}$. We introduce the subalgebra L of $\mathfrak{S}_r \ltimes \mathsf{U}\mathcal{T}_r$ generated by s, u and Y, and $\mathcal{L}$ the Lie subalgebra of $\mathcal{T}_r$ generated by u and Y. Note that they commute with $\mathsf{U}\mathcal{T}_{r-2}$, and that the image of L in $\mathrm{End}_{\mathsf{U}\mathfrak{g}}(V(0,\alpha)^{\otimes r})$ is the same as the image of $\mathsf{U}\mathcal{L} \subset \mathsf{U}\mathcal{T}_r$, as the image of s is a polynomial (depending on α) of the image of u, at least for generic values of α (this last assertion needs only to be checked for $r = 2$, and in this case it follows from the spectral decomposition of $V(0,\alpha)^{\otimes 2}$).

Now, the restriction to $\mathsf{U}\mathcal{T}_{r-2} \times \mathsf{U}\mathcal{L}$ of $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, V(0,\alpha)^{\otimes r})$ is

$$\bigoplus_{b,l} \mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([b,l]_{r-2}, V(0,\alpha)^{\otimes(r-2)}) \otimes \mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, [b,l]_{r-2} \otimes V(0,\alpha)^{\otimes 2})$$

We first prove that is sufficient to show that the $\mathsf{U}\mathcal{L}$-modules $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, [b,l]_{r-2} \otimes V(0,\alpha)^{\otimes 2})$ are irreducible. Indeed, any stable subspace of $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, V(0,\alpha)^{\otimes r})$ need to have for basis a subset $\mathcal{B}_0$ of our given (standard) basis $\mathcal{B}$. Note that this standard basis is naturally indexed by sequences of $\mathsf{U}\mathfrak{g}$-modules of the form $([a_1,k_1]_1, [a_2,k_2]_2, \ldots, [a_r = a, k_r = k]_r)$ or, more combinatorially, by paths in the Bratteli diagram.

By induction on r, we can assume that all the $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([b,l]_{r-1}, V(0,\alpha)^{\otimes(r-1)})$ are irreducible under the action of $\mathsf{U}\mathcal{T}_{r-1}$. Since the restriction of $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, V(0,\alpha)^{\otimes r})$ to $\mathsf{U}\mathcal{T}_{r-1}$ is

$$\bigoplus_{b,l} \mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([b,l]_{r-1}, V(0,\alpha)^{\otimes(r-1)}) \otimes \mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, [b,l]_{r-1} \otimes V(0,\alpha))$$

(with $\mathsf{U}\mathcal{T}_{r-1}$ acting trivially on the second tensor factor) this implies that, if some $\gamma = (\gamma_1, \ldots, \gamma_r = [a,k]_r)$ belongs to $\mathcal{B}_0$, then all the elements of $\mathcal{B}$ with the same γ_{r-1} belong to $\mathcal{B}_0$. From the Bratteli diagram it is clear that, if $\mathcal{B}_0 \ne \mathcal{B}$, there exist $\delta \in \mathcal{B} \setminus \mathcal{B}_0$ and $\gamma \in \mathcal{B}_0$ which differ only at the place $r-1$, that is $\gamma_i = \delta_i$ for $i \ne r-1$. But if we know in addition that the $\mathsf{U}\mathcal{L}$-action on $\mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, \gamma_{r-2} \otimes V(0,\alpha)^{\otimes 2}) = \mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, \delta_{r-2} \otimes V(0,\alpha)^{\otimes 2})$ is irreducible, there exists an element of $\mathsf{U}\mathcal{L}$ that maps γ and δ, which proves the contradiction that we want.

We now prove that every $M = \mathrm{Hom}_{\mathsf{U}\mathfrak{g}}([a,k]_r, [b,l]_{r-2} \otimes V(0,\alpha)^{\otimes 2})$ is an irreducible $\mathsf{U}\mathcal{L}$-module, or equivalently an irreducible L-module. There are several cases to consider. We contend ourselves with the most difficult (4-dimensional) one, the other ones being similar and easier (and basically already dealt with in [M1], annexe). This is the case where $[b,l]_{r-2} = [a, k-1]_{r-2}$, with $a \ge 1$. For the action of u and s, M can be decomposed as $D_1 \oplus D_2 \oplus U$

with $\dim D_1 = \dim D_2 = 1$, $\dim U = 2$, and

$$
\begin{aligned}
D_1 &= \operatorname{Hom}_{U\mathfrak{g}}([a,k]_r, [a,k-1]_{r-2} \otimes V(0,2\alpha)) \\
D_2 &= \operatorname{Hom}_{U\mathfrak{g}}([a,k]_r, [a,k-1]_{r-2} \otimes V(0,2\alpha+1)) \\
U &= \operatorname{Hom}_{U\mathfrak{g}}([a,k]_r, [a,k-1]_{r-2} \otimes V(1,2\alpha))
\end{aligned}
$$

We choose a basis of M compatible with this decomposition. In such a basis, we have

$$
u = -2 \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & (\alpha+1)^2 & 0 & 0 \\ 0 & 0 & \alpha(\alpha+1) & 0 \\ 0 & 0 & 0 & \alpha(\alpha+1) \end{pmatrix} \quad s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
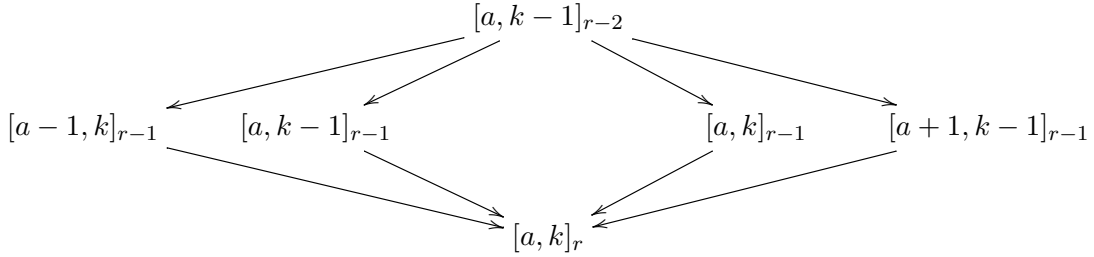$$

Now, we note that s, u and Y are related by the (easy-to-check) relation $Y + sYs = 2u$. This relation implies that Y has the form

$$
\begin{pmatrix} D & M \\ N & v\mathrm{Id} \end{pmatrix}
$$

with $M, N, D, \mathrm{Id} \in Mat_2(\Bbbk)$,

$$
D = -2 \begin{pmatrix} \alpha^2 & 0 \\ 0 & (\alpha+1)^2 \end{pmatrix} \quad \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

and $v = -2\alpha(\alpha+1)$. We now compute the eigenvalues of Y, using lemma 2.1. Recall the part of the Bratteli diagram that is involved here.
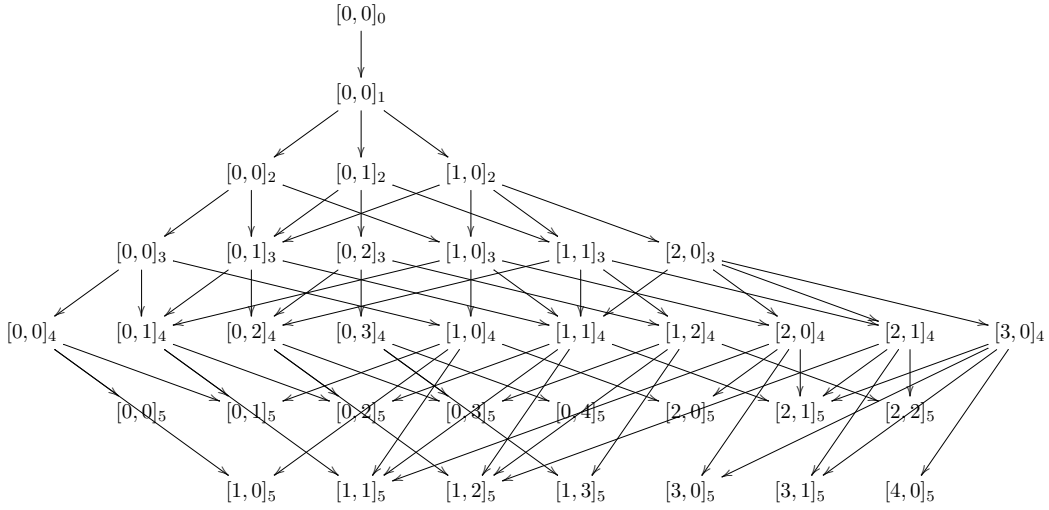

By lemma 2.1, the eigenvalues of Y are the ones of $(\Delta_r(C)+\Delta_{r-2}(C)\otimes 1\otimes 1-2\Delta_{r-1}(C)\otimes 1)/2$, that is $-2(\alpha^2+\alpha-\frac{a}{2})$, $-2(\alpha^2+r\alpha+k+\frac{a}{2})$, $-2(\alpha^2+(2-r)\alpha-\frac{a}{2}-k)$, $-2(\alpha^2+\alpha+\frac{a}{2}+1)$. Since $a \geq 1$ and $r \geq 3$, we note that Y and u share no common eigenvalue for generic α. This implies the following

- $\det M \neq 0$, for otherwise there would exist some $m = (x,y) \in \Bbbk^2$ in $\operatorname{Ker} M$, hence $Y\tilde{m} = vY$ for $\tilde{m} = (0,0,x,y)$ contradicting $v \notin \operatorname{Sp}(Y)$.
- the columns of N are non-zero.

Since Y has the same eigenvalues as its transpose, this implies also that $\det N \neq 0$ and that the rows of M are non-zero. It is then straightforward to check that these conditions imply the irreducibility of M under u and Y, unless Y has one of the following two forms

$$
Y = -2 \begin{pmatrix} \alpha^2 & 0 & x & 0 \\ 0 & (\alpha+1)^2 & 0 & y \\ z & 0 & \alpha(\alpha+1) & 0 \\ 0 & t & 0 & \alpha(\alpha+1) \end{pmatrix} \quad \text{or } Y = -2 \begin{pmatrix} \alpha^2 & 0 & 0 & x \\ 0 & (\alpha+1)^2 & y & 0 \\ 0 & z & \alpha(\alpha+1) & 0 \\ t & 0 & 0 & \alpha(\alpha+1) \end{pmatrix}
$$

But in these two cases, Y would have two distinct eigenvalues whose sum is $-2(\alpha^2+\alpha(\alpha+1)) = -2(2\alpha^2+\alpha)$ (the two others having sum $-2((\alpha+1)^2+\alpha(\alpha+1)) = -2(2\alpha^2+3\alpha+1)$). It is

$$[0,0]_0$$
$$[0,0]_1$$
$$[0,0]_2 \quad [0,1]_2 \quad [1,0]_2$$
$$[0,0]_3 \quad [0,1]_3 \quad [0,2]_3 \quad [1,0]_3 \quad [1,1]_3 \quad [2,0]_3$$
$$[0,0]_4 \quad [0,1]_4 \quad [0,2]_4 \quad [0,3]_4 \quad [1,0]_4 \quad [1,1]_4 \quad [1,2]_4 \quad [2,0]_4 \quad [2,1]_4 \quad [3,0]_4$$
$$[0,0]_5 \quad [0,1]_5 \quad [0,2]_5 \quad [0,3]_5 \quad [0,4]_5 \quad [2,0]_5 \quad [2,1]_5 \quad [2,2]_5$$
$$[1,0]_5 \quad [1,1]_5 \quad [1,2]_5 \quad [1,3]_5 \quad [3,0]_5 \quad [3,1]_5 \quad [4,0]_5$$

TABLE 1. Bratteli diagram for $r \leq 5$: labels

immediately checked that, when $r \geq 3$, this cannot happen for generic α, and this concludes the proof. $\qquad \square$

This theorem implies the following.

Corollary 4.2. *For generic values of α, the monodromy morphisms $\mathbb{C}((h))P_n \to LG_n(\alpha)$ are surjective, where P_n denotes the pure braid group on n strands.*

Proof. The monodromy construction provides a morphism $AP_n \to LG_n(\alpha)$ where $A = \mathbb{C}[[h]]$. The image of P_n is generated by elements whose image is $1 + 2h\Omega_{ij} +$ higher terms . The conclusion is then an elementary application of Nakayama's lemma. $\qquad \square$

Remark 4.3. *In the case of ordinary Lie algebras, the list of cases for which we have a similar property has been obtained in* [LZ].

Since, for generic $\alpha \in \Bbbk$, the Bratteli diagram does not depend on α, we can define a generic version of the split semisimple (tower of) algebra(s) $LG_n(\alpha)$, defined over an arbitrary field K, that we denote LG_n. By the R-matrix construction, we get a morphism $KB_n \to LG_n$, with K as in section 3. By the Kohno-Drinfeld theorem of N. Geer, this morphism is conjugated, up to a convenient embedding $K \hookrightarrow \mathbb{C}((h))$, to the monodromy morphisms considered above. This embedding maps $q \mapsto e^{-h}$, $t_0 \mapsto q^{-2\alpha}$, $t_1 \mapsto q^{2(\alpha+1)}$, $a \mapsto (-1)q^{2\alpha(\alpha+1)} = -\exp(-2h\alpha(\alpha+1))$, $b = -at_0 \mapsto \exp(-2h\alpha^2)$, $c = -at_1 \mapsto \exp(-2h(\alpha+1)^2)$.

The previous corollary thus implies the following one.

Corollary 4.4. *The morphism $KP_n \to LG_n$ is surjective, where P_n denotes the pure braid group on n strands.*

5. Cubic Hecke algebras and their representations

5.1. **Definition and general properties.** Let K be a field of characteristic 0, and $a, b, c \in K^\times$. The cubic Hecke algebra $H_n = H_n(a, b, c)$ is the quotient of the group algebra KB_n of
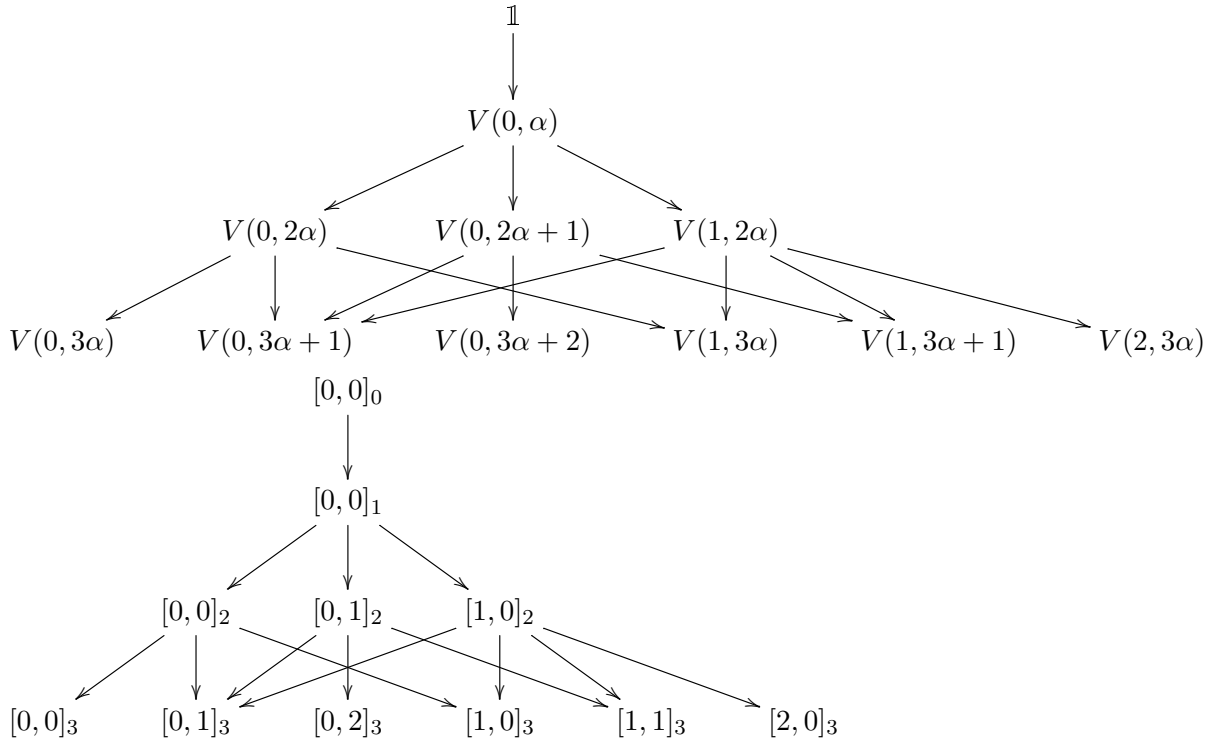
FIGURE 3. Generic encoding of the $\mathfrak{sl}(2|1)$-modules on the Bratteli diagram for $r \leq 3$.
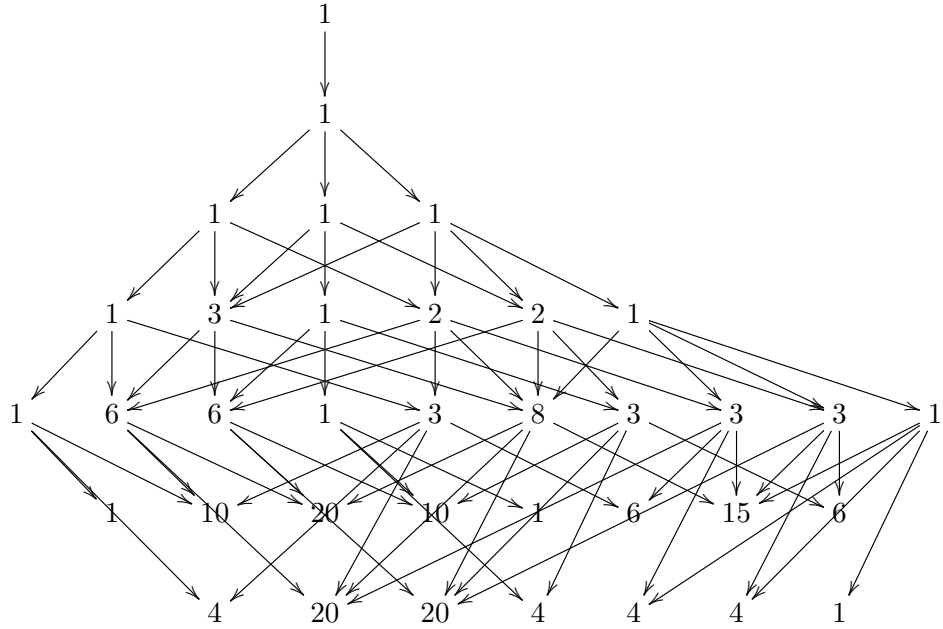

TABLE 2. Bratteli diagram for $r \leq 5$: dimensions

the braid group by the relations $(s_i - a)(s_i - b)(s_i - c) = 0$ or, equivalently, by the single relation $(s_1 - a)(s_1 - b)(s_1 - c) = 0$, since all s_i's are conjugated in B_n. In case a, b, c are three distinct roots of 1, H_n is the group algebra of the group $\Gamma_n = B_n/s_i^3 = B_n/s_1^3$, which is known to be finite if and only if $n \leq 5$, by a theorem of Coxeter (see [Co]).

We recall from [M5] the following theorem

Theorem 5.1. *Let $\overline{K}$ denote the algebraic closure of K. If a, b, c are algebraically independant over $\mathbb{Q}$, then $H_n \simeq \overline{K}\Gamma_n$ for $n \leq 5$. Moreover, these isomorphisms can be chosen so that the natural diagrams commute*

$$
\begin{array}{ccccccc}
H_2 & \longrightarrow & H_3 & \longrightarrow & H_4 & \longrightarrow & H_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{K}\Gamma_2 & \longrightarrow & \overline{K}\Gamma_3 & \longrightarrow & \overline{K}\Gamma_4 & \longrightarrow & \overline{K}\Gamma_5
\end{array}
$$

and they are uniquely defined up to inner automorphism. In particular the correspondence between irreducible representations of $\overline{K}\Gamma_n$ and H_n is canonical.

A consequence of this theorem is that the algebras H_n are semisimple, and thus isomorphic (over $\overline{K}$) to a direct sum of matrix algebras, each of the matrix algebras corresponding to an irreducible character of Γ_n.

Note that, inside $LG_n(\alpha)$, s_1 acts as a conjugate of $\tau \exp h\Omega_{12}$, with τ, Ω_{12} as in section 2.5, and thus it acts semisimply with eigenvalues $b = \exp -2h\alpha^2, c = \exp -2h(\alpha + 1)^2, a = -\exp -2h\alpha(\alpha + 1)$. For generic values, $1, \alpha$ and α^2 are linearly independent over $\mathbb{Q}$, and thus these eigenvalues are algebraically independent. We can thus consider LG_n as a quotient of H_n, that is $LG_n = H_n/\mathfrak{I}_n$ for an ideal $\mathfrak{I}_n$ over H_n. By the above theorem, $\mathfrak{I}_n$ is itself a sum of matrix algebras, and is uniquely determined by the irreducible representations of H_n which are not annihilated by $\mathfrak{I}_n$.

Recall from [M5] that $H_3 = H_2 + H_2 s_2 H_2 + H_2 s_2^{-1} H_2 + H_2 s_2 s_1^{-1} s_2$, that is $H_3 = H_2 + H_2 s_2 H_2 + H_2 s_2^{-1} H_2 + K s_2 s_1^{-1} s_2 + K s_1 s_2 s_1^{-1} s_2 + K s_1^{-1} s_2 s_1^{-1} s_2$. In particular a Markov trace on H_3 is uniquely determined by its value on H_2 and its value on $s_1^{-1} s_2 s_1^{-1} s_2$, since $tr((s_1 s_2 s_1^{-1}) s_2) = tr((s_2 s_1 s_2) s_1^{-1}) = tr(s_1 s_2 s_1 s_1^{-1}) = tr(s_1 s_2)$. Notice that the value on $s_1^{-1} s_2 s_1^{-1} s_2$ is the value of the corresponding link invariant on the figure-eight knot 4_1.

5.2. **The cubic algebra on 3 strands and Ishii's relations.** We recall from [Co] that Γ_3 is a semidirect product $Q_8 \rtimes (\mathbb{Z}/3)$, where Q_8 is a quaternion group of order 8. Its deformation H_3 admits

- three 1-dimensional representations S_x for $x \in \{a, b, c\}$, defined by $s_i \mapsto x$.
- three 2-dimensional representations $T_{x,y}$ indexed by the subsets $\{x, y\} \subset \{a, b, c\}$ of cardinality 2,
- one 3-dimensional representation V.

The spectrums of the generators are $\mathrm{Sp}(S_x(s_i)) = \{x\}$, $\mathrm{Sp}(T_{x,y}(s_i)) = \{x, y\}$, $\mathrm{Sp}(V(s_i)) = \{a, b, c\}$. In particular, these irreducible representations of B_3 (or H_3) can be uniquely identified through their restriction to B_2. One has matrix models over K of these irreducible representations (see below), hence H_3 is split semisimple, and we have

$$H_3 \simeq K \times K \times K \times M_2(K) \times M_2(K) \times M_2(K) \times M_3(K)$$

and every ideal of H_3 is a sum of some of these matrix algebras. For instance, the ideal corresponding to the BMW algebra has the form $K \times M_2(K) \times M_3(K)$, whereas the ideal

corresponding to the Funar-Bellingeri quotient of [BF] (see also [CM], [M5]) is $Z(H_3) = K \times K \times K$. The mere fact that $\dim LG_3 = 20 = 24 - 2^2$ imposes that the ideal defining LG_3 as a quotient of H_3 is one of the $M_2(K)$, precisely the only one which does not correspond to a representation of LG_3. From the Bratteli diagram of $LG_3(\alpha)$ we get that our choice of parameters imposes that it is $T_{b,c}$. Since it is a simple ideal, it is clearly generated by either one of the non-trivial Ishii's relation of [I2]. The fact that at least one of Ishii's relation is non trivial in H_3 is easily checked, and can be used to provide another quick proof of theorem 1 of [I2]. We choose one of these non-trivial relations and call it r_2.

5.3. **Description of the representations of H_4.** A description of the irreducible representations of H_4 can be found in [M1]. We use the same notation here. There are

- three 1-dimensional representations S_x for $x \in \{a, b, c\}$, defined by $s_i \mapsto x$.
- three 2-dimensional representations $T_{x,y}$ indexed by the subsets $\{x, y\} \subset \{a, b, c\}$ of cardinality 2, which factorize through the special morphism $B_4 \to B_3$ (hence through H_3).
- one 3-dimensional representation V, factorizing through B_3.
- six 3-dimensional representations $U_{x,y}$ for each tuple (x, y) with $x \neq y$ and $x, y \in \{a, b, c\}$.
- six 6-dimensional representations $V_{x,y,z}$ for each tuple (x, y, z) with $\{x, y, z\} = \{a, b, c\}$
- three 8-dimensional representations W_x for $x \in \{a, b, c\}$
- two 9-dimensional representations X, X'.

Except for X, X', they are uniquely defined by their restriction to B_3. If one let $S_x, T_{x,y}, V$ also denote the (irreducible) restriction to B_4 of the representations $S_x, T_{x,y}, V$, we have

$$
\begin{array}{rcl}
\mathrm{Res}U_{x,y} & = & S_x + T_{x,y} \\
\mathrm{Res}V_{x,y,z} & = & S_x + T_{x,y} + V \\
\mathrm{Res}W_x & = & S_x + T_{x,y} + T_{x,z} + V \\
\mathrm{Res}X & = & T_{x,y} + T_{x,z} + T_{y,z} + V
\end{array}
$$

The representations $T_{x,y}$ of B_3 are well-determined by their restriction to B_2 : with obvious notations, $Res_{B_2}T_{x,y} = S_x + S_y$.

A complete set of matrices for these representations was first found by Broué and Malle in [BrMa]. Other constructions were subsequently given, in [M1] and [MM]. The latter ones have been included in the development version of the CHEVIE package for GAP3, and the order in which they are stored in this package at the present time is $S_a, S_b, S_c, T_{b,c}, T_{a,b}, T_{a,c}, V$, $U_{b,a}, U_{a,c}, U_{c,b}, U_{c,a}, U_{a,b}, U_{b,c}, V_{c,a,b}, V_{b,c,a}, V_{a,b,c}, V_{b,a,c}, V_{c,b,a}, V_{a,c,b}, W_a, W_c, W_b, X, X'$.

For the convenience of the reader, we provide these models in tables 3 and 4.

5.4. **Representations of H_4 which factorize through LG_4.** Since LG_3 is the quotient of H_3 by the ideal corresponding to its representation $T_{b,c}$, It follows that the quotient of the group algebra of B_4 by the cubic relation and (r_2) can be identified with the quotient of H_4 by the ideal J corresponding to all irreducible representations whose restriction to H_3 contains an irreducible component of type $T_{b,c}$. Viewed the other way round, this proves that the quotient algebra H_4/J is a semisimple algebra whose irreducible representations are the irreducible representations of H_4 whose restriction to H_3 do *not* contain an irreducible component of type $T_{b,c}$.

From the description of the branching rule we get that these representations are the S_x for $x \in \{a, b, c\}$, $T_{a,b}, T_{a,c}, V, U_{a,b}, U_{b,a}, U_{a,c}, U_{c,a}, V_{a,b,c}, V_{b,a,c}, V_{a,c,b}, V_{c,a,b}, W_a$, hence the

$$T_a : s_1 \mapsto \begin{pmatrix} b & 0 \\ bc & c \end{pmatrix} \quad s_2 \mapsto \begin{pmatrix} c & -1 \\ 0 & b \end{pmatrix} \quad s_3 \mapsto \begin{pmatrix} b & 0 \\ bc & c \end{pmatrix}$$

$$V : s_1 \mapsto \begin{pmatrix} c & 0 & 0 \\ ac+b^2 & b & 0 \\ b & 1 & a \end{pmatrix} \quad s_2 \mapsto \begin{pmatrix} a & -1 & b \\ 0 & b & -ac-b^2 \\ 0 & 0 & c \end{pmatrix} \quad s_3 \mapsto \begin{pmatrix} c & 0 & 0 \\ ac+b^2 & b & 0 \\ b & 1 & a \end{pmatrix}$$

$$U_{a,b} : s_1 \mapsto \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ a & 0 & a \end{pmatrix} \quad s_2 \mapsto \begin{pmatrix} a & -a+b & -b \\ 0 & b & -b \\ 0 & 0 & a \end{pmatrix} \quad s_3 \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ -a & 2a & b \end{pmatrix}$$

TABLE 3. Representations of H_4, first part.

$V_{a,b,c}$

$$\begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
ac+b^2 & b & 0 & 0 & 0 & 0 \\
b & 1 & c & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 1 & 0 & b
\end{pmatrix}$$

$$\begin{pmatrix}
c & -1 & b & -1 & -b & a \\
0 & b & -ac-b^2 & 0 & 0 & -ab \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & b & ac+b^2 & -ab \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & a
\end{pmatrix}$$

$$\begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
ac+b^2 & 0 & 0 & b & 0 & 0 \\
-b & 0 & 0 & -1 & c & 0 \\
0 & 1 & 0 & 0 & 0 & b
\end{pmatrix}$$

W_a

$$\begin{pmatrix}
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
c & ab+c^2 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
-1 & -c & 0 & b & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & b & 0 & -1 & a & 0 \\
0 & 0 & 0 & 0 & 0 & b-c & 0 & a
\end{pmatrix}$$

$$\begin{pmatrix}
a & -ab-c^2 & c & 0 & 0 & 0 & 0 & 0 \\
0 & c & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & -a & 0 & a & -a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\
0 & 0 & -a & 0 & -ac & a & ac & 0 \\
0 & 0 & -1 & 0 & b-c & 0 & c & 0 \\
0 & 0 & -c & 0 & bc-c^2 & 0 & ab+c^2 & a
\end{pmatrix}$$

$$\begin{pmatrix}
a & 0 & 0 & bc-c^2 & 0 & 0 & 0 & 0 \\
0 & a & 0 & c & 0 & -1 & 0 & 0 \\
0 & 0 & a & 0 & 0 & b-c & ac+c^2 & -c \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 1 & -a & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b
\end{pmatrix}$$

X

$$\begin{pmatrix}
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
ac+b^2 & b & 0 & 0 & 0 & 0 & -j^2bc & 0 & 0 \\
b & 1 & a & 0 & 0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & -c & jc & a+j^2b \\
j^2a-b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
j^2a & 0 & 0 & 0 & b & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & j^2c & b
\end{pmatrix}$$

$$\begin{pmatrix}
a & -1 & b & -jb & 0 & 0 & 0 & 0 & b \\
0 & b & -ac-b^2 & -ac+jb^2 & 0 & 0 & 0 & 0 & -ac-b^2 \\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & a & a & -a & 0 & 0 & 0 \\
0 & 0 & jb & 0 & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & a & a & a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & -jb \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c
\end{pmatrix}$$

$$\begin{pmatrix}
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
ac+b^2 & b & 0 & 0 & -j^2bc & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & -j^2c & 0 & 0 & 0 & 0 \\
0 & 0 & a+j^2b & a & -c & -jc & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\
j^2a-b & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\
-j^2a & 0 & 0 & 0 & 0 & 0 & -b & a & 0 \\
b & 1 & 0 & 0 & c & 0 & 0 & 0 & a
\end{pmatrix}$$

Images of s_1, s_2, s_3 (from top to bottom) under $V_{a,b,c}$, W_a and X.
In these formulas, j denotes a third root of 1.

TABLE 4. Representations of H_4, second part.

corresponding quotient has dimension $1^2 \times 3 + 2^2 \times 2 + 3^2 + 3^2 \times 4 + 6^2 \times 4 + 8^2 \times 1 = 264$. From the Bratteli diagrams of section 4 we can compute $\dim LG_n$ for arbitrary n, and we get that LG_4 has dimension 175. This proves that *Ishii's relations are not sufficient* to define LG_n. More precisely :

Proposition 5.2. *The quotient of KB_4 by a generic cubic relation r_1 and Ishii's relation r_2 has dimension* $\dim KB_4/(r_1, r_2) = \dim H_4/(J) = 264 > \dim LG_4 = 175$.

Remark 5.3.

   (i) *The following argument, by contradiction, could alternatively be used to show that $KB_4/(J) \neq LG_4$. Up to some permutation of the parameters, BMW_3 is a quotient of LG_3. Since the relations of BMW_n are on 3 strands, this implies that BMW_4 is a quotient of $KB_4/(J)$. But it is not a quotient of LG_4, and actually even the (ordinary, quadratic) Hecke algebra on 4 strands is not a quotient of LG_4 ; this can indeed be inferred from the fact that the Bratteli diagram for the representation of the Hecke algebra traditionnally indexed by the partition $[2, 2]$ does not appear as a sub-diagram inside the Brattelli diagram of LG_4.*
   (ii) *By the same method, one can get the dimension of $KB_5/(r_1, r_2)$. It is 6490.*

We now explain how to get a matrix description of LG_4. Since each irreducible representation of H_4 is well-determined from its restriction to H_3, the Bratteli diagram obtained before determines the irreducible representation of LG_4 ; they are the $S_a, S_b, S_c, U_{b,a}, U_{a,c}, U_{c,a}, U_{a,b}, V_{c,a,b}, V_{b,a,c}, W_a$. This provides an explicit morphism $\Phi : KB_4 \twoheadrightarrow K^3 \times M_3(K)^4 \times M_6(K)^2 \times M_8(K) \simeq K^{175}$ factorizing though H_4 whose image can be identified with LG_4.
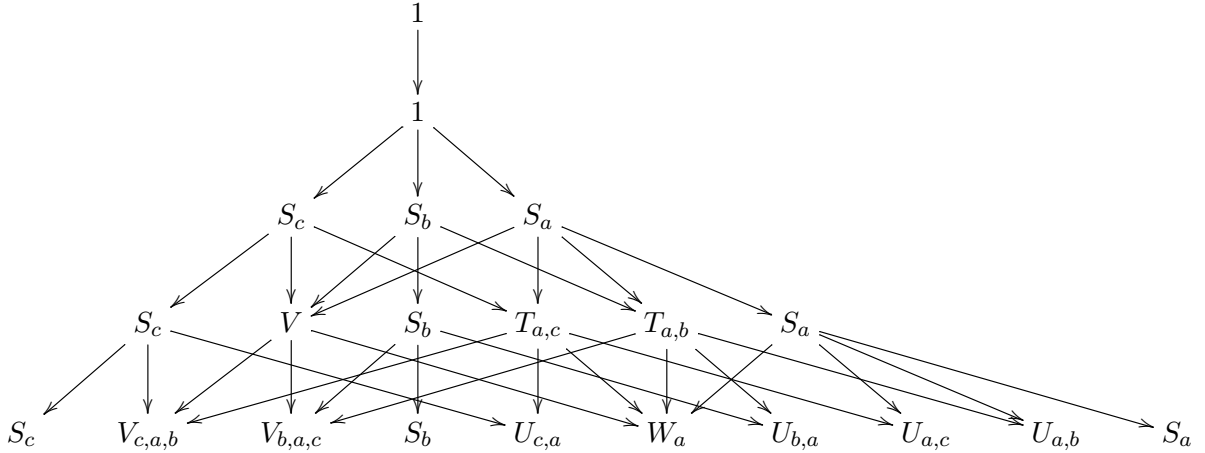
In the forecoming section we will use this matrix description to get a inductive properties of this algebra. Before that, we conclude the present section by describing the part of LG_n which factorizes through ordinary quadratic Hecke algebras.

There is clearly one quotient of LG_n which factors through a quadratic Hecke algebra for each irreducible component $[u, v]_2$ of $V(0, \alpha)^{\otimes 2}$: if $\langle u, v \rangle_2$ denotes the simple ideal of LG_2 corresponding to it, then $LG_n/\langle u, v \rangle_2$ is a quotient of a quadratic Hecke algebra. Any quotient of a quadratic (generic) Hecke algebra is uniquely determined by its irreducible representations, namely the irreducible representations of the Hecke algebra (indexed as usual by partitions of n) which factor through it. It follows that this quotient can be uniquely

identified by its Bratteli diagram, which is the part of the Bratteli diagram of LG_n made of the paths which do *not* pass through $[u, v]_2$. When $[u, v]_2 = [1, 0]_2$ we easily get from this description that the irreducible representations of $LG_n/\langle 1, 0 \rangle_2$ are the 1-dimensional representations corresponding to $[0, 0]_n$ and $[0, n-1]_n$, hence $LG_n/\langle 1, 0 \rangle_2 \simeq K^2$. Under the correspondence between irreducible representations of the quadratic Hecke algebras and partitions, which is canonical up to the operation of transposing all Young diagrams at the same time, we have for instance $[0, 0]_n \mapsto [n]$, $[0, n-1]_n \mapsto [1^n]$. When $[u, v]_2 = [0, 0]_2$, we get that the irreducible representations of $LG_n/\langle 0, 0 \rangle_2$ are the $[n-m, m]_n$ for $1 \leq m \leq n$, and that the correspondence with partitions can be chosen to be $[n-m, m]_n \mapsto [n-m+1, 1^{m-1}]$. The situation is similar with $[u, v]_2 = [0, 1]_2$, the correspondence being given by $[m, 0]_n \mapsto [n-m, 1^m]$. Since the quotient of the quadratic Hecke algebra whose irreductible representations are indexed by hook partitions, which is a centralizer algebra for $\mathfrak{gl}(1|1)$ inside $\mathrm{End}(U^{\otimes n})$ with U its standard 2-dimensional representation (see e.g.[BeMo]), is a defining algebra for the Alexander polynomial (see [RoSa, V]), this gives another explanation for the well-known connection between the Links-Gould and Alexander polynomials (see [I3, GP]).

6. Inductive description of LG_n

We investigate the image of braid words in the algebras LG_3 and LG_4. We first determine a basis made of braid words for LG_3 (section 6.1), then of LG_4 (section 6.2). This first basis enables us to get a new relation in LG_4. We now start again from the basis of LG_3 obtained earlier and get in section 6.3 a new basis which is a more suitable for induction. We use it to prove a decomposition of LG_4 as a LG_3-bimodule, and then more generally, in section 6.4, a decomposition of LG_{n+1} as a LG_n-bimodule.

6.1. **A basis for** $n = 3$. The braid words with at most 3 crossings and avoiding the pattern s_i^r for $|r| \geq 2$ are 1, $s_1^{\pm 1}$, $s_2^{\pm 1}$, $s_1^{\pm 1} s_2^{\pm 1}$, $s_2^{\pm 1} s_1^{\pm 1}$, $s_1^{\pm 1} s_2^{\pm 1} s_1^{\pm 1}$, $s_2^{\pm 1} s_1^{\pm 1} s_2^{\pm 1}$, that is 29 words, whose image span LG_3. Among them, there are 13 words ($13 = 1 + 4 + 2 \times 2 + 2 \times 2$) with at most two crossings. These 13 words have linearly independent images in LG_3. We denote $\mathcal{B}_0 = \mathcal{B}_0^{(3)}$ these images, V_0 the subspace of LG_3 that they span (see table 5).

$$\mathcal{B}_0 = [1, s_1, s_1^{-1}, s_2, s_2^{-1}, s_1 s_2, s_1 s_2^{-1}, s_1^{-1} s_2, s_1^{-1} s_2^{-1}, s_2 s_1, s_2 s_1^{-1}, s_2^{-1} s_1, s_2^{-1} s_1^{-1}]$$

We let V_1 denote the span of the 8 words of the form $s_1^{\pm 1} s_2^{\pm 1} s_1^{\pm 1}$. We have $\dim V_0 + V_1 = 19$. Somewhat arbitrarily, we choose for basis of LG_3 the following set (see table 6)

$$\mathcal{B}^{(3)} = \mathcal{B} = \mathcal{B}_0 \sqcup [s_1^{-1} s_2^{-1} s_1^{-1}, s_1^{-1} s_2^{-1} s_1, s_1^{-1} s_2 s_1^{-1}, s_1 s_2^{-1} s_1^{-1}, s_1 s_2^{-1} s_1, s_1 s_2 s_1^{-1}, s_2^{-1} s_1 s_2^{-1}];$$

This basis is made of all possible braid words with at most 2 crossings with in addition 6 of the 8 words of the form $s_1^{\pm 1} s_2^{\pm 1} s_1^{\pm 1}$ (the words $s_1 s_2 s_1$ and $s_1^{-1} s_2 s_1$ are removed), which together form a basis of $V_0 + V_1$, and one of the form $s_2^{\pm 1} s_1^{\pm 1} s_2^{\pm 1}$. Since, according to [M5], the words $s_1^{\pm} s_2^{\pm} s_1^{\pm}$ are linearly independent in H_3, there should be a relation arising from the expression of $s_1 s_2 s_1$ and $s_1^{-1} s_2 s_1$.

We get the following :

$$\begin{aligned}
s_1^{-1} s_2 s_1 &= \tfrac{-1}{a} s_1 s_2 + a s_1 s_2^{-1} + a s_1^{-1} s_2 - a^3 s_1^{-1} s_2^{-1} + a^{-1} s_2 s_1 - a s_2 s_1^{-1} - a s_2^{-1} s_1 + a^3 s_2^{-1} s_1^{-1} \\
&\quad + a^2 s_1^{-1} s_2^{-1} s_1 - a^2 s_1 s_2^{-1} s_1^{-1} + s_1 s_2 s_1^{-1}
\end{aligned}$$

so, in particular, $s_1^{-1} s_2 s_1 \equiv s_1 s_2 s_1^{-1} + a^2 (s_1^{-1} s_2^{-1} s_1 - s_1 s_2^{-1} s_1^{-1}) \mod V_0$, that is
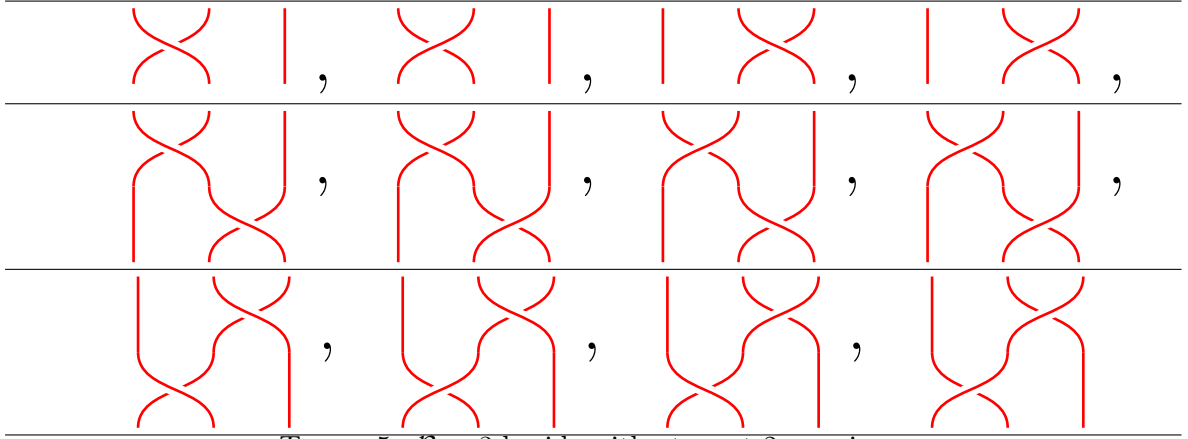
TABLE 5. $\mathcal{B}_0$: 3-braids with at most 2 crossings.
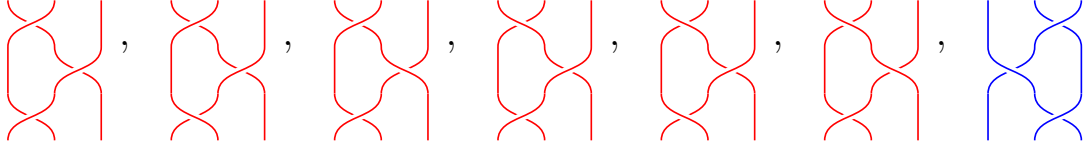
TABLE 6. The $6 + 1$ braid words added to $\mathcal{B}_0$ to form $\mathcal{B}$

We also have

$$
\begin{aligned}
s_1 s_2 s_1 \;=\; & \left(-ab - ac - bc - a^2\right) s_2 + \left(a^2 bc + a^3 b + a^3 c + a^4\right) s_2^{-1} + (a + b + c)\, s_1 s_2 \\
& + \left(-a^2 b - a^2 c - a^3\right) s_1 s_2^{-1} + (abc)\, s_1^{-1} s_2 + \left(-a^3 bc\right) s_1^{-1} s_2^{-1} + (a)\, s_2 s_1 \\
& + \left(abc + a^2 b + a^2 c\right) s_2 s_1^{-1} + \left(-a^3\right) s_2^{-1} s_1 + \left(-a^3 bc - a^4 b - a^4 c\right) s_2^{-1} s_1^{-1} \\
& + \left(a^4 bc\right) s_1^{-1} s_2^{-1} s_1^{-1} + \left(-a^2 bc\right) s_1^{-1} s_2 s_1^{-1} + \left(a^3 b + a^3 c\right) s_1 s_2^{-1} s_1^{-1} \\
& + \left(a^2\right) s_1 s_2^{-1} s_1 + (-ab - ac)\, s_1 s_2 s_1^{-1}
\end{aligned}
$$

hence

$$
s_1 s_2 s_1 \equiv a^3 (abc) s_1^{-1} s_2^{-1} s_1^{-1} - a(abc) s_1^{-1} s_2 s_1^{-1} + a^3 (b+c) s_1 s_2^{-1} s_1^{-1} + a^2 s_1 s_2^{-1} s_1 - a(b+c) s_1 s_2 s_1^{-1} \quad \text{mod } V_0
$$

6.2. **Case $n = 4$: new relations over 4 crossings.** We denote $\mathcal{B}_0^{(4)}$ the set of $1 + 2 \times 3 + 2^2 \times 5 = 27$ braids with at most 2 crossings, which correspond to the patterns described in table 7 (plus the trivial braid, with 0 crossings), $\mathcal{B}_1^{(4)}$ the set of braids with 3 crossings described by the patterns of table 8. Because of the study of LG_3 and of the cubic relation, the image in LG_4 of every braid with at most 3 crossings can be written as a linear combination of $\mathcal{B}_0^{(4)} \sqcup \mathcal{B}_1^{(4)}$. We check that these $8 \times 4 + 2 \times 7 + 27 = 73$ elements are indeed linearly independent in LG_4, and we let V_3 denote the subspace they span.
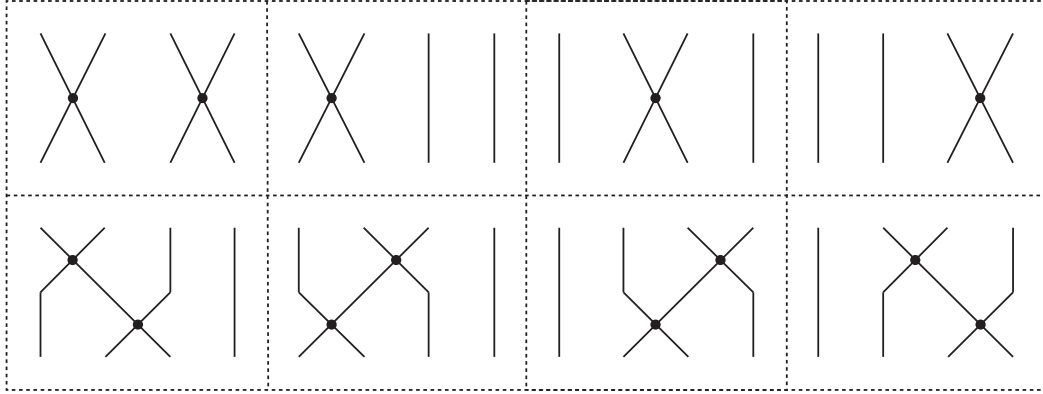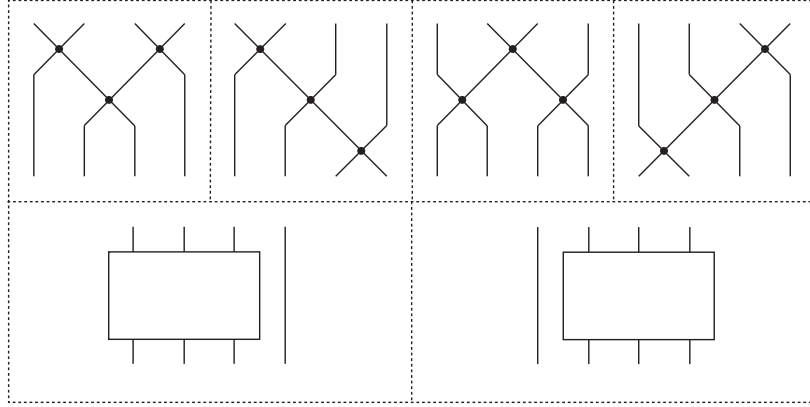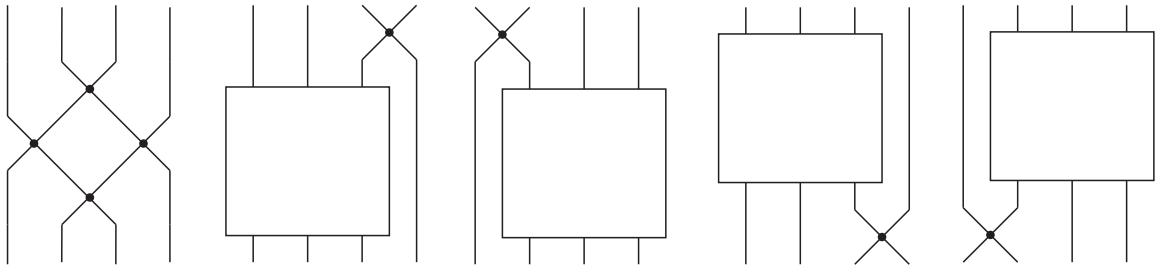
TABLE 7. Patterns for 1 and 2 crossings, on 4 strands

The box represents arbitrary elements of $\mathcal{B}^{(3)} \setminus \mathcal{B}_0^{(3)}$.
TABLE 8. Patterns for 3 crossings, on 4 strands.

The box represents arbitrary elements of $\mathcal{B}^{(3)} \setminus \mathcal{B}_0^{(3)}$.
TABLE 9. Patterns for 4 crossings, on 4 strands.

We denote $\mathcal{B}_2^{(4)}$ the set of $2^4 + 7 \times 2 \times 4 = 72$ braids with 4 crossings, which correspond to the patterns described in table 9. Their images in LG_4 are linearly independent, that is they span a subspace V_4 of dimension 72. We get $\dim(V_3 + V_4) = 141 < 145 = 72 + 73$.

Among these 145 words in s_i and s_i^{-1}, only 5 of them contain a pattern $s_3^{\pm} s_2^{\pm} s_3^{\pm}$. Besides $s_3^{-1} s_2 s_3^{-1}$, they are the $\mathcal{A}_{3,U}^{(4)} = \{s_1^{\varepsilon} s_3^{-1} s_2 s_3^{-1} \mid \varepsilon \in \{-1,1\}\}$ and $\mathcal{A}_{3,D}^{(4)} = \{s_3^{-1} s_2 s_3^{-1} s_1^{\varepsilon} \mid \varepsilon \in \{-1,1\}\}$.

Let $\mathcal{B}_3^{(4)} = (\mathcal{B}_0^{(4)} \sqcup \mathcal{B}_1^{(4)} \sqcup \mathcal{B}_2^{(4)}) \setminus (\mathcal{A}_{3,D}^{(4)} \cup \mathcal{A}_{3,U}^{(4)})$, and $V_4^{(0)}$ the subspace of LG_4 spanned by its image. We let $\mathcal{B}_{3,D}^{(4)} = \mathcal{B}_3^{(4)} \sqcup \mathcal{A}_{3,D}^{(4)}$, $\mathcal{B}_{3,U}^{(4)} = \mathcal{B}_3^{(4)} \sqcup \mathcal{A}_{3,U}^{(4)}$, and $V_4^{(D)}$, $V_4^{(U)}$ the subspace spanned by their images, respectively.

We have $|\mathcal{B}_3^{(4)}| = 141$ and $\dim V_4^{(0)} = 139$. Moreover, we get $\dim V_4^{(D)} = \dim V_4^{(U)} = 141$ hence $V_4^{(D)} = V_4^{(U)} = V_3 + V_4$. We choose for basis of this subspace a rather arbitrary subset of $\mathcal{B}_{3,D}^{(4)}$ of size 141, namely $\mathcal{B}_4^{(4)} = \mathcal{B}_{3,D}^{(4)} \setminus \{s_1^{-1} s_2^{-1} s_3^{-1} s_2^{-1}, s_1 s_2^{-1} s_3^{-1} s_2^{-1}\}$, which we check to be linearly independent.

In terms of the combinatorics of words, this enables one to pass a $s_1^{\pm}$ from over to under a pattern $s_3^{\pm} s_2^{\pm} s_3^{\pm}$, modulo terms containing no more than one $s_3^{\pm}$.

Drop this. Keep this.
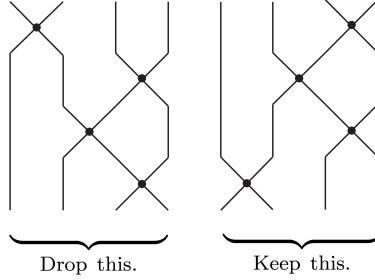
More precisely, by explicitly computing in this basis of $V_3 + V_4$, we get the following

$$
\begin{aligned}
s_1^{-1}(s_3^{-1} s_2 s_3^{-1}) &\equiv (s_3^{-1} s_2 s_3^{-1}) s_1^{-1} && \mod LG_3 s_3 LG_3 + LG_3 s_3^{-1} LG_3 + LG_3 \\
s_1(s_3^{-1} s_2 s_3^{-1}) &\equiv (s_3^{-1} s_2 s_3^{-1}) s_1 && \mod LG_3 s_3 LG_3 + LG_3 s_3^{-1} LG_3 + LG_3
\end{aligned}
$$

We let $F^{\pm} \in H_4$ denote the image in H_4 of the expression of $s_1^{\pm}(s_3^{-1} s_2 s_3^{-1}) - (s_3^{-1} s_2 s_3^{-1}) s_1^{\pm} - L^{\pm}$ with L^+, L^- are the only linear combinations of the words of $\mathcal{B}_4^{(4)}$ such that $F^+, F^- \mapsto 0$ in LG_4. We claim that $LG_4 = H_4/(r_2, F^+, F^-)$, but we actually prove more. Indeed, we check that both these elements are non-zero exactly in all the irreducible representations of H_4 which do not factorize through LG_4. This has for immediate consequence the following

Proposition 6.1. $LG_4 = H_4/(F^+) = H_4/(F^-)$.

We denote r_3 the lifting in KB_4 of one of the relations F^+ and F^-, so that $LG_4 = KB_4/(r_1, r_2, r_3) = KB_4/(r_1, r_3)$.
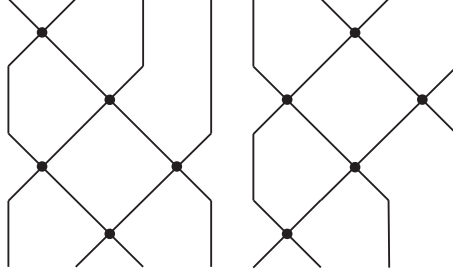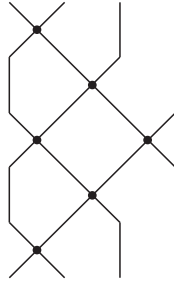
FIGURE 4. Patterns with 5 crossings

FIGURE 5. Pattern with 6 crossings

6.3. Case $n = 4$: a spanning set suitable for induction. We start again from the collection $\mathcal{B}_3^{(4)}$, from which we remove the words $\{s_1^{-1}s_2^{-1}s_3^{-1}s_2^{-1}, s_1 s_2^{-1}s_3^{-1}s_2^{-1}\}$, and thus get a family $\mathcal{B}_{3,0}^{(4)}$ of 139 elements. Their image in LG_4 is linearly independent. The words contained in $\mathcal{B}_{3,0}^{(4)}$ all contain at most one crossing $s_3^{\pm}$ between the third and fourth strand, except for the word $s_3^{-1}s_2 s_3^{-1}$. We want to expand this family into a basis of LG_4 which has the same property. For this, we add braids with 5 and 6 crossings.

We first add the 48 words corresponding to the patterns $s_1^{\alpha}s_2^{\beta}s_1^{\gamma}s_3^{\delta}s_2^{\varepsilon}$ and $s_2^{\delta}s_3^{\varepsilon}s_1^{\alpha}s_2^{\beta}s_1^{\gamma}$ with $\alpha, \beta, \gamma, \delta, \varepsilon \in \{-1, 1\}$, which are described in figure 4, with the additional property that we avoid the subwords $s_1^{\alpha}s_2^{\beta}s_1^{\gamma} \in \{s_1 s_2 s_1, s_1^{-1}s_2 s_1\}$. This means that the subwords $s_1^{\alpha}s_2^{\beta}s_1^{\gamma}$ must belong to the image of $\mathcal{B}^{(3)}$; in other words, we avoid the patterns $s_1 s_2 s_1$ and $s_1^{-1}s_2 s_1$. This indeed provides $2 \times 2^2 \times 6 = 48$ new words.

We thus get a family $\mathcal{B}_5^{(4)}$ of size $139 + 48 = 187 > 175 = \dim LG_4$. The linear span of $\mathcal{B}_5^{(4)}$, which we denote V_5, actually has dimension 168. We first look for braids with 6 crossings which complete the previous family into a spanning set of LG_4. We consider the pattern described in figure 5. We are looking for a set of 7 braids whose image complete the spanning of LG_4. A suitable set is given by the following family

$$\mathcal{A}_6^{(4)} = \{ \quad s_1 s_2^{-1} s_1 s_3^{-1} s_2 s_1, s_1 s_2^{-1} s_1 s_3 s_2^{-1} s_1, s_1 s_2 s_1 s_3^{-1} s_2^{-1} s_1, s_1 s_2 s_1 s_3^{-1} s_2 s_1, s_1 s_2 s_1 s_3 s_2^{-1} s_1,$$
$$s_1^{-1} s_2 s_1^{-1} s_3^{-1} s_2 s_1^{-1}, s_1^{-1} s_2 s_1^{-1} s_3 s_2^{-1} s_1^{-1} \}$$
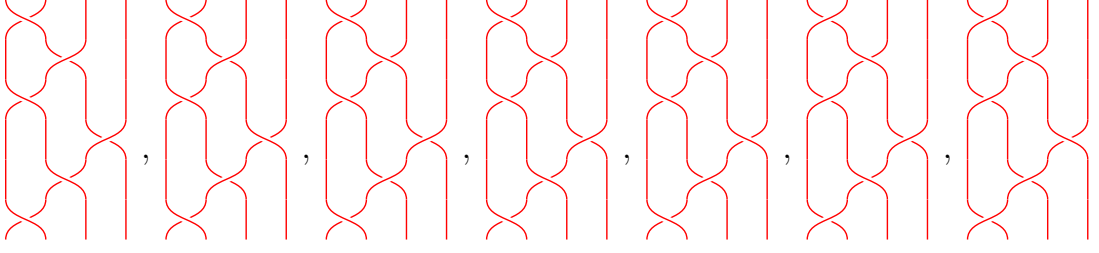
We now select a suitable subset of the already chosen 48 words which correspond to the patterns described in figure 4. We keep all the braids corresponding to the pattern on the right-hand side of figure 4 *but* the words

$$\mathcal{D}_{5,R}^{(4)} = \{s_2 s_3 s_1^{-1} s_2^{-1} s_1^{-1}, s_2 s_3 s_1^{-1} s_2^{-1} s_1, s_2 s_3 s_1 s_2^{-1} s_1^{-1}, s_2 s_3 s_1 s_2 s_1^{-1}\}.$$

We denote $\mathcal{A}_{5,R}^{(4)}$ the corresponding set of $24 - 4 = 20$ words. Finally, among the 24 words corresponding to the pattern on the left-hand side of figure 4, we keep only the 9 words

$$\begin{aligned}
\mathcal{A}_{5,L}^{(4)} = \quad &\{s_1^{-1} s_2^{-1} s_1^{-1} s_3 s_2, s_1^{-1} s_2^{-1} s_1 s_3 s_2, s_1^{-1} s_2 s_1^{-1} s_3 s_2, s_1 s_2^{-1} s_1^{-1} s_3 s_2, s_1 s_2^{-1} s_1 s_3 s_2, \\
&s_1 s_2 s_1^{-1} s_3 s_2, s_1^{-1} s_2^{-1} s_1^{-1} s_3 s_2^{-1}, s_1^{-1} s_2 s_1^{-1} s_3 s_2^{-1}, s_1 s_2^{-1} s_1^{-1} s_3 s_2^{-1}\}.
\end{aligned}$$

The final collection $\mathcal{B}_6^{(4)} = \mathcal{B}_{3,0}^{(4)} \sqcup \mathcal{A}_{5,L}^{(4)} \sqcup \mathcal{A}_{5,R}^{(4)} \sqcup \mathcal{A}_6^{(4)}$ has cardinality 175. By computer calculation one can then check the following.

Proposition 6.2. *The collection $\mathcal{B}_6^{(4)}$ is a basis for LG_4. The collection of the $b_1 s_3^r b_1$ for $b_1, b_2 \in \mathcal{B}^{(3)}$ and $r \in \{-1, 0, 1\}$ spans in LG_4 a subset of dimension $174 = \dim LG_4 - 1$. In particular, we have*

$$LG_4 = \left(\sum_{r \in \{-1,0,1\}} LG_3 s_3^r LG_3 \right) \oplus K s_3^{-1} s_2 s_3^{-1}$$

6.4. LG_{n+1} as LG_n-bimodule.

Lemma 6.3.
(i) $LG_{n-1} s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1} LG_{n-1} \subset \sum_r LG_{n-1} s_{n-1}^r LG_{n-1} + LG_{n-3} s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}$
(ii) $LG_{n-1} s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1} LG_{n-1} s_{n-1}^{\pm 1} \subset \sum_r LG_{n-1} s_{n-1}^r LG_{n-1} s_{n-1}^{\pm 1} + LG_{n-3} s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}$

Proof. Part (ii) is a straightforward consequence of (i) and of the choice of basis for LG_3. We prove part (i). When $n = 4$ this is a consequence of our choice of basis. Let $A = \sum_r LG_{n-1} s_{n-1}^r LG_{n-1} + LG_{n-3} s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1} \subset LG_n$. Since $s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1} \in A$, we only need to prove that A is a LG_{n-1}-submodule on both sides. Because LG_{n-3} commutes with $s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}$ this amounts to saying that A is stable by multiplication on both sides by $s_{n-2}^{\pm 1}$ and $s_{n-3}^{\pm 1}$, that is $s_{n-2}^{\pm 1}(s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}) \in A$, $s_{n-3}^{\pm 1}(s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}) \in A$, $(s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}) s_{n-2}^{\pm 1} \in A$, $(s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1}) s_{n-3}^{\pm 1} \in A$. These eight braids are conjugates of braids whose image lie in LG_4 and we can use the result for $n = 4$ and $n = 3$, that we obtained above, in order to conclude the proof.

$\square$
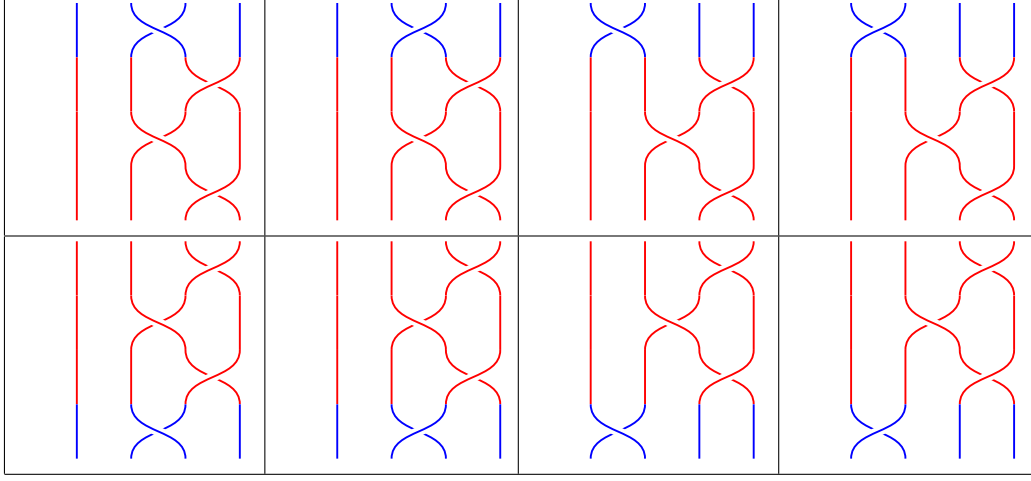
Theorem 6.4. *For $n \geq 3$ we have*

(i) $LG_n = LG_{n-1}s_{n-1}^{\pm 1}LG_{n-1} + \sum_{k+\ell=n}LG_kLG_l$

(ii) $LG_n = \sum_r LG_{n-1}s_{n-1}^rLG_{n-1} + LG_{n-3}(s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1})$

Proof. (second formula). By induction on n, the cases $n = 3$ and $n = 4$ being already done. Let $A = \sum_r LG_{n-1}s_{n-1}^rLG_{n-1} + LG_{n-3}(s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1}) \subset LG_n$. Since $1 \in A$, we only need to prove that $s_k^{\pm 1}A \subset A$ for $k \in \{n-1, n-2, n-3\}$. For $k < n-1$ this is a straightforward consequence of lemma 6.3 (i), so we can assume $k = n-1$. We have, for some u, $s_{n-1}^{\pm 1}LG_{n-3}(s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1}) = LG_{n-3}(s_{n-1}^us_{n-2}s_{n-1}^{-1}) \subset LG_{n-3}s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1} + \sum_r LG_{n-1}s_{n-1}^rLG_{n-1} \subset A$ because of the chosen basis for LG_3. We then only need to prove $s_{n-1}^{\pm}LG_{n-1}s_{n-1}^rLG_{n-1} \subset A$. We use the induction assumption $LG_{n-1} \subset \sum_{r'}LG_{n-2}s_{n-2}^{r'}LG_{n-2} + LG_{n-4}(s_{n-2}^{-1}s_{n-3}s_{n-2}^{-1})$ hence

$$
\begin{aligned}
s_{n-1}^{\pm 1}LG_{n-1}s_{n-1}^rLG_{n-1} &\subset \sum_{r'} s_{n-1}^{\pm 1}LG_{n-2}s_{n-2}^{r'}LG_{n-2}s_{n-1}^rLG_{n-1} \\
&\quad + s_{n-1}^{\pm 1}LG_{n-4}(s_{n-2}^{-1}s_{n-3}s_{n-2}^{-1})s_{n-1}^rLG_{n-1} \\
&\subset \sum_{r'} LG_{n-2}s_{n-1}^{\pm 1}s_{n-2}^{r'}s_{n-1}^rLG_{n-2}LG_{n-1} \\
&\quad + LG_{n-4}s_{n-1}^{\pm 1}(s_{n-2}^{-1}s_{n-3}s_{n-2}^{-1})s_{n-1}^rLG_{n-1}
\end{aligned}
$$

and, by the case $n = 3$, $s_{n-1}^{\pm 1}s_{n-2}^{r'}s_{n-1}^r \in \sum_u LG_{n-1}s_{n-1}^uLG_{n-1} + LG_{n-1}s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1}LG_{n-1}$, hence $LG_{n-1}s_{n-1}^{\pm 1}s_{n-2}^{r'}s_{n-1}^rLG_{n-1}$. Moreover, $s_{n-1}^{\pm 1}(s_{n-2}^{-1}s_{n-3}s_{n-2}^{-1})s_{n-1}^r \in \sum_u LG_{n-1}s_{n-1}^uLG_{n-1} + LG_{n-1}s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1}LG_{n-1}$ because of the chosen basis in LG_4, hence

$$s_{n-1}^{\pm 1}LG_{n-1}s_{n-1}^rLG_{n-1} \subset \sum_u LG_{n-1}s_{n-1}^uLG_{n-1} + LG_{n-1}s_{n-1}^{-1}s_{n-2}s_{n-1}^{-1}LG_{n-1} \subset A$$

by lemma 6.3. $\square$

7. Markov traces

Using the careful analysis of the previous section, we define a quotient A_n of the braid group algebra KB_n of n strands by a cubical relation r_1 as well as one relation r_2 on three strands and one relation r_3 on four strands. Notice that we conjecture (see conjecture 1.4) that this algebra is isomorphic for all n to the centralizer algebra LG_n.

Since $A_4 \simeq LG_4$, the proof of theorem 6.4 can be adapted immediately to yield the following statement.

Theorem 7.1. *For all $n \geq 3$,*
$$A_{n+1} = A_n + A_n s_n A_n + A_n s_n^{-1} A_n + A_{n-2} s_{n-1}^{-1} s_n s_{n-1}^{-1}.$$

This implies immediately that A_n is finite dimensional for all n. The precise dimension of A_n is the content of conjecture 1.5.

Using the methods of section 5.4, the character table of Γ_5 and the fact (see [M5]) that H_5 is a flat deformation of $K\Gamma_5$ enable us to get the list of irreducible representations of H_5 which factor through H_5. This yields $\dim A_5 = 1764 = \dim LG_5$, whence the following evidence for conjecture 1.4.

Theorem 7.2. $A_5 \simeq LG_5$.

The main result of this section is that the tower of algebras $(A_n)_{n \geq 1}$ can be endowed with a unique trace Tr_n which computes the Links-Gould invariant. In addition we prove that the relations r_1, r_2 and r_3 are a complete set of relations for the Links-Gould invariant, i.e. one can recursively compute the Links-Gould invariant using this relations.

Given an integer $n \geq 1$, consider the natural embedding of B_n into B_{n+1}. Denote by ϕ_n its extension to an homomorphism from A_n to A_{n+1}.

Theorem 7.3. *For $z \in K$, there exists a family of traces $Tr_n : A_n \to K$, $n \geq 1$, such that*
- *$Tr_{n+1}(\phi_n(\beta)) = zTr_n(\beta)$ for all $\beta \in A_n$.*
- *$Tr_n(\alpha\beta) = Tr_n(\beta\alpha)$ for all α, $\beta \in A_n$.*
- *$Tr_{n+1}(\phi_n(\beta)s_n^{\pm 1}) = Tr_n(\beta)$ for all $\beta \in A_n$ ('Markov property').*
- *$Tr_1 = 1$*

if and only if $z = 0$. If $z = 0$, this family is unique. The same statements hold with (A_n) replaced by (LG_n).

Proof. First we prove that if the trace exists then z is equal to zero. Since A_4 is semi-simple, it is isomorphic to a direct sum of ten matrix algebras and therefore a trace on A_4 is a linear combination of matrix traces. By using the Markov properties above, we get that the value of Tr_4 on the family $\mathcal{F} = (s_3, s_1 s_3, s_1^{-1} s_3, s_2 s_3, s_2^{-1} s_3, s_1 s_2 s_3, s_1 s_2^{-1} s_3, s_1^{-1} s_2 s_3, s_1^{-1} s_2^{-1} s_3, s_3^{-1})$ is $(z^2, z, z, z, z, 1, 1, 1, 1, z^2)$. On the other hand, we check by computer that the values of the 10 matrix traces on this family provide an invertible 10×10 matrix. As a consequence, the values of Tr_4 on this family determines its value on arbitrary elements of B_4, as polynomial functions of z. This enables us to compute the value of Tr_4 on $s_1^{-1} s_3^{-1}$ and $s_2^{-1} s_3^{-1}$. Since its value has to be z in both cases by the Markov property, we get two equations on z, which have the form $\alpha' z(z - \alpha) = 0$ and $\beta' z(z - \beta) = 0$ for some $\alpha, \alpha', \beta, \beta' \in K^{\times}$, with $\alpha \neq \beta$. This clearly implies $z = 0$.

Existence follows from the existence of the Links-Gould invariant and unicity from the careful analysis of the previous section. In more details, define T_n by $T_n(\beta) = \mathfrak{LG}(\widehat{\beta})$ for all

β and $n \geq 1$. First T_n is well defined on A_n since the relations r_1, r_2 and r_3 are satisfied by the Links-Gould invariant. In addition since the Links-Gould invariant vanishes on split links, it implies that $T_n(\beta) = 0$ for all $\beta \in \text{Im}(\phi_{n-1})$ and $T_1 = 1$ is by the normalization of the Links-Gould invariant (one on the unknot). We have also $T_{n+1}(\beta s_n^{\pm 1}) = T_n(\beta)$ for all $\beta \in \text{Im}(\phi_n)$, since the Links-Gould invariant is invariant under the first Reidemeister move. It remains to say that $T_n(\beta \alpha) = T_n(\alpha \beta)$ is satisfied because the Links-Gould invariant is trully an invariant of links, it does in particular not depend on where you open the link to compute it (see Remark 5.1 in [GP]). Given two braids α and β consider the topological partial closure of the braids $\alpha\beta$ and $\beta\alpha$ decribed in Figure 6. It can be easily seen that these are two different openings of the topological closure of $\alpha\beta$ (which is of course isotopic to the topological closure of $\beta\alpha$.) This finishes the proof of the existence of Tr_n.
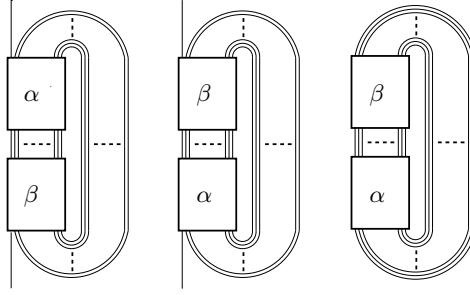

FIGURE 6. Partial closure of $\alpha\beta$ and $\beta\alpha$ and closure of $\alpha\beta$.

The algebra A_3 and A_4 are respectively isomorphic to the algebras LG_3 and LG_4.

We use the bimodule decomposition afforded by theorem 7.1. Suppose by induction that Tr_k is unique for $k \leq n$. Let γ be an element of in $A_{n+1} = A_n s_n^r A_n + A_{n-2} s_n^{-1} s_{n-1} s_n^{-1}$ ($r \in \{-1, 0, +1\}$) we can suppose that either $\gamma = \alpha_1 s_n^r \alpha_2$ or $\gamma = \beta s_n^{-1} s_{n-1} s_n^{-1}$ with α_1, α_2 $\in A_n$ and $\beta \in A_{n-2}$. In the first case if $r = 0$ we have $Tr_{n+1}(\gamma) = 0$ and if $r = \pm 1$ we have $Tr_{n+1}(\gamma) = Tr_n(\alpha_1 \alpha_2)$. In the second case we have $Tr_{n+1}(\gamma) = Tr_{n+1}(\beta s_n^{-2} s_{n-1}) = Tr_{n+1}(s_{n-1} \beta s_n^{-2})$. By applying the cubic relation r_1 to the factor s_n^{-2} we reduce to the previous case. Hence Tr_{n+1} is unique. It is direct computation given the basis for A_1, A_2 and A_3 to prove the trace is unique for these algebras. This finishes the proof of unicity.

The case of LG_n is similar, since $LG_4 = A_4$. $\square$

Remark 7.4. *Given an integer $n \geq 1$, for all $1 \leq k \leq n$ consider the natural embedding of $B_k \times B_{n-k}$ into B_n (see Figure (7)). Denote by ϕ_k its extension to an homomorphism from $A_k \otimes A_{n-k}$ to A_n. Define I_n the subvector space of A_n generated by the images of the ϕ_k ($1 \leq k \leq n$).*

The fact that z is equal to zero and an induction argument shows that the unique trace Tr_n on A_n vanishes on I_n. This implies that the Links-Gould invariant vanishes on split links (see [I1] for a different proof).
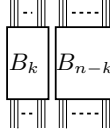
FIGURE 7. Injection of $B_k \times B_{n-k}$ into B_n.

Corollary 7.5. *The relations r_1, r_2 and r_3 are a complete set of skein relations for the Links-Gould invariant.*

Proof. It is an immediate consequence of Theorem 7.3. $\square$

Remark 7.6. *Notice first that the relations r_1, r_2 and r_3 are sufficient to compute the Links-Gould invariant. In addition using the representations of section 5, one can deduce relations expressing the elements $s_3^{\pm} s_2^{-1} s_1 s_2^{-1} s_3^{\pm}$, in the chosen basis of A_4 which could in pratice simplify a recursive computation. All these relations are of course consequences of r_1, r_2 and r_3*

As noticed in the proof of the theorem, the Markov trace on A_4 is a linear combination of the matrix traces on the irreducible representations of A_4, namely

$$Tr_4 = \sum_{i=1}^{10} a_{\chi_i} \operatorname{tr}_{\chi_i}$$

where $i \in \{1, \ldots, 10\}$, $a_{\chi_i} \in K$, and χ_i is the i-th irreducible representation of A_4, following the order chosen at the end of section 5. For comparaison with other traces factoring through H_4, we provide these coefficients in table 10.

8. ON THE IMAGE AND KERNEL OF $B_n \to LG_n(\alpha)^\times$

We recall that $\mathcal{LG}_n(\alpha)$ has been defined in section 4 as the *Lie* subalgebra of $\operatorname{End}(V(0, \alpha)^{\otimes n})$ generated by the $\Omega'_{ij}s$.

8.1. **Structure of $\mathcal{LG}_n(\alpha)$ and Zariski closure of B_n inside $LG_n(\alpha)^\times$.** The proof of the following proposition is parallel to its analogue for BMW (see [M4] prop. 5.1), the algebra $LG_n(\alpha)$ playing the role of the Birman-Wenzl-Murakami algebra thanks to theorem 4.1.

Proposition 8.1. *For generic values of α, the Lie algebra $\mathcal{LG}_n(\alpha)$ is reductive with center $\Bbbk T_n$.*

For $\lambda = [a, k]_r$ we let $\rho_\lambda : \mathcal{LG}_n(\alpha) \to \mathfrak{gl}_{d(\lambda)}(\Bbbk)$ denote the corresponding representation of $\mathcal{LG}_n(\alpha)$, and ρ'_λ its restriction to the derived Lie algebra $\mathcal{LG}'_n(\alpha)$. The proof of the following proposition is parallel to propositions 5.5 and 5.6 of [M4].

Proposition 8.2. *For generic α, ρ'_{λ_1} is isomorphic to ρ'_{λ_2} if and only if $\lambda_1 = \lambda_2$ or $d(\lambda_1) = d(\lambda_2) = 1$. If $d(\lambda_1), d(\lambda_2) > 1$, then the dual representation of ρ_{λ_2} cannot be isomorphic to ρ_{λ_1}.*

Proof. As in [M4], $\mathcal{LG}'_n$ is generated by the $t'_{ij} = t_{ij} - 2T_n/n(n-1)$ with $T_n = \sum_{1 \le r, s \le n} t_{rs}$ and T_n acts by a scalar on a given irreducible representation. For ρ'_{λ_1} and ρ'_{λ_2} to be isomorphic, $\rho'_{\lambda_1}(t'_{12})$ and $\rho'_{\lambda_2}(t'_{12})$ should be conjugate, hence $(\rho_{\lambda_1}(T_n) - \rho_{\lambda_2}(T_n))/n(n-1)$ should then belong to $X - X$, where $X = \{-2\alpha^2, -2(\alpha+1)^2, -2\alpha(\alpha+1)\}$ is the set of possible eigenvalues

$$a_{\chi_1} = \frac{c^2a^4b^2 + a^4bc + a^4 - c^2a^3b^3 - a^3c^3b^2 + b^3c^3a^2 - a^2c^2b^2 + ac^2b^3 + ac^3b^2 + b^2c^2}{a\,(a-b)\,(a-c)\,(a^2+c^2)\,(b^2+a^2)}$$

$$a_{\chi_2} = -\frac{(ab+1)\,(a^3b - ca^2b - ab + c^2)\,c^2}{(a-c)\,(b-c)\,(a^2+c^2)\,(ba^2-c^3)}, \qquad a_{\chi_3} = \frac{(ac+1)\,(a^3c - ca^2b - ac + b^2)\,b^2}{(a-b)\,(b-c)\,(b^2+a^2)\,(-b^3+ca^2)}$$

$$a_{\chi_4} = -\frac{(ac+1)\,(ca^2 - cab - a - c)\,b^2}{a\,(a-b)\,(b-c)\,(b+c)\,(b^2+a^2)}$$

$$a_{\chi_5} = -\frac{(ab+1)\,(a^4bc - bc^2a^3 + a^3b - a^3c - b^3a^2c + a^2c^2b^2 - ca^2b + ac^2b^3 - ac^3b^2 + b^2ac + cb^3 - b^2c^2)\,c}{a\,(a-c)\,(b-c)\,(a^2+c^2)\,(-b^3+ca^2)}$$

$$a_{\chi_6} = \frac{(ab+1)\,(ba^2 - cab - a - b)\,c^2}{a\,(a-c)\,(b-c)\,(b+c)\,(a^2+c^2)}$$

$$a_{\chi_7} = \frac{(ac+1)\,(a^4bc - ca^3b^2 + a^3c - a^3b - a^2c^3b + a^2c^2b^2 - ca^2b + ac^3b^2 + ac^2b - ac^2b^3 - b^2c^2 + c^3b)\,b}{a\,(a-b)\,(b-c)\,(b^2+a^2)\,(ba^2-c^3)}$$

$$a_{\chi_8} = \frac{c^2b\,(ac+1)\,(ab+1)}{a\,(b-c)\,(b+c)\,(ba^2-c^3)}, \qquad a_{\chi_9} = -\frac{b^2c\,(ac+1)\,(ab+1)}{a\,(b-c)\,(b+c)\,(-b^3+ca^2)}$$

$$a_{\chi_{10}} = \frac{cb\,(ac+1)\,(ab+1)\,(bc+a^2)}{a\,(ba^2-c^3)\,(-b^3+ca^2)}$$

TABLE 10. Coefficients of the unique Markov trace on $A_4 = LG_4$.

for $\rho_{\lambda_1}(t_{ij})$ and $\rho_{\lambda_2}(t_{ij})$. For generic α the set $X + (-X) = \{0\} \sqcup (-2) \times \{2\alpha + 1, \alpha + 1, \alpha, -\alpha, -\alpha - 1, -2\alpha - 1\}$ has cardinality 7. This implies that either $(\rho_{\lambda_1}(T_n) - \rho_{\lambda_2}(T_n))/n(n-1) = 0$, or both $\rho_{\lambda_1}(t_{12})$ and $\rho_{\lambda_2}(t_{12})$ have a single eigenvalue (as, for $x, y \in X$, $x - y$ then determines x and y when $x - y \neq 0$). By semisimplicity of the action of t_{12} this implies that the images of the t_{ij}'s are scalars in both ρ_{λ_1} and ρ_{λ_2}. By irreducibility this implies that $\dim \rho_{\lambda_1} = \dim \rho_{\lambda_2} = 1$. The case of the dual representations is similar, using that, for $x \in X$, $2x \in X_+^= = (-2) \times \{2\alpha^2, 2\alpha^2 + 2\alpha, 2\alpha^2 + 4\alpha + 1\}$ and, for $x, y \in X$ with $x \neq y$, $x + y \in X_+^{\neq} = (-2) \times \{2\alpha^2 + 2\alpha + 1, 2\alpha^2 + \alpha, 2\alpha^2 + 3\alpha + 1\}$; again, for generic α, $X_+^{\neq} \cap X_+^= = \emptyset$, $X_+^{\neq} = 3$, and we get a contradiction unless $\dim \rho_{\lambda_1} = \dim \rho_{\lambda_2} = 1$. $\square$

We let $V_{[a,k]_r}$ denote the $[a, k]_r$-component of LG_n.

Theorem 8.3. *For generic α, the image of $\mathcal{LG}'_n(\alpha)$ inside $LG_n(\alpha) = \bigoplus \mathrm{End}(V_{[a,k]_r})$ is $\bigoplus \mathfrak{sl}(V_{[a,k]_r})$.*

Proof. The proof is by induction on r, and follows the same general pattern as in [M4]. We will use freely the Lie-theoretic results of [M3]. Assuming the result known for $r - 1$ (and the cases $r \leq 2$ being trivial), we only need to check that, for $\lambda = [a, k]_r$, $\rho_\lambda(\mathcal{LG}_n) = \mathfrak{sl}(V_{[a,k]_r})$, by the same argument as in [M4, M2]. By abuse, we denote $\lambda = \rho_\lambda$, Res_i the restriction from $\mathcal{LG}_r$ to $\mathcal{LG}_i$ for $i \leq r$, and $\mathrm{Res} = \mathrm{Res}_{r-1}$. First note that, if $\mathrm{Res}\lambda$ has s irreducible components $\mu_1, \ldots, \mu_s$ occuring with multiplicity one, then by the induction assumption the rank of $\rho_\lambda(\mathcal{LG}_{r-1})$ is $(\sum \dim \mu_j) - s = \dim V_\lambda - s$, hence the rank of $\rho_\lambda(\mathcal{LG}_{r-1})$ is at least

$\dim V_\lambda - s > (\dim V_\lambda)/2$ as soon as $\dim V_\lambda > 2s$. By [M3] lemma 3.1 this implies the conclusion $\rho_\lambda(\mathcal{LG}_{r-1})$.

We will show that this assumption is almost always satisfied, and we will check separately the remaining cases. First note that the restriction is always multiplicity free, and that the number of components is at most 4. More precisely, for $a \leq r-1$ and $k \leq r-a-1$, $\mathrm{Res}[a,k]_r$ contains

    (i) $[a+1, k-1]_{r-1}$ if $a \leq r-3$ and $1 \leq k \leq r-a-2$
    (ii) $[a, k]_{r-1}$ if $a \leq r-2$ and $k \leq r-a-2$
    (iii) $[a, k-1]_{r-1}$ if $a \leq r-2$ and $1 \leq k$
    (iv) $[a-1, k]_{r-1}$ if $a \geq 1$

Moreover we notice that the assumptions of lemma 3.3 (I) of [M3] are satisfied as soon as $\dim \lambda < (1 + \mathrm{rk}\mathfrak{h})^2$: under this condition, this lemma implies that $\mathfrak{g}$ is simple.

We first deal with a few special cases. For a given λ, we denote $\mathfrak{g}$ the image of $\mathcal{LG}'_r$ and $\mathfrak{h}$ the image of $\mathcal{LG}'_{r-1}$.

The cases $[a, 0]_r$, $[r-2, 1]_r$ and $[1, r-2]_r$.
The 1-dimensional cases $[0, 0]_r$ and $[r-1, 0]_r$ are trivial. We have $\mathrm{Res}[1, 0]_r = [0, 0]_{r-1} + [1, 0]_{r-1}$, which implies $\dim[1, 0]_r = r-1$. Let $\lambda = [1, 0]_3$. Since $\dim \lambda = 2$, $\mathfrak{g} \subset \mathfrak{sl}_2$ and $\mathfrak{g}$ is semisimple hence $\mathfrak{g} = \mathfrak{sl}_2$. Let $\lambda = [1, 0]_4$. Then $\mathfrak{h} = \mathfrak{sl}_2 \subset \mathfrak{g} \subset \mathfrak{sl}_3$. If $\mathfrak{g}$ has rank $2 > 3/2$ then $\mathfrak{g} = \mathfrak{sl}_3$ by [M3] lemma 3.1 ; otherwise $\mathfrak{g}$ has rank 1, that is $\mathfrak{g} \simeq \mathfrak{sl}_2$. But the only 3-dimensional irreducible representation of $\mathfrak{sl}_2$ is selfdual, which excludes this case. Let $\lambda = [1, 0]_5$. Then $\mathfrak{h} = \mathfrak{sl}_3 \subset \mathfrak{g} \subset \mathfrak{sl}_4$. If $\mathrm{rk}\mathfrak{g} > 2$ then $\mathfrak{g} = \mathfrak{sl}_4$ by [M3] lemma 3.1 ; otherwise $\mathfrak{g}$ has rank 2; since $4 < (2+1)^2$, by lemma 3.3 of [M3] $\mathfrak{g}$ is simple. Since the 4-dimensional representations of $\mathfrak{so}_5$ and $\mathfrak{sp}_4$ are selfdual, this implies $\mathfrak{g} = \mathfrak{sl}_4$. Let now $\lambda = [1, 0]_r$ for $r > 5$. Then $\mathrm{rk}\mathfrak{h} = r-3 > (r-1)/2 = (\dim \lambda)/2$ and $\mathfrak{g} = \mathfrak{sl}_{r-1}$, which concludes the case of $[1, 0]_r$. The case of $[r-2, 0]_r$ is similar.

Let now $\lambda = [a, 0]_r$ for $a > 1$ and $a < r-2$, which implies $r \geq 5$. Then $\mathrm{Res}\lambda = [a-1, 0]_{r-1} + [a, 0]_{r-1}$ hence we need to show that $\dim \lambda > 4$ in order to apply lemma 3.1 of [M3]. It is easily checked that $\mathrm{Res}_4[a, 0]_r$ always contains $[1, 0]_4 + [2, 0]_4$, of dimension $3 + 3 = 6$, and this concludes the case of the $[a, 0]_r$.

The same arguments provide the case of the $[r-2, 1]_r$ and of the $[1, r-2]_r$.

The cases of $[0, 1]_3$, $[0, 1]_4$, $[0, 2]_4$.
Let $\lambda = [0, 1]_3$, of dimension 3. Then $\mathfrak{g} \subset \mathfrak{sl}_3$; if $\mathrm{rk}\mathfrak{g} = 1$ then $\mathfrak{g} \simeq \mathfrak{sl}_2 \simeq \mathfrak{so}_3$, but the 3-dimensional irreducible representation of $\mathfrak{g}$ is selfdual, a contradiction. Thus $\mathrm{rk}\mathfrak{g} = 2 > 3/2$ and $\mathfrak{g} \simeq \mathfrak{sl}_3$ by lemma 3.1 of [M3].

Let $\lambda = [0, 1]_4$, of dimension 6. We have $\mathrm{Res}\lambda = [0, 0]_3 + [0, 1]_3 + [1, 0]_3$ hence $\mathfrak{h} \simeq \mathfrak{sl}_3 \times \mathfrak{sl}_2$ and $\mathrm{rk}\mathfrak{g} \geq 3$. If $\mathrm{rk}\mathfrak{g} > 3 = 6/2$ we are done, so we need to exclude the case $\mathrm{rk}\mathfrak{g} = 3$. In that case, by [M3] lemma 3.3, we get that $\mathfrak{g}$ is simple, and by [M3] lemma 3.4 $\mathfrak{g} \simeq \mathfrak{sl}_4$ and λ is selfdual, a contradiction. The case of $\lambda = [0, 2]_4$ is similar to $[0, 1]_4$.

The case of $[a, r-a-1]_r$ for $1 \leq a \leq r-3$.
This case implies $r \geq 4$. Let $\lambda = [a, r-a-1]_r$. Then $\mathrm{Res}\lambda = [a, r-a-2]_{r-1} + [a-1, r-a-1]_{r-1}$, and we have the conclusion as soon as $\dim \lambda > 4$. Note that if $a \leq r-4$ (resp. $a \geq 2$) then $[a, r-a-2]_{r-1}$ belongs to the case under study for $r-1$: by induction this shows that $\dim \lambda > 4$ for $r \geq 6$, as $[1, 4]_6$, $[2, 3]_6$, $[3, 2]_6$ have for dimensions 5, 10 and 19. There remains to study the cases of $[1, 2]_4$ and $[1, 3]_5$, since $\dim[2, 2]_5 = 6 > 4$. For $\lambda = [1, 2]_4$ we have

$\dim \lambda = 3$ and the case is similar to the one of $[0,1]_3$ above. For $\lambda = [1,3]_5$ we have $\dim \lambda = 4$ and the case is similar to the one of $[1,0]_5$ above. This concludes these cases.

Since the case $k = 0$ has been done, we can assume $k \geq 1$. If $a = r - 1$, this implies $k = 0$, so we can also assume $a \leq r - 2$. If $a = r - 2$, then $k \leq 1$ hence $k = 1$, and this is the case of the $[1, r-2]_r$, which is also done. Thus $a \leq r - 3$, $k \geq 1$.

Assume for a while $a = 0$. Then $k \leq r - 1$; the 1-dimensional case $[0, r-1]_r$ is trivial hence we can assume $k \leq r - 2 = r - a - 2$. Then $\mathrm{Res}\lambda$ has 3 simple components (1), (2) and (3), hence we are done if we know that $\dim \lambda > 6$. For $r \geq 6$, it is easily checked that $\mathrm{Res}_5[0,k]_5$ contains one of the $[0,l]_5$ for $1 \leq l \leq 3$, which have dimension > 6. The remaining cases are $[0,1]_3$, $[0,1]_4$, $[0,2]_4$, which have already been tackled.

We can now assume $1 \leq a \leq r - 3$ and $k \geq 1$. Since the case $k = r - a - 1$ has been tackled we can assume $k \leq r - a - 2$. In that case, $\mathrm{Res}\lambda$ admits the 4 components (1)-(4), so we are done if $\dim \lambda > 8$. Note that, under our conditions $1 \leq a \leq r - 3$ and $1 \leq k \leq r - a - 2$, for $r \geq 5$, at least one of the components (2),(3) and (4) fulfills the same conditions for $r - 1$. Since $\dim[1,1]_5 = \dim[1,2]_5 = 20$ and $\dim[2,1]_5 = 15$, this shows by induction that $\dim \lambda > 8$ for $r \geq 5$. The only remaining case is for $\lambda = [1,1]_4$. We have $\mathrm{rk}\mathfrak{h} = 4$ hence $\mathrm{rk}\mathfrak{g} \geq 4$. If $\mathrm{rk}\mathfrak{g} > 4 = (\dim \lambda)/2$ we are done, so we need to exclude the case $\mathrm{rk}\mathfrak{g} = 4$. By lemma 3.3 (I) of [M3] we know that $\mathfrak{g}$ is simple, and by lemma 3.4 of [M3] the fact that λ is not selfdual provides a contradiction. This concludes the proof of the theorem. $\square$

Theorem 8.3 implies the 'Zariski-closure' part of theorem 1.7 of the introduction, as in [M4] for the BMW-algebra and [M2] for the Hecke algebra. The remaining part of the theorem is proved in the section below.

8.2. **Faithfulness of $B_n \to LG_n(\alpha)^\times$.** Here we assume that K is a field of characteristic 0 and $a, b, c \in K^\times$ three algebraically independent elements. For convenience we moreover assume that -1, a and b admit square roots in K. We let S_n denote the 1-dimensional KB_n module defined by $s_i \mapsto a$, U_n denote the $(n-1)$-dimensional KB_n module afforded by the reduced Burau representation (convention : the image of s_i has eigenvalues a with multiplicity $(n-2)$ and b with multiplicity 1). Recall that the Krammer representation Kr_n defined in [Kr] is a $n(n-1)/2$-dimensional irreducible representation of B_n over $\mathbb{Q}(q,t)$, such that the image of s_i has 3 eigenvalues $1, -q, tq^2$. This provides a faithful representation of B_n. Up to renormalization ($s_i \mapsto \lambda s_i$) and change of parameter we can assume instead that the three eigenvalues are a, b, c and that this representation is defined over K, without affecting the faithfulness property.

This representation factors through the BMW-algebra. Usually this algebra is defined over the field $\mathbb{Q}(s, \alpha)$ of rational fractions (see e.g. [M4]), however under the same renormalization process $s_i \mapsto \lambda s_i$ one can define it over K in such a way that the image of s_1 has eigenvalues a, b, c (explicitely, $a = \lambda s, b = -\lambda s^{-1}, c = -\lambda \alpha^{-1}$, and conversely $\lambda = \sqrt{-ab}, s = \sqrt{-a/b}, \alpha = -c^{-1}\sqrt{-ab}$, whence our assumptions on K). The algebra BMW_n is semisimple, with irreducible representations parametrized by partitions of m for $0 \leq m \leq n$ and $n - m$ an even integer. For $n = 2$, the empty partition $\emptyset$ of $m = 0$ corresponds to $s_1 \mapsto c$ while $[2]$ and $[1,1]$ correspond to $s_1 \mapsto a$ and $s_1 \mapsto b$, respectively. Under this convention, Kr_n is the irreducible component labelled by the partition $[n-2]$, and the reduced Burau representation is labelled by $[n-1, 1]$.

It thus has the property that Kr_2 is $s_1 \mapsto c$ and the restriction of Kr_{n+1} to $B_n \subset B_{n+1}$ is $S_n + U_n + Kr_n$. In diagrammatic terms, Kr_n has a Bratteli diagram of the form
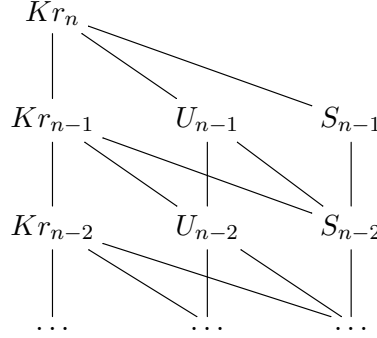
We prove that this property characterizes the Krammer representation.

Proposition 8.4. *Let Kr_n for $n \geq 2$ denote a family of irreducible KB_n-modules, with the property that the restriction of Kr_{n+1} to $KB_n \subset KB_{n+1}$ is $S_n + U_n + Kr_n$, and that Kr_2 is the 1-dimensional KB_2-module $s_i \mapsto c$. Then Kr_n is isomorphic to the Krammer representation.*

Proof. When $n = 2$ there is nothing to prove, and for $n = 3$ we know from the description of H_3 that there is up to isomorphism only one irreducible 3-dimensional representation of B_3 where s_1 has 3 distinct eigenvalues a, b, c, so we can assume $n \geq 4$. Then the restriction of K_n to B_3 is a direct sum of irreducible representations which factorize through the Birman-Wenzl-Murakami algebra (up to a change of parameters and renormalization), and it thus needs to factorize through the BMW-algebra ; since the relations for the BMW-algebra are generated by relations in B_3 this proves that K_n itself factorizes through the BMW-algebra. It is then a simple combinatorial task to check that the only irreducible representations of the BMW-algebra for $n \geq 4$ with this Bratteli diagram corresponds to the partition $[n-2]$. $\square$

Corollary 8.5. *For $n \geq 2$, the Krammer representation factorizes through LG_n.*

Proof. The KB_n-module $V_{[0,n-2]_n}$ obviously satisfies the assumption, and factors through LG_n. $\square$

Corollary 8.6. *For $n \geq 2$, the morphism $B_n \to LG_n^\times$ is into.*

Proof. Immediate consequence of the faithfulness of the Krammer representation. $\square$

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