INFINITESIMAL HECKE ALGEBRAS II

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Abstract. For W a finite (2-)reflection group and B its (generalized) braid group, we determine the Zariski closure of the image of B inside the corresponding Iwahori-Hecke algebra. The Lie algebra of this closure is reductive and generated in the group algebra $\mathbb{C}W$ by the reflections of W. We determine its decomposition in simple factors. In case W is a Coxeter group, we prove that the representations involved are unitarizable when the parameters of the representations have modulus 1 and are close to 1. We consequently determine the topological closure in this case.

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1. Introduction

Let V be a finite-dimensional complex vector space. A *reflection* of V is an element of $\mathrm{GL}(V)$ of order 2 whose fixed points form an hyperplane of V, called the reflecting hyperplane of the reflection. A central (finite) hyperplane arrangement $\mathcal{A}$ in V is called a reflection arrangement if there exists a set $\mathcal{R}$ of reflections, whose set of reflection hyperplanes is $\mathcal{A}$, and which generate a finite group W. If this is the case, $\mathcal{R}$ and W are uniquely determined by $\mathcal{A}$ (see [Ma09] prop. 2.1).

Topological structures associated to such a reflection arrangement include the fundamental group P of the complement $X = V \setminus \bigcup \mathcal{A}$ and $B = \pi_1(X/W)$. In case $\mathcal{A}$ is the complexification of a real arrangement, W is a finite Coxeter group and conversely every finite Coxeter group occur, through their classical reflection representation. In this case, B is the so-called Artin-Tits or generalized braid group associated to W. If W is of Coxeter type A_{n-1} then B is the classical braid group on n strands.

Another well-known algebraic structure associated to a central arrangement of hyperplanes is its holonomy Lie algebra $\mathfrak{g}$. It admits a nice presentation discovered by Kohno, with one generator t_H for each hyperplane $H \in \mathcal{A}$. By definition, W-equivariant representations of $\mathfrak{g}$ yields by monodromy (linear) representations of B. The representations induced by the natural projection $B \to W$ correspond to trivial representations of $\mathfrak{g}$.

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A seminal remark of Cherednik is that these representations can be deformed into a 1-parameter family by making t_H act as hs_H, where h is a formal parameter and $s_H \in \mathcal{R}$ denotes the reflection associated to $H \in \mathcal{A}$. It has been shown in most cases that one gets in this way all representations of the generic Iwahori-Hecke algebra associated to W, as monodromy representations of B over the field $K_0 = \mathbb{C}((h))$ of Laurent series. The remarkable phenomenon here is that the reflections fulfill in the group algebra $\mathbb{C}W$ the defining Lie-theoretic relations of $\mathfrak{g}$.

A natural object to consider is thus the *Lie* subalgebra of $\mathbb{C}W$ generated by $\mathcal{R}$, that we call the *infinitesimal Iwahori-Hecke algebra* $\mathcal{H}$ associated to the reflection group W — or to $\mathcal{A}$, since $\mathcal{A}$ determines $(W, \mathcal{R})$. It turns out to be a reductive Lie algebra, whose center is easily determined. The determination of its simple factors enables to determine the Zariski hull of P and B in the corresponding representations (see §6).

This task has been already accomplished for W of dihedral type and for the symmetric group (see [Ma03b, Ma07a]). More precisely, infinitesimal Iwahori-Hecke algebras were first introduced in [Ma03b] in case W is a finite Coxeter group, as a generalization of the Lie algebra of transpositions introduced in [Ma01] and fully decomposed in [Ma07a].

The purpose of this work is to extend this achievement to arbitrary reflection arrangement.

1.1. **Structure of infinitesimal Hecke algebras.** We let $\epsilon : W \to \{\pm 1\}$ denote the sign character of W, defined by $\epsilon(s) = -1$ for $s \in \mathcal{R}$, or equivalently as the restriction to W of the determinant $\mathrm{GL}(V) \to \mathbb{C}^\times$, and let $\Bbbk \subset \mathbb{C}$ denote the field of definition of W, namely the algebraic number field generated by the traces of elements of W on $\mathrm{GL}(V)$. It is well-known (see [Bd, Bes, MM]) that all the ordinary representations of W are defined over $\Bbbk$. The group W is a Coxeter group iff $\Bbbk \subset \mathbb{R}$, and a crystallographic group iff $\Bbbk = \mathbb{Q}$.

An intermediate Lie algebra between $\mathcal{H}$ and $\Bbbk W$ is the Lie subalgebra $\mathcal{L}_\epsilon(W)$ of $\Bbbk W$ spanned by the $g - \epsilon(g)g^{-1}$ for $g \in W$, introduced and decomposed in [Ma08]. We have $\mathcal{H} \subset \mathcal{L}_\epsilon(W) \subset \Bbbk W$ since $s - \epsilon(s)s^{-1} = 2s$ for $s \in \mathcal{R}$.

A common feature of the three Lie algebras $\mathcal{H}$, $\mathcal{L}_\epsilon(W)$ and $\Bbbk W$ is that they are reductive and that we get irreducible representations for them by restriction of the irreducible representations of W. Moreover

(1) the corresponding Lie ideals are simple
(2) all simple Lie ideals are obtained this way

This is shown in [Ma08] in the case of $\mathcal{L}_\epsilon(W)$ and will be a consequence of our results in the case of $\mathcal{H}$. A consequence is that the simple Lie ideals of these Lie algebras define an equivalence relation on the irreducible representations of W, hence a partition of the set $\mathrm{Irr}(W)$ of irreducible representations of W.

In case of $\mathcal{L}_\epsilon(W)$, this equivalence relation is defined by $\rho \sim \rho^* \otimes \epsilon$, where ρ^* denotes the dual representation of ρ. The ideal $\mathcal{L}_0(\rho)$ of $\mathcal{L}_\epsilon(W)$ attached to the class of $\rho \in \mathrm{Irr}(W)$ is either of linear or orthosymplectic type, depending on whether ρ is isomorphic to $\rho^* \otimes \epsilon$ or not. Denoting V_ρ

the vector space attached to $\rho \in \mathrm{Irr}(W)$, we thus have

$$\Bbbk W \simeq \bigoplus_{\rho\in \mathrm{Irr}(W)} \mathfrak{gl}(V_\rho), \quad \mathcal{L}_\epsilon(W) \simeq \bigoplus_{\rho\in \mathrm{Irr}(W)/\sim} \mathcal{L}_0(\rho),$$

In order to describe the decomposition of $\mathcal{H}$, we need to define a set $\mathrm{Ref}(W)$ of what we call reflection representations (see def. 2.12),

$$\begin{aligned} \mathrm{QRef} &= \{\eta\otimes\rho \mid \rho\in \mathrm{Ref}, \eta \in \mathrm{Hom}(W,\{\pm 1\})\} \\ \Lambda\mathrm{Ref} &= \{\eta \otimes \Lambda^k \rho \mid \rho \in \mathrm{Ref}, \eta\in\mathrm{Hom}(W,\{\pm1\}), k\geq 0\}\end{aligned}$$

and we define an equivalence relation $\approx$ on $\mathrm{Irr}(W)$ by

$$\rho_1 \approx \rho_2 \Leftrightarrow \rho_2 \in \{\rho_1\otimes \eta, \rho_1^*\otimes\eta\otimes\epsilon \mid \eta\in \mathrm{X}(\rho_1)\}$$

with

$$\mathrm{X}(\rho) = \{\eta \in \mathrm{Hom}(W,\{\pm1\}) \mid \forall s\in\mathcal{R} \ \ \eta(s) = -1 \Rightarrow \rho(s) = \pm 1\}.$$

In case $\rho^*\otimes\epsilon\simeq\rho\otimes\eta$ for some $\eta\in\mathrm{X}(\rho)$, then $\epsilon\otimes\eta$ embeds in either $S^2\rho^*$ or $\Lambda^2\rho^*$. This defines an orthogonal or symplectic Lie algebra $\mathfrak{osp}(V_\rho)\subset \mathfrak{gl}(V_\rho)$ that contains $\rho(\mathcal{H}')$. We denote $\mathcal{L}(\rho) = \mathfrak{osp}(V_\rho)$ in this case, and $\mathcal{L}(\rho) = \mathfrak{gl}(V_\rho)$ otherwise.

Note that $\rho_1\approx\rho_2$ iff $\rho_1\sim\rho_2$, and $\mathcal{L}(\rho)\simeq\mathcal{L}_0(\rho)$, whenever there is a single conjugacy class of reflection, notably when W is a Coxeter group of type ADE

We let $\mathrm{Irr}'(W) = \mathrm{Irr}(W)\setminus\Lambda\mathrm{Ref}(W)$. For a special situation that occur only then W has type H_4, we actually need a refinement $\approx'$ of $\approx$.

Theorem 1. *The Lie algebra $\mathcal{H}$ is reductive, its center has for dimension the number of reflection classes $\mathcal{R}/W$, and*

$$\mathcal{H}\simeq \Bbbk^{\mathcal{R}/W} \oplus \left(\bigoplus_{\rho\in\mathrm{QRef}/\approx'} \mathfrak{sl}(V_\rho)\right) \oplus \left(\bigoplus_{\rho\in\mathrm{Irr}'(W)/\approx'}\mathcal{L}(\rho)\right)$$

The special situation occuring for H_4 is related simultaneaously to triality and to the fact that H_4 is the only complex reflection group which admits $\rho\in\mathrm{Irr}(W)$ of dimension 8 whose symmetric square contains ϵ. The two representations having this property are equivalent under $\approx'$. For all the other groups, $\approx'$ and the relation $\approx$ described above coincide.

We finally show that the *real* Lie subalgebra $\mathcal{H}^c$ of $\mathbb{C}W$ generated by the $\sqrt{-1}s, s\in\mathcal{R}$ is a compact form of $\mathcal{H}$.

The sketch of proof of theorem 1 is as follows. In §2 we prove that it is only needed to prove it with $\Bbbk$ replaced by $\mathbb{C}$, and that it is true provided that $\mathcal{H}(\rho) = \mathcal{L}(\rho)$ for all $\rho\in\mathrm{Irr}(W)\setminus\Lambda\mathrm{Ref}$. Since in general $\mathcal{H}(\rho)\subset\mathcal{L}(\rho)\subset\mathfrak{sl}_N$ where $N=\dim\rho$, we need several Lie-theoretic lemmas, proved in §3, in order to deal with the situation $\mathfrak{h}\subset\mathfrak{g}\subset\mathfrak{sl}_N$, where $\mathfrak{h},\mathfrak{g}$ are complex semisimple Lie algebras acting irreductibly on $\mathbb{C}^N$, $\mathfrak{h}$ being known and $\mathfrak{g}$ having to be described. Then §4 establishes basic facts on the representation theory of W when W belongs to the main general series $G(e,e,n)$, and in particular determines $\Lambda\mathrm{Ref}$ in this case.

The core of the proof is then §5, which sets up the induction process, assuming that W contains a proper reflection subgroup W_0 for which the theorem is known. We prove there this induction step when W_0 has a single class of reflections, under some conditions on the branching rule of the pair (W, W_0) ; we use that to reduce our problem to the case where W has a single class of reflections, and then apply these results to the general series, with $W_0 = G(e,e,n)$ and $W = G(e,e,n+1)$. The case of exceptional groups (except H_4) is then tackled, using a combination of the previous results and ad-hoc (sometimes computer-aided) methods.

We then turn (§6) to the applications described below and also deal with the special case of H_4. The proofs of a few technical results needed in the proof of theorem 1 are postponed at the end of the paper (§7).

1.2. **Applications.** By the Cherednik monodromy construction, to each $\rho \in \mathrm{Irr}(W), \rho : W \to \mathrm{GL}(V_\rho)$ is associated a linear representation R over $K_0 = \mathbb{C}((h))$ of the (generalized) braid group B attached to W. Letting P denote the pure braid group $\mathrm{Ker}\,(B \twoheadrightarrow W)$, a first consequence of this work is the following.

Theorem 2. *Let $R : B \to GL_N(K_0)$ be the monodromy representation associated to $\rho \in \mathrm{Irr}(W)$. Then the Zariski closure of $R(P)$ is connected with Lie algebra $\rho(\mathcal{H})$.*

We actually make more precise the above statement by computing the closure of $R(B)$. Depending on ρ, it coincides with the closure of $R(P)$ or contains it as an index 2 subgroup.

Such representations of B factorizes through a morphism $B \to H(q)^\times$ where $H(q)$ denotes the so-called Hecke algebra of W, with (transcendant) parameter $q = \exp(\mathrm{i}\pi h) \in K_0$. It is a natural quotient of the group algebra $K_0 B$. In the cases where this Hecke algebra is known to be isomorphic to the group algebra $K_0 W$ we more generally compute the Zariski closure of the image of P and B inside $H(q)^\times \simeq (K_0 W)^\times$.

We say that the groups for which this assumption is already known to hold are *tackled*. This includes all Coxeter groups by Tits theorem.

Theorem 3. *(see theorem 6.9) Assume that W is irreducible and tackled. The Zariski closure of the image of P inside $H(q)^\times$ is connected and has Lie algebra $\mathcal{H} \otimes K_0$. It is contained in the Zarisi closure of the image of B as an index 2 subgroup.*

We now assume that W is a *Coxeter group*, and consider the representation $R : B \to \mathrm{GL}_N(K_0)$ of $H(q)$ attached to some $\rho \in \mathrm{Irr}(W)$. Such a ρ is a complexification of a real representation ρ_0 of W. It is known that R admits a matrix model over $\mathbb{R}[q, q^{-1}]$, hence any $u \in \mathbb{C}^\times$ defines a specialized representation $R_u : B \to \mathrm{GL}_N(\mathbb{C})$. We prove the following (theorem 6.12).

Theorem 4. *For $u \in \mathbb{C}$ with $|u| = 1$ and u close enough to 1, the representation R_u is unitarizable. If in addition u is a transcendental number, then the topological closure of $R_u(P)$ is a connected compact Lie group, with Lie algebra $\rho(\mathcal{H}^c)$.*

In particular, for the case $W = \mathfrak{S}_n$, we recover the results of M. Freedman, M. Larsen, Z. Wang in [FLW] (for generic parameters). Recall that, in this case, the unitarizability is known by Wenzl unitary matrix models. Such models can be obtained in type A by using a rational Drinfeld associator, as in [Ma06].

1.3. **Generalizations.** In case W has several conjugacy classes of reflections, the Hecke algebra can be defined with one parameter q_c for each class. In the generic case, these parameters are distinct indeterminates. Such representations are obtained by monodromy construction, with $q_c = \exp(\mathrm{i}\pi\lambda_c h)$, the λ_c being complex numbers linearly independant over $\mathbb{Q}$ (then the $q_c \in K_0$ are algebraically independant over $\mathbb{C}$). In particular $\lambda_c \neq 0$. Then the relevant Lie algebra, being spanned by the $\lambda_c s$, for $s \in c \subset \mathcal{R}$, is still $\mathcal{H}$, and the results above can be easily extended to this case.

The case of non-transcendantal parameters (or algebraically dependant parameters in the case of several parameters), and the specially interesting case of roots of 1, cannot a priori be dealt with by the methods used here (although the knowledge of the generic case should be useful for dealing with the specialized ones).

The natural generalization of this work is thus to deal with the pseudo-reflection groups. In that case, several parameters immediately arise, and the general preliminary study is more complicated than the (2-)reflection case. We will deal with this more general setting in a forthcoming paper (which will built on the combinatorical case-by-case arguments and the results exposed here).

Throughout this paper, the name CHEVIE refers to the corresponding GAP package, which can be downloaded at `http://www.math.jussieu.fr/~jmichel`.

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2. Infinitesimal Iwahori-Hecke algebras

Let W be a finite complex (2-)reflection group, and $\epsilon : W \to \{\pm 1\}$ its sign character, afforded by the determinant.

2.1. **Definitions and basic properties.** For now, we only assume that $\Bbbk$ is a field of characteristic 0.

Definition 2.1. *The infinitesimal Iwahori-Hecke algebra associated to a finite complex (2-)reflection group W is the Lie subalgebra $\mathcal{H}_W$ of $\Bbbk W$ generated by the set of reflections of W.*

Note that, if W_0 is a reflection subgroup of W, that is if W_0 is generated by reflections of W, then $\mathcal{H}_{W_0}$ embeds in $\mathcal{H}_W$ as a Lie subalgebra. If $W \simeq W_1 \times W_2$ as complex reflection groups, then $\mathcal{H}_W \simeq \mathcal{H}_{W_1} \times \mathcal{H}_{W_2}$.

Let $\mathcal{R}$ denote the set of all reflections in W, and $\mathcal{S} \subset \mathcal{R}$ a subset satisfying

1. $\mathcal{S}$ generates W
2. Every reflection of W is conjugated in W to some element of $\mathcal{S}$.

A typical example for $\mathcal{S}$ is the set of simple reflections attached to some Weyl chamber when W is a Coxeter group. We let W act on $\mathcal{R}$ by conjugation, and $\mathcal{R}/W$ the corresponding set of equivalence classes. It is the set of conjugacy classes of W whose elements are reflections. For $c \in \mathcal{R}/W$ we denote $T_c = \sum_{s\in c} s \in \Bbbk W$, and introduce $p : \Bbbk G \to Z(\Bbbk G)$ defined by $p(x) = (1/\#G)\sum_{g\in G} gxg^{-1}$.

For an arbitrary finite group G, Lie subalgebras of $\Bbbk G$ generated by generating sets of G have noticeable properties, that we recall from [Ma04], lemme 3.

Proposition 2.2. *Let S be a generating set of the finite group G. Then the Lie subalgebra $\mathcal{L}$ of $\Bbbk G$ generated by S is reductive, and every irreducible representation of G is irreducible for the action of $\mathcal{L}$. Moreover, the center of $\mathcal{L}$ is contained in the sub-vector space of $\mathcal{L}$ spanned by the T_c for $T_c \in L$ and $c\cap S \neq \emptyset$. If S is the union of some conjugacy classes, then $Z(\mathcal{L}) = p(\mathcal{L})$ and $\mathcal{L}'$ is generated by the elements $s - T_{c(s)}/\#c(s)$ for $s \in S$.*

Proof. The fact that irreducible representations of G gives rise to irreducible representations of $\mathcal{L}$ comes from the fact that S generates both $\mathcal{L}$ (as a Lie algebra) and G (as a group). Now any faithful representation of $\Bbbk G$ restricts to a faithful, semisimple representation of $\mathcal{L}$, so $\mathcal{L}$ is reductive. Another consequence of the fact that S generates both $\mathcal{L}$ and G is that $Z(\mathcal{L}) \subset Z(\Bbbk G)$.

We define $p : \Bbbk G \twoheadrightarrow Z(\Bbbk G)$ by $p(x) = (1/\#G)\sum_{g\in G} gxg^{-1}$. It is easily checked (see e.g. [Ma08]) that $(\Bbbk G)' = \mathrm{Ker}\, p = \cap_c \mathrm{Ker}\,\delta_c$ for c a conjugacy class of G and δ_c the characteristic linear form associated to it. Let E be the vector space spanned by S and $\mathcal{L}'$ the vector space generated by iterated brackets of elements of $\mathcal{L}$. Obviously $\mathcal{L} = E + \mathcal{L}'$ and $\mathcal{L}' \subset (\Bbbk G)' = \cap \mathrm{Ker}\,\delta_c$ so $\delta_c(\mathcal{L}) = \delta_c(E)$ for any conjugacy class c. It follows that $\delta_c(\mathcal{L}) = \{0\}$ for every class such that $c \cap S = \emptyset$. Since $Z(\Bbbk G)$ is spanned by the elements T_c the conclusion follows.

Finally, if S is the union of conjugacy classes then $\mathcal{L}$ is stable under conjugation by W hence p restricts to a linear endomorphism of $\mathcal{L}$. Since $Z(\mathcal{L}) \subset Z(\Bbbk G)$ we know that p acts identically on $Z(\mathcal{L})$. From $\mathcal{L}' \subset (\Bbbk G)' = \mathrm{Ker}\, p$ and $\mathcal{L} = Z(\mathcal{L}) \oplus \mathcal{L}'$ we thus deduce $p(\mathcal{L}) = Z(\mathcal{L})$. Now the collection $s - T_{c(s)}/\#c(s)$ for $s \in S$ generates a Lie algebra that clearly contains $\mathcal{L}'$ (because each $T_{c(s)}$ lies in $Z(\Bbbk G)$) and is contained in $\mathcal{L}$. Since $p(s - T_{c(s)}/\#c(s)) = 0$ it is contained in $\mathcal{L}'$, which concludes the proof. □

The following facts are mostly recollected from [Ma03b]. We provide a proof for the convenience of the reader. For $s \in \mathcal{R}$ we let $s' = s - T_{c(s)}/\#c(s)$, where $c(s) \in \mathcal{R}/W$ denotes the conjugacy class of s, and recall from the introduction that $\mathcal{L}_\epsilon(W)$ is the Lie subalgebra of $\Bbbk W$ spanned by the $w - \epsilon(w)w^{-1}$, $w \in W$.

Proposition 2.3. *Let W be a finite complex reflection group. Then*

1. *$\mathcal{H}_W$ is a reductive Lie algebra.*
2. *The center of $\mathcal{H}_W$ has dimension $\#\mathcal{R}/W$ with basis $\{T_c; c \in \mathcal{R}/W\}$.*
3. *The derived Lie algebra $\mathcal{H}'_W$ of $\mathcal{H}_W$ is generated by the $s' = s - T_{c(s)}/\#c(s)$ for $s \in \mathcal{R}$.*
4. *$\mathcal{H}_W \subset \mathcal{L}_\epsilon(W)$*
5. *$\mathcal{H}_W$ is stable in $\Bbbk W$ for the adjoint action of W, and is generated by $\mathcal{S}$.*

(6) *$\mathcal{H}'_W$ is stable in $\Bbbk W$ for the adjoint action of W, and is generated by $\{s' \mid s \in \mathcal{S}\}$.*

Proof. The first three claims are consequences of proposition 2.2, since W is generated by $\mathcal{R}$. If $s \in \mathcal{R}$, then $\epsilon(s) = -1$ hence $s = \frac{s-\epsilon(s)}{2} \in \mathcal{L}_\epsilon(W)$. It follows that $\mathcal{H}_W \subset \mathcal{L}_\epsilon(W)$. Finally, since $\mathcal{R}$ is stable under conjugation then so is $\mathcal{H}_W$. For $s \in \mathcal{R}$ let $\mathrm{Ad}(s) : y \mapsto sys^{-1} = sys$ and $\mathrm{ad}(s) : y \mapsto [s,y]$ associated endomorphisms of $\mathcal{H}_W$. It is easily checked that, for all $s \in \mathcal{R}$, $\mathrm{ad}(s)^2 = 2(\mathrm{Id} - \mathrm{Ad}(s))$. In particular, since $\mathcal{S}$ generates W, the Lie algebra generated by $\mathcal{S}$ is stable under the action of W. Since the orbit under conjugation of $\mathcal{S}$ is $\mathcal{R}$ it follows that this Lie algebra contains $\mathcal{R}$ hence equals $\mathcal{H}_W$. The proof of (7) goes along the same lines. □

From now on, we assume that W is irreducible, and that $\Bbbk$ contains the field of definition of W. Multiplicative characters will play a special role. We recall a consequence of Stanley theorem (see [Sta], theorem 3.1), denoting $\{\pm 1\}^{\mathcal{R}/W}$ the set of maps from the set $\mathcal{R}/W$ of reflection classes to $\{\pm 1\}$.

Lemma 2.4. *There is a natural bijection $\{\pm 1\}^{\mathcal{R}/W} \to \mathrm{Hom}(W, \{\pm 1\})$ which associates to each $f \in \{\pm 1\}^{\mathcal{R}/W}$ a character $\eta : W \to \{\pm 1\}$ such that $\eta(s) = f(c)$ whenever $c \in \mathcal{R}/W$ and $s \in c$.*

Although we do not use it now, we recall that a consequence of the Shephard-Todd classification is that $\#\mathcal{R}/W \leq 3$ when W is irreducible, with moreover $\#\mathcal{R}/W \leq 2$ if $\mathrm{rk}(W) \geq 3$. For $\rho \in \mathrm{Irr}(W)$ we let

$$\mathrm{X}(\rho) = \{\eta \in \mathrm{Hom}(W, \{\pm 1\}) \mid \forall s \in \mathcal{R} \ \ \eta(s) = -1 \Rightarrow \rho(s) = \pm 1\}.$$

We first need a lemma

Lemma 2.5. *Let E be a finite-dimensional $\Bbbk$-vector space, for an arbitrary characteristic 0 field $\Bbbk$. If E admits a nondegenerate symmetric bilinear form $< \, , \, >$ and an involutive skew-isometry, i.e. $s \in \mathrm{End}(E)$ with $s^2 = 1$ and $< sx, sy >= - < x, y >$ for every $x, y \in E$, then $< \, , \, >$ is hyperbolic.*

Proof. Whenever $x \neq 0$ we have $< sx, x >=< sx, ssx >= - < x, sx >$ hence $< sx, x >= 0$. Since $< \, , \, >$ is nondegenerate, it admits a non-isotropic $x \in E$ with $< x, x > \neq 0$. If $sx = \lambda x$ for some $\lambda \in \Bbbk$, then $- < x, x >=< sx, sx >= \lambda^2 < x, x >$ and $\lambda^2 = -1$, contradicting $s^2 = 1$. Thus x, sx are linearly independant. Let $u = x + sx$, $v = (x - sx)/2 < x, x >$. The plane H spanned by u, v is hyperbolic, E is an orthogonal direct sum of H and its orthogonal, which is s-invariant because H is so. The conclusion then follows by induction on the dimension of E. □

Proposition 2.6. *If there exists $\eta \in \mathrm{X}(\rho)$ such that $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$, this η is unique. The associated embedding $\epsilon \otimes \eta \hookrightarrow \rho^* \otimes \rho^*$ defines (up to scalar) a bilinear form on V_ρ, and we denote by $\mathfrak{osp}(V_\rho) \subset \mathfrak{sl}(V_\rho)$ the Lie subalgebra preserving it. We have $\rho(\mathcal{H}') \subset \mathfrak{osp}(V_\rho)$. If the bilinear form is symmetric, then it is hyperbolic (provided $\dim \rho > 1$).*

Proof. Let $\eta_1, \eta_2 \in \mathrm{X}(\rho)$ with $\rho \otimes \eta_1 \simeq \rho \otimes \eta_2$, and let $s \in \mathcal{R}$. If $\rho(s) \notin \{\pm 1\}$ then $\eta_1(s) = 1 = \eta_2(s)$. If $\rho(s) \in \{\pm 1\}$ then $\rho(s)\eta_1(s) = \rho(s)\eta_2(s)$ hence $\eta_1(s) = \eta_2(s)$. Since $\mathcal{R}$ generates W we get $\eta_1 = \eta_2$.

Let now $\eta \in \mathrm{X}(\rho)$ with $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$ and $< , >$ the corresponding bilinear form on V_ρ. Recall that $\rho(\mathcal{H}') \subset \mathfrak{sl}(V_\rho)$ is generated by the $\rho(s'), s \in \mathcal{R}$. For all $s \in \mathcal{R}$ we have $< \rho(s)x, \rho(s)y >= \epsilon(s)\eta(s) < x, y >$ for all $x, y \in V_\rho$. Since $s^2 = 1$ this means $< \rho(s)x, y >= \epsilon(s)\eta(s) < x, \rho(s)y >$. When $\rho(s) \notin \{\pm 1\}$ we get $< \rho(s)x, y > + < x, \rho(s)y >= 0$ hence $< \rho(s')x, y > + < x, \rho(s')y >= 0$ and $\rho(s') \in \mathfrak{osp}(V_\rho)$.. When $\rho(s) \in \pm 1$ we have $\rho(s') = 0$ hence again $\rho(s') \in \mathfrak{osp}(V_\rho)$ and $\rho(\mathcal{H}') \subset \mathfrak{osp}(V_\rho)$. The last assertion is a consequence of lemma 2.5 as involutive skew-isometries are provided by the $s \in \mathcal{R}$ with $\eta(s) = 1$. If not such s exists, then $\rho(W)$ is abelian and $\dim V_\rho = 1$. □

Obviously $\mathfrak{osp}(V_\rho)$ is either a special orthogonal Lie algebra, when $\epsilon \otimes \eta \hookrightarrow S^2\rho^*$, or a symplectic one, when $\epsilon \otimes \eta \hookrightarrow \Lambda^2\rho^*$. Note that the hyperbolicity property implies that the orthogonal Lie algebras possibly involved here are even ones.

Any representation ρ of W restricts to a representation $\rho_\mathcal{H}$ of $\mathcal{H}_W$, and in particular to a representation $\rho_{\mathcal{H}'}$ of $\mathcal{H}'_W$. We denote $\mathrm{Irr}(\mathcal{H}_W)$ and $\mathrm{Irr}(\mathcal{H}'_W)$ the set of (classes of) irreducible representations of $\mathcal{H}_W$ and $\mathcal{H}'_W$, respectively.

The following proposition is a generalization of [Ma07a], prop. 2.

Proposition 2.7. *Let W be a finite reflection group. Then*

(1) *If $\rho \in \mathrm{Irr}(W)$ then $\rho_\mathcal{H} \in \mathrm{Irr}(\mathcal{H}_W)$.*
(2) *If $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ then $\rho^1 \simeq \rho^2 \Leftrightarrow \rho^1_\mathcal{H} \simeq \rho^2_\mathcal{H}$.*
(3) *If $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ with $\dim \rho^1 > 1$ then $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$ if and only if $\rho^1 \simeq \rho^2 \otimes \eta$ for some $\eta \in \mathrm{X}(\rho_2) = \mathrm{X}(\rho_1)$.*
(4) *$(\rho \otimes \epsilon)_\mathcal{H} \simeq (\rho_\mathcal{H})^*$.*

Proof. For $\mathcal{R}/W = \{\mathcal{R}_1, \dots, \mathcal{R}_k\}$, we let $T_i = T_{\mathcal{R}_i}$ and let $\eta_i : W \to \{\pm 1\}$ be defined by $\eta_i(\mathcal{R}_i) = -1$ and $\eta_i(\mathcal{R}_j) = 1$ for $j \neq i$.

The first two claims are immediate consequences of the fact that $\mathcal{R}$ generates W. The last one is a consequence of $\mathcal{H} \subset \mathcal{L}_\epsilon(W)$ (see prop. 2.3), and of the corresponding statement for $\mathcal{L}_\epsilon(W)$.

Now assume $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$. $\mathcal{H}'_W$ is generated by the elements $s - T_i/\#\mathcal{R}_i$ $1 \leq i \leq \#\mathcal{R}/W$ and $s \in \mathcal{R}_i$. Since all irreducible representations of W over $\Bbbk$ are absolutely irreducible, we know that $\rho^1(T_i)$ and $\rho^2(T_i)$ are scalars. It follows that there exists $P \in \mathrm{Hom}(\rho^1, \rho^2)$ such that $P\rho^1(s)P^{-1} = \rho^2(s) + \omega_i$, with $\omega_i = \frac{\rho^2(T_i) - \rho^1(T_i)}{\#\mathcal{R}_i} \in \Bbbk$ for all $1 \leq i \leq \#\mathcal{R}/W$ and $s \in \mathcal{R}_i$. From $s^2 = 1$ we get $1 = 1 + 2\omega_i\rho^2(s) + \omega_i^2$. Let $I = \{i \mid \omega_i \neq 0\}$. For $i \in I$ we have $\rho^2(s) = \frac{-\omega_i}{2} \in \Bbbk$ hence $\rho^1(s) = \frac{\omega_i}{2} = -\rho^2(s)$ and $P\rho^1(s)P^{-1} = -\rho^2(s)$. For $i \notin I$ we have $P\rho^1(s)P^{-1} = \rho^2(s)$ hence $\rho^1 \simeq \rho^2 \otimes \eta$ where $\eta = \prod_{i \in I} \eta_i$, because $\mathcal{R}$ generates W. Moreover, if $\omega_i \neq 0$ and $s \in \mathcal{R}_i$, then $s^2 = 1$ implies $\omega_i = \pm 2$ and this concludes the proof. Conversely, if $\rho^2 \simeq \rho^1 \otimes \eta$ and $\rho^1(s) = \pm 1$ whenever $\eta(s) = -1$, then for such s we have that $s - T_{c(s)}/\#c(s)$ has zero image under both ρ^1 and ρ^2, and these images are obviously conjugated if $\eta(s) = 1$. It follows that $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$. □

The following consequence is obvious.

Corollary 2.8. *If W admits a single conjugacy class of reflections and $\rho^1, \rho^2 \in \mathrm{Irr}(W)$, then $\rho^1 \simeq \rho^2$ iff $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$.*

We use $\mathrm{X}(\rho)$ to define an equivalence relation on $\mathrm{Irr}(W)$.

Definition 2.9. *For ρ_1, ρ_2 we define $\rho_1 \approx \rho_2$ by $\rho_2 \in \{\rho_1 \otimes \eta, \rho_1^* \otimes \eta \otimes \epsilon \mid \eta \in \mathrm{X}(\rho_1)\}$.*

Notice that $\mathrm{X}(\rho)$ is a subgroup of $\mathrm{Hom}(W, \{\pm 1\})$ and that $\rho_1 \approx \rho_2 \Rightarrow \mathrm{X}(\rho_1) = \mathrm{X}(\rho_2)$. It follows that $\approx$ is an equivalence relation on $\mathrm{Irr}(W)$, in general coarser than the relation $\sim$ of the introduction, generated by $\rho^* \otimes \epsilon \sim \rho$. Also notice that $\epsilon \in \mathrm{X}(\rho)$ if and only if $\rho(s) = \pm 1$ for all $s \in \mathcal{R}$. This is possible only when $\dim \rho = 1$. In particular, if W has a single class of reflections and $\rho^1, \rho^2 \in \mathrm{Irr}(W)$ whith $\dim \rho^i > 1$, then $\rho^1 \approx \rho^2$ iff $\rho^1 \sim \rho^2$.

Definition 2.10. *If $\rho \in \mathrm{Irr}(W)$, we let $\mathcal{H}(\rho)$ denote the orthogonal of $\mathrm{Ker}\, \rho_{\mathcal{H}'}$ with respect to the Killing form of $\mathcal{H}'$, and let $\mathcal{L}(\rho) = \mathfrak{sl}(V_\rho)$ if $\rho \otimes \eta \not\simeq \rho^* \otimes \epsilon$ for every $\eta \in \mathrm{X}(\rho)$, and $\mathcal{L}(\rho) = \mathfrak{osp}(V_\rho)$ otherwise.*

It is clear that $\mathcal{H}(\rho)$ is an ideal of $\mathcal{H}'$, isomorphic to $\rho(\mathcal{H}')$ as a Lie algebra. We prove that all $\mathcal{L}(\rho)$ are semisimple. By their definition, this amounts to saying that the exceptional case $\mathcal{L}(\rho) \simeq \mathfrak{so}_2 \simeq \Bbbk$ with $\dim \rho = 2$ never occurs (however the non-simple case $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$ *do* occur, see lemma 2.22 below). The reason is that, if $\dim \rho = 2$, then $\Lambda^2 \rho^* = \det \circ \rho^* \in \mathrm{Hom}(W, \{\pm 1\})$ can be written as $\epsilon \otimes \eta$ for some $\eta \in \mathrm{Hom}(W, \{\pm 1\})$, and we have $\eta \in \mathrm{X}(\rho)$, as $\rho(s) \neq \pm 1$ implies $\det \rho(s) = -1 = \epsilon(s)$. Also notice that $\rho(\mathcal{H}')$ is semisimple and nonzero, otherwise $\rho(s) \in \{\pm 1\}$ for all $s \in \mathcal{R}$ and $\dim \rho = 1$. By $\mathfrak{sl}_2(\Bbbk) \simeq \mathfrak{sp}_2(\Bbbk)$ this implies the following.

Lemma 2.11. *Let $\rho \in \mathrm{Irr}(W)$ with $\dim(\rho) = 2$. Then $\mathcal{L}(\rho) = \mathfrak{sl}(V_\rho) = \rho(\mathcal{H}') \simeq \mathcal{H}(\rho)$.*

2.2. **Reflection representations.** From now on we assume that W is an irreducible reflection group of rank at least 2. We need a slightly nonstandard definition.

Definition 2.12. *An irreducible representation $R : W \to \mathrm{GL}(V_R)$ with $\dim R \geq 2$ is called a* reflection representation *if, for all $s \in \mathcal{R}$, if $R(s) \neq \mathrm{Id}$ then $R(s)$ is a reflection of V_R.*

By the irreducibility assumption on W, such representations exist as soon as $\mathrm{rk}(W) \geq 2$, and at least one of them is faithful. All faithful reflection representations of W have for dimension the rank of W by the classification theorem of Shephard and Todd, and they are all deduced from the defining representation by Galois action (see [MM]).

It is a classical fact due to Steinberg (see [Bo456], ch. V §2 exercice 3) that the alternating powers $\Lambda^k R$ of R for $1 \leq k \leq \dim R$ of a reflection representation are distinct irreducible representations of W, thus leading to distinct simple ideals of $\Bbbk W$ as an associative algebra ; indeed, it is straightforward to check that the proof of [Bo456] applies to our more general definition of a reflection representation.

Taking k-th alternating powers gives rise to two representations of $\mathcal{H}_W$ on $\Lambda^k V_R$, namely $(\Lambda^k R)_{\mathcal{H}}$ and $\Lambda^k(R_{\mathcal{H}})$. Likewise, suppose we are given $\eta \in \mathrm{Hom}(W, \{\pm 1\})$. Then, to any $x \in \Bbbk$ one can associate a character γ_x^η of $\mathcal{H}_W$ defined by $s \mapsto \eta(s)(x-1)$ for $s \in \mathcal{R}$. Letting $R^\eta = R \otimes \eta$, one may then define two twisted representations $(\eta \otimes \Lambda^k R)_{\mathcal{H}}$ and $\Lambda^k(R^\eta_{\mathcal{H}})$. Identifying

$\Bbbk \otimes \Lambda^k V_R$ with $\Lambda^k V_R$, all these representations as well as $\gamma_x^\eta \otimes (\eta \otimes \Lambda^k R)_{\mathcal{H}}$ act on $\Lambda^k V_R$.

Proposition 2.13. *Let $R : W \to GL(V_R)$ be a reflection representation of a finite irreducible reflection group W and $\eta \in \mathrm{Hom}(W, \{\pm 1\})$. For any $k \in [0, \dim V_R]$,*

(1) $\Lambda^k(R^\eta_{\mathcal{H}}) = \gamma_k^\eta \otimes (\eta \otimes \Lambda^k R)_{\mathcal{H}}$
(2) $\Lambda^k(R^\eta_{\mathcal{H}'}) = (\eta \otimes \Lambda^k R)_{\mathcal{H}'}$

and, in particular, $\Lambda^k R_{\mathcal{H}'} = (\Lambda^k R)_{\mathcal{H}'}$.

Proof. Let $s \in \mathcal{R}$. If $R(s) = 1$, then $(\eta \otimes \Lambda^k R)(s) = \eta(s)$ and $(\Lambda^k R^\eta_{\mathcal{H}})(s) = k\eta(s) = \eta(s)(k-1)\mathrm{Id} + (\eta \otimes \Lambda^k R)(s)$. Otherwise, there exists a basis $e_1, \dots, e_n$ of V_R such that $s.e_1 = -e_1$ and $s.e_i = e_i$ if $i \neq 1$, where we make W act on V_R through R. One has a natural basis (e_I) of $\Lambda^k V_R$ which is indexed by the subsets of size k in $\{1, \dots, n\}$: if $I = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$ then $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then

$$\left\{ \begin{array}{rcll} (\eta \otimes \Lambda^k R)(s)(e_I) & = & \eta(s) e_I & \text{if } 1 \notin I \\ & = & -\eta(s) e_I & \text{if } 1 \in I \end{array} \right.$$

$$\left\{ \begin{array}{rcll} (\Lambda^k R^\eta_{\mathcal{H}})(s)(e_I) & = & k\eta(s) e_I & \text{if } 1 \notin I \\ & = & (k-2)\eta(s) e_I & \text{if } 1 \in I \end{array} \right.$$

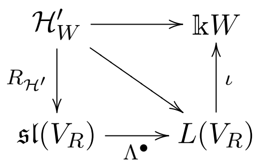
and it follows that $(\Lambda^k R^\eta_{\mathcal{H}})(s) = \eta(s)(k-1)\mathrm{Id} + (\eta \otimes \Lambda^k R)(s)$, thus proving (1). Statement (2) follows immediately. □

Remark 2.14.

It is a standard fact that, if R is faithful, then $\epsilon \otimes \Lambda^k R^* = \Lambda^{n-k} R$, where $n = \dim V_R$. In general, we easily check that $\Lambda^{n-k} R = \eta \otimes \epsilon \otimes \Lambda^k R^*$ with $\eta \otimes \epsilon(g) = \det R(g)^{-1}$ for $g \in W$. For $s \in \mathcal{R}$, this means that $\eta(s) = -1$ iff $\det R(s) = 1$. In particular, $\eta \in \mathrm{X}(R)$.

Let $L(V_R) = \bigoplus_{k=0}^{\infty} \mathfrak{sl}(\Lambda^k V_R)$, and $\Lambda^\bullet : \mathfrak{sl}(V_R) \to L(V_R)$ be the natural representation of $\mathfrak{sl}(V_R)$ on the exterior algebra of V_R. By considering $\Lambda^k V_R$ as the underlying vector space of $\Lambda^k R$, there is a natural embedding $\iota : L(V_R) \hookrightarrow (\Bbbk W)' \subset \Bbbk W$ and, more generally, to any $\eta \in \mathrm{Hom}(W, \{\pm 1\})$ is associated a natural embedding $\iota_\eta : L(V_R) \hookrightarrow (\Bbbk W)'$ by identification of $\Lambda^k V_R$ with the underlying vector space of $\eta \otimes \Lambda^k R$.

Proposition 2.13 leads to considering the following diagram



where the morphism $\mathcal{H}'_W \to L(V_R)$ is defined in the obvious way such that the upper right triangle commutes. Proposition 2.13 implies that the lower left triangle also commutes. We showed in [Ma04] by a case-by-case analysis that the morphism $\mathcal{H}'_W \to \mathfrak{sl}(V)$ is surjective for finite irreducible Coxeter groups. More generally, we have the following result.

Proposition 2.15. *Let $R : W \to GL(V_R)$ be a reflection representation of a finite irreducible reflection group W and $\eta \in \mathrm{Hom}(W, \{\pm 1\})$. Then $R^{\eta}_{\mathcal{H}'} : \mathcal{H}'_W \to \mathfrak{sl}(V_R)$ is surjective.*

To prove this, we will need the following classical lemmas.

Lemma 2.16. *A root system of type A_n admits no proper subsystem of rank n.*

Proof. We realize a root system of type A_n as the elements $e_i - e_j$ in the euclidean vector space $\mathbb{R}^{n+1}$ with basis $e_1, \dots, e_{n+1}$. We let s_{ij} denote the reflection corresponding to the root $e_i - e_j$. Let E be a subsystem of rank n and let $\mathcal{E} = \{i \mid e_1 - e_i \in E\}$, $\mathcal{F} = \{2, \dots, n+1\} \setminus \mathcal{E}$. Assume $\mathcal{F} \neq \emptyset$. Then $e_i - e_j \notin E$ for all $i \in \mathcal{E}$ and $j \in \mathcal{F}$ otherwise $s_{1i}(e_i - e_j) = e_1 - e_j \in E$ and $j \in \mathcal{E} \cap \mathcal{F} = \emptyset$. Let $\mathcal{E}' = \mathcal{E} \cup \{1\}$. Then E is contained in the vector space generated by the $e_i - e_j$ for $i, j \in \mathcal{E}'$ or $i, j \in \mathcal{F}$, whose dimension is at most $n - 1$. By contradiction, it follows that $\mathcal{F} = \emptyset$ and E contains all $e_1 - e_i$. Since the permutations $(1\ i)$ generate $\mathfrak{S}_{n+1}$, the Weyl groups of the two subsystems are the same. Since this Weyl group acts transitively on the set of roots and $E \neq \emptyset$ it follows that E is not proper. □

For V a finite-dimensional $\mathbb{C}$-vector space and $W \subset \mathrm{GL}(V)$ a finite group acting irreducibly, we endow V with a W-invariant hermitian scalar product. If $H \subset V$ is a hyperplane with orthogonal spanned by $e_H \in V$, we let $W_H = \{w \in W \mid w.e_H = e_H\}$. We recall the following lemma from [Ma09].

Lemma 2.17. *(see* [Ma09]*, cor. 3.2) Let $W \subset \mathrm{GL}(V)$ be a finite group generated by pseudo-reflections and acting irreducibly on V. Let $\mathcal{R}$ denote its set of pseudo-reflections. There exists an hyperplane H of V such that W_H acts irreducibly on H and such that its image in* $\mathrm{GL}(H)$ *is generated by the images of $\mathcal{R} \cap W_H$.*

We are now in position to prove proposition 2.15.

Proof. Since, for all $s \in \mathcal{R}$, we have $R^{\eta}(s) = \eta(s)R(s)$, it suffices to show that $R_{\mathcal{H}'}$ is surjective. We will thus assume $\eta = 1$. Since $R(W) \subset \mathrm{GL}(V_R)$ is an irreducible reflection group admitting as reflections the image of $\mathcal{R}$, we can assume that R is faithful and is the defining representation of W, that is $V_R = V$ and $W \subset \mathrm{GL}(V)$.

Under these assumptions, we prove the proposition by induction on $n = \mathrm{rk}\, W = \dim V$. For convenience here, we drop the implicit assumption $\dim V \geq 2$ in the statement and start the induction at the trivial case $n = 1$. We proceed by assuming $n \geq 2$. Let $H \subset V$ an hyperplane and $W_0 = W_H \subset W$ being afforded by lemma 2.17. We have $\mathcal{H}_{W_0} \subset \mathcal{H}_W$ hence $\mathcal{H}'_{W_0} \subset \mathcal{H}'_W$ and $\mathrm{Im}\, R_{\mathcal{H}'} \supset \mathrm{Im}\, R_{\mathcal{H}'_{W_0}} = \mathfrak{sl}(H)$ by the induction hypothesis. It follows that there exists a Cartan subalgebra of rank $n-2$ of $\mathrm{Im}\, R_{\mathcal{H}'}$ inside $\mathfrak{sl}(H)$. We are going to exhibit a semisimple element $x \in \mathfrak{sl}(V) \setminus \mathfrak{sl}(H)$ centralizing $\mathfrak{sl}(H)$ that belongs to the image of $\mathcal{H}'_W$: this will provide a Cartan subalgebra of rank $n-1$ for this image, which then equals $\mathfrak{sl}(V)$ since it is a semisimple Lie subalgebra of $\mathfrak{sl}(V)$ and the root system of type A_{n-1}, corresponding to $\mathfrak{sl}(V)$, contains no proper subsystem of rank $n-1$ by lemma 2.16.

We now construct x. Let $\mathcal{R}_0$ denote the set of reflections of W_0, and define the following elements in $\mathcal{H}_W$

$$T = \sum_{s \in \mathcal{R}} s, \quad T_0 = \sum_{s \in \mathcal{R}_0} s, \quad X = (\#\mathcal{R})T_0 - (\#\mathcal{R}_0)T$$

It is clear that X belongs to $\mathcal{H}_W$ and that it commutes to $\mathcal{H}_{W_0}$. Let $x = R_{\mathcal{H}}(X)$. We know that T acts on V by the scalar $\frac{n-2}{n}\#\mathcal{R}$, because $trR(s) = n-2$ for all $s \in \mathcal{R}$ and R is irreducible. Similarly, T_0 acts on H by $\frac{n-3}{n-1}\#\mathcal{R}_0$ and on D by $\#\mathcal{R}_0$. It follows that x acts on H by $(\#\mathcal{R})(\#\mathcal{R}_0)\frac{-2}{n(n-1)}$ and by $(\#\mathcal{R})(\#\mathcal{R}_0)\frac{2}{n}$ on D. Thus $x \notin \mathfrak{sl}(H)$ but $x \in \mathfrak{sl}(V)$ and x is semisimple. Since $\mathcal{H}_W$ is reductive, the intersection of $\mathfrak{sl}(V)$ with its image in $\mathfrak{gl}(V)$ is the image of $\mathcal{H}'_W$. It follows that x belongs to the image of $\mathcal{H}'_W$, which concludes the proof. □

2.3. **Reflection ideals.** We define the following subsets of $\mathrm{Irr}(W)$.

$$\begin{aligned} &\mathrm{Ref} = \{\text{reflection representations}\} \\ &\mathrm{QRef} = \{\eta \otimes \rho \mid \rho \in \mathrm{Ref}, \eta \in \mathrm{Hom}(W, \{\pm 1\})\} \\ &\Lambda\mathrm{Ref} = \{\eta \otimes \Lambda^k \rho \mid \rho \in \mathrm{Ref}, \eta \in \mathrm{Hom}(W, \{\pm 1\}), k \geq 0\} \end{aligned}$$

and we let $\overline{\mathrm{QRef}} = \mathrm{QRef}/\approx$. By choosing an arbitrary system of representatives, we identify $\overline{\mathrm{QRef}}$ with a subset of QRef. Notice that, since $\mathrm{Ref} \neq \emptyset$, then $\mathrm{Hom}(W, \{\pm 1\}) \subset \Lambda\mathrm{Ref}$. More generally, representations of small dimension belong to ΛRef.

Lemma 2.18. *Let* $\rho \in \mathrm{Irr}(W)$. *If* $\dim \rho \leq 3$ *then* $\rho \in \Lambda\mathrm{Ref}$. *If moreover* $\dim \rho \neq 1$ *then* $\rho \in \mathrm{QRef}$.

Proof. Since $\mathrm{Hom}(W, \{\pm 1\}) \subset \Lambda\mathrm{Ref}$ it is sufficient to show the second part of the statement. Assume $\dim \rho = 2$. Let $\mathcal{R}_0 = \{s \in \mathcal{R} \mid \rho(s) = \mathrm{Id}\}, \mathcal{R}_1 = \{s \in \mathcal{R} \mid \rho(s) = -\mathrm{Id}\}, \mathcal{R}_2 = \{s \in \mathcal{R} \mid Sp\rho(s) = \{-1, 1\}\}$. Since $\rho(s)^2 = \mathrm{Id}$ for all $s \in \mathcal{R}$ we have $\mathcal{R} = \mathcal{R}_0 \sqcup \mathcal{R}_1 \sqcup \mathcal{R}_2$, and each $\mathcal{R}_i$ is stable by conjugation. Using lemma 2.4 we define $f : \mathcal{R} \to \{\pm 1\}$ by $f(s) = 1$ if $s \in \mathcal{R}_0 \sqcup \mathcal{R}_2$, $f(s) = -1$ is $s \in \mathcal{R}_1$, and denote $\eta \in \mathrm{Hom}(W, \{\pm 1\}$ the corresponding character. Then $\eta \otimes \rho \in \mathrm{Ref}$ hence $\rho \in \mathrm{QRef}$. Similarly, if $\dim \rho = 3$, letting $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2$ as before, we define $\mathcal{R}_2^{\pm}$ to be the set of all $s \in \mathcal{R}_2$ such that ± 1 appear in the spectrum of $\rho(s)$ with multiplicity 2 ; we define f as before on $\mathcal{R}_0 \sqcup \mathcal{R}_1$, $f(s) = \pm 1$ for $s \in \mathcal{R}_2^{\pm}$, and again $\eta \otimes \rho \in \mathrm{Ref}$ hence $\rho \in \mathrm{QRef}$. □

Definition 2.19. *The* reflection ideal *of* $\mathcal{H}'$ *is*

$$\mathcal{I} = \mathcal{H}' \cap \bigoplus_{\rho \in \Lambda\mathrm{Ref}} \mathrm{End}(V_\rho).$$

It is clear that $\mathcal{I}$ is an ideal of $\mathcal{H}'$, and we have a natural projection $\mathcal{H}' \to \mathcal{I}$ factorizing the inclusion $\mathcal{H}' \to \Bbbk W$. Let $\rho \in \mathrm{QRef}$. We have $\mathcal{H}(\rho) \simeq \mathfrak{sl}(V_\rho)$ by proposition 2.15. Now, proposition 2.7 and proposition 2.13 imply that $\mathcal{I}$ is the sum of the ideals $\mathcal{H}(\rho)$ for $\rho \in \overline{\mathrm{QRef}}$, since semi-simple Lie algebras are direct sums of their simple ideals and $\mathfrak{sl}(V_\rho)$ is a simple Lie algebra. Finally, if $\rho^1, \rho^2 \in \overline{\mathrm{QRef}}$, $\mathcal{H}(\rho^1) = \mathcal{H}(\rho^2)$ implies $\rho^1_{\mathcal{H}'} \simeq \rho^2_{\mathcal{H}'}$ or $\rho^1_{\mathcal{H}'} \simeq (\rho^2_{\mathcal{H}'})^*$, since $\mathfrak{sl}_n$ admits at most two irreducible representations of dimension n. It follows that $\mathcal{H}(\rho^1) = \mathcal{H}(\rho^2) \Rightarrow \rho^1 \approx \rho^2 \Rightarrow \rho^1 = \rho^2$ by definition of $\overline{\mathrm{QRef}}$, hence $\mathcal{I}$ is the direct sum of the ideals $\mathcal{H}(\rho) \simeq \mathfrak{sl}(V_\rho)$ for $\rho \in \overline{\mathrm{QRef}}$. In particular we get

Proposition 2.20. *The reflection ideal $\mathcal{I}$ is isomorphic to*

$$\bigoplus_{\rho \in \overline{\mathrm{QRef}}} \mathfrak{sl}(V_\rho).$$

Example : the reflection ideal of G_{13}. We illustrate this decomposition for W of type G_{13}. This group has 4 multiplicative characters $\mathbb{1}, \epsilon, \eta, \eta \otimes \epsilon$, among 16 irreducible characters. Its field of definition is $\Bbbk = \mathbb{Q}(\mu_{24})$. A computer computation shows that $\dim \mathcal{H}' = 51$.

This group admits 4 faithful 2-dimensional reflection representations, numbered $\rho_5, \rho_7, \rho_8, \rho_{10}$ in CHEVIE, and two non-faithful ones, ρ_6 in dimension 2 and ρ_{12} in dimension 3. We have $\rho_6(W) \simeq G(1,1,3) \simeq \mathfrak{S}_3$ and $\rho_{12}(W)$ is the Coxeter group of type B_3.

There are two natural actions on the 4 faithful ones, the action by tensor product of $\mathrm{Hom}(W, \{\pm 1\})$ and the action of $\mathrm{Gal}(\Bbbk|\mathbb{Q})$. Both are transitive, and we have $\rho \simeq \rho^* \otimes \epsilon$ and $\mathrm{X}(\rho) = \{\mathbb{1}\}$ for all of them. It follows that they all belong to $\overline{\mathrm{QRef}}$, and that no other representation of QRef arise from them. Moreover, we have $\iota(L(V_\rho)) = \mathfrak{sl}(V_\rho)$ for all of them.

In dimension 2, the other one is $\rho_6 \simeq \rho_6 \otimes \eta \otimes \epsilon$. We have $\rho_9 \simeq \rho_6 \otimes \epsilon \in$ QRef, and both are defined over $\mathbb{Q}$. In particular $\rho_9 \simeq \epsilon \otimes \rho_6^*$ and we can assume $\overline{\mathrm{QRef}} \cap \{\rho_6, \rho_9\} = \{\rho_6\}$. Another reason for $\rho_9 \approx \rho_6$ is that $\rho_9 = \rho_6 \otimes \eta$ and $\mathrm{X}(\rho_6) = \{\mathbb{1}, \eta\}$.

Finally, the 3-dimensional one has a real-valued character and induces 3 other representations $\rho_{11} = \rho_{12} \otimes \eta$, $\rho_{13} = \rho_{12} \otimes \epsilon$, $\rho_{14} = \rho_{12} \otimes \eta \otimes \epsilon$. Since $\mathrm{X}(\rho_{12}) = \{\mathbb{1}\}$, we can choose $\overline{\mathrm{QRef}} = \{\rho_5, \rho_6, \rho_7, \rho_8, \rho_{10}, \rho_{11}, \rho_{12}\}$. It follows that $\mathcal{I} = \mathfrak{sl}_2^5 \times \mathfrak{sl}_3^2$ and $\dim \mathcal{I} = 31$

This example shows in particular that we cannot assume $\overline{\mathrm{QRef}} \subset \mathrm{Ref}$ in general, since $\rho_{11}, \rho_{14} \notin \mathrm{Ref}$. Moreover, note that these four last representations of W have the same alternating square, although they may represent different simple ideals of $\mathcal{H}'$.

2.4. **Decomposition in semisimple components.** We define the following subsets of $\mathrm{Irr}(W)$.

$$\mathcal{E} = \{\rho \in \mathrm{Irr}(W) \mid \rho \notin \Lambda\mathrm{Ref} \text{ and } \forall \eta \in \mathrm{X}(\rho) \ \ \rho^* \otimes \epsilon \not\simeq \rho \otimes \eta\}$$
$$\mathcal{F} = \{\rho \in \mathrm{Irr}(W) \mid \rho \notin \Lambda\mathrm{Ref} \text{ and } \exists \eta \in \mathrm{X}(\rho) \ \ \rho^* \otimes \epsilon \simeq \rho \otimes \eta\}$$

We identify $\mathcal{E}/\approx$ and $\mathcal{F}/\approx$ with subsets of $\mathcal{E}$ and $\mathcal{F}$, respectively.

Proposition 2.7 and proposition 2.13 imply that the inclusion $\mathcal{H}' \subset (\Bbbk W)'$ factorizes through an injective morphism

$$\Phi : \mathcal{H}' \hookrightarrow \mathcal{I} \oplus \left(\bigoplus_{\rho \in \mathcal{E}/\approx} \mathfrak{sl}(V_\rho) \right) \oplus \left(\bigoplus_{\rho \in \mathcal{F}/\approx} \mathfrak{osp}(V_\rho) \right).$$

The central result of this article is the following

Theorem 2.21. *Unless if $W = H_4$ the morphism Φ is an isomorphism.*

For dihedral goups, all irreducible representations except the 1-dimensional ones are 2-dimensional reflection representations, hence $\mathcal{H}' \simeq \mathcal{I}$. This theorem has been proved for Coxeter types A_n in [Ma07a], and $I_2(m)$ in [Ma04]. It is sufficient to prove this theorem for $\Bbbk = \mathbb{C}$, which we assume from now on.

We will show that, in order to prove this theorem for a given $W \neq F_4$, it is sufficient to show the following : for all $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$ such that $\dim \rho > 1$, we have $\rho(\mathcal{H}') = \mathcal{L}(\rho)$ (see corollary 2.24). For this we need a few preliminary results.

Using the classification of complex reflection groups and of their representations, we will provide later a proof of the following lemma (see lemmas 7.3, 7.6, 7.4).

Lemma 2.22. *Let* $\rho \in \mathrm{Irr}(W)$*.*

(1) *If* $\dim \rho = 4$ *and* $\eta \otimes \epsilon \hookrightarrow S^2\rho^*$ *for some* $\eta \in \mathrm{X}(\rho)$ *then* W *is a Coxeter group of type* F_4*.*
(2) *If* $\dim \rho = 8$ *and* $\eta \otimes \epsilon \hookrightarrow S^2\rho^*$ *for some* $\eta \in \mathrm{X}(\rho)$ *then* W *is a Coxeter group of type* H_4*.*

Moreover, if $\dim \rho = 6$ *and* $\eta \otimes \epsilon \hookrightarrow S^2\rho^*$ *for some* $\eta \in \mathrm{X}(\rho)$*, then* W *has rank 4 and* $\rho \in \Lambda\mathrm{Ref}$*.*

Proposition 2.23. *Assume* $W \not\in \{F_4, H_4\}$*. If* $\rho \in \mathrm{Irr}(W)$ *satisfies* $\mathcal{L}(\rho) \simeq \mathcal{H}(\rho)$ *then* $\mathcal{H}(\rho)$ *is a simple ideal. If* $\rho_1, \rho_2 \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$ *satisfy* $\mathcal{L}(\rho_i) \simeq \mathcal{H}(\rho_i)$ *for* $i \in \{1, 2\}$ *, then* $\rho_1 \approx \rho_2 \Leftrightarrow \mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$*.*

Proof. The first part of the statement comes from the fact that the exceptional (nonsimple) cases $\mathfrak{so}_2$ and $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$ do not occur when W is not of type F_4, by lemmas 2.11 and 2.22. Assume now $\rho_1, \rho_2 \not\in \Lambda\mathrm{Ref}$. If $\rho_1 \approx \rho_2$ then $(\rho_1)_{\mathcal{H}'} \simeq (\rho_2)_{\mathcal{H}'}$ or $(\rho_1)_{\mathcal{H}'} \simeq ((\rho_2)_{\mathcal{H}'})^*$ by proposition 2.7 hence $\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$. Assume now $\mathcal{L}(\rho_i) \simeq \mathcal{H}(\rho_i)$ and $\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$. Then $\dim \rho_1 = \dim \rho_2 = N$, because the exceptional case $\mathfrak{so}_6 \simeq \mathfrak{sl}_3$ is excluded by the condition $\rho_i \not\in \Lambda\mathrm{Ref}$, by lemma 2.22, and the case $\mathfrak{so}_3 \simeq \mathfrak{sl}_2$ is also excluded because only $\mathfrak{so}_{2n}$ occur, by the hyperbolicity property. We first consider the case $\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2) \simeq \mathfrak{sl}_N(\Bbbk)$ with $N \geq 3$. This Lie algebra admits only two non isomorphic irreducible representations of dimension N. But $\rho_1, \rho_1^* \otimes \epsilon$ and ρ_2 are non-isomorphic irreducible representations of $\mathcal{H}'$ that factorize through $\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$, unless $\rho_1 \approx \rho_2$. We now assume $N = 2$ or $\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$ is either orthogonal or symplectic. Then this Lie algebra admits only one irreducible representation of dimension N, hence $\rho_1 \approx \rho_2$, except if it is of Cartan type D_4. But this does not happen here by lemma 2.22, since W is not of type H_4. ☐

Corollary 2.24. *Assume* $W \not\in \{F_4, H_4\}$*. If* $\forall \rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$ $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$*, then* Φ *is an isomorphism.*

Proof. Being semisimple, $\mathcal{H}'$ is a direct sum of its simple ideals. Moreover Φ is injective and factorizes, by definition of $\approx$, through the sum of the $\mathcal{H}(\rho)$, for $\rho \in \mathrm{Irr}(W)$. We first prove that, if $\rho_1 \in \Lambda\mathrm{Ref}$ and $\rho_2 \not\in \Lambda\mathrm{Ref}$, then $\mathcal{H}(\rho_1) \neq \mathcal{H}(\rho_2)$. Firstly we have $\rho_1 \not\approx \rho_2$ since, by remark 2.14, $\Lambda\mathrm{Ref}$ is the union of equivalence classes for $\approx$. By definition of $\Lambda\mathrm{Ref}$ and proposition 2.13 we can assume $\rho_1 \in \mathrm{QRef}$, hence $\mathcal{H}(\rho_1) \simeq \mathfrak{sl}_n$ with $n = \dim \rho_1$. If $\mathcal{H}(\rho_2) \simeq$

$\mathfrak{sl}_n$ then $n = \dim \rho_2$ since by lemma 2.22 the situation $\mathfrak{sl}_3 \simeq \mathfrak{so}_6$ is excluded by $\rho_2 \notin \Lambda\mathrm{Ref}$. Since $\mathfrak{sl}_n$ admits at most two irreducible representations of dimension n, $\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$ implies that, as a representation of $\mathcal{H}'$, ρ_2 is isomorphic either to ρ_1 or its dual, hence $\rho_1 \approx \rho_2$, a contradiction.

If $\rho_1, \rho_2 \notin \Lambda\mathrm{Ref}$, then proposition 2.23 says $\rho_1 \approx \rho_2 \Leftrightarrow \mathcal{H}(\rho_1) = \mathcal{H}(\rho_2)$. By proposition 2.20 it follows that $\mathcal{H}'$ is the direct sum of the $\mathcal{H}(\rho)$ for $\rho \in \overline{\mathrm{QRef}} \sqcup (\mathcal{E}/\approx) \sqcup (\mathcal{F}/\approx)$. Since $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ for these ρ by proposition 2.20 and the hypothesis, we get that $\mathcal{H}'$ is isomorphic to the direct sum of these $\mathcal{L}(\rho)$. By equality of dimension this proves that Φ is an isomorphism. □

2.5. **Further Lie-theoretic properties.** The following general lemma on $\mathcal{H}(\rho)$ will be technically useful for our purpose.

Lemma 2.25. *Let W be an irreducible complex reflection group whose reflections are all conjugated. Assume furthermore that, for all $\rho \in \mathrm{Irr}(W)$, $\dim \rho \leq 3$ implies $\dim \rho = 1$. Then, for all $\rho \in \mathrm{Irr}(W)$, $\mathcal{H}(\rho)$ does not admit a simple ideal of rank 1.*

Proof. Let $\rho : W \to \mathrm{GL}(V_\rho)$ be an irreducible representation of W. We let $\mathfrak{g} = \rho(\mathcal{H}') \simeq \mathcal{H}(\rho)$, and assume by contradiction that $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{g}_0$ as a Lie algebra. We thus have $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{g} \subset \mathfrak{gl}(V_\rho)$. Under the action of W, the Lie algebra $\mathfrak{g}$ is an invariant subspace of $\mathfrak{gl}(V_\rho) = V_\rho \otimes V_\rho^*$. Moreover, since $\mathfrak{sl}_2(\mathbb{C})$ is an ideal of $\mathfrak{g}$, it is invariant by the adjoint action of $\mathfrak{g}$ hence by W because of the relation $\mathrm{ad}(s)^2 = 2(\mathrm{Id} - \mathrm{Ad}(s))$ for $s \in \mathcal{R}$. It follows that $\mathfrak{sl}_2(\mathbb{C})$ is a 3-dimensional representation of W. By assumption, it is a direct sum of 1-dimensional ones. On the other hand, the multiplicity of a 1-dimensional representation of W in $V_\rho \otimes V_\rho^*$ is at most 1 by irreducibility of V_ρ and Schur's lemma. It follows that $\mathfrak{sl}_2(\mathbb{C})$ can be decomposed as a direct sum of 3 distinct 1-dimensional representations of W. But there are at most two such representations since $\#\mathrm{Hom}(W, \{\pm 1\}) = 2$, hence a contradiction. □

The argument to prove the theorem is by induction. For this we need to relate the induction table between two reflection groups with the induction table of the corresponding Lie algebras. This is partly done by the following general lemma. Recall that, when W has a single (conjugacy) class of reflections, then $\mathrm{X}(\rho) = \{\mathbb{1}\}$ as soon as $\dim \rho > 1$.

Lemma 2.26. *Let W be an irreducible reflection group. Assume that it admits an irreducible reflection subgroup $W_0 \subset W$, different from H_4 and F_4, which has a single class of reflections, and for which the statement of theorem 2.21 holds true. Let $\rho \in \mathrm{Irr}(W)$ and $r \geq 1$ such that $\forall \varphi \in \mathrm{Irr}(W_0)$ $(\mathrm{Res}_{W_0}\rho | \varphi) \leq r$. Then, for any simple Lie ideal $\mathfrak{h}$ of $\rho(\mathcal{H}'_{W_0})$, there exists $J \in \mathrm{Irr}(\mathfrak{h})$ such that $\dim \mathrm{Hom}_{\mathfrak{h}}(J, \mathrm{Res}_{\mathfrak{h}} \rho_{\mathcal{H}'}) \leq r$.*

Proof. For convenience we denote here $\mathcal{H}'_0 = \mathcal{H}'_{W_0}$, $\mathcal{H}' = \mathcal{H}_W$ and let $\mathcal{H}_0(\varphi)$ denote the orthogonal of $\mathrm{Ker}\,(\varphi_{\mathcal{H}'_0})$ for the Killing form of $\mathcal{H}'$, for $\varphi \in \mathrm{Irr}(W_0)$. Let $\mathfrak{h}$ be a simple Lie ideal of $\rho(\mathcal{H}'_0)$. We let $a_1\rho^1 + \dots a_t\rho^t$ be a decomposition of $\mathrm{Res}_{W_0}\rho$ in nonisomorphic simple components, with $1 \leq a_i \leq r$. By proposition 2.7 (1) we know that each $\rho^i_{\mathcal{H}'_0}$ is irreducible. We can assume that $\dim \rho^i > 1$ for $i \leq s$ and $\dim \rho^i = 1$ for $s < i \leq t$. The simple

Lie ideals of $\rho(\mathcal{H}'_0)$ are the $\mathcal{H}_0(\rho^i)$ for $i \leq s$ by proposition 2.23, since $W_0 \notin \{F_4, H_4\}$ and theorem 2.21 holds for W_0. Assuming that there exists a simple Lie ideal $\mathfrak{h}$ of $\rho(\mathcal{H}'_0)$, we have $s \geq 1$ and we can assume $\mathfrak{h} = \mathcal{H}_0(\rho^1)$. We have $\rho^1_{\mathcal{H}'_0} \simeq \rho^i_{\mathcal{H}'_0}$ iff $\rho^i = \rho^1$ by proposition 2.7 (3), since $\mathrm{Hom}(W_0, \{\pm 1\}) = \{\mathbb{1}, \epsilon\}$. Since $\rho^1_{\mathcal{H}'_0}$ is irreducible and $\rho^i_{\mathcal{H}'_0}$ factorizes through a well-defined simple Lie ideal when $i \leq s$, it follows that $\dim \mathrm{Hom}_{\mathfrak{h}}(\mathrm{Res}_{\mathfrak{h}} \rho^1_{\mathcal{H}'_0}, \mathrm{Res}_{\mathfrak{h}} \rho^i_{\mathcal{H}'_0}) = 0$ for $i \neq 1$. In, particular, letting J be an irreducible component of $\mathrm{Res}_{\mathfrak{h}} \rho^1$, we get $\dim \mathrm{Hom}_{\mathfrak{h}}(J, \mathrm{Res}_{\mathfrak{h}} \rho_{\mathcal{H}'}) = a_1 \leq r$. ☐

2.6. **Compact form.** In this section, for $\Bbbk = \mathbb{C}$, we investigate the compact real form of the reductive Lie algebra $\mathcal{H}_W$.

Let $\mathcal{H}^{\mathbb{R}}_W \subset \mathbb{R}W$ denote the infinitesimal Hecke algebra defined over $\mathbb{R}$, and $\mathcal{H}^c_W$ the *real* Lie subalgebra of $\mathbb{C}W$ generated by the elements $\mathrm{i}s$, for $s \in \mathcal{R}$. Note that conjugation by $s \in \mathcal{R}$ can be written $\mathrm{Id} + (1/2)\mathrm{ad}(\mathrm{i}s)^2$, hence $\mathcal{H}^c_W$ is also generated by the elements $\mathrm{i}s$ for $s \in \mathcal{S}$, for $\mathcal{S} \subset \mathcal{R}$ as above. In particular, if W is a Coxeter group, it is also generated by the $\mathrm{i}s$ for s running among the simple reflections w.r.t. a chosen Weyl chamber.

We let $W^{\pm} = \{w \in W \mid \epsilon(W) = \pm w\}$. Clearly, $W = W^+ \sqcup W^-$ and $\mathbb{R}W = \mathbb{R}W^+ \oplus \mathbb{R}W^-$.

Proposition 2.27. *$\mathcal{H}^c_W$ is a compact real form of $\mathcal{H}_W$, and*

$$\mathcal{H}^c_W = \left(\mathcal{H}^{\mathbb{R}}_W \cap \mathbb{R}W^+\right) \oplus \mathrm{i}\left(\mathcal{H}^{\mathbb{R}}_W \cap \mathbb{R}W^-\right).$$

Proof. Let $\mathcal{H}^{c'}_W = (\mathcal{H}^{\mathbb{R}}_W \cap \mathbb{R}W^+) \oplus \mathrm{i}(\mathcal{H}^{\mathbb{R}}_W \cap \mathbb{R}W^-)$, and let $\mathcal{L}$ be the real linear span of the $w - \epsilon(w)w^{-1}, w \in W$ inside $\mathbb{R}W$. It is easily checked that $\mathcal{L}$ is a Lie subalgebra of $\mathbb{R}W$. One clearly has $\mathcal{L} = (\mathcal{L} \cap \mathbb{R}W^+) \oplus (\mathcal{L} \cap \mathbb{R}W^-)$. The Lie algebra $\mathcal{H}_W$ is spanned over $\mathbb{C}$ by elements of the form $[s_1, [s_2, \dots, [s_{r-1}, s_r] \dots]$ for $s_i \in \mathcal{R}$. Such elements belong either to $\mathbb{R}W^+$ (r even) or $\mathbb{R}W^-$ (r odd). We thus get $\mathcal{H}^{\mathbb{R}}_W = (\mathcal{H}^{\mathbb{R}}_W \cap \mathbb{R}W^+) \oplus (\mathcal{H}^{\mathbb{R}}_W \cap \mathbb{R}W^-)$ and $\mathcal{H}^{\mathbb{R}}_W \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H}^{c'}_W \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H}_W$. Moreover, $\mathcal{H}^c_W \subset \mathcal{H}^{c'}_W$. Since the $\mathrm{i}s$ for $s \in \mathcal{R}$ generate $\mathcal{H}_W$ over $\mathbb{C}$, we get equality of dimensions hence $\mathcal{H}^c_W = \mathcal{H}^{c'}_W$. It remains to show that this real Lie algebra is compact.

Consider the natural W-invariant quadratic form on $\mathbb{R}W$ given by $(w_1, w_2) = \delta_{w_1, w_2}$ if $w_1, w_2 \in W$. We extend it to an hermitian form still denoted $(\ ,\ )$ over $\mathbb{C}W$. It defines a real bilinear form over $\mathbb{C}W$, for which $\mathbb{C}W^+$ and $\mathbb{C}W^-$ are orthogonal (real) subspaces. Finally, we restrict it to $\mathbb{R}W^+ \oplus \mathrm{i}\mathbb{R}W^-$. It is easily checked to be symmetric and positive definite. In order to prove that $\mathcal{H}^c_W$ is compact, we prove that this form is invariant under the adjoint action of $\mathcal{H}^c_W \subset \mathbb{R}W^+ \oplus \mathrm{i}\mathbb{R}W^-$.

Since $\mathcal{H}^c_W \subset (\mathcal{L} \cap \mathbb{R}W^+) \oplus \mathrm{i}(\mathcal{L} \cap \mathbb{R}W^-)$, it is enough to prove its invariance under the elements of the Lie algebra $\mathbb{C}W$ of the form $w - w^{-1}$ and $\mathrm{i}(w + w^{-1})$, for $w \in W$. This calculation is straightforward and left to the reader. ☐

3. Inductive properties of semisimple Lie algebras

Let $\mathfrak{h}, \mathfrak{g}$ be semisimple complex Lie algebras such that $\mathfrak{h} \subset \mathfrak{g}$, and choose an irreducible *faithful* representation of $\mathfrak{g}$. Equivalently, one may assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{sl}(U)$, for some finite-dimensional complex vector space U. Note that this implies $\operatorname{rk}\mathfrak{g} \leq \dim(U) - 1$, where $\operatorname{rk}\mathfrak{g}$ denotes the semisimple rank of $\mathfrak{g}$.

We will use the following lemma to show that $\mathfrak{g} = \mathfrak{sl}(U)$.

Lemma 3.1. *(see* [Ma07b]*, proposition 3.8) Let U be a finite-dimensional complex vector space, and $\mathfrak{g}$ be a sub-Lie-algebra of $\mathfrak{sl}(U)$ acting irreductibly on U. If $\operatorname{rk}(\mathfrak{g}) > \frac{\dim(U)}{2}$ then $\mathfrak{g} = \mathfrak{sl}(U)$.*

In order to identify $\mathfrak{g}$ with an orthogonal or symplectic Lie algebra, we will first use the following two lemmas in order to show that $\mathfrak{g}$ is simple, and then use the classification of simple Lie algebras and of their representations.

Lemma 3.2. *(see* [Ma07a]*, lemme 15) Let U be a finite-dimensional $\mathbb{C}$-vector space, and $\mathfrak{h} \subset \mathfrak{g}$ be two semisimple Lie subalgebras of $\mathfrak{sl}(U)$. Assume the following properties*

1. *U is irreducible as a $\mathfrak{g}$-module.*
2. *The restriction of U to every simple ideal of $\mathfrak{h}$ admits an irreducible component of multiplicity 1.*
3. *One has $\operatorname{rk}(\mathfrak{g}) < 2\operatorname{rk}(\mathfrak{h})$.*

Then $\mathfrak{g}$ is a simple Lie algebra.

In more subtle cases, we will use the following stronger form.

Lemma 3.3. *Let U be a finite-dimensional $\mathbb{C}$-vector space, and $\mathfrak{h} \subset \mathfrak{g}$ be two semisimple Lie subalgebras of $\mathfrak{sl}(U)$. We let $r = \operatorname{rk}\mathfrak{h}$, and $\mathfrak{h} = \prod_{j\in J}\mathfrak{h}_j$ the decomposition of $\mathfrak{h}$ as a product of its simple ideals. For all $j \in J$, we let*

$$m_j = \gcd\{\dim \operatorname{Hom}_{\mathfrak{h}_j}(R, \operatorname{Res}_{\mathfrak{h}_j} U) \ \mid \ R \in \operatorname{Irr}(\mathfrak{h}_j)\}, r_j = \operatorname{rk}\mathfrak{h}_j$$

If U is an irreducible $\mathfrak{g}$-module and $\dim(U) < (r+1)^2$, then $\mathfrak{g}$ is a simple Lie algebra if either

$$(I)\forall j \in J \ \ m_j = 1 \qquad or\ (II)\left\{\begin{array}{ll} 1) & \forall j \in J \ \ m_j \leq 2 \\ 2) & \mathfrak{g} \text{ does not have a simple ideal of rank 1}\end{array}\right.$$

Proof. We decompose $\mathfrak{g} = \prod_{i\in I}\mathfrak{g}_i$ as a product of simple ideals. We let q_i be the composite of $\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_i$, and we let $\mathfrak{g}^{(i)} = \prod_{j\neq i}\mathfrak{g}_j$. We say that $\mathfrak{g}_i$ is *special* if $\operatorname{Ker} q_i \neq \{0\}$.

Assume that there are no special ideal. Then $\mathfrak{h}$ embeds into every $\mathfrak{g}_i$. By decomposing the irreducible $\mathfrak{g}$-module U as a tensor product of irreducible $\mathfrak{g}_i$-modules U_i, we get $\dim U_i \geq \operatorname{rk}\mathfrak{g}_i + 1 \geq r+1$ hence

$$\dim U = \prod_{i\in I}\dim U_i \geq (r+1)^{\#I} \Rightarrow \#I \leq \frac{\log\dim U}{\log(r+1)} < \frac{\log(r+1)^2}{\log(r+1)} = 2.$$

It follows that $\#I = 1$, which shows that $\mathfrak{g}$ is a simple Lie algebra. We are thus reduced to show that $\mathfrak{g}$ has no special ideal.

If $\mathfrak{g}_i$ is such a special ideal, then $\operatorname{Ker} q_i$ contains one of the simple ideals $\mathfrak{h}_j$ of $\mathfrak{h}$. Since U is an irreducible $\mathfrak{g}$-module, it can be decomposed as $U \simeq U_i \otimes U^{(i)}$ with U_i and $U^{(i)}$ irreducible faithful representations of $\mathfrak{g}_i$ and $\mathfrak{g}^{(i)}$, respectively. In particular, $\dim U_i \geq 2$. Moreover,the image of $\mathfrak{h}_j$ in $\mathfrak{g}_i$ is $\{0\}$, thus $\mathfrak{h}_j$ act trivially on U_i and $\operatorname{Res}_{\mathfrak{h}_j} U = (\dim U_i)\operatorname{Res}_{\mathfrak{h}_j} U^{(i)}$. It follows that

$\dim U_i$ divides every $\dim \mathrm{Hom}_{\mathfrak{h}_j}(R, \mathrm{Res}_{\mathfrak{h}_j} U)$ for R an irreducible $\mathfrak{h}_j$-module, hence their gcd m_j.

Under assumption (I), this implies $\dim U_i = 1$, a contradiction. Hence (I) implies that there are no special ideal. We now assume (II). We have $\dim U_i = 2$. But only rank 1 simple Lie algebras admit an irreducible representations of dimension 2, so we get a contradiction.

□

Our conventions on the numbering of heighest weights are the ones of [FH].

Lemma 3.4. *(see* [Ma07a]*, lemme 14) The couples $(\mathfrak{g}, \varpi)$ with $\mathfrak{g}$ a rank n simple complex Lie algebra and ϖ the highest weight of a N-dimensional irreducible representation of $\mathfrak{g}$ such that $2n \leq N < 4n$ are, for all $n \geq 2$, (B_n, ϖ_1) if $N = 2n+1$, (C_n, ϖ_1) if $N = 2n$, and, for all $n \geq 3$, (D_n, ϖ_1) if $N = 2n$ plus, for $n \leq 6$, the following exceptional couples.*

Rank 1 : (A_1, ϖ_1) *for* $N = 2$, $(A_1, 2\varpi_1)$ *for* $N = 3$.
Rank 2 : (G_2, ϖ_1) *for* $N = 7$, (B_2, ϖ_2) *for* $N = 4$, (C_2, ϖ_1) *for* $N = 5$.
Rank 3 : (B_3, ϖ_3) *for* $N = 8$, (A_3, ϖ_2) *for* $N = 6$.
Rank 4 : (D_4, ϖ_3) *and* (D_4, ϖ_4) *for* $N = 8$, (A_4, ϖ_2) *and* (A_4, ϖ_3) *for* $N = 10$.
Rank 5 : (A_5, ϖ_2) *and* (A_5, ϖ_4) *for* $N = 15$, (D_5, ϖ_4) *and* (D_5, ϖ_5) *for* $N = 16$.
Rank 6 : (A_6, ϖ_2) *and* (A_6, ϖ_5) *for* $N = 21$.

For further use, we note the following facts in the above list of exceptions. Starting from rank 4 we have $N \geq 8$, and the representations are not selfdual except for (D_4, ϖ_3) and (D_4, ϖ_4) : their dual is a different exception also noted in the list. On the contrary, before rank 4 we have $N \leq 8$ and all representations are selfdual. In particular, if the representation is self dual then the rank is at most 4 and $N \leq 8$, otherwise the rank is at least 4 and $N \geq 10$.

Moreover, note that some of the exceptions noted above are simply reminiscent from the well-known exceptional isomorphisms $A_3 \simeq D_3$, $B_2 \simeq C_2$.

When we know in advance that the rank of $\mathfrak{g}$ is large enough, we will use the following lemma.

Lemma 3.5. *Let $\mathfrak{g}$ be a simple (complex) Lie subalgebra of $\mathfrak{sl}(U)$ which acts irreductibly on U, and such that $\mathrm{rk}\,\mathfrak{g} > \frac{\dim U}{5}$. Then*

1. *if $\mathrm{rk}\,\mathfrak{g} \geq 10$ and $U \not\simeq U^*$, then $\mathfrak{g} = \mathfrak{sl}(U)$;*
2. *if $\mathrm{rk}\,\mathfrak{g} \geq 6$ and $U \simeq U^*$, then $\mathfrak{g} \simeq \mathfrak{sp}(U)$ or $\mathfrak{g} \simeq \mathfrak{so}(U)$.*

Proof. It is easily checked that, for $n \geq 11$, $\mathfrak{sl}_n(\mathbb{C})$ admits no other irreducible representation of dimension less than $5(n+1)$ than $\mathbb{C}^n$ and its dual. Similarly, only the n-dimensional irreducible representations of the Lie algebras of Cartan type B_n, C_n, D_n have dimension less than $5n$, provided that $n \geq 6$; moreover, these representations are selfdual. This proves (1), as there are no exceptional simple Lie algebra of rank at least 10. Moreover, the irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$, $6 \leq n \leq 9$, which have dimension at most $5(n+1)$, are easily checked not to be selfdual. In order to prove (2) we thus only need to exclude the exceptional types E_6, E_7, E_8. The representations of E_7 and E_8 corresponding to the fundamental weights have dimension at least $56 > 5 \times 8$, so these are excluded. The simple Lie algebra of type E_6 admits exactly two representations in this range, which are dual one to the other, and thus do not satisfy the assumptions of (2). □

In small rank, we will need a specific result for 6-dimensional representations.

Lemma 3.6. *Let $\mathfrak{g} \to \mathfrak{sl}(U)$ be a faithful 6-dimensional irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. Assume that there exists $\mathfrak{h} \subset \mathfrak{g}$ with $\mathfrak{h} \simeq \mathfrak{sl}_2$ such that $\mathrm{Res}_{\mathfrak{h}} U$ admits multiplicities and also an irreducible component occuring with multiplicity 1. Then $\mathfrak{g}$ is simple. Moreover, $\mathfrak{g} \to \mathfrak{sl}(U)$ is not selfdual iff $\mathfrak{g} \simeq \mathfrak{sl}(U)$ and otherwise either $\mathfrak{g} \simeq \mathfrak{so}(U)$ or $\mathfrak{g} \simeq \mathfrak{sp}(U)$.*

Proof. Let $\mathfrak{g} = \mathfrak{g}^1 \times \cdots \times \mathfrak{g}^p$ be a decomposition of $\mathfrak{g}$ in simple factors. Then U can be decomposed as $U_1 \otimes \cdots \otimes U_p$ with $\mathfrak{g}^i$ acting faithfully and irreducibly on U_i. Since $\mathfrak{g}^i$ is simple we have $\dim U_i \geq 2$ and get $\dim U \geq 2^p$ hence $p = 2$ if $\mathfrak{g}$ is not simple. In that case, since $\dim U = 6$ we can assume $\dim U_1 = 2$ and $\dim U_2 = 3$. It follows that $\mathfrak{g}^1 \simeq \mathfrak{sl}_2$. Let $\pi_i : \mathfrak{g} \twoheadrightarrow \mathfrak{g}^i$ and $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$ denote the canonical morphisms, and let $\varphi_i = \pi_i \circ \iota$. If $\varphi_1 = 0$, we would have $\mathfrak{h} \subset \mathfrak{g}^2$ and

$$\mathrm{Res}_{\mathfrak{h}} U = \mathrm{Res}_{\mathfrak{h}} \mathrm{Res}_{\mathfrak{g}^2} U = \mathrm{Res}_{\mathfrak{h}} (\dim U_1) U_2 = (\dim U_1) \mathrm{Res}_{\mathfrak{h}} U_2$$

contradicting the existence of a multiplicity 1 component. It follows that $\varphi_1 \neq 0$ and similarly we show $\varphi_2 \neq 0$. Since $\mathfrak{h} \simeq \mathfrak{sl}_2$ is simple it follows that φ_1 and φ_2 are injective.

We let $[p]$ denote the $(p+1)$-dimensional irreducible representation of $\mathfrak{h} \simeq \mathfrak{sl}_2$. Since $\mathrm{Res}_{\mathfrak{h}} U_1$ and $\mathrm{Res}_{\mathfrak{h}} U_2$ are faithful we have $\mathrm{Res}_{\mathfrak{h}} U_1 = [1]$ and $\mathrm{Res}_{\mathfrak{h}} U_2 \in \{[2], [0] + [1]\}$, hence

$$\mathrm{Res}_{\mathfrak{h}} U_1 \otimes U_2 \in \{[1] \otimes [2], [1] \otimes ([0] + [1])\} = \{[1] + [3], [1] + [0] + [2]\}$$

contradicting the presence of multiplicities in $\mathrm{Res}_{\mathfrak{h}} \rho$.

By contradiction, it follows that $\mathfrak{g}$ is simple. Since $\mathfrak{g} \subset \mathfrak{sl}_6$ we have $1 \leq \mathrm{rk}\, \mathfrak{g} \leq 5$. If $\mathrm{rk}\, \mathfrak{g} = 1$ we would have $\mathfrak{g} = \mathfrak{h}$ contradicting either the irreducibility of the action of $\mathfrak{g}$ or the presence of multiplicities in $\mathrm{Res}_{\mathfrak{h}} U$. If $\mathrm{rk}\, \mathfrak{g} > 3$ then $\mathfrak{g} = \mathfrak{sl}(U)$ by lemma 3.1 and we get the conclusion.

We can thus assume $\mathrm{rk}\, \mathfrak{g} \in \{2, 3\}$. If $\mathrm{rk}\, \mathfrak{g} = 3$, $\mathfrak{g}$ cannot be of type B_3 because $\mathfrak{so}_7$ does not admit an irreducible representation of dimension 6, hence is either of Cartan type $A_3 \simeq D_3$ or C_3. The only possibilities for U to be 6-dimensional and irreducible imply $U \simeq U^*$ as $\mathfrak{g}$-modules and $\mathfrak{g} \simeq \mathfrak{so}(U)$ or $\mathfrak{g} \simeq \mathfrak{sp}(U)$.

There remains to rule out the case $\mathrm{rk}\, \mathfrak{g} = 2$. The cases G_2 and $C_2 = B_2$ are excluded because they do not admit 6-dimensional irreducible representations. We thus have $\mathfrak{g} \simeq \mathfrak{sl}_3$. Now an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3$ corresponds to a 3-dimensional faithful representation of $\mathfrak{sl}_2$, either $[2]$ or $[0] + [1]$. Letting U_0 denote one of the standard (3-dimensional) representations of $\mathfrak{sl}_3 \simeq \mathfrak{g}$ we have $\mathrm{Res}_{\mathfrak{h}} U_0 \in \{[2], [0] + [1]\}$. On the other hand, 6-dimensional irreducible representations of $\mathfrak{sl}_3$ are isomorphic either to $S^2 U_0$ or its dual. But $S^2[2] = [4] + [0]$ and

$$S^2\left([0] + [1]\right) = S^2[0] + [0] \otimes [1] + S^2[1] = [0] + [1] + [2]$$

are both multiplicity-free, contradicting our assumption. This concludes the proof of the lemma. □

4. Representations of $G(e,1,r), G(e,e,r)$ and $G(2e,e,r)$

Recall from the Shephard-Todd classification that an irreducible (finite) *pseudo*-reflection group W of rank at least 2 belongs either to a finite list of 34 exceptions, labelled from G_4 to G_{37}, or to an infinite series $G(de,e,r)$ with 3 integral parameters with $(d,e,r) \neq (1,2,2)$. The group $G(de,e,r)$ is defined to be the set of $r \times r$ monomial complex matrices with entries in $\mu_{de} = \mu_{de}(\mathbb{C})$ whose product of nonzero entries belongs to μ_d. It has order $r! d^r e^{r-1}$, and rank r if $(d,e) \neq (1,1)$.

The reflection groups in this infinite series are the groups $G(e,e,r)$ and $G(2e,e,r)$. The pseudo-reflection groups $G(e,1,r)$ are used as auxiliary tools in the construction of their representations, as $G(e,e,r)$ is a normal index e subgroup of $G(e,1,r)$. Similarly, $G(2e,2e,r)$ is an index 2 subgroup of $G(2e,e,r)$, which has index e in $G(2e,1,r)$. The number of conjugacy classes is given by the following lemma, which is standard and easy to check.

Lemma 4.1. *If W is an irreducible reflection group of type $G(e,e,r)$ then $\#\mathcal{R}/W \leq 2$, and $\#\mathcal{R}/W = 1$ for $r \geq 3$. If it has type $G(2e,e,r)$ then $\#\mathcal{R}/W \leq 3$ and $\#\mathcal{R}/W \leq 2$ if $r \geq 3$.*

4.1. **Preliminaries.** Let $r \geq 1$. As usual, we label irreducible representations of the symmetric group $\mathfrak{S}_r$ by partitions $\lambda = [\lambda_1, \dots, \lambda_s]$ with $\lambda_i \in \mathbb{Z}_{>0}$ and $\lambda_i \geq \lambda_{i+1}$ of total size $|\lambda| = \sum \lambda_i = r$, choosing for convention that $[r]$ labels the trivial representation of $\mathfrak{S}_r$. We let λ' denote the conjugate partition of λ, defined by $\lambda'_i = \#\{j \ ; \ \lambda_j \geq i\}$. We denote $\emptyset$ the only partition of r, identify when needed a partition with its Young diagram, using the convention that the diagram $[3,2]$ has two rows and three columns. When convenient, we also identify partitions with the irreducible representation of the adequate symmetric group labelled by it.

Letting λ, μ be two partitions, we use the notation $\mu \subset \lambda$ if $\forall i \ \mu_i \leq \lambda_i$ and $\mu \nearrow \lambda$ for $\mu \subset \lambda$ and $|\lambda| = |\mu| + 1$. In terms of Young diagrams, this means that μ is deduced from λ by removing one box. Young's rule states that, under the usual inclusion $\mathfrak{S}_r \subset \mathfrak{S}_{r+1}$, the restriction of an irreducible representation λ of $\mathfrak{S}_{r+1}$ to $\mathfrak{S}_r$ is the direct sum of all $\mu \nearrow \lambda$, and in particular the number of irreducible components of this restriction is $\delta(\lambda) = \#\{i \mid \lambda_i > \lambda_{i+1}\}$.

Let then $e \geq 1$. The group is isomorphic to the wreath product $\mathbb{Z}/e\mathbb{Z} \wr \mathfrak{S}_r = \mathfrak{S}_r \ltimes (\mathbb{Z}/e\mathbb{Z})^r$, whence its irreducible representations are indexed by multipartitions $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{e-1})$ of total size $|\boldsymbol{\lambda}| = |\lambda^0| + \dots + |\lambda^{e-1}| = r$. We let $p(\boldsymbol{\lambda}) = \#\{i \mid 0 \leq i \leq e-1, \lambda^i \neq \emptyset\}$, and define $\delta(\boldsymbol{\lambda}) = \sum \delta(\lambda^i)$. It is the number of irreducible compoents in the (multiplicity-free) restriction of $\boldsymbol{\lambda}$ to its natural subgroup $G(e,e,r-1)$, if $|\boldsymbol{\lambda}| = r$. Indeed, these components corresponds to multipartitions $\boldsymbol{\mu}$ of $r-1$ such that $\mu^i = \lambda^i$ for all but one i, for which $\mu^i \nearrow \lambda^i$. We use the notation $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$ in that case.

The cyclic group $\mathbb{Z}/e\mathbb{Z}$ acts on the set of irreducibles of $G(e,1,r)$ by cyclically permuting the parts of a multipartion $\boldsymbol{\lambda}$, and so does its subgroup $2\mathbb{Z}/e\mathbb{Z}$ when e is an even integer. We denote $\mathrm{Aut}(\boldsymbol{\lambda})$ the fixer of $\boldsymbol{\lambda}$ under the former action, and $\mathrm{Aut}^0(\boldsymbol{\lambda})$ the fixer under the latter. Let $A(\boldsymbol{\lambda})$ and $B(\boldsymbol{\lambda})$ denote the orders of $\mathrm{Aut}(\boldsymbol{\lambda})$ and $\mathrm{Aut}^0(\boldsymbol{\lambda})$, respectively. By Clifford theory, $A(\boldsymbol{\lambda})$ is the number of irreducible components of the restriction to $G(e,e,r)$ of the irreducible representation $\boldsymbol{\lambda}$ of $G(e,1,r)$. Similarly, if $\boldsymbol{\lambda}$ is an irreducible representation of $G(2e,1,r)$, then $B(\boldsymbol{\lambda})$ is the number of irreducible components of its restriction to $G(2e,e,r)$. Starting from $\boldsymbol{\lambda} \in \mathrm{Irr}(G(2e,1,r))$, we clearly have $A(\boldsymbol{\lambda})/B(\boldsymbol{\lambda}) \in \{1,2\}$. Moreover, if ρ is an irreducible component of the restriction of $\boldsymbol{\lambda}$ to $G(2e,e,r)$, then $B(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda})$ if and only if ρ has irreducible restriction to $G(2e,2e,r)$, and otherwise has two distinct components.

If ρ is an irreducible representation of $G(e,e,r)$, it is an irreducible component of the restriction of some $\boldsymbol{\lambda} = (\lambda^0,\dots,\lambda^{e-1})$ of $G(e,1,r)$. We leave to the reader to check that $\rho^* \otimes \epsilon$ is then an irreducible component of $((\lambda^{e-1})',\dots,(\lambda^0)')$.

4.2. **Basic facts.** We fix $e \geq 1$. In order to make estimates of dimensions, we will make special use of *binary representations* associated to a subset I of $\{0,1,\dots,e-1\}$. The corresponding representation $b(I) = \boldsymbol{\lambda}$ of $G(e,1,\#I)$ is defined by $\lambda^i = [1]$ is $i \in I$ and $\lambda^i = \emptyset$ otherwise. Letting $r = \#I$, the restriction of $b(I)$ to $G(e,1,r-1)$ is the sum of all $b(I')$ for $I' \subset I$ with $\#I' = r-1$. By induction we get $\dim b(I) = r! = (\#I)!$. We define a partial order $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ on multipartitions by $\mu^i \subset \lambda^i$ for all $0 \leq i \leq i-1$.

From the branching rule, the following three facts are clear. Removing the empty parts of $\boldsymbol{\lambda}$, we get a representation of $G(p(\boldsymbol{\lambda}),1,|\boldsymbol{\lambda}|)$ of the same dimension ; also, the dimension increases with respect to $\subset$; finally, since any $\boldsymbol{\lambda}$ contains a (unique) binary multipartition $b(I)$ with $\#I = p(\boldsymbol{\lambda})$, we get $\dim \boldsymbol{\lambda} \geq p(\boldsymbol{\lambda})!$ by restriction to $G(e,1,p(\boldsymbol{\lambda}))$, with equality only if $\boldsymbol{\lambda}$ itself is a binary multipartition. Similarly, if $0 \leq i \leq e-1$ is such that $\lambda^i \neq \emptyset$, then the restriction to $G(e,1,|\lambda^i|)$ of $\boldsymbol{\lambda}$ contains at least $(p(\boldsymbol{\lambda})-1)!$ copies of $\boldsymbol{\mu}$ such that $\mu^i = \lambda^i$ and $\mu^j = \emptyset$ if $j \neq i$. In particular $\dim(\boldsymbol{\lambda}) \geq (p(\boldsymbol{\lambda})-1)!\dim(\lambda^i)$.

In order to deal with the automorphism group of $\boldsymbol{\lambda}$, we will need the following lemma.

Lemma 4.2. *Let $\boldsymbol{\lambda}$ be a irreducible representation of $G(e,1,r)$.*

(1) *$A(\boldsymbol{\lambda})$ divides e, r and $p(\boldsymbol{\lambda})$.*
(2) *If $A(\boldsymbol{\lambda}) \neq \{1\}$ and $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$ then $A(\boldsymbol{\mu}) = \{1\}$*

Proof. $A(\boldsymbol{\lambda})$ divides e because $\mathrm{Aut}(\boldsymbol{\lambda})$ is a subgroup of $\mathbb{Z}/e\mathbb{Z}$. Let $A(\boldsymbol{\lambda}) = e/b$. Then $\lambda^{i+b} = \lambda^i$ hence $r = A(\boldsymbol{\lambda})\sum_{i=0}^{b-1}|\lambda^i|$ and $p(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda})\#\{i \mid 0 \leq i|eqb-1, \lambda^i \neq \emptyset\}$, whence the conclusion of (1). Part (2) is proved in [Ma07c], prop. 3.1. ☐

As a consequence, we get the following rough estimates on the dimensions of irreducible representations of $G(e,e,r)$.

Lemma 4.3. *Let ρ be an irreducible component of the restriction to $G(e,e,r)$ of a representation $\boldsymbol{\lambda}$ of $G(e,1,r)$. Then $\dim\rho \geq (p(\boldsymbol{\lambda})-1)!$ and $\dim\rho \geq p(\boldsymbol{\lambda})!$ as soon as $\exists i\ \lambda^i \notin \{\emptyset,[1]\}$.*

Proof. If $A(\boldsymbol{\lambda}) = 1$ we have $\dim\rho = \dim\boldsymbol{\lambda} \geq p(\boldsymbol{\lambda})!$ and we are done. We thus can assume $A(\boldsymbol{\lambda}) \neq 1$.

e	r	$\dim \rho$	$A(\boldsymbol{\lambda})$	$p(\boldsymbol{\lambda})$	λ^{i_1} or $(\lambda^{i_1})'$	λ^{i_2} or $(\lambda^{i_2})'$	λ^{i_3} or $(\lambda^{i_3})'$
$e \geq 1$	$r \geq 3$	$r-1$	1	1	$[r-1,1]$		
$e \geq 2$	$r \geq 3$	r	1	2	$[r-1]$	$[1]$	
$e \geq 2$	4	2	1	1	$[2,2]$		
$3\|e$	3	2	3	3	$[1]$	$[1]$	$[1]$
$2\|e$	4	3	2	2	$[2]$	$[2]$	

TABLE 1. Quasi-reflection representations of $G(e,e,r)$ for $r \geq 3$.

By lemma 4.2 for all $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$ we have $A(\boldsymbol{\mu}) = 1$. It follows that the restriction to $G(e,e,r-1)$ of ρ contains the restriction of all these $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$, for which $\dim(\boldsymbol{\mu}) \geq p(\boldsymbol{\mu})!$. Moreover, $p(\boldsymbol{\mu}) = p(\boldsymbol{\lambda})$ unless $\boldsymbol{\mu}$ is deduced from $\boldsymbol{\lambda}$ by removing a box at some place i with $\lambda^i = [1]$, in which case $p(\boldsymbol{\mu}) = p(\boldsymbol{\lambda}) - 1$. The conclusion follows. ☐

Finally, the following is lemma is standard and easily proved.

Lemma 4.4. *If W has type $G(e,e,2)$ or $G(2e,e,2)$, then $\dim \rho \leq 2$ for all $\rho \in \mathrm{Irr}(W)$. In particular $\mathrm{Irr}(W) = \Lambda\mathrm{Ref}(W)$.*

4.3. **Quasi-reflection representations of $G(e,e,r)$, $r \geq 3$.** Recall from lemma 4.1 that $G(e,e,r)$ admits only one conjugacy class of reflections for $n \geq 3$. The goal of this section is to determine the representations in $\Lambda\mathrm{Ref}$ for the groups $G(e,e,r)$.

We first recall that the groups $G(e,e,r)$ admit a distinguish set of generating reflections $s_1', s_1, s_2, \dots, s_r$, where s_k is the permutation matrix corresponding to the transposition $(k, k+1)$ and s_1' is the conjugate of s_1 by the diagonal matrix $\mathrm{diag}(\exp(\frac{2\mathrm{i}\pi}{e}), 1, \dots, 1)$. If e_1 divides e_2, there exists a natural morphism from $G(e_2,e_2,r)$ to $G(e_1,e_1,r)$ that maps each generator to the generator with the same name. In matrix terms it is deduced from the ring homomorphism of $\mathbb{Z}[\zeta]$ that maps ζ to $\zeta^{\frac{e_2}{e_1}}$, where $\zeta = \exp(2\mathrm{i}\pi/e_2)$. If $\boldsymbol{\lambda}$ is an irreducible representation of $G(e_1,1,r)$ which is restricted to $G(e_1,e_1,r)$, then the representation of $G(e_2,e_2,r)$ deduced from this morphism is the restriction of $\boldsymbol{\mu} \in \mathrm{Irr}G(e_2,1,r)$ described by $\mu^{\frac{e_2}{e_1}k} = \lambda^k$ for all k, and $\mu^k = \emptyset$ if k is not divisible by e_2/e_1. Conversely, the restriction of $\boldsymbol{\mu} \in \mathrm{Irr}G(e_2,1,r)$ factorizes through $G(e_1,e_1,r)$ iff $j-i$ is divisible by e_2/e_1 whenever $\mu^i, \mu^j \neq \emptyset$. All this is easily checked from the explicit formulas of [Ar] or [MM] (see formulas (3.2) there). Obviously, through these natural morphisms, elements of $\Lambda\mathrm{Ref}$ for $G(e_1,e_1,r)$ induce elements of $\Lambda\mathrm{Ref}$ for $G(e_2,e_2,r)$.

We prove the following.

Proposition 4.5. *The quasi-reflection representations ρ of $G(e,e,r)$ for $r \geq 3$ are given in table 1, where $\boldsymbol{\lambda}$ is an irreducible representation of $G(e,1,r)$ whose restriction to $G(e,e,r)$ contains ρ.*

Proof. If ρ is a representation of $G(e,e,r)$, we let $\mathtt{a}(\rho)$ and $\mathtt{b}(\rho)$ denote the multiplicity of 1 and -1, respectively, in the spectrum of a reflection. By abuse of notation, if $\boldsymbol{\lambda}$ is a representation of $G(e,1,r)$, we denote $\mathtt{a}(\boldsymbol{\lambda})$ and $\mathtt{b}(\boldsymbol{\lambda})$ the multiplicities for the restriction to $G(e,e,r)$.

First note that $\mathtt{a}(\boldsymbol{\lambda})$ and $\mathtt{b}(\boldsymbol{\lambda})$ are increasing functions of $\boldsymbol{\lambda}$ for the order $\subset$. Also note that they only depend on the collection of non-empty partitions (with multiplicities) that form $\boldsymbol{\lambda}$. Finally, note that, if ρ_1, ρ_2 are two components of the same $< \boldsymbol{\lambda} >$, then it readily follows from the formulas 3.3 of [MM] that $\mathtt{a}(\rho_1) = \mathtt{a}(\rho_2)$ and $\mathtt{b}(\rho_1) = \mathtt{b}(\rho_2)$.

We assume that ρ is a quasi-reflection representation. If $\rho =< \boldsymbol{\lambda} >$ is irreducible then this case has been dealt with in [MM] – specifically, the arguments in §5 there justify the first three lines of table 1. Otherwise, ρ embeds in some representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ with $a = A(\boldsymbol{\lambda}) \neq 1$. We let $\alpha = e/a$, hence $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{\alpha-1}, \lambda^0, \dots)$.

Assume that there exists distincts $i, j \in [0, \alpha-1]$ such that $\lambda^i \neq \emptyset$ and $\lambda^j \neq \emptyset$. We refer to [ArKo] or [MM] for the definitions of multitableaux and standard multitableaux of shape $\boldsymbol{\lambda}$, and recall that the representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ has a basis indexed by all standard multitableaux of shape $\boldsymbol{\lambda}$. We also recall that s_1 acts by 1 (resp. -1) on such a standard multitableau $\mathbf{T}$ if 1 and 2 are placed on the same line (resp. the same column) of some part of $\mathbf{T}$, and that otherwise the multitableau $\mathbf{T}'$ deduced from $\mathbf{T}$ by exchanging 1 and 2 is also standard, the plane spanned by $\mathbf{T}$ and $\mathbf{T}'$ is stable under s_1, which acts with eigenvalues $1, -1$ on this plane.

We define 2α standard multitableaux $\mathbf{T}^{u,\pm}$ of shape $\boldsymbol{\lambda}$, for $u \in [0, a-1]$, in the following way. For $\mathbf{T}^{u,+}$, we place 1 in position $i + \alpha u$ and 2 in position $j + \alpha u$. For $\mathbf{T}^{u,-}$, we place 1 in position $j + \alpha u$ and 2 in position $i + \alpha u$. Then we fill in the remaining boxes, with numbers from 3 to r, in a uniform way. The subspace of $\boldsymbol{\lambda}$ generated by these is stable under $s_1 \in G(e,1,r)$, and $1, -1$ are eigenvalues with multiplicity a. Since $a \geq 2$ we can also choose multitableaux $\mathbf{T}^+$ and $\mathbf{T}^-$ such that 1 lies in position i and 2 in position $i + \alpha$ for $\mathbf{T}^+$ and in the reverse order for $\mathbf{T}^-$ – the filling for the remaining numbers being the same. The action of s_1 on the plane generated by $\mathbf{T}^+$ and $\mathbf{T}^-$ has two eigenvalues, 1 and -1. Since this plane is in direct sum with the precedingly defined subspace, it follows that $\mathtt{a}(\boldsymbol{\lambda}) \geq a+1$ and $\mathtt{b}(\boldsymbol{\lambda}) \geq a+1$. In particular $\mathtt{a}(\rho) = \mathtt{a}(\boldsymbol{\lambda})/a \geq 1 + \frac{1}{a}$ and similarly $\mathtt{b}(\rho) \geq 1 + \frac{1}{a}$, contradicting the assumption that ρ is a quasireflection representation.

We thus can assume that only one $i \in [0, \alpha-1]$ satifies $\lambda^i = \lambda \neq \emptyset$. Assume $\lambda \supset [2,1]$ and consider 2α multitableaux $\mathbf{T}^{u,\pm}$ for $u \in [0, a-1]$ defined by placing 1 and 2 in position $i + \alpha u$, the number 2 being placed above or on the right of 1 depending on $\pm$, the filling for the remaining numbers being uniform. As before, we get 1 and -1 as eigenvalues for the action of s_1 on this subspace with multiplicity a. Considering the plane generated by $\mathbf{T}^+$ and $\mathbf{T}^-$ defined in the same way, we get the same contradiction.

It follows that, up to tensorization by the sign character, we can assume $\lambda = [m]$ for some $m \geq 1$. In that case there are two types of multitableaux $\mathbf{T}$ of shape $\boldsymbol{\lambda}$:

(1) 1 and 2 are placed in the same position $i + \alpha u$ for some $u \in [0, a-1]$
(2) 1 and 2 are placed in positions $i + \alpha u$ and $i + \alpha u'$ with $u, u' \in [0, a-1]$ and $u \neq u'$.

Multitableaux of type (1) are fixed par s_1. The multiplicities of 1 and -1 for the action of s_1 on the subspace generated by all multitableaux of type (2) are the same. If $m \geq 2$, define $\boldsymbol{\lambda}'$ by replacing $[m]$ with $[m-2]$ in position i

of $\boldsymbol{\lambda}$. If $m=1$ we note $\dim \boldsymbol{\lambda}' = 0$. Define $\boldsymbol{\lambda}''$ by replacing $[m]$ with $[m-1]$ in position i and $i+\alpha$ in $\boldsymbol{\lambda}$. The number of multitableaux of type (1) is $a \dim \boldsymbol{\lambda}'$, and the number of multitableaux of type (2) is $a(a-1)\dim \boldsymbol{\lambda}''$. It follows that $\mathtt{a}(\rho) = \frac{a-1}{2}\dim \boldsymbol{\lambda}'' + \dim \boldsymbol{\lambda}'$ and $\mathtt{b}(\rho) = \frac{a-1}{2}\dim \boldsymbol{\lambda}''$. In particular $\mathtt{a}(\rho) \geq \mathtt{b}(\rho)$ and ρ is a quasireflection representation iff $\mathtt{b}(\rho) = 1$.

If $m=1$, then $\boldsymbol{\lambda}''$ is a binary representation of size $a-2$, hence $(a-1)\dim \boldsymbol{\lambda} = (a-1)!$ and $\mathtt{b}(\rho) = 1$ iff $(a-1)! = 2$, that is $a=3$ hence $r=3$. It follows that ρ factorizes through one of the (2-dimensional) irreducible components of the restriction to $G(3,3,3)$ of $([1],[1],[1])$. If $m=2$, then $p(\boldsymbol{\lambda}'') = p(\boldsymbol{\lambda}) = a$ hence $(a-1)!\dim \boldsymbol{\lambda}'' \geq a!(a-1) = 2$ iff $a=2$ and $\boldsymbol{\lambda}''$ is a binary representation, whence $m=2, r=4$ and ρ factorizes through one of the (3-dimensional) components of the restriction of $([2],[2])$ to $G(2,2,4)$. □

We are now able to determine $\Lambda\mathrm{Ref}$.

Proposition 4.6. *If ρ is an irreducible representation of $G(e,e,r)$ for $r \geq 3$ with $\dim \rho > 1$ which belongs to $\Lambda\mathrm{Ref} \setminus \mathrm{QRef}$, then ρ is the restriction of an irreducible representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ and, either $p(\boldsymbol{\lambda}) = 1$ and there exists $\lambda^i = [r-p, 1^p]$ for some $p \in [2, r-3]$, or $p(\boldsymbol{\lambda}) = 2$ and there exist $i \neq j$ with $\lambda^i = [r-p]$, $\lambda^j = [1^p]$ for some $p \in [2, r-2]$.*

Proof. Let R be a reflection representation of $G(e,e,r)$ and $\rho = \eta \otimes \Lambda^k R \in \Lambda\mathrm{Ref}$. Assuming $\rho \notin \mathrm{QRef}$ and $\dim \rho > 1$, this implies $\dim R \geq 4$. By table 1 it follows that R is the restriction of some representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ with $A(\boldsymbol{\lambda}) = 1$, $p(\boldsymbol{\lambda}) \leq 2$ and $\dim \boldsymbol{\lambda} \in \{r-1, r\}$, which implies $r \geq 4$. We can assume $\lambda^0 \neq \emptyset$. There are two cases to consider. If $p(\boldsymbol{\lambda}) = 1$, then R factors through the representation of $G(1,1,r) = \mathfrak{S}_r$ corresponding to the partition $\lambda^0 = [r-1,1]$, and it is well-known that $\Lambda^k[r-1,1] = [r-k, 1^k]$ as representation of $\mathfrak{S}_r$, which proves half of the proposition. If $p(\boldsymbol{\lambda}) = 2$, we can assume $\lambda^0 = [r-1]$ and $\lambda^s = [1]$ for some $1 \leq s \leq e-1$.

We let $\boldsymbol{\mu}(k,r,s)$ denote the multipartition $\boldsymbol{\mu}$ with $|\boldsymbol{\mu}| = r$, $p(\boldsymbol{\mu}) \leq 2$ such that $\mu^0 = [r-k]$, $\mu^s = [1^k]$. In particular, $\boldsymbol{\lambda} = \boldsymbol{\mu}(1,r,s)$. We prove that, as a representation of $G(e,1,r)$, $\Lambda^k \boldsymbol{\mu}(1,r,s) = \boldsymbol{\mu}(k,r,s)$ for $0 \leq k \leq r$. This will prove the second part of the proposition. Since $\boldsymbol{\mu}(0,r,s)$ is the trivial representation of $G(e,1,r)$, we can assume $k \geq 1$. The proof is then by induction on r, the case $r=2$ being easily checked from the matrix models of [ArKo] or [MM]. We thus assume $r \geq 3$. If $k < r$, then by the branching rule the restriction to $G(e,1,r-1)$ of $\boldsymbol{\mu}(1,r,s)$ is $\mathbb{1} + \boldsymbol{\mu}(1,r-1,s)$, hence the restriction of $\Lambda^k \boldsymbol{\mu}(1,r,s)$ is $\Lambda^k \boldsymbol{\mu}(1,r-1,s) + \Lambda^{k-1}\boldsymbol{\mu}(1,r-1,s)$, which is $\boldsymbol{\mu}(k,r-1,s) + \boldsymbol{\mu}(k-1,r-1,s)$ by the induction hypothesis. On the other hand, by Steinberg theorem we know that $\Lambda^k \boldsymbol{\mu}(1,r,s)$ is irreducible hence corresponds to some multipartition of size r that contains both $\boldsymbol{\mu}(k,r-1,s)$ and $\boldsymbol{\mu}(k-1,r-1,s)$. The only possibility being $\boldsymbol{\mu}(k,r,s)$ this concludes the proof, the case $k=r$ being similar and left to the reader. □

5. Induction process

In this section we assume that W is an irreducible reflection group and W_0 is a proper reflection subgroup of W, meaning that W_0 is a proper subgroup of W which is generated by reflections of W, for which the theorem holds. We let $\mathcal{R}_0 = W_0 \cap \mathcal{R}$ denote the set of reflections of W_0.

For convenience, for $\rho \in \mathrm{Irr}(W)$ we let $\overline{\rho}$ or $\mathrm{Res}\rho$ denote the restriction of ρ to W_0. We denote $(\ |\ )$ the standard scalar product on the representation ring, and the notation $\varphi \nearrow \rho$ means $(\overline{\rho}|\varphi) \geq 1$ for $\rho \in \mathrm{Irr}(W)$ and $\varphi \in \mathrm{Irr}(W_0)$. Note that this notation is consistent with the one introduced in the previous section for (multi-)partitions, when $W = G(d,1,r)$ and $W_0 = G(d,1,r-1)$. We let $\Lambda\mathrm{Ref} = \Lambda\mathrm{Ref}(W)$ and denote $\mathrm{Ind}\Lambda\mathrm{Ref} = \{\varphi \in \mathrm{Irr}(W) \mid \exists \varphi \in \Lambda\mathrm{Ref}(W_0)\ \varphi \nearrow \rho\}$. If $\overline{\rho} = a_1\rho_1 + \cdots + a_r\rho_r$, with $a_i \in \mathbb{Z}_{>0}$ and ρ_i non-isomorphic irreducible representations of W_0, we let $soc(\overline{\rho})$ denote $\rho_1 + \cdots + \rho_r$. We use the notation $\mathcal{H}'_0, \mathcal{H}_0(\varphi)$ for $\mathcal{H}'_{W_0}, \mathcal{H}_{W_0}(\varphi)$ and extend the definition 2.10 of $\mathcal{H}_0(\varphi)$ to non-irreducible representations so that, if $\overline{\rho}$ is decomposed as above, then $\mathcal{H}_0(\overline{\rho}) = \mathcal{H}_0(soc(\overline{\rho})) = \mathcal{H}_0(\rho_1) + \cdots + \mathcal{H}_0(\rho_r) \subset \mathcal{H}'_0$. We still have $\mathcal{H}_0(\overline{\rho}) \simeq \overline{\rho}(\mathcal{H}'_0)$, and $\mathcal{H}_0(\overline{\rho})$ naturally embeds into $\mathcal{H}(\rho)$.

In §1 we prove several basic induction lemmas under the assumption that W_0 admits a single class of reflections. In §2 we use them to prove that, once we know how to prove the theorem for any group W with a single class of reflections, then we can prove the general case. This enables us to assume that, if W belongs to the infinite series, then W has type $G(e,e,r)$ with $r \geq 3$. In §3 we show a few preliminary results for specific induction patterns and in §4 we deal with the case $r = 3$, so that we can assume $r \geq 4$ in the sequel. Then we put $W_0 = G(e,e,r-1)$ and assume that the theorem holds for W_0. We prove under this assumption that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$, first for $\rho \in \mathrm{Ind}\Lambda\mathrm{Ref} \setminus \Lambda\mathrm{Ref}$ in §5, then for $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$ in §6. This implies that the theorem holds for W by corollary 2.24, and proves the theorem for the groups $G(e,e,r)$ by induction on the rank. Finally, §7 deals with the 15 exceptional groups.

5.1. **Induction results in the single case.** In the following lemmas we assume that W_0 admits only one conjugacy class of reflections. A consequence of this assumption is that non-isomorphic irreducible representations of W_0 of dimension at least 2 correspond to non-isomorphic irreducible representations of $\mathcal{H}_{W_0}$ by corollary 2.8. Another consequence is that W_0 is generated by the conjugacy class in W_0 of any reflection $s \in W_0$.

Lemma 5.1. *Assume that W_0 admits a single conjugacy class of reflections. Then $\Lambda\mathrm{Ref} \subset \mathrm{Ind}\Lambda\mathrm{Ref}$.*

Proof. Let $\rho \in \Lambda\mathrm{Ref}$ with $\dim\rho > 1$, the case $\dim\rho = 1$ being obvious. We have $\rho = \eta \otimes \Lambda^k\rho_0$ for some $\rho_0 \in \mathrm{Ref}$, $\eta : W \to \{\pm 1\}$ and $1 \leq k < \dim\rho_0$. The restriction $\overline{\eta}$ of η to W_0 is a morphism from W_0 to $\{\pm 1\}$. We choose $s \in \mathcal{R}_0$ and let $\overline{\rho_0}$ denote the restriction of ρ_0 to W_0.

Let us write $\overline{\rho_0} = \rho_1 \oplus \cdots \oplus \rho_r$ with $\rho_1, \ldots, \rho_r$ irreducible. We can assume $\dim\rho_1 \geq \dim\rho_i$ for $i \geq 1$. Since $\rho_0(s)$ is either a reflection or a scalar, there exists at most one i such that $\rho_i(s) \notin \{-1, 1\}$. Moreover, if $\rho_i(s) = \pm 1$ then $\dim\rho_i = 1$ since W_0 is generated by the conjugacy class of s. It follows that $\dim\rho_2 = \dim\rho_3 = \cdots = \dim\rho_r = 1$. We need to find a subrepresentation of $\overline{\rho}$ which belongs to $\Lambda\mathrm{Ref}(W_0)$.

We let $\mathbb{C}e_i$, U, and U_0 denote the underlying vector spaces of ρ_i for $i \geq 2$, ρ_1 and ρ_0, respectively. If $k \leq r-1$ then the vector $e_2 \wedge \cdots \wedge e_{k+1} \in \Lambda^k U_0$ is non-zero, and W_0 acts on it by $\overline{\eta} \otimes (\rho_2 \otimes \cdots \otimes \rho_{k+1})$, which is 1-dimensional hence belongs to $\Lambda\mathrm{Ref}(W_0)$. If $k \geq r$ then the subspace $\Lambda^{k+1-r} U \wedge e_2 \wedge \cdots \wedge e_r$ of $\Lambda^k U_0$ is non-zero ; indeed, we have $k+1-r \geq 1$ and $k \leq \dim U_0 = \dim U + r - 1 \Rightarrow k+1-r \leq \dim U$. Moreover, it is W_0-stable and W_0 acts on it by $(\Lambda^{k+1-r}\rho_1) \otimes (\overline{\eta} \otimes \rho_2 \otimes \cdots \otimes \rho_r)$, which belongs to $\Lambda\mathrm{Ref}(W_0)$. It follows that $\rho \in \mathrm{Ind}\Lambda\mathrm{Ref}$ in both cases. ☐

Lemma 5.2. *Assume that W_0 admits a single class of reflections. Let $\rho \in \mathrm{Irr}(W) \setminus \mathrm{Ind}\Lambda\mathrm{Ref}$. If $\forall \rho' \in \mathrm{Irr}(W_0)$ $(\overline{\rho}|\rho') \leq 1$ then $\mathcal{H}(\rho)$ is a simple Lie algebra. The same conclusion holds if $\forall \rho' \in \mathrm{Irr}(W_0)$ $(\overline{\rho}|\rho') \leq 2$ provided that W admits a single conjugacy class of reflections and $\forall \rho \in \mathrm{Irr}(W)$ $\dim \rho \leq 3 \Rightarrow \dim \rho = 1$.*

Proof. Let $soc(\overline{\rho})$ denote the (direct) sum of the irreducible components of $\overline{\rho}$ and $r = \mathrm{rk}\,\mathcal{H}_0(\overline{\rho})$. By assumption we have $\dim \rho \leq 2 \dim soc(\overline{\rho})$. Let $\mathfrak{h}$ denote one of the simple Lie ideals of $\mathcal{H}_0(\overline{\rho})$. By hypothesis on W_0 we have $\mathfrak{h} = \mathcal{H}_0(\varphi)$ for some $\varphi \nearrow \rho$. If $\varphi \simeq \varphi^* \otimes \epsilon$, we have $\dim \varphi = 2\,\mathrm{rk}\,\mathfrak{h}$. Otherwise, either $\varphi^* \otimes \epsilon \not\nearrow \rho$ and $\dim \varphi = \mathrm{rk}\,\mathfrak{h} + 1 \leq 2\,\mathrm{rk}\,\mathfrak{h}$ since $\mathrm{rk}\,\mathfrak{h} \geq 1$, or $\varphi^* \otimes \epsilon \nearrow \rho$ and $\dim \varphi + \dim \varphi^* \otimes \epsilon = 2\,\mathrm{rk}\,\mathfrak{h} + 2$. It follows that $\dim soc(\overline{\rho}) \leq 2r + d$ where $d = \#\{\varphi \nearrow \rho \mid \varphi^* \otimes \epsilon \not\simeq \varphi \text{ and } \varphi^* \otimes \epsilon \nearrow \rho\}$. Recall from lemma 2.18 that the 2-dimensional irreducible representations of W are always quasi-reflection representations. Thus the assumption $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$ implies $\dim \varphi \geq 3$ for all $\varphi \nearrow \rho$. In particular $\dim soc(\overline{\rho}) \geq 3d$ hence, by using $\dim soc(\overline{\rho}) \leq 2r + d$, we get $\frac{2}{3} \dim soc(\overline{\rho}) \leq 2r$. In case $\forall \rho' \in \mathrm{Irr}(W_0)$ $(\overline{\rho}|\rho') \leq 1$ it follows that $\dim \rho \leq 3r < (r+1)^2$, and the conclusion follows from lemma 3.3 using assumption (I). In the other case, if in addition $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) \geq 4$, we also get $\dim \rho \leq 6r < (r+1)^2$ and the conclusion follows from lemma 3.3 using assumption (II) and lemma 2.25.

We can thus assume that $r \leq 3$, that $\#\mathcal{R}/W = 1$ and that $\forall \rho \in \mathrm{Irr}(W)$ $\dim \rho \leq 3 \Rightarrow \dim \rho = 1$. We notice that, if $\mathfrak{h} = \mathcal{H}_0(\varphi)$ is a simple Lie ideal of $\mathcal{H}_0(\overline{\rho})$ afforded by $\varphi \nearrow \rho$, we have $\mathrm{rk}\,\mathcal{H}_0(\varphi) \geq 2$ because otherwise by the hypothesis on W_0 we would have $\dim \varphi = 2$ contradicting $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$. Since $r \leq 3$ it follows that $\mathcal{H}_0(\overline{\rho}) = \mathfrak{h}$ is simple and that $r \geq 2$. In particular $soc(\overline{\rho}) = \varphi$ or $soc(\overline{\rho}) = \varphi + \varphi^* \otimes \epsilon$ for some $\varphi \in \mathrm{Irr}(W_0)$ with $\dim \varphi \geq 3$. It follows that $\dim \rho \leq 4 \dim \varphi$. On the other hand, since $\varphi \notin \Lambda\mathrm{Ref}$ we have $\dim \varphi \in \{r+1, 2r\}$ hence $\dim \varphi \leq 2r$ and $\dim \rho \leq 8r \leq 24$.

Now, if $\mathcal{H}(\rho)$ is not simple, since it cannot have simple Lie ideals of rank 1 by lemma 2.25, its rank is at least 4. If we decompose $\mathcal{H}(\rho) \simeq \mathfrak{h}_1 \times \mathfrak{h}_2 \times \cdots \times \mathfrak{h}_m$ with $m \geq 2$, $\mathfrak{h}_i$ simple and $\mathrm{rk}\,\mathfrak{h}_i \geq 2$, then $\rho_{\mathcal{H}'} \simeq V_1 \otimes \cdots \otimes V_m$ with V_i an irreducible faithful representation of $\mathfrak{h}_i$. Since $\mathrm{rk}\,\mathfrak{h}_i \geq 2$ we have $\dim V_i \geq 3$ and $\dim \rho \geq 3^m$. Since $m \geq 2$ and $\dim \rho \leq 24$ it follows that $m = 2$. Moreover, $\rho \simeq \rho^* \otimes \epsilon \Leftrightarrow V_1 \simeq V_1^*$ and $V_2 \simeq V_2^*$. In that case, we actually have $\dim V_i \geq 4$, hence $\dim \rho \geq 16$. We will also use the following elementary facts : since $\mathrm{rk}\,\mathfrak{h}_i \geq 2$, if $\dim V_i = 3$ then $\mathfrak{h}_i$ has Cartan type A_2 ; if $\dim V_i = 4$, then either $\mathfrak{h}_i$ has Cartan type $B_2 = C_2$ if $V_i \simeq V_i^*$ or it has Cartan type A_3.

We denote by $\psi = \psi_1 \times \psi_2 : \mathcal{H}_0(\overline{\rho}) \to \mathcal{H}(\rho) \simeq \mathfrak{h}_1 \times \mathfrak{h}_2$ the natural inclusion. Notice that, since $\mathcal{H}_0(\overline{\rho})$ is simple, $\dim \mathcal{H}_0(\overline{\rho}) > \dim \mathfrak{h}_i$ or $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) > \mathrm{rk}\,\mathfrak{h}_i$ imply $\psi_i = 0$. On the other hand, $\overline{\rho}$ admits irreducible components with multiplicity at most 2 by assumption. Since W_0 has a single class of reflections, the same is true for $\overline{\rho}_{\mathcal{H}'_0} \simeq \mathrm{Res}_{\mathcal{H}'_0} V_1 \otimes V_2$. Let $i \in \{1,2\}$. Since $\dim V_i \geq 3$ it follows that $\mathcal{H}_0(\overline{\rho})$ is not included in any simple ideal of $\mathcal{H}(\rho)$, otherwise $\overline{\rho}$ would admit irreducible components of multiplicity at least 3, hence ψ_i is non-zero, hence injective. It follows that $\dim \mathcal{H}_0(\overline{\rho}) \leq \dim \mathfrak{h}_i$ and $\mathrm{rk}\, \mathcal{H}_0(\overline{\rho}) \leq \mathrm{rk}\, \mathfrak{h}_i$. In particular, if $r = 3$ we have $\mathrm{rk}\, \mathfrak{h}_i \geq 3$ hence $\dim V_i \geq 4$ and $\dim \rho \geq 16$.

Since the cases $\overline{\rho} = \varphi$ and $\overline{\rho} = \varphi + \varphi^* \otimes \epsilon$ with $\varphi \not\simeq \varphi^* \otimes \epsilon$ have already been tackled, only the following possibilities still have to be investigated : $\overline{\rho} = 2\varphi$ or $\overline{\rho} \in \{2\varphi + 2\varphi^* \otimes \epsilon, 2\varphi + \varphi^* \otimes \epsilon\}$ with $\varphi \not\simeq \varphi^* \otimes \epsilon$.

We will use the well-known fact that, if U_1, U_2 are n-dimensional irreducible representations of $\mathfrak{sl}_n$, namely the standard representation or its dual, then $U_1 \otimes U_2$ has 2 distinct irreducible components. It follows that, if $\mathcal{H}_0(\overline{\rho})$ has type A_r, then the case $\dim V_1 = \dim V_2 = r+1$ is excluded. Moreover, in that case the possibility $\dim V_1 = r+1$, $\dim V_2 = r+2$ is also excluded because a $(n+1)$-dimensional faithful representation of $\mathfrak{sl}_n$ has to be a direct sum of the trivial representation and of a n-dimensional irreducible representation, thus leading to 3 irreducible components.

We can now proceed to the separate study of the special cases.

- $\overline{\rho} = 2\varphi$. If $\varphi \not\simeq \varphi^* \otimes \epsilon$ we have $\dim \rho = 2r + 2 \leq 8 < 9 \leq (r+1)^2$ hence this case is handled by lemma 3.3 (II) for $\mathcal{H}_0(\overline{\rho})$. We thus can assume $\varphi \simeq \varphi^* \otimes \epsilon$, hence $\dim \rho = 4r \leq 12$. If $r = 2$ we still have $4r = 8 < 9 = (r+1)^2$ and we conclude as before. There only remains the case $r = 3$ and $\dim \rho = 12 < 16$, a contradiction.
- $\overline{\rho} = 2\varphi + \varphi^* \otimes \epsilon$ with $\varphi \not\simeq \varphi^* \otimes \epsilon$. Since $\varphi \not\simeq \varphi^* \otimes \epsilon$ we have $\dim \rho = 3(r+1)$. If $r = 3$, $\dim \rho < (r+1)^2$ and we conclude by lemma 3.3 (II). We thus assume $r = 2$. Then $\dim \rho = 9$ and $\dim V_1 = \dim V_2 = 3 = r+1$. It follows that $\mathcal{H}(\rho)$ has Cartan type $A_2 \times A_2$. Its Lie subalgebra $\mathcal{H}_0(\overline{\rho})$ has Cartan type A_2, thus leading to a contradiction by the remark above.
- $\overline{\rho} = 2\varphi + 2\varphi^* \otimes \epsilon$. If $r = 2$, we have $\dim \rho = 12$, hence we can assume $\dim V_1 = 3$, hence $\mathfrak{h}_1 \simeq \mathfrak{sl}_3$, and $\dim V_2 = 4$; moreover, $\mathcal{H}_0(\overline{\rho})$ has Cartan type A_2. Since ψ_1, ψ_2 are injective, ψ_1 is an isomorphism. Let E, E^* denote the two smallest faithful representations of $\mathfrak{sl}_3$. Since they have dimension 3, as a representation of $\mathcal{H}_0(\overline{\rho}) \simeq \mathfrak{sl}_3$ the representation V_2 can be decomposed as the trivial representation plus either E or E^*. Similarly, as a representation of $\mathcal{H}_0(\overline{\rho})$, $V_1 \simeq E$ or $V_1 \simeq E^*$. It follows that $\overline{\rho}$ should have 3 irreducible components, a contradiction.

  We thus can assume $r = 3$, hence $\dim \rho = 16$, $\dim V_1 = \dim V_2 = 4$. Moreover $\mathcal{H}_0(\overline{\rho})$ has Cartan type A_3, hence $\mathrm{rk}\, \mathfrak{h}_i \geq 3$ and $\mathfrak{h}_i \simeq \mathfrak{sl}_4$. It follows that the restriction of $V_1 \otimes V_2$ to $\mathcal{H}_0(\overline{\rho})$ is one of the inner tensor products $F \otimes F$, $F \otimes F^*$, $F^* \otimes F^*$ with F the standard representation of $\mathfrak{sl}_4$. The fact that these tensor products have only two irreducible components yields a contradiction.

□

Lemma 5.3. *Assume that W_0 admits a single class of reflections. Let $\rho \in \mathrm{Irr}(W) \setminus \mathrm{Ind}\Lambda\mathrm{Ref}$. If $\forall \rho' \in \mathrm{Irr}(W_0)$ we have $(\overline{\rho}|\rho') \leq 1$, then $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$.*

Proof. We decompose $\overline{\rho}$ in irreducible components

$$\overline{\rho} = \sum_{i=1}^{r} \rho_i + \sum_{i=1}^{s} \varphi_i + \varphi_i^* \otimes \epsilon + \sum_{i=1}^{t} \psi_i$$

with $\rho_i^* \otimes \epsilon \not\simeq \rho$, $\varphi_i \not\simeq \varphi^* \otimes \epsilon$ and $\psi_i \simeq \psi_i^* \otimes \epsilon$. We know that $\mathcal{H}(\rho)$ is simple by lemma 5.2. Moreover,

$$\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) = \sum_{i=1}^{r} (\dim \rho_i - 1) + \sum_{i=1}^{s} (\dim \varphi_i - 1) + \sum_{i=1}^{t} \frac{\dim \psi_i}{2} = \frac{\dim \rho}{2} + \left(\sum_{i=1}^{r} \frac{\dim \rho_i}{2} \right) - r - s.$$

It follows that $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) > \frac{\dim \rho}{4}$ as soon as

$$\frac{\dim \rho}{4} > r + s - \frac{1}{2} \sum_{i=1}^{r} \dim \rho_i$$

Since $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$ we have $\dim \rho_i \geq 3$ thus $\frac{\dim \rho_i}{2} \geq \frac{3}{2} > 1$ hence $r - \frac{1}{2} \sum_{i=1}^{r} \dim \rho_i < 0$ if $r \geq 1$. Similarly $\dim \varphi_i \geq 3$ hence $\dim \rho \geq 2 \sum_{i=1}^{s} \dim \varphi_i \geq 6s$ and $s \leq \frac{\dim \rho}{6} < \frac{\dim \rho}{4}$. It follows that $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) > \frac{\dim \rho}{4}$. Then $\mathcal{H}(\rho) = \mathcal{L}(\rho)$ by lemmas 3.1 and 3.4, at least if $\dim \rho \geq 22$ or $\mathrm{rk}\,\mathcal{H}(\rho) \geq 7$. To complete the proof, we can thus assume that $\dim \rho \leq 21$ and $\mathrm{rk}\,\mathcal{H}(\rho) \leq 6$. We first assume that $\rho_{\mathcal{H}'}$ is *not* selfdual. In particular $\rho \not\simeq \rho^* \otimes \epsilon$.

Our first task is to exclude the case $r = 0$, that is $\overline{\rho} \simeq \overline{\rho}^* \otimes \epsilon$. Assume by contradiction that $r = 0$. A first consequence is that, since each ψ_i has even dimension, then $\dim \rho$ is also even. Since $\rho_{\mathcal{H}'}$ is not selfdual, the remaining exceptions are

(1) $\mathrm{rk}\,\mathcal{H}(\rho) = 5$, $\mathcal{H}(\rho) \simeq \mathfrak{so}_{10}$, $\dim \rho = 16$.
(2) $\mathrm{rk}\,\mathcal{H}(\rho) = 4$, $\mathcal{H}(\rho) \simeq \mathfrak{sl}_5$, $\dim \rho = 10$.

Moreover, we have $2(s+t) \leq \mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) \leq 5$ hence $s + t \leq 2$. Note that $s + t$ is the number of simple Lie ideals of $\mathcal{H}_0(\overline{\rho})$. We first consider the case $s + t = 2$, that is $\mathcal{H}_0(\overline{\rho}) \simeq \mathfrak{h}_1 \times \mathfrak{h}_2$. We have $\mathrm{rk}\,\mathfrak{h}_i \in \{2, 3\}$ and $\mathrm{rk}\,\mathfrak{h}_1 + \mathrm{rk}\,\mathfrak{h}_2 \in \{4, 5\}$. Exhausting all possibilities we find that this is not possible if $\dim \rho = 16$, and that the only possibilities for $\dim \rho = 10$ are of Cartan type $C_3 \times C_2$, $D_3 \times C_2$ and $A_2 \times C_2$. Since $\dim \rho = 10 \Leftrightarrow \mathrm{rk}\,\mathcal{H}(\rho) = 4$ and $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) \leq \mathrm{rk}\,\mathcal{H}(\rho) = 4$, we have $\mathcal{H}_0(\overline{\rho}) \simeq \mathfrak{sl}_3 \times \mathfrak{sp}_4$. But $\mathfrak{sl}_3 \times \mathfrak{sp}_4$ does not embed into $\mathfrak{sl}_5$ (for instance because the smallest faithful representation of $\mathfrak{sl}_3 \times \mathfrak{sp}_4$ has dimension $3 + 4 = 7$), a contradiction.

Now we know that $r \geq 1$. By the calculation above we have $\mathrm{rk}\,\mathcal{H}(\rho) > \frac{\dim \rho}{2}$ as soon as $\sum_i \dim \rho_i > 2(r + s)$. Since $r \geq 1$ and $\dim \rho_i \geq 3$, if $s = 0$ then $\mathcal{H}(\rho) = \mathfrak{sl}(\rho) = \mathcal{L}(\rho)$ by lemma 3.1. We thus have $r \geq 1, s \geq 1$. Since every representations of W_0 involved here have dimension at least 3 we have $\dim \rho \geq 9$. On the other hand, since $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$ we know that $\mathcal{H}_0(\overline{\rho})$ does not contain Lie ideals of rank 1 hence $\mathrm{rk}\,\mathcal{H}(\rho) \geq \mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) \geq 2(r + s + t)$. On the other hand $\mathrm{rk}\,\mathcal{H}(\rho) \leq 6$ hence $r + s + t \leq 3$. Since $r, s \geq 1$ we also have $\mathrm{rk}\,\mathcal{H}(\rho) \geq 4$. Moreover the case $\mathrm{rk}\,\mathcal{H}(\rho) = 4$ can be ruled out, for it implies $r = s = 1$ and $t = 0$, thus $r + s = 2 < \frac{\dim \rho}{2}$ except if $\dim \rho_1 = 4$. But $\dim \rho_1 = 4$ implies $\mathrm{rk}\,\mathcal{H}(\rho) \geq \mathrm{rk}\,\mathcal{H}(\rho_1) + \mathrm{rk}\,\mathcal{H}(\rho_2) \geq 3 + 2 = 5$, a contradiction.

We are left with two cases : either $\mathrm{rk}\,\mathcal{H}(\rho) = 5$ or $\mathrm{rk}\,\mathcal{H}(\rho) = 6$. We first deal with the former one. If $r \geq 2$ or $s \geq 2$ or $t \geq 1$ we would have $\mathrm{rk}\,\mathcal{H}(\rho) \geq 6$, so we know that $r = 1, s = 1, t = 0$, with $\mathrm{rk}\,\mathcal{H}(\rho_1) = \mathcal{H}(\rho_2) = 2$, hence $\dim\rho = 3+3+3 = 9$. But there are no 9-dimensional representations in the list of lemma 3.4.

It follows that $\mathrm{rk}\,\mathcal{H}(\rho) = 6$ hence $\mathcal{H}(\rho) \simeq \mathfrak{sl}_7$ and $\dim\rho = 21$. If $t \neq 0$, from $6 \geq 2(r+s+t)$ we get $r = s = t = 1$ and $\mathrm{rk}\,\mathcal{H}(\rho_1) = \mathcal{H}(\varphi_1) = \mathcal{H}(\psi_1) = 2$ whence $\dim\rho = 3 + 2\times 3 + 4 = 13 \neq 21$. Hence $t = 0$.If $r + s = 3$ we also have $\mathrm{rk}\,\mathcal{H}(\rho_i) = \mathcal{H}(\varphi_j) = 2$ hence $\dim\rho_i = \dim\varphi_j = 3$ and $\dim\rho \leq 3(r+2s) \leq 9+3s \leq 15 < 21$ since $r \geq 1$ and $s \leq 2$. It follows that $r = s = 1$ and $t = 0$. Letting $\alpha = \mathrm{rk}\,\mathcal{H}(\rho_1)$ and $\beta = \mathrm{rk}\,\mathcal{H}(\varphi_1)$ we have $4 \leq \alpha+\beta \leq 6$ and $\dim\rho = \alpha+1+2(\beta+1) \leq 2(\alpha+\beta)+3 \leq 15 < 21$, again a contradiction.

We now assume that $\rho_{\mathcal{H}'}$ is selfdual. This implies that $\overline{\rho}_{\mathcal{H}'}$ is selfdual, hence $\overline{\rho} \simeq \overline{\rho}^* \otimes \epsilon$, since W_0 has a single class of reflections, and $r = 0$, $\mathcal{H}(\rho)$ is included either in $\mathfrak{sp}_N$ or $\mathfrak{so}_N$ with $N = \dim\rho$. The only selfdual representations in the list of lemma 3.4 are for $\dim\rho \leq 8$ and $\mathrm{rk}\,\mathcal{H}(\rho) \leq 4$. Moreover, if $\dim\rho = 8$ or $\mathrm{rk}\,\mathcal{H}(\rho) = 4$, this means that $\mathcal{H}(\rho)$ has type D_4 and in that case again $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ since it is included in $\mathfrak{so}_8$ or $\mathfrak{sp}_8$ and there are no inclusion from $\mathfrak{so}_8$ to $\mathfrak{sp}_8$. It follows that $\dim\rho \leq 6$ and $\mathrm{rk}\,\mathcal{H}(\rho) \leq 3$. Since $\mathcal{H}_0(\overline{\rho})$ has no ideal of rank 1 it follows that $s + t = 1$.

If $s = 1$, that is $\overline{\rho} = \varphi_1 + \varphi_1^* \otimes \epsilon$, from $\dim\rho \leq 6$ we get $\dim\varphi_1 \leq 3$ hence $\varphi_1 \in \Lambda\mathrm{Ref}(W_0)$ by lemma 2.18, and $\rho \in \mathrm{Ind}\Lambda\mathrm{Ref}$, a contradiction. We thus have $s = 0, t = 1, \overline{\rho} = \psi_1$.

If $\mathrm{rk}\,\mathcal{H}_0(\psi_1) = 2$ we have $\dim\psi_1 = \dim\rho = 4$, and the only exception is of type B_2, which is a fake exception since $B_2 \simeq C_2$. If $\mathrm{rk}\,\mathcal{H}_0(\psi_1) = 3$ we have $\dim\psi_1 = \dim\rho = 6$. But the only 6-dimensional irreducible representation of $\mathfrak{sl}_4$ is orthogonal, hence $\mathcal{H}(\rho) \simeq \mathfrak{so}_6 \simeq \mathfrak{sl}_4$, hence $\mathcal{L}(\rho) \simeq \mathcal{H}(\rho)$ in that case too. ☐

If $\overline{\rho} = a_1\rho_1 + \cdots + a_r\rho_r$ with $\rho_i \neq \rho_j$, we denote by $\delta(\rho) = a_1 + \cdots + a_r$ the number of irreducible components (counting multiplicities) of $\overline{\rho}$.

Lemma 5.4. *Asssume that W and W_0 admits a single class of reflections and that $\forall\rho \in \mathrm{Irr}(W)\ \dim\rho \leq 3 \Rightarrow \dim\rho = 1$. Let $\rho \in \mathrm{Irr}(W) \setminus \mathrm{Ind}\Lambda\mathrm{Ref}$, and $\zeta \nearrow \rho$. Assume $\delta(\rho) \geq 3$, $(\zeta|\overline{\rho}) \leq 2$, and $\rho' \nearrow \rho \Rightarrow \dim\rho' \geq 5$. If $\forall\rho' \in \mathrm{Irr}(W_0)\ \rho' \not\simeq \zeta \Rightarrow (\rho'|\overline{\rho}) \leq 1$ then $\mathcal{H}(\rho) = \mathcal{L}(\rho)$.*

Proof. If $(\zeta|\overline{\rho}) = 1$ then $\mathcal{H}(\rho) = \mathcal{L}(\rho)$ by lemma 5.3, so we can assume $(\zeta|\overline{\rho}) = 2$. By lemma 5.2 we know that $\mathcal{H}(\rho)$ is a simple Lie algebra. We decompose

$$\overline{\rho} = \zeta + \sum_{i=1}^{r}\rho_i + \sum_{i=1}^{s}(\varphi_i + \varphi_i^* \otimes \epsilon) + \sum_{i=1}^{t}\psi_i$$

with ζ isomorphic to one of the $\rho_i, \varphi_i, \psi_i$, with $\rho_i \not\simeq \rho_i^* \otimes \epsilon$, $\varphi_i \not\simeq \varphi_i^* \otimes \epsilon$, $\rho_i^* \otimes \epsilon \not\nearrow \rho$ and $\psi_i \simeq \psi_i^* \otimes \epsilon$. By assumption, we have $r + s + t \geq 2$.

Like in the proof of lemma 5.3, we have $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho}) > \frac{\dim\rho}{4}$ iff $\frac{\dim\rho}{4} > A$ where

$$A = r + s + \frac{\dim\zeta}{2} - \frac{1}{2}\sum_{i=1}^{r}\dim\rho_i.$$

From this proof, we also recall that $r - \frac{1}{2}\sum_{i=1}^{r} \dim \rho_i < 0$ if $r \geq 1$, since $\dim \rho_i \geq 5 \geq 3$. Likewise, $\dim \rho \geq 10s$ since $\dim \varphi_i \geq 5$ hence $s \leq \frac{\dim \rho}{10}$. Finally, $\dim \rho \geq 15$ since $\delta(\rho) \geq 3$.

There are three possibilities

(1) $\zeta \simeq \rho_{i_0}$ for some $i_0 \in [1, r]$. Since $\dim \rho_i \geq 5$ we have

$$r + \frac{\dim \zeta}{2} - \frac{1}{2}\sum_{i=1}^{r} \dim \rho_i = r - \frac{1}{2}\sum_{i \neq i_0} \dim \rho_i \leqslant r - \frac{5}{2}(r-1) = \frac{-3}{2}r + \frac{5}{2} = \frac{5-3r}{2} < 0$$

as soon as $r \geq 2$. In this case, $A < s \leq \frac{\dim \rho}{10} < \frac{\dim \rho}{4}$. If $r = 1$, then $A = s + 1 \leq 1 + \frac{\dim \rho}{10} < \frac{\dim \rho}{4}$ since $\dim \rho \geq 15$.

(2) $\zeta \simeq \psi_{i_0}$ for some $i_0 \in [1, t]$. In that case, $\dim \rho \geq 2 \dim \zeta + 10s$ hence $\frac{\dim \rho}{4} \geq \frac{\dim \zeta}{2} + s + \frac{3s}{2}$ and $A < \frac{\dim \rho}{4}$ as soon as $s > 0$ or $r > 0$. If $s = r = 0$ then $t \geq 2$, $\dim \rho \geq 2 \dim \zeta + 5 > 2 \dim \zeta$ and $A = \frac{\dim \zeta}{2} < \frac{\dim \rho}{4}$.

(3) $\zeta \simeq \varphi_{i_0}$ for some $i_0 \in [1, s]$. Then $\zeta^* \otimes \epsilon \nearrow \rho$ and $\dim \rho \geq 3 \dim \zeta + 10(s-1)$. It follows that

$$\frac{\dim \rho}{6} \geqslant \frac{\dim \zeta}{2} + \frac{5}{3}(s-1) = \frac{\dim \zeta}{2} + s + \frac{2}{3}s - \frac{5}{3} = s + \frac{\dim \zeta}{2} + \frac{2s-5}{3}$$

hence $A \leq \frac{\dim \rho}{6} + \frac{5-2s}{3} < \frac{\dim \rho}{4}$ since $\dim \rho \geq 15$ and $s \geq 1$.

To conclude we apply lemmas 3.1 and 3.4. The exceptions in lemma 3.4 that we have to rule out are for $\dim \rho \in \{15, 16, 21\}$, with $\rho \simeq \rho^* \otimes \epsilon$, and $\operatorname{rk} \mathcal{H}(\rho) \in \{5, 6\}$. Since $r + s + t \geq 2$, we have $\operatorname{rk} \mathcal{H}(\rho) \geq 4r + 4s + 3t > 3(r + s + t) \geq 6$ as soon as $(r, s) \neq (0, 0)$. If $(r, s) = (0, 0)$ then $\operatorname{rk} \mathcal{H}_0(\overline{\rho}) \geq 3t \geq 6$, with equality holding iff $t = 2$, $\operatorname{rk} \mathcal{H}(\psi_i) = 3$, $\dim \psi_i = 6$ and $\dim \rho = 18$, contradicting $\dim \rho \in \{15, 16, 21\}$. □

5.2. **Reduction to one conjugacy class.** The goal of this section is to show that, using the lemmas above, we can assume that W admits a single conjugacy class of reflections (except when W is a Coxeter group of type F_4 or H_4).

We use the Shephard-Todd classification to prove the following.

Lemma 5.5. *If Φ is an isomorphism for every irreducible $W \neq H_4$ with $\#\mathcal{R}/W = 1$, then Φ is an isomorphism for every irreducible $W \notin \{F_4, H_4\}$.*

Proof. Let $W \notin \{F_4, H_4\}$ with $\#\mathcal{R}/W > 1$. The only case in the exceptional series is for W of type G_{13}, for which we check by a direct computation of the dimensions that Φ is surjective. We thus assume that W belongs to the infinite series $G(2e, e, r)$ or $G(e, e, r)$. When $r = 2$, lemma 4.4 states that $\operatorname{Irr}(W) = \Lambda\mathrm{Ref}(W)$, and we know that Φ is an isomorphism in that case. Assuming $r \geq 3$, it follows from lemma 4.1 that W has type $G(2e, e, r)$ for some $e \geq 1$. Let W_0 be its natural reflection subgroup of type $G(2e, 2e, r)$, for which the theorem is assumed to hold. By corollary 2.24 we have to prove that, for any $\rho \in \operatorname{Irr}(W) \setminus \Lambda\mathrm{Ref}$, then $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$. Now W_0 has index 2 in W, hence by Clifford theory $\overline{\rho}$ has at most two irreducible components, and this decomposition is multiplicity-free. If $\rho \notin \mathrm{Ind}\Lambda\mathrm{Ref}$ the conclusion follows from lemma 5.3. We thus assume $\rho \in \mathrm{Ind}\Lambda\mathrm{Ref}$ and $\rho \notin \Lambda\mathrm{Ref}$.

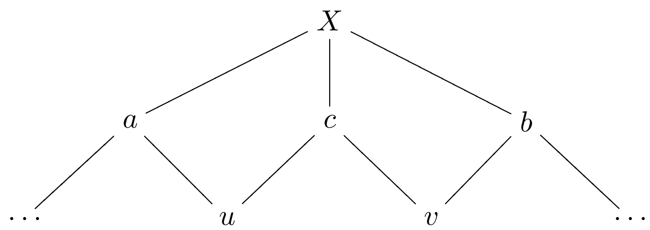
Let $\boldsymbol{\lambda}$ be an irreducible representation of $G(2e,1,r)$ whose restrictions contains ρ. In particular, its restriction to $G(2e,2e,r)$ belongs to $\Lambda\mathrm{Ref}(W_0)$, and is thus listed by table 1 and proposition 4.6. We check that, for $r \geq 5$, this implies $\rho \in \Lambda\mathrm{Ref}$, and that the only cases to consider are when $\boldsymbol{\lambda}$ has one of the two special shapes at the bottom of table 1 (the last two lines).

The first shape to deal with is for $r = 3$, in which case we have $A(\boldsymbol{\lambda}) = 3$. Since $A(\boldsymbol{\lambda})/B(\boldsymbol{\lambda}) \in \{1,2\}$ it follows that $B(\boldsymbol{\lambda}) = 3$ hence ρ has dimension 2 and belongs to $\mathrm{QRef} \subset \Lambda\mathrm{Ref}$. The second one is for $r = 4$, in which case we have $A(\boldsymbol{\lambda}) = 2$ and $B(\boldsymbol{\lambda}) \in \{1,2\}$. First assume $B(\boldsymbol{\lambda}) = 1$, meaning that the restriction ρ of $\boldsymbol{\lambda}$ to W is irreducible. Then $\overline{\rho}$ is the sum of two quasireflection representations ρ_1, ρ_2 of dimension 3 of W_0. Since W_0 has a single reflection class, these correspond to distinct simple ideals provided that $\rho_2 \not\simeq \rho_1^* \otimes \epsilon$. In order to check this it is enough to check $\overline{\rho} \not\simeq \overline{\rho}^* \otimes \epsilon$, which is true since $\overline{\rho}^* \otimes \epsilon$ is the restriction of an irreducible representation $\boldsymbol{\mu}$ of $G(2e,1,r)$ with $\{\mu^i\} = \{(\lambda_i)'\}$, and either $\{\lambda^i\} = \{\emptyset, [2]\}$ or $\{\lambda^i\} = \{\emptyset, [1,1]\}$. Thus $\mathrm{rk}\,\mathcal{H}(\rho) \geq 4 > 6/2$ and $\mathcal{H}(\rho) \simeq \mathfrak{sl}(V_\rho) \simeq \mathcal{L}(\rho)$ by lemma 3.1. Now assume $B(\boldsymbol{\lambda}) = 2$. Then $\overline{\rho}$ is an irreducible quasireflection representation of W_0, hence $\mathcal{H}(\rho) \simeq \mathfrak{sl}(\rho) \simeq \mathcal{H}_0(\overline{\rho})$, which concludes the proof. □

5.3. **Induction process for special cases.**

5.3.1. *Three irreducible components.* We assume that $W_1 \subset W_0 \subset W$ is a chain of inclusions between irreducible reflection groups which respect the reflections, and such that W_1, W_0 and W admit a single conjugacy class of reflections. We assume that the theorem holds for W_0.

Let $\rho \in \mathrm{Irr}(W)$, and assume that the restriction diagram of ρ with respect to $W_1 \subset W_0 \subset W$ has the following form



meaning that the restriction of ρ to W_0 has (exactly) three (irreducible) distinct components a, c, b, that the restriction of c to W_1 has two distinct components u, v, that u is a component of the restriction of a to W_1, that v is a component of the restriction of b to W_1, and that the restrictions of a and b to W_1 are not irreducible.

Lemma 5.6. *In the configuration above,*

(1) *if* $\mathcal{H}_0(a) \simeq \mathfrak{sl}(V_a)$ *and* $\mathcal{H}_0(b) \simeq \mathfrak{sl}(V_b)$, *with* $\mathcal{H}_0(a)$, $\mathcal{H}_0(b)$ *and* $\mathcal{H}_0(c)$ *corresponding to distinct ideals of* $\mathcal{H}_0'$, *then* $\mathcal{H}(\rho) \simeq \mathfrak{sl}(V_\rho)$ *as soon as* $\mathrm{rk}\,\mathcal{H}_0(c) \geq 2$.
(2) *if* $\mathcal{H}_0(a) \simeq \mathfrak{sl}(V_a)$, $(\rho_{\mathcal{H}'})^* \simeq \rho_{\mathcal{H}'}$, $b_{\mathcal{H}_0'} = (a_{\mathcal{H}_0'})^*$, *and* $\mathcal{H}_0(a)$, $\mathcal{H}_0(c)$ *are distinct ideals of* $\mathcal{H}_0'$, *then* $\mathcal{H}(\rho) \simeq \mathfrak{osp}(V_\rho)$ *as soon as* $\mathrm{rk}\,\mathcal{H}_0(a) + \mathrm{rk}\,\mathcal{H}_0(c) \geq 5$ *and* $\mathrm{rk}\,\mathcal{H}(c) \geq 1$.

Proof. By lemma 3.3 we know that $\mathcal{H}(\rho)$ is a simple Lie algebra. We first prove (1). Since $\mathcal{H}(\rho)$ contains a copy of $\mathcal{H}_0(a) \times \mathcal{H}_0(b) \times \mathcal{H}_0(c)$ one has

$$\begin{array}{rcl} \mathrm{rk}\,\mathcal{H}(\rho) & \geqslant & \mathrm{rk}(\mathcal{H}_0(a)) + \mathrm{rk}(\mathcal{H}_0(b)) + \mathrm{rk}(\mathcal{H}_0(c)) \geqslant \dim(a) + \dim(b) + \mathrm{rk}(\mathcal{H}_0(c)) - 2 \\ & \geqslant & \dim(a) + \dim(b) = \dim(\rho) - \dim(c) \end{array}$$

Since $\dim(\rho) = \dim(a) + \dim(b) + \dim(c) \geq \dim(c) + \dim(u) + \dim(v) + 2 = 2\dim(c)$ it follows that $\mathrm{rk}\,\mathcal{H}(\rho) > \frac{\dim(\rho)}{2}$ hence $\mathcal{H}(\rho) = \mathfrak{sl}(V_\rho)$ by lemma 3.1.

We then prove (2). We have

$$\begin{array}{rcl} \mathrm{rk}\,\mathcal{H}(\rho) & \geqslant & \mathrm{rk}(\mathcal{H}_0(a)) + \mathrm{rk}(\mathcal{H}_0(c)) \geqslant \dim(a) + \mathrm{rk}(\mathcal{H}_0(c)) - 1 \\ & \geqslant & \dim(a) > \frac{\dim(\rho)}{4} \end{array}$$

because

$$\begin{array}{rcl} \dim(\rho) & = & \dim(a) + \dim(b) + \dim(c) = 2\dim(a) + \dim(u) + \dim(v) \\ & \leqslant & 2\dim(a) + \dim(a) + \dim(b) - 2 < 4\dim(a). \end{array}$$

This implies $\dim\rho < 4(r+1)$ where $r = \mathrm{rk}\,\rho(\mathcal{H}'_0)$, hence $\dim\rho < (r+1)^2$ since $r \geq \mathrm{rk}\,\mathcal{H}_0(a) + \mathrm{rk}\,\mathcal{H}_0(c) \geq 5$. Then lemma 3.3 (I) claims that $\mathcal{H}(\rho)$ is simple and, since $\mathrm{rk}\,\mathcal{H}(\rho) \geq r \geq 5$, lemma 3.4 with $(\rho_{\mathcal{H}'})^* \simeq \rho_{\mathcal{H}'}$ together imply $\mathcal{H}(\rho) \simeq \mathfrak{osp}(V_\rho)$.

□

For future reference, we state as a lemma the following simple fact.

Lemma 5.7. *If* $\dim(\rho) > 4$ *and the restriction of* ρ *to* W_0 *has exactly two irreducible components* a, b *corresponding to distinct ideals* $\mathcal{H}_0(a) \simeq \mathfrak{sl}(V_a)$, $\mathcal{H}_0(b) \simeq \mathfrak{sl}(V_b)$, *then* $\mathcal{H}(\rho) \simeq \mathfrak{sl}(V_\rho)$.

Proof. We have $\mathrm{rk}\,\mathcal{H}(\rho) \geq \dim(a) + \dim(b) - 2 = \dim(\rho) - 2$ and $\dim(\rho) - 2 > \frac{\dim(\rho)}{2}$ as soon as $\dim(\rho) > 4$. The conclusion follows from lemma 3.1. □

5.4. **Groups** $G(e,e,r)$ **of small rank.** The case $r = 2$ is known by lemma 4.4. Also note that we have $\rho \simeq \rho^* \otimes \epsilon$ for all 2-dimensional irreducible representations of $G(e,e,2)$; hence distinct such representations define distinct ideals of $\mathcal{H}$.

We now assume $r = 3$. Let ρ be an irreducible representation of $G(e,e,3)$, and $\boldsymbol{\lambda}$ a representation of $G(e,1,3)$ such that ρ embeds in the restriction of $\boldsymbol{\lambda}$. If $p(\boldsymbol{\lambda}) \leq 2$, then $\boldsymbol{\lambda}$ is a quasi-reflection representation and so is ρ, whence $\mathcal{H}(\rho) = \mathcal{L}(\rho)$. We thus can assume $p(\boldsymbol{\lambda}) = 3$, hence $\dim\boldsymbol{\lambda} = 6$. Since $A(\boldsymbol{\lambda})$ divides $p(\boldsymbol{\lambda})$ it follows that $A(\boldsymbol{\lambda}) \in \{1,3\}$. If $A(\boldsymbol{\lambda}) = 3$, then $\dim\rho = 2$ and ρ is a quasi-reflection representation. Hence we can assume $A(\boldsymbol{\lambda}) = 1$ and $\dim\rho = 6$. We can assume $\lambda^0 = \lambda^i = \lambda^j = [1]$ with $\#\{0,i,j\} = 3$. The restriction of $\boldsymbol{\lambda}$ to $G(e,1,2)$ is the sum of three irreducible components $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$ of dimension 2, with $p(\boldsymbol{\lambda}_i) = 2$. In particular $A(\boldsymbol{\lambda}_i) \in \{1,2\}$. Let $\rho_i =< \boldsymbol{\lambda}^i >$. Notice that $\mathcal{H}(\rho)$ cannot have Cartan type C_2, as $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$ does not admit an irreducible 6-dimensional representation. It cannot be of type G_2 or B_3 for the same reason. We separately consider the following possibilities.

- If $A(\boldsymbol{\lambda}_1) = A(\boldsymbol{\lambda}_2) = A(\boldsymbol{\lambda}_3) = 1$ and the restrictions to $G(e,e,3)$ of the $\boldsymbol{\lambda}_i$ are irreducible representations. Then the restriction of ρ to $G(e,e,2)$ is multiplicity free, hence by lemma 3.3 (I) $\mathcal{H}(\rho)$ is simple. If $\mathrm{rk}\,\mathcal{H}(\rho) \geq 4$ then $\mathcal{H}(\rho) = \mathfrak{sl}(V_\rho)$ by lemma 3.1 and we are done.

On the other hand, $\mathcal{H}(\rho) \supset \mathfrak{sl}_2^3$ hence we can assume $\mathrm{rk}\,\mathcal{H}(\rho) = 3$. Since $(\mathfrak{sl}_2)^3$ does not embed in $\mathfrak{sl}_4$ or in a Lie algebra of Cartan type B_3, we get that $\mathcal{H}(\rho)$ must have Cartan type C_3. Since $\dim \rho = 6$ it follows that $\rho_{\mathcal{H}'}$ is selfdual hence $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$.

- If $A(\boldsymbol{\lambda}_1) = A(\boldsymbol{\lambda}_2) = 1$, $A(\boldsymbol{\lambda}_3) = 2$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ have isomorphic restriction to $G(e,e,3)$. This situation can occur only if $e = 4g$ for some integer g, and we can assume $i = g, j = 2g$. But then ρ factorises through the morphism $G(4g,4g,3) \twoheadrightarrow G(4,4,3)$, and it is sufficient to check that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ for the restriction to $G(4,4,3)$ of the representation $([1],[1],[1],\emptyset)$, which we do by computer.

In all other cases, the restriction of $\boldsymbol{\mu}$ admits multiplicities, and also contains some 2-dimensional irreducible component, whose corresponding ideal is isomorphic to $\mathfrak{sl}_2$, with multiplicity 1. Lemma 3.6 tackles these cases, hence the theorem holds for $r = 3$ too.

5.5. **Induction process for $W = G(e,e,r)$, $\rho \in \mathrm{Ind}\Lambda\mathrm{Ref}$.** We first consider the case where the restriction of ρ to $G(e,e,r-1)$ contains one of the exceptional quasireflection representations of table 1. The first case is when $r-1 = 4$, and e is even. In this case, up to $\rho \rightsquigarrow \rho^* \otimes \epsilon$, wa can assume $\lambda^0 = \lambda^g = [2]$, for $e = 2g$. Notice that the corresponding 3-dimensional representations are not selfdual, nor dual of each other, as representations of $\mathcal{H}'_0$. The second case is when $r-1 = 3$, and e is divisible by 3. Then the corresponding representations of $\mathcal{H}'_0$, being 2-dimensional, are selfdual and correspond to distinct ideals of $\mathcal{H}'_0$, since $\#\mathcal{R}_0/W_0 = 1$.

5.5.1. *The special cases of $G(2g,2g,4)$ and $G(3g,3g,3)$.*
$G(2g,2g,4)$. Let ρ_0, ρ_1 denote the two irreducible (3-dimensional) components of the restriction to $G(2g,2g,4)$ of the representation $\boldsymbol{\lambda}$ of $G(2g,1,4)$ defined by $\lambda^0 = [2]$, $\lambda^g = [2]$. Note that $\rho_1 \not\simeq \rho_0^* \otimes \epsilon$ hence $\mathcal{H}(\rho_0) \neq \mathcal{H}(\rho_1)$. Assume that ρ is an irreducible representation of $G(2g,2g,5)$ such that ρ_0 embeds in ρ. The exists an irreducible representation $\boldsymbol{\mu}$ of $G(2g,1,5)$ with $\boldsymbol{\lambda} \nearrow \boldsymbol{\mu}$ such that ρ embeds in its restriction to $G(2g,2g,5)$. We have $p(\boldsymbol{\mu}) \in \{2,3\}$. By lemma 4.2, $A(\boldsymbol{\mu})$ has to divide 5 and $p(\boldsymbol{\mu})$, hence $A(\boldsymbol{\mu}) = 1$ and $\dim \rho = \dim \boldsymbol{\mu}$.

If $p(\boldsymbol{\mu}) = 2$, first assume $\mu^0 = [3]$ and $\mu^g = [2]$, and the restriction of ρ to $G(2g,2g,4)$ decomposes as $\rho_0 \oplus \rho_1 \oplus \rho_2$ with ρ_2 a 4-dimensional reflection representation of $G(2g,2g,4)$. If follows that $\mathrm{rk}\,\mathcal{H}(\rho) \geq 2+2+3 = 7$. Since $\dim \rho = 10$ lemma 3.1 implies $\mathcal{H}(\rho) = \mathfrak{sl}(V_\rho)$. The other case is for $\mu^0 = [2,1]$, $\mu^g = [2]$. Then the restriction of ρ to $G(2g,2g,4)$ decomposes as $\rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3$, with $\rho_2^* \otimes \epsilon \simeq \rho_2$, $\rho_3^* \otimes \epsilon \simeq \rho_3$ with $\rho_2, \rho_3 \notin \Lambda\mathrm{Ref}$, $\dim \rho_2 = 6$, $\dim \rho_3 = 8$. It follows that $\mathrm{rk}\,\mathcal{H}(\rho) \geq 11$, hence $\mathcal{H}(\rho) \simeq \mathfrak{sl}(V_\rho)$ by lemma 3.1, since $\dim \rho = 20$ in this case.

If $p(\boldsymbol{\mu}) = 3$ then we can assume $\mu^0 = \mu^g = [2]$ and $\mu^i = [1]$ for some $i \not\equiv 0 \bmod g$. It follows that the restriction of ρ to $G(2g,2g,4)$ decomposes as $\rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3$ with ρ_2, ρ_3 distinct 12-dimensional representations of $G(2g,2g,4)$ such that $\rho_3 \not\simeq \rho_2^* \otimes \epsilon$. By the induction hypothesis it follows that $\mathrm{rk}\,\mathcal{H}(\rho) \geq 2+2+11+11 = 26$. Since $\dim \rho = 30$ lemma 3.1 implies $\mathcal{H}(\rho) \simeq \mathfrak{sl}(\rho)$.

$G(3g,3g,3)$. Let ρ_0,ρ_1,ρ_2 denote the three irreducible components of the restriction to $G(3g,3g,3)$ of the representation $\boldsymbol{\lambda}$ of $G(3g,1,3)$ defined by $\lambda_0=\lambda_g=\lambda_{2g}=[1]$. Assume that ρ is an irreducible representation of $G(3g,3g,4)$ such that ρ_0 embeds in ρ. Let $\boldsymbol{\mu}$ denote a representation of $G(3g,1,4)$ such that ρ embeds in its restriction. By lemma 4.2 we have $A(\boldsymbol{\mu})=1$ We have $\boldsymbol{\lambda}\nearrow\boldsymbol{\mu}$, and $p(\boldsymbol{\mu})\in\{3,4\}$.

If $p(\boldsymbol{\mu})=3$, then we may assume, up to tensorization by the sign character, that $\mu^0=[2]$ and $\mu^g=\mu^{2g}=[1]$. Then the restriction of ρ equals $\rho_0\oplus\rho_1\oplus\rho_2$ plus two irreducible components, restriction of the representations $\boldsymbol{\mu}_1,\boldsymbol{\mu}_2$ of $G(3g,1,4)$ defined by $\mu_1^0=\mu_2^0=[2]$, $\mu_1^g=\mu_2^{2g}=[1]$, $\mu_1^{2g}=\mu_2^g=\emptyset$. It is easily checked that these five components correspond to distinct ideals, and that the sum of their rank is $7>\frac{12}{2}=\frac{\dim\rho}{2}$ hence $\mathcal{H}(\rho)\simeq\mathfrak{sl}(\rho)$ by lemma 3.1.

If $p(\boldsymbol{\mu})=4$, then we may assume that $\mu_0=\mu_g=\mu_{2g}=\mu_i=[1]$ for some $0<i<g$. The restriction of ρ equals $\rho_0\oplus\rho_1\oplus\rho_2$ plus three irreducible 6-dimensional components $\tilde{\rho}_1,\tilde{\rho}_2,\tilde{\rho}_3$, restriction of the representations $\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\mu}_3$ of $G(3g,1,4)$ defined by $\mu_1^i=\mu_2^i=\mu_3^i=[1]$, $\mu_1^0=\mu_1^g=\mu_2^0=\mu_2^{2g}=\mu_3^g=\mu_3^{2g}=[1]$. As representations of $\mathcal{H}_0'$, $\tilde{\rho}_1$ may be selfdual, $\tilde{\rho}_2$ may be the dual of $\tilde{\rho}_3$, but in any case the ideal corresponding to $\tilde{\rho}_2$ is isomorphic to $\mathfrak{sl}_6$ and different from the ones afforded by $\widetilde{rho}_1$ or ρ_0,ρ_1,ρ_2. It follows that $\mathrm{rk}\,\mathcal{H}_0(\overline{\rho})\geq 1+1+1+3+5=11$. Since $\dim\rho=24$, lemma 3.3 (I) implies that $\mathcal{H}(\rho)$ is simple. Then parts (1) and (2) of lemma 3.5 (1) imply that $\mathcal{H}(\rho)\simeq\mathcal{L}(\rho)$.

5.5.2. *General case.* By §4 we know that the representations that we are going to deal with correspond to very special multipartitions $\boldsymbol{\lambda}$, so many information on $\boldsymbol{\lambda}$ will clear from the context. For brevity, we use the following shortcuts. If λ is some partition, we denote by (λ) a multipartition $\boldsymbol{\lambda}$ with $p(\boldsymbol{\lambda})=1$ such that $\lambda^i=\lambda$ for some i. If λ,μ are two partitions, we also denote $(\lambda;\mu)$ a multipartition $\boldsymbol{\lambda}$ with $p(\boldsymbol{\lambda})=2$ and $\lambda^i=\lambda$, $\lambda^j=\mu$ for some $i\neq j$. In rare cases, we similarly use the notation $(\lambda;\mu;\nu)$ for a 3-parts multipartition. For $\boldsymbol{\lambda}=(\lambda;\mu)$ and $A(\boldsymbol{\lambda})=2$, we denote $(\lambda;\lambda)^{\pm}$ the irreducible components of the restriction to $G(e,e,r)$. When the restriction is irreducible and the context clear, we identify $\boldsymbol{\lambda}$ with its restriction to $G(e,e,r)$. Notice that, using these shortcuts, we may write $(\lambda;\mu)^*\otimes\epsilon=(\mu';\lambda')$. In particular, we can assume $\lambda\geq\lambda'$ (for the lexicographic order) when needed.

If $r\geq 4$, for the representations c of $G(e,e,r-1)$ in $\Lambda\mathrm{Ref}\setminus\mathrm{QRef}$ listed in proposition 4.6, we can apply lemma 5.6 with respect to the chain of inclusions $G(e,e,r-2)\subset G(e,e,r-1)\subset G(e,e,r)$ to all representations ρ of $G(e,e,r)$ in $\mathrm{Ind}\Lambda\mathrm{Ref}$ such that $c\nearrow\rho$. The corresponding representations a,b,u,v are listed in table 2. To save space in the table, we let $r'=r-1$. If $c=[r'-p,1^p]$ we have $u=[r'-p,1^{p-1}]$ and $v=[r'-p-1,1^p]$; if $c=[r'-p],[1^p]$ we have $u=[r'-p-1],[1^p]$ and $v=[r'-p],[1^{p-1}]$. Note that, when $c=[r'-p,1^p]$ and $r'=2p+1$, or when $c=([r'-p];[1^p])$, then the part (2) of this lemma has to be applied.

We now assume that $c\in\mathrm{QRef}$, and that $\boldsymbol{\lambda}$ is a representation of $G(e,1,r)$ whose restriction contains c. We can assume $r\geq 4$, since the case of $G(e,e,3)$ has already been settled. If $c\in\{([r'-1];[1]),([r'-1,1])\}$, then any ρ with

p	c	ρ	a	b
$2 \leq p \leq r'-3$	$([r'-p, 1^p])$	$([r'-p, 1^p]; [1])$	$([r'-p, 1^{p-1}]; [1])$	$([r'-p-1, 1^p]; [1])$
		$([r'-p, 2, 1^{p-1}])$	$([r'-p, 2, 1^{p-2}])$	$([r'-p-1, 2, 1^{p-1}])$
$2 \leq p \leq r'-2$	$([r'-p]; [1^p])$	$([r'-p, 1]; [1^p])$	$([r'-p-1, 1]; [1^p])$	$([r'-p, 1]; [1^{p-1}])$
		$([r'-p]; [2, 1^{p-1}])$	$([r'-p-1]; [2, 1^{p-1}])$	$([r'-p]; [2, 1^{p-2}])$
		$([r'-p]; [1^p]; [1])$	$([r'-p-1]; [1^p]; [1])$	$([r'-p]; [1^{p-1}]; [1])$

TABLE 2. Restrictions for IndΛRef of $G(e, e, r)$, $r \geq 4$, $r' = r-1$.

	r	c	X	a	b
(1)	$r' \geq 4$	$([r'-1]; [1])$	$([r'-1, 1]; [1])$	$([r'-2, 1]; [1])$	$([r'-1, 1])$
			$([r'-1]; [2])$	$([r'-2]; [2])$	
			$([r'-1]; [1]; [1])$	$([r'-2]; [1]; [1])$	$([r'-1]; \emptyset; [1])$
(2)	$r' \geq 3$	$([r'-1, 1])$	$([r'-1, 2])$	$([r'-2, 2])$	
(3)		$([2, 2])$	$([3, 2])$	$([3, 1])$	
(4)		$([2]; [2])^{\pm}$	$([3]; [2])$	$([3]; [1])$	$[2][2]^{\mp}$

TABLE 3. Restrictions for IndQRef of $G(e, e, r)$, $r \geq 4$, $r' = r-1$.

ρ	$([2, 1]; [1])$		$([2]; [2])$	$([2]; [1]; [1])$
$\dim \rho$	9		6	12
$\mathrm{rk}\, \mathcal{H}_0(\overline{\rho})$	4	6	4	9

TABLE 4. Special cases.

$c \nearrow \rho$ which is not a quasireflection representation is handled by lemma 5.6 or lemma 5.7, along the patterns described in table 3, (1) and (2), provided that $\mathrm{rk}\, \mathcal{H}_0(c) \geq 2$. This is the case for $r \geq 5$ by the induction hypothesis. If $r = 5$ and c is not of the above types, either we have $A(\boldsymbol{\lambda}) = 2$ and we are in the special case for $G(2g, 2g, 4)$ already described, or $c = ([2, 2])$ and we can use lemma 5.7 along the pattern (3) of table 3.

If $r = 4$, then ρ is of one of types listed in table 4 (up to taking conjugate parts). The two possible results for $\mathrm{rk}\, \mathcal{H}_0(\overline{\rho})$ given in the table for $\rho = ([2, 1]; [1])$ depend on whether $\rho \simeq \rho^* \otimes \epsilon$ (first column) or not. The lemmas of §3 show that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ in these cases.

5.6. **Induction for the groups $G(e, e, r)$.** Here we prove the theorem for the groups $G(e, e, r)$. The cases $r \in \{2, 3\}$ have been done separately, so we can start the induction at $G(e, e, r)$ for $r \geq 4$. Let $\rho \in \mathrm{Irr} G(e, e, r)$, a component of $\boldsymbol{\lambda} \in \mathrm{Irr} G(e, 1, r)$. If $\rho \in$ IndΛRef we already proved that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$, so we can assume $\rho \notin$ IndΛRef. If the restriction of ρ to $G(e, e, r-1)$ is multiplicity-free we have $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ by lemma 5.3.

It is easily checked that, if $r = 4$, then the only possibility for the restriction $\overline{\rho}$ of ρ to have multiplicities is when $e = 5g$ for some integer g and $\boldsymbol{\lambda}$ can be taken to be $\lambda^0 = \emptyset$, $\lambda^g = \lambda^{2g} = \lambda^{3g} = \lambda^{4g} = [1]$ with $p(\boldsymbol{\lambda}) = 4$. In that case, ρ factorizes through a representation of $G(5,5,4)$, and we check that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ by computer for this representation. We can thus assume $r > 4$. This implies that $G(e,e,r)$ does not admit irreducible representations of dimensions 2 or 3 (see proposition 7.2). By [Ma07c] theorem 1 we know that the multiplicities in $\overline{\rho}$ are at most 2. We need a lemma.

Lemma 5.8. *If ρ is an irreducible representation of $G(e,e,r)$ with $r \geq 2$ whose restriction $\overline{\rho}$ to $G(e,e,r-1)$ admits an irreducible component with multiplicity 2, then $\overline{\rho}$ admits, counting multiplicities, at least 3 irreducible components.*

Proof. By [Ma07c] theorem 3.3, the assumption implies that ρ is the restriction of some irreducible representation $\boldsymbol{\lambda}$ of $G(e,1,r)$. Let ρ_1 be an irreducible component of $\overline{\rho}$ occuring with multiplicity 2. By [Ma07c] proposition 3.4, ρ_1 is the restriction of some $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$, and there is some $\boldsymbol{\mu}^+ \nearrow \boldsymbol{\lambda}$ with $\boldsymbol{\mu} \neq \boldsymbol{\mu}^+$ whose restriction to $G(e,e,r-1)$ is isomorphic to ρ_1. We argue by contradiction. If there were less than 3 irreducible components, then the restriction of $\boldsymbol{\lambda}$ to $G(e,1,r-1)$ would be isomorphic to $\boldsymbol{\mu} \oplus \boldsymbol{\mu}^+$. In particular, $p(\boldsymbol{\lambda}) \leq 2$. Moreover, since $\boldsymbol{\mu}^+$ should be a nontrivial cyclic permutation of $\boldsymbol{\mu}$, one has $p(\boldsymbol{\lambda}) > 1$ hence $p(\boldsymbol{\lambda}) = 2$. Letting $\boldsymbol{\lambda} = (a;b)$ with a,b nontrivial partitions at suitable places $0 \leq i < j \leq e-1$, we can assume $\boldsymbol{\mu} = (a^0;b)$, $\boldsymbol{\mu}^+ = (a;b^0)$ with $a^0 \nearrow a$, $b^0 \nearrow b$. Since $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^+$ are cyclically permuted one from the other, we have $2(j-i) = e$. But this implies for $\boldsymbol{\lambda} = (a;a)$ that $A(\boldsymbol{\lambda}) = 2$, contradicting the irreducibility assumption on ρ. □

By lemma 5.4 we have $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ as soon as $\overline{\rho}$ has only one component of multiplicity 2. We thus assume that $\overline{\rho}$ admits at least 2 multiplicity-two components. Then every irreducible component of $\overline{\rho}$ has dimension at least 12, by the following lemma.

Lemma 5.9. *If $r > 4$ and $\overline{\rho}$ admits at least 2 multiplicity-two components, then $\rho' \nearrow \rho \Rightarrow \dim \rho' \geq 12$.*

Proof. ρ' is a component of some $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$, and $\dim \rho' = (\dim \boldsymbol{\mu})/A(\boldsymbol{\mu})$. Since $\dim \boldsymbol{\mu} \geq p(\boldsymbol{\mu})!$, $A(\boldsymbol{\mu})$ divides $p(\boldsymbol{\mu})$ and $p(\boldsymbol{\mu}) \geq p(\boldsymbol{\lambda}) - 1$, first note that $\dim \rho' \geq (p(\boldsymbol{\lambda}) - 2)!$. By [Ma07c] (see end of §3 and proposition 7.3 there) the assumption on the multiplicity-two components of $\overline{\rho}$ implies that $\boldsymbol{\lambda}$ can be chosen such that there exists m dividing e with $u = e/m \geq 5$ and

(1) λ^i only depends on the class of i modulo m if m does not divide i
(2) $\lambda^0 = a$, $\lambda^i = b$ for i a nonzero multiple of m, with $a \nearrow b$.

Condition (2) implies $p(\boldsymbol{\lambda}) \geq 4$. Moreover, if there exists o with $m \not| i$ and $\lambda^i \neq \emptyset$, then condition (1) implies $p(\boldsymbol{\lambda}) \geq 9$ hence $\dim \rho' \geq 7! \geq 12$. Assuming that this is not the case, then ρ factors through $G(u,u,r)$, so that we can assume $m = 1$ hence $e \geq 5$. If $e \geq 7$ then $p(\boldsymbol{\lambda}) \geq 6$ and $\dim \rho' \geq 4! \geq 12$, so we can assume $e \in \{5,6\}$. If $a \neq \emptyset$ then $|b| \geq 2$ and by the branching rule $\dim \boldsymbol{\mu} \geq \dim([1];[1];[2];[2];[2]) = 5040$. Since $A(\boldsymbol{\mu}) \leq e \leq 6$ this implies $\dim \rho' \geq 5040/6 \geq 12$. We thus can assume $a = \emptyset$, $b = [1]$, $r = e-1$ and $\dim \boldsymbol{\mu} = (e-2)!$. Since $r > 4$ we have $e = 6$, $\dim \boldsymbol{\mu} = 24$. Now $A(\boldsymbol{\mu})$ divides both $p(\boldsymbol{\mu}) = 4$ and $e = 6$ by lemma 4.2 hence $A(\boldsymbol{\mu}) \leq 2$ and $\dim \rho' \geq 24/2 = 12$. □

Now write

$$\overline{\rho} = \sum_{i=1}^{r} \alpha_i \rho_i + \sum_{i=1}^{s} (\beta_i \varphi_i + \gamma_i \varphi_i^* \otimes \epsilon) + \sum_{i=1}^{t} \eta_i \psi_i$$

with $\alpha_i, \beta_i, \gamma_i \in \{1, 2\}$ and the usual conventions on $\rho_i, \varphi_i, \psi_i$. Since $\dim \rho' \geq 3$ for $\rho' \nearrow \rho$ we have $\dim \rho' - 1 \geq \frac{\dim \rho'}{2}$ and $\operatorname{rk} \mathcal{H}_0(\overline{\rho}) \geq \dim soc(\overline{\rho})/2 - s$, where

$$soc(\overline{\rho}) = \sum_{i=1}^{r} \rho_i + \sum_{i=1}^{s} (\varphi_i + \varphi^* \otimes \epsilon) + \sum_{i=1}^{t} \psi_i$$

hence

$$\operatorname{rk} \mathcal{H}_0(\overline{\rho}) \geqslant \frac{\dim \rho}{4} - s > \frac{\dim \rho}{5}$$

as soon as $s < \dim \rho/20$. In particular, since $\dim \varphi_i \geq 11$ for all $i \in [1, s]$ we have $\operatorname{rk} \mathcal{H}_0(\overline{\rho}) > \frac{\dim \rho}{5}$ and $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ by lemma 3.5, since $\operatorname{rk} \mathcal{H}(\rho) \geq \operatorname{rk} \mathcal{H}(\varphi_i) = \dim \varphi_i - 1 \geq 10$. This concludes the proof of the theorem for the infinite series.

5.7. **Exceptional groups.** In order to conclude the proof of the theorem for $W \neq H_4$. we need to show that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ for all irreducible representations not in $\Lambda\mathrm{Ref}$ for these groups. This holds true also for W of Coxeter type H_4.

Proposition 5.10. *If W is an irreducible exceptional reflection group and $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$, then $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$.*

This section is devoted to the proof of this proposition. We first deal with groups of small rank, before using the above induction lemmas to prove the proposition for groups of higher rank.

5.7.1. *Groups of small rank.* For these groups, we used a computer to show that $\rho(\mathcal{H}')$ and $\mathcal{L}(\rho)$ have the same dimension. For the groups labelled G_{12}, G_{13}, G_{22}, $G_{23} = H_3$, the representations have dimension small enough to do the computations directly over the field of definition of these groups, and matrix models of all representations are known. For some representations of the groups G_{24}, G_{27}, $G_{28} = F_4$, and $G_{30} = H_4$, we used reduction of the coefficients. Notice that it is enough to show that $\dim \rho(\mathcal{H}') \geq \dim \mathcal{L}(\rho)$.

When ρ admits a model over $\mathbb{Q}$, we consider a Lie subalgebra A_ρ of $\mathfrak{sl}(V_\rho)$ generated by nonzero multiples of the $\rho(s'), s \in \rho$, so that A_ρ is defined over $\mathbb{Z}$ and $\operatorname{rk}_{\mathbb{Z}} A_\rho = \dim_{\mathbb{Q}} \rho(\mathcal{H}') = \dim_{\mathbb{Q}} \mathcal{H}(\rho)$. It is then sufficient to find a prime p such that $\dim_{\mathbb{F}_p} A_\rho \otimes_{\mathbb{Z}} \mathbb{F}_p \geq \dim_{\mathbb{Q}} \mathcal{L}(\rho)$ to prove this inequality. In most cases, it was enough to take $p = 11$. This solves the cases of the groups G_{24} and $G_{28} = F_4$. For G_{24} we had to find matrix models of 3 rational representations which were not available before. We used for this decomposition of tensor products of already known representations and methods [MM]). These new models have since been included in CHEVIE.

Matrix models for the representations of H_4 were found by Alvis, and are included in CHEVIE. In the case of G_{27} we computed matrix models for the representations not in $\Lambda\mathrm{Ref}$ (now also included in CHEVIE). As predicted by a theorem of Benard (see [Bd]), the nonrational representations among them can be realized over a quadratic field, either $\mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\zeta_3)$. As in

Group	Type of subgroup	Reflection subgroup
G_{29}	$G(4,4,3)$	$< t,u,v >$
G_{31}	G_{29}	$< s,t,v,w >$
G_{33}	$G(3,3,4)$	$< s,t,u,w >$
G_{34}	G_{33}	$< s,t,u,v,w >$
E_6	D_5	$< s_1,\dots,s_5 >$
E_7	E_6	$< s_1,\dots,s_6 >$
E_8	E_7	$< s_1,\dots,s_7 >$

TABLE 5. Induction patterns for exceptional groups.

the rational case, the dimension of $\rho(\mathcal{H}')$ is equal to the rank of some Lie algebra A_ρ defined (by chasing denominators) over the ring of integers of the corresponding quadratic field. It is then enough to compute the dimension of the reduction of A_ρ modulo a suitable place. Recall that, when $\mathbb{Z}[u]$ is a principal quadratic ring and $\pi = \alpha + \beta u \in \mathbb{Z}[u]$ has norm a prime integer p, then a morphism $\mathbb{Z}[u] \twoheadrightarrow \mathbb{F}_p$ can be defined by sending u to $-\alpha\beta^{-1}$. Using these explicit reductions, we prove by computer that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ for these additional two groups.

5.7.2. *Groups of higher rank.* The remaining exceptional groups are labelled $G_{29}, G_{31}, G_{33}, G_{34}, G_{35} = E_6, G_{36} = E_7$ and $G_{37} = E_8$ in the Shephard-Todd classification. We use the character and induction tables implemented in CHEVIE to check the following properties. All of them have a single class of reflections, and the dimensions greater than 1 of their irreducible representations are at least 4. By lemma 2.25, it follows that, for all $\rho \in \mathrm{Irr}(W)$, $\mathcal{H}(\rho)$ admits no simple ideal of rank 1. We choose for $W_0 \subset W$ the subgroups described in table 5, where the generators correspond to the tables of [BMR].

Notice that they all such W_0 are irreducible, that they have only one class of reflections, and all differ from F_4 and H_4. Assuming that the theorem holds true for W_0, we check that the condition $\dim V < (r+1)^2$ of lemma 3.3 holds for the pairs $\mathfrak{h} = \rho(\mathcal{H}'_{W_0})$ and $\mathfrak{g} = \rho(\mathcal{H}')$ for all $\rho \in \mathrm{Irr}(\rho) \setminus \Lambda\mathrm{Ref}$. Using the induction table, this is easily done by computer. Now the induction table for these pairs does not contain multiplicities more than 2. Then lemma 2.26 shows that condition (II) of lemma 3.3 is fulfilled, which proves that $\mathcal{H}(\rho)$ is a simple Lie algebra for all $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$.

We check that $\dim \rho < 4\,\mathrm{rk}\,\rho(\mathcal{H}'_0)$ for all $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$. This proves that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ for all these by lemma 3.4, provided we are not in the exceptions listed there. For the groups here of rang at least 6, we check that $\mathrm{rk}\,\rho(\mathcal{H}'_0) \geq 14 > 6$, hence $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$, for all $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$. This proves the theorem for G_{34}, E_6, E_7, E_8 by corollary 2.24.

For G_{29}, G_{31} and G_{33} and W_0 as in the table, the following situations appear, for $\rho \notin \Lambda\mathrm{Ref}$. Here we marked "*" the cases where $\rho^* \otimes \epsilon \simeq \rho$.

G_{29}	dim	5	6	10	10	15	15	16	20
	$\mathrm{rk}\,\rho(\mathcal{H}'_0)$	3	3	5	6	8	9	9	10
	sym.								
G_{31}	dim	5	9	10	15	16	20	20	20
	$\mathrm{rk}\,\rho(\mathcal{H}'_0)$	4	7	9	14	7	9	18	19
	sym.					$*$	$*$		
G_{33}	dim	6	15	15	20				
	$\mathrm{rk}\,\rho(\mathcal{H}'_0)$	5	11	13	17				
	sym.								

We leave to the reader to check that none of these belong to the exceptions of lemma 3.4, hereby proving $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ (and the theorem) for these groups.

6. Closures inside Iwahori-Hecke algebras

Recall that $W \subset \mathrm{GL}_n(\mathbb{C})$ be a finite irreducible reflection group of rank n, and that we mean by reflection an involutive element of $\mathrm{GL}_n(\mathbb{C})$ distinct from the identity which fixes an hyperplane of $\mathbb{C}^n$.

6.1. **Braid groups and holonomy Lie algebras.** Let $\mathcal{R}$ be the set of reflections of W, and $\mathcal{A}$ the associated (central) hyperplane arrangement in $\mathbb{C}^n$. We let $X = \mathbb{C}^n \setminus \bigcup \mathcal{A}$ denote its complement. The (generalized) braid group associated to W is defined by $B = \pi_1(X/W)$. It fits into a short exact sequence $1 \to P \to B \to W \to 1$ where $P = \pi_1(X)$. To every $H \in \mathcal{A}$ we associate the closed 1-form $\frac{d\alpha_H}{\alpha_H}$, where α_H is an arbitrary linear form on $\mathbb{C}^n$ with kernel H.

We recall from [Ko] the construction of the holonomy Lie algebra in this setting. The transposed of the cup-product gives a morphism $\delta : H_2(X,\mathbb{Q}) \to \Lambda^2(H_1(X,\mathbb{Q}))$. On the other hand, $\Lambda^2(H_1(X,\mathbb{Q}))$ can be identified to the homogenous part of degree 2 of the free Lie algebra on the vector space $H_1(X,\mathbb{Q})$. The *holonomy Lie algebra* $\mathcal{T}$ of X is defined as the quotient of this Lie algebra by the image $\delta\,(H_2(X,\mathbb{Q}))$ of δ. It it thus a graded quadratic Lie algebra, defined such that the class of $\omega \wedge \omega$ in $H^2(X,\mathbb{Q})$ vanishes, where ω is the following closed form

$$\omega = \sum_{H \in \mathcal{A}} t_H \omega_H \in \Omega_1(X) \otimes \mathcal{T}.$$

By a classical result of Brieskorn [Br] the algebra generated by the forms ω_H for $H \in \mathcal{A}$ embeds in the cohomology algebra, hence $\omega \wedge \omega = 0$.

Let us consider the action $w.t_{w(H)}$ of W on $\mathcal{T}$ and introduct the semidirect product $\mathfrak{B} = \mathbb{C}W \ltimes \mathsf{U}\mathcal{T}$ where $\mathsf{U}\mathcal{T}$ denotes the universal envelopping algebra of $\mathcal{T}$. This algebra is graded, by $\deg(t_H) = 1$ and $\deg(w) = 0$ if $w \in W$. We denote $\widehat{\mathfrak{B}}$ its completion with respect to the grading, $A = \mathbb{C}[[h]]$ the ring of formal series in one variable, $K_0 = \mathbb{C}((h))$ its field of fractions, $K = \bigcup_{n \geq 0} \mathbb{C}((h^{\frac{1}{n}}))$ the field of (formal) Puiseux series. Recall that K is an algebraic closure of K_0. Any (finite dimensional) linear representation $\rho : \mathfrak{B} \to M_N(\mathbb{C})$ defines a closed integrable 1-form $\omega_\rho = h \sum_{H \in \mathcal{A}} \rho(t_H)\omega_H \in$

$\Omega^1(X) \otimes M_N(A)$. This 1-form is W-equivariant, hence induces an integrable 1-form on X/W.

We now fix a base point $\underline{z} \in X/W$, and consider the differential equation $dF = \omega_\rho F$. The monodromy of this equation gives a representation $R : B \to \mathrm{GL}_N(A)$. We say that an element of $B = \pi(X/W)$ is a braided reflection ('generator-of-the-monodromy' in [BMR]) if it is the composition of a path γ from $\underline{z}$ to some (neighborhood of an) hyperplane H, followed by a positive half turn around H in X, and by γ in the opposite direction. In particular the image in W of such an element is the reflection around H.

We let $\mathcal{T}_\rho = \rho(\mathcal{T}) \otimes_{\mathbb{C}} A$, and $\mathrm{GL}^0_N(K) = \exp(h\mathrm{End}(V_\rho)) \subset \mathrm{GL}_N(A)$. We will use the following facts, for which we refer to [Ma07d] and [Ma05].

- $R(B) \subset \rho(W) \ltimes \mathrm{GL}^0_N(K)$ and $R(P) \subset \mathrm{GL}^0_N(K)$. More generally, the monodromy of ω_ρ along any path in X belongs to $\exp h\mathcal{T}_\rho$.
- If $b \in B$ is a braided reflection around $H \in \mathcal{A}$, then $R(b)$ is conjugated to $\rho(s)\exp(\mathrm{i}\pi h\rho(t_H))$ by an element of $\exp h\mathcal{T}_\rho \subset \mathrm{GL}^0_N(K)$ and $R(b^2) \equiv 1 + 2\mathrm{i}\pi h\rho(t_H)$ modulo h.
- If the restriction of ρ to $\mathcal{T}$ is irreducible, then the restriction of R to P is irreducible.
- The Lie algebra of the Zariski-closure $\overline{R(P)}$ of $R(P)$ in $\mathrm{GL}_N(K)$ contains $\rho(\mathcal{T}) \otimes K$.

We will also need the following facts, which are elementary consequences of $R(P) \subset \exp h\mathcal{T}_\rho$, and of the fact that $\mathcal{T}$ is generated by the $t_H, H \in \mathcal{A}$.

- If there exists a bilinear form $< \, , \, >$ on V_ρ such that $\rho(t_H)$ is antisymmetric, then the induced bilinear form on $V_\rho \otimes A$ is invariant under R
- If the $\rho(t_H)$ have zero trace, then $R(P) \subset \mathrm{SL}_N(K)$, and $\det R(g) = \det \rho(\pi(g))$ for all $g \in B$.

Finally, the correspondance $\rho \rightsquigarrow R$ is functorial (see [Ma05]). In particular, if ρ^1, ρ^2 are two representations of $\mathcal{T}$ and R_1, R_2 the corresponding monodromy representations of P, then

(1) $R_1 \otimes R_2$ is the monodromy represention of $\rho_1 \otimes \rho_2$.
(2) $R_1 \simeq R_2$ iff $\rho_1 \simeq \rho_2$ (see [Ma05], proposition 5).

6.2. **Hecke algebras.** The (generic) Iwahori-Hecke algebra $H_W(q)$ associated to W can be defined as the quotient of the group algebra KB by the ideal generated by all $(\sigma - q)(\sigma + q^{-1}) = 0$, for σ a braided reflection, with q some transcendental constant in K. Here we take $q = e^{\mathrm{i}\pi h}$. We recall the following facts from [BMR]. It is known for Coxeter groups and reflection groups in the infinite series, plus a few other ones, that these algebras are semisimple with representations afforded by the following monodromy construction, originally due to I. Cherednik : given a representation ρ of W, it can be extended to a representation of $\mathfrak{B}$ by letting t_H act as $\rho(s_H)$, where $s_H \in W$ is the reflection corresponding to $H \in \mathcal{A}$; the monodromy representation associated to ρ factors through an irreducible representation of $H_W(q)$ by the remarks above. In particular we denote $S_\eta : B \to K^\times$ the 1-dimensional representation associated to $\eta \in \mathrm{Hom}(W, \{\pm 1\}$.

We say that the reflection groups W for which this property has been proved are *tackled*. Recall from [BMR] that this property is conjectured to hold for any reflection group.

We define $V_\rho^h = V_\rho \otimes_{\mathbb{C}} K$, $\mathrm{SL}(V_\rho^h) = \{x \in \mathrm{GL}(V_\rho^h) \mid \det(x) = 1\}$ and $\widetilde{\mathrm{SL}}(V_\rho^h) = \{x \in \mathrm{GL}(V_\rho^h) \mid \det(x) = \pm 1\}$. Recall from [BMR] that B is generated by braided reflections. Let σ be such a braided reflection, and $s = \pi(\sigma) \in \mathcal{R}$. Since $R(\sigma)$ is conjugated to $\rho(s)\exp(\mathrm{i}\pi\rho(s))$, if $\mathrm{tr}\,\rho(s) = 0$ for all $s \in \mathcal{R}$, then $R(P) \subset \mathrm{SL}(V_\rho^h)$, $R(B) \subset \widetilde{\mathrm{SL}}(V_\rho^h)$ and the following diagram commutes, with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{SL}(V_\rho^h) & \longrightarrow & \widetilde{\mathrm{SL}}(V_\rho^h) & \xrightarrow{\det} & \{\pm 1\} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \det\rho} & & \\ 1 & \longrightarrow & P & \longrightarrow & B & \xrightarrow{\pi} & W & \longrightarrow & 1 \end{array}$$

Notice that, if $\mathrm{tr}\,\rho(s) = 0$ for all $s \in \mathcal{R}$, then, since $s^2 = 1$, we have $\det\rho(s) = (-1)^{\frac{\dim\rho}{2}}$ for all $s \in \mathcal{R}$. In particular, $\det\rho = \mathbb{1}$ or $\det\rho = \epsilon$ in this case. We let $B^2 = \mathrm{Ker}\,\epsilon \circ \pi$. We have $P \subset B^2 \subset B$. Recalling that the Lie algebra of $\overline{R(P)}$ contains $\rho(\mathcal{H}) \otimes_{\mathbb{C}} K$, the following is a consequence of proposition 2.15, as $\mathrm{GL}(V_\rho^h)$ and $\mathrm{SL}(V_\rho^h)$ are connected algebraic groups.

Proposition 6.1. *Let $\rho \in \mathrm{QRef}$. If $\exists s \in \mathcal{R}$ $\mathrm{tr}\,\rho(s) \neq 0$ then $\overline{R(B)} = \overline{R(P)} = \mathrm{GL}(V_\rho^h)$. Otherwise, this implies $\dim\rho = 2$ and we have $\overline{R(P)} = \overline{R(B^2)} = \mathrm{SL}(V_\rho^h)$, $\overline{R(B)} = \widetilde{\mathrm{SL}}(V_\rho^h)$.*

Similarly, the following is a consequence of proposition 2.13 (notice that the morphisms defined there are W-equivariant, hence are isomorphisms of $\mathfrak{B}$-modules).

Proposition 6.2. *Let $\rho \in \Lambda\mathrm{Ref} \setminus \mathrm{QRef}$. If $\dim\rho = 1$, then $R = S_\rho$ and $\overline{R(P)} = \overline{R(B)} = K^\times$. If $\dim\rho > 1$, that is $\rho = \eta \otimes \Lambda^k\rho_0$ with $\rho_0 \in \mathrm{Ref}$, $\eta \in \mathrm{Hom}(W, \{\pm 1\})$, $\dim\rho_0 > k \geq 2$, then we have $R \simeq S_\eta^{-k} \otimes \Lambda^k R_{0,\eta}$, with $R_{0,\eta}$ the monodromy representation of B associated to $\eta \otimes \rho_0 \in \mathrm{Irr}(W)$. In particular, $\overline{R(P)} = \overline{R(B)} \simeq \mathrm{GL}(V_{\rho_0})$*

If $\rho \in \mathrm{Irr}(W)$ satisfies $\rho \simeq \rho^* \otimes \epsilon$, then we denote $\mathrm{OSP}(V_\rho^h)$ the algebraic group over K of direct (determinant one) isometries of the bilinear form $(\ |\ )$ associated to $\epsilon \hookrightarrow \rho^* \otimes \rho^*$. It has for Lie algebra $\mathfrak{osp}(V_\rho) \otimes K$. By the remarks above, $R(P) \subset \mathrm{OSP}(V_\rho^h)$. Recall from proposition 2.6 that, when $\epsilon \hookrightarrow S^2\rho^*$, then the corresponding symmetric bilinear form is hyperbolic, hence defined over $\mathbb{Q}$.

We define $\widehat{\mathrm{OSP}}(V_\rho^h) = \{x \in \mathrm{GL}_N(K) \mid \exists \alpha_1, \alpha_2 \in \{\pm 1\}\ x^{-1} = \alpha_1 x^+, \det(x) = \alpha_2\}$ where x^+ denotes the adjoint of x with respect to $(\ |\ )$. We have $\mathrm{OSP}(V_\rho^h) \subset \widehat{\mathrm{OSP}}(V_\rho^h)$, with index 2 if $\epsilon \hookrightarrow \Lambda^2\rho^*$, and index 2 or 4 if $\epsilon \hookrightarrow S^2\rho^*$, depending on $\dim\rho$ modulo 4. We denote $p : \widehat{\mathrm{OSP}}(V_\rho^h) \to \{\pm 1\}^2$ the morphism defined by (α_1, α_2). Since every reflection in $\mathcal{R}$ is antisymmetric w.r.t. $(\ |\ )$ and has trace 0, we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{OSP}(V_\rho^h) & \longrightarrow & \widehat{\mathrm{OSP}}(V_\rho^h) & \xrightarrow{p} & \{\pm 1\}^2 \\ \uparrow R & & \uparrow R & & \uparrow (\epsilon, \det \rho) \\ P & \longrightarrow & B & \xrightarrow[\pi]{} & W \end{array}$$

and define $\widetilde{\mathrm{OSP}}(V_\rho^h) = p^{-1}((\epsilon, \det \rho)(W))$. By definition, $R(B) \subset \widetilde{\mathrm{OSP}}(V_\rho^h)$. Moreover, since $\rho \simeq \rho^* \otimes \epsilon$ then $\mathrm{tr}\, \rho(s) = 0$ for all $s \in \mathcal{R}$ hence $\det \rho \in \{\mathbb{1}, \epsilon\}$. It follows that $\mathrm{OSP}(V_\rho^h)$ always has index 2 in $\widetilde{\mathrm{OSP}}(V_\rho^h)$ and that $R(B^2) \subset \mathrm{OSP}(V_\rho^h)$. Denoting $p_1 : \widetilde{\mathrm{OSP}}(V_\rho^h) \to \{\pm 1\}$ the map $x \mapsto \alpha_1$, this leads to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{OSP}(V_\rho^h) & \longrightarrow & \widetilde{\mathrm{OSP}}(V_\rho^h) & \xrightarrow{p_1} & \{\pm 1\} & \longrightarrow & 1 \\ & & \uparrow R & & \uparrow R & & \uparrow \epsilon & & \\ 1 & \longrightarrow & P & \longrightarrow & B & \xrightarrow[\pi]{} & W & \longrightarrow & 1 \end{array}$$

For any $\rho \notin \Lambda\mathrm{Ref}$, we proved in §5 that $\mathcal{H}(\rho) \simeq \rho(\mathcal{H}') \simeq \mathcal{L}(\rho)$, which is isomorphic to $\mathfrak{osp}(V_\rho)$ if $\rho^* \otimes \epsilon \simeq \rho$, and to $\mathfrak{sl}(V_\rho)$ otherwise. Also note that $\rho(\mathcal{H}) = \mathrm{tr}(\rho(\mathcal{H})) + \rho(\mathcal{H}') = \mathrm{tr}(\rho(\mathcal{R})) + \rho(\mathcal{H}')$. From this, the proof of the following proposition is straightforward, and follows the lines of [Ma07a], theorem B.

Proposition 6.3. *Let $\rho \notin \Lambda\mathrm{Ref}$. If $\rho^* \otimes \epsilon \simeq \rho$, then $\overline{R(P)} = \mathrm{OSP}(V_\rho^h)$ and $\overline{R(B)} = \widetilde{\mathrm{OSP}}(V_\rho^h)$. If $\rho^* \otimes \epsilon \not\simeq \rho \otimes \eta$ for every $\eta \in \mathrm{X}(\rho)$, we have the following cases :*

(1) *if there exists $s \in \mathcal{R}$ with $\mathrm{tr}\, \rho(s) \neq 0$, then $\overline{R(P)} = \overline{R(B)} = \mathrm{GL}(V_\rho^h)$*
(2) *if $\forall s \in \mathcal{R}$ $\mathrm{tr}\, \rho(s) = 0$, then $\overline{R(P)} = \mathrm{SL}(V_\rho^h)$. If $\det \rho = \mathbb{1}$ then $\overline{R(B)} = \mathrm{SL}(V_\rho^h)$, otherwise $\overline{R(B)} = \widetilde{SL}(V_\rho^h)$.*

The last case to consider is when $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$ for some $\eta \in \mathrm{X}(\rho)$ with $\eta \neq \mathbb{1}$. We define as before the bilinear form $(\mid)$ associated to $\epsilon \otimes \eta \hookrightarrow \rho^* \otimes \rho^*$ and $x \mapsto x^+$ the adjunction operation. Since $\eta \neq 1$, there exists $s \in \mathcal{R}$ with $\rho(s) \neq \pm 1$, and in particular $\mathrm{tr}\, \rho(s) \neq 0$. Moreover, any braided reflection associated to s is mapped by R to the scalar $\pm q^{\pm 1}$ with $q = \exp(\mathrm{i}\pi h)$, its square to $q^{\pm 2}$, and the subgroups generated by either image has $K^\times$ for algebraic closure.

Let $\widehat{\mathrm{GOSP}}(V_\rho^h)$ denote the algebraic group $\{x \in \mathrm{GL}(V_\rho^h) \mid x^+ x \in K^\times\}$, $\varphi : \widehat{\mathrm{GOSP}}(V_\rho^h) \to K^\times$ the character $\varphi(x) = x^+ x$, and $d : \widehat{\mathrm{GOSP}}(V_\rho^h) \to K^\times$ its determinant character. Recall that $\dim V_\rho = 2u$ is even, and let $\psi = \varphi^u d^{-1} : \widehat{\mathrm{GOSP}}(V_\rho^h) \to K^\times$. We define $\mathrm{GOSP}(V_\rho^h) = \mathrm{Ker}\, \psi$.

It is easily checked that $\mathrm{GOSP}(V_\rho^h)$ is connected, with Lie algebra $K \oplus \mathfrak{osp}(V_\rho) \otimes K$, and contains both $K^\times$ and $\mathrm{OSP}(V_\rho^h)$. Every braided reflection is conjugated to some $y = \rho(s) \exp(\mathrm{i}\pi h \rho(s)) \in \widehat{\mathrm{GOSP}}(V_\rho^h)$ by some element in $\mathrm{OSP}(V_\rho^h)$. If $\rho(s) \neq \pm 1$ we have $\mathrm{tr}\, \rho(s) = 0$ as before (since $\rho(s)$ is then conjugated to $-\rho(s)$) hence $\det y = \det \rho(s) = (-1)^u$, and $\varphi(y) = \varphi(\rho(s)) = -1$, where $\psi(y) = 1$. If $\rho(s) = \pm 1$ then $d(y) = (\pm 1)^{2u} q^{\pm 2u}$ for

$q = \exp(\mathrm{i}\pi h)$, and $\varphi(y) = q^{\pm 2}$, whence $\psi(y) = 1$. It follows that $\overline{R(P)} \subset \overline{R(B)} \subset \mathrm{GOSP}(V_\rho^h)$ and by the same arguments as above we get that $\overline{R(P)} = \overline{R(B)} = \mathrm{GOSP}(V_\rho^h)$.

Proposition 6.4. *Let $\rho \notin \Lambda\mathrm{Ref}$ with $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$ for some $\eta \in \mathrm{X}(\rho) \setminus \{\mathbb{1}\}$. Then $\overline{R(P)} = \overline{R(B)} = \mathrm{GOSP}(V_\rho^h)$.*

The four propositions above together imply theorem 2 of the introduction (notice that the algebraic groups considered here are defined over K_0 and have a dense set of K_0-points). It is worth noticing that all situations above actually occur, most of them already for $W = \mathfrak{S}_n$ (see [Ma07a]).

If $\rho \notin \Lambda\mathrm{Ref}$, we let $G(\rho) = \mathrm{OSP}(V_\rho^h)$, $\tilde{G}(\rho) = \widetilde{\mathrm{OSP}}(V_\rho^h)$ if $\rho^* \otimes \epsilon \simeq \rho$, $\tilde{G}(\rho) = \mathrm{GOSP}(V_\rho^h)$ if $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$ for some $\eta \in \mathrm{X}(\rho) \setminus \{\mathbb{1}\}$, and $\tilde{G}(\rho) = \mathrm{GL}(V_\rho^h)$ otherwise. We assume that W is *tackled*. Then $H_W^\times$ is naturally isomorphic to $\prod_{\rho \in \mathrm{Irr}(W)} \mathrm{GL}(V_\rho^h)$. Under this isomorphism the results above prove that the morphism $B \to H_W^\times$ factors through the algebraic group

$$\prod_{\rho \in \mathrm{Hom}(W, \{\pm 1\})} K^\times \times \prod_{\rho \in \overline{\mathrm{QRef}}} \mathrm{GL}(V_\rho^h) \times \prod_{\rho \in (\mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref})/\approx} \tilde{G}(\rho)$$

We let $\mathcal{R}/W = \{\mathcal{R}_1, \dots, \mathcal{R}_k\}$ and denote $\eta_i : W \to \{\pm 1\}$ such that $\eta_i(\mathcal{R}_j) = \{-1\}$ if $j = i$ and $\{1\}$ if $j \neq i$. We let $\chi_\rho^i = (\dim \rho - \mathrm{tr}\, \rho(s_i))/2$ for $s_i \in \mathcal{R}_i$. If $\rho \notin \Lambda\mathrm{Ref}$ is such that $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$ for some $\eta \in \mathrm{X}(\rho) \setminus \{\mathbb{1}\}$, we notice that η is uniquely determined by $\eta(s) = -1 \Leftrightarrow \rho(s) \in \{\pm 1\}$, and we write $\eta = \prod_{i \in I_\rho} \eta_i$ for some nonempty $I_\rho \subset \{1, \dots, k\}$. For $i \in I_\rho$ we let $\rho(\mathcal{R}_i)$ denote the common value of the $\rho(s)$ for $s \in \mathcal{R}_i$.

We introduce the subgroup $\check{G}$ of tuples (a_ρ) as above such that

$$(a_{\mathbb{1}} a_\epsilon)^2 = 1, \quad \det a_\rho = a_{\mathbb{1}}^{\dim \rho} \prod_{i=1}^{k} (a_{\eta_i} a_{\mathbb{1}}^{-1})^{\chi_\rho^i},$$

and

$$\phi(a_\rho) = (a_\epsilon a_{un})^{-1} \prod_{i \in I_\rho} (a_{\eta_i} a_{\mathbb{1}}^{-1})^{-\rho(\mathcal{R}_i)}$$

for $\varphi : \mathrm{GOSP}(V_\rho^h) \to K^\times$ defined above. Since a braided reflection σ with $\pi(\sigma) = s$ has image conjugated to $\rho(s) \exp(\mathrm{i}\pi h \rho(s))$, and since B is generated by braided reflections, it is easily checked that the image in $H_W^\times$ of B lies inside $\check{G}$. There is a morphism $p_\epsilon : \check{G} \to \{\pm 1\}$ given by $(a_\rho) \mapsto a_{\mathbb{1}} a_\epsilon$, and also one $p_\rho : \check{G} \to \{\pm 1\}$ for each $\rho \notin \Lambda\mathrm{Ref}$ such that $\rho^* \otimes \epsilon \simeq \rho$, which is inherited from $p_1 : \widetilde{\mathrm{OSP}}(V_\rho^h) \to \{\pm 1\}$. We define $\tilde{G} \subset \check{G}$ by

$$\tilde{G} = \{g \in \check{G} \mid p_\epsilon(g) = p_\rho(g), \rho \notin \Lambda\mathrm{Ref}, \rho^* \otimes \epsilon \simeq \rho\}$$

and denote $\pi_\epsilon : \tilde{G} \to \{\pm 1\}$ the morphism induced by p_ϵ. The image of B is contained in $\tilde{G}$, while the image of B^2 is contained in $G = \mathrm{Ker}\, \pi_\epsilon$.

The following follows from theorem 2.21, and is a generalization to (almost all) irreducible complex reflection groups of theorem C of [Ma07a].

Proposition 6.5. *Assume that W is an irreducible complex reflection group which is* tackled *with $W \neq H_4$. The image of B is Zariski-dense in $\tilde{G}$. The* *connected component of $\tilde{G}$ is G, which contains the images of B^2 and P as Zariski-dense subgroups. The short exact sequence $1 \to G \to \tilde{G} \to \{\pm 1\} \to 1$ defined by π_ϵ is split.*

Proof. Since $\tilde{G}$ contains the image of B and G contains the image of $B^2 \supset P$, it is sufficient to show that the Zariski-closure of the image of P has the same Lie algebra as G, as G is clearly connected. Actually, since the former Lie algebra is obviously included in the latter, one only needs to know that these two Lie algebras are isomorphic. This is the content of theorem 2.21, which concludes the proof of the main part of the proposition. The extension is split by $-1 \to (\rho(s))_\rho$, where s is some reflection is s. Indeed, letting σ denote a braided reflection with $\pi(\sigma) = s$, its image in $H_W^\times$ belongs to $\tilde{G} \setminus G$ and is conjugated to the collection of $(\rho(s)e^{i\pi h\rho(s)})_\rho$ by some element of G. By checking the definition of G we see that $e^{i\pi h\rho(s)}$ belongs to G, hence $(\rho(s))_\rho$ belongs to $\tilde{G} \setminus G$. ☐

6.3. **The case of $W = H_4$.** The decomposition of $\mathcal{H}$ is similar to the other cases, except that one has to decompose the factor $\mathcal{H}(\rho)$ for $\rho = \rho^a + \rho^b$ where ρ^a, ρ^b are the two 8-dimensional irreducible representations of W. We have $\rho^i \notin \Lambda\mathrm{Ref}$, $\epsilon \hookrightarrow S^2\rho^i$, hence $\mathcal{H}(\rho^i) \simeq \rho^i(\mathcal{H})' \simeq \mathfrak{so}_8(\mathbb{k})$ for $i \in \{a, b\}$. Let W_0 be a maximal parabolic subgroup of W of Coxeter type H_3. We have $\mathrm{Res}_{W_0}\rho^a \simeq \varphi + \varphi^* \otimes \epsilon \simeq \mathrm{Res}_{W_0}\rho^b$ where $\varphi \in \mathrm{Irr}(W_0) \setminus \Lambda\mathrm{Ref}(W_0)$ is 4-dimensional, defined over $\mathbb{Q}$ and such that $\varphi \not\simeq \varphi^* \otimes \epsilon$.

6.3.1. *The Lie algebra $\rho(\mathcal{H}')$.* In particular, $\varphi(\mathcal{H}_0') \simeq \mathfrak{sl}_4(\mathbb{k})$. We thus can choose matrix models for ρ^a, ρ^b in hyperbolic $\mathfrak{so}_8(\mathbb{k})$ with *equal* restrictions to $\mathcal{H}_0'$, such that $(\varphi + \varphi^* \otimes \epsilon)(\mathcal{H}_0')$ is an isotropic $\mathfrak{sl}_4$ inside $\mathfrak{so}_8$. By computer we check that $\dim \rho(\mathcal{H}') = \dim \mathfrak{so}_8(\mathbb{k})$. Since we proved $\mathcal{H}(\rho^i) \simeq \mathfrak{so}_8(\mathbb{k})$ in proposition 5.10, the following follows readily from the simplicity of the Lie algebra $\mathfrak{so}_8(\mathbb{k})$.

Lemma 6.6. *Under the models above, $\rho(\mathcal{H}') = \{(x, \psi(x) \mid x \in \mathfrak{so}_8(\mathbb{k})\}$ for some $\psi \in \mathrm{Aut}(\mathfrak{so}_8(\mathbb{k}))$ which is not induced by $\mathrm{GL}_8(\mathbb{k})$-conjugation and which pointwise stabilizes the isotropic $\mathfrak{sl}_4(\mathbb{k})$.*

The fact that ψ is not a conjugation automorphism is derived from the fact that $\rho^a_{\mathcal{H}'} \not\simeq \rho^b_{\mathcal{H}'}$. The existence of such an automorphism is reminiscent from the triality phenomenon. We describe it in more (matrix) detail. Assuming that the quadratic form is in hyperbolic shape, elements of $\mathfrak{so}_8(\mathbb{k})$ have the form $\begin{pmatrix} s+a & b \\ c & -s-{}^t a \end{pmatrix}$, with $s \in \mathbb{k}, a \in \mathfrak{sl}_4(\mathbb{k})$ and $b, c \in \mathfrak{so}_4(\mathbb{k})$, where $\mathfrak{so}_4(\mathbb{k})$ designates skew-symmetric matrices. We leave to the reader to check that any automorphism ψ satisfying the conditions of the lemma has the form

$$\psi\left(\begin{pmatrix} s+a & b \\ c & -s-{}^t a \end{pmatrix}\right) = \begin{pmatrix} -s+a & \lambda^{-1}c' \\ \lambda b' & s-{}^t a \end{pmatrix}$$

where λ is some fixed nonzero scalar, and $M \mapsto M'$ is defined by

$$\begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & b_4 & b_5 \\ -b_2 & -b_4 & 0 & b_6 \\ -b_3 & -b_5 & -b_6 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & b_6 & -b_5 & b_4 \\ -b_6 & 0 & b_3 & -b_2 \\ b_5 & -b_3 & 0 & b_1 \\ -b_4 & b_2 & -b_1 & 0 \end{pmatrix}$$

In particular, $\psi^2 = \mathrm{Id}$, and the matrix model can be chosen such that $\lambda = 1$.

6.3.2. *Zariski closure of* $R(P)$. For $x \in \mathrm{SO}_8(K)$, we denote $\overline{x}$ its image in $\mathrm{PSO}_8(K)$. We know compute the Zariski closure $\Gamma \subset \mathrm{SO}_8(K) \times \mathrm{SO}_8(K)$ of $R(P)$. We denote $\Gamma_i \simeq \mathrm{SO}_8(K)$ the Zariski closure of $R_i(P)$. We let p_i denote the projection of $\mathrm{SO}_8(K) \times \mathrm{SO}_8(K)$ onto its i-th factor. It is a morphism of algebraic groups, hence it maps Γ onto a closed subgroup of Γ_i and its restriction $\hat{p}_i$ to Γ is also a morphism of algebraic groups. Since $R(P)$ is Zariski-dense in Γ it follows that $R_i(P) = \hat{p}_i \circ R(P)$ is Zariski-dense in $p_i(\Gamma)$ hence $p_i(\Gamma) = \Gamma_i$.

We now let $\Gamma^i = \mathrm{Ker}\,\hat{p}_j \subset \Gamma$ for $\{i,j\} = \{1,2\}$. Then p_i identifies Γ^i with a normal subgroup of Γ_i, wich is connected. Since Γ and the Γ_i, Γ^i are linear algebraic groups, we have a natural isomorphism of algebraic groups $(\Gamma/\Gamma^i)/(\Gamma^i\Gamma^j/\Gamma^i) \simeq (\Gamma/\Gamma^j)/(\Gamma^i\Gamma^j/\Gamma^j)$ (see e.g. [Sp] 5.5.11). Since by definition $\Gamma_j = \mathrm{Ker}\,\hat{p}_i$ we have $\Gamma/\Gamma^j \simeq p_i(\Gamma) = \Gamma_i$ and similarly $\Gamma/\Gamma^i \simeq \Gamma_j$. Finally, since $\Gamma^i \cap \Gamma^j = \{1\}$ we have $\Gamma^i\Gamma^j/\Gamma^j \simeq \Gamma^i$. It follows that this natural isomorphism defines an isomorphism $\Psi : \Gamma_1/\Gamma^1 \to \Gamma_2/\Gamma^2$.

Since $\mathrm{SO}_8(K)$ does not admit any proper connected normal subgroup, the connected component of $p_1(\Gamma^1)$ is either trivial or equal to Γ_1. The latter case implies $p_1(\Gamma^1) = \Gamma_1$, $\mathrm{Ker}\,\hat{p}_1 = \{1\} \times \mathrm{SO}_8(K)$. Since $p_1(\Gamma) = \mathrm{SO}_8(K)$ this implies $\Gamma = \mathrm{SO}_8(K) \times \mathrm{SO}_8(K)$. In this case, $R_1 \otimes R_2$ would be irreducible under the action of P, meaning that the tensor product $(\rho^a)_{\mathcal{T}} \otimes (\rho^b)_{\mathcal{T}}$ would be irreducible under the action of the holonomy Lie algebra $\mathcal{T}$. A direct computation shows that this is not the case, for instance because $\dim(\rho^a)_{\mathcal{T}} \otimes (\rho^b)_{\mathcal{T}}(\mathsf{U}\mathcal{T}) < 64^2$. If follows that the connected components of $p_1(\Gamma^1)$, and similarly of $p_2(\Gamma^2)$, are trivial. It follows that each Γ^i is a subgroup of $Z(\Gamma_i) = \{\pm 1\}$. By $\Gamma_1/\Gamma^1 \simeq \Gamma_2/\Gamma^2$ it the same subgroup for $i \in \{1,2\}$, as $\mathrm{SO}_8(K) \not\simeq \mathrm{PSO}_8(K) = \mathrm{SO}_8(K)/\{\pm 1\}$.

We prove that this subgroup is $\{\pm 1\}$, because otherwise Ψ would induce an automorphism of $\mathrm{SO}_8(K)$; since automorphisms of $\mathrm{SO}_8(K)$ are conjugation by some element in $\mathrm{O}_8(K)$, this would provide an intertwinner between the representations R_1, R_2, hence also between the representations $(\rho^a)_{\mathcal{H}}, (\rho^b)_{\mathcal{H}}$. By proposition 2.7 this implies that $\rho^a, \rho^b \in \mathrm{Irr}(W)$ are isomorphic, which is not the case.

In particular Ψ is an automorphism $\mathrm{PSO}_8(K) \simeq \Gamma_1/\{\pm 1\} \to \Gamma_2/\{\pm 1\} \simeq \mathrm{PSO}_8(K)$, and $\Gamma = \{(x,y) \in \mathrm{SO}_8(L)^2 \mid \overline{y} = \Psi(\overline{x})\}$. This automorphism cannot be induced by $\mathrm{PO}_8(K)$, otherwise there could exist $x \in \mathrm{GL}_8(K)$ such taht $xR_2(g)x^{-1} = \pm R_1(g)$ for all $g \in P$, whence $x\rho^b(g)x^{-1} = \pm\rho^a(g)$ for all $g \in \mathcal{R}$, hence for all $g \in W$. Then $\eta : W \to \{\pm 1\}$ defined by $x\rho^b(g)x^{-1} = \eta(g)\rho^a(g)$ is a character of W such that $\rho^b \simeq \rho^a \otimes \eta$. But $\mathrm{Hom}(W, \{\pm 1\}) = \{\mathbb{1}, \epsilon\}$ and $\rho^b \not\simeq \rho^a, \rho^b \not\simeq \epsilon \otimes \rho^a$, a contradiction. It is thus a triality automorphism. Moreover, the matrix models of ρ^a, ρ^b have been chosen such that $\rho^a(w) = \rho^b(w)$ for all $w \in W_0$. It follows that Ψ pointwise stabilizes the isotropic $\mathrm{SL}_4(K)$, hence the induced automorphism ψ of $\mathrm{Aut}(\mathfrak{so}_8(K))$ belongs to the ones determined by the lemma above, and has order 2. Since $\mathrm{PSO}_8(K)$ is connected it follows that $\Psi^2 = \mathrm{Id}$.

From this description and the classification of reductive algebraic groups, it is clear that Γ is isomorphic to $\mathrm{Spin}_8(K)$, as Γ is connected with Lie algebra $\mathfrak{so}_8(K)$ and has at least the 4 elements $(\pm 1, \pm 1)$ in its center. We thus proved the following.

Proposition 6.7. *There exists* $\Psi \in \mathrm{Aut}(\mathrm{PSO}_8(K))$ *inducing* $\psi \in \mathrm{Aut}(\mathfrak{so}_8(K))$, *such that the Zariski-closure of* $R(P)$ *is* $\{(x,y) \in \mathrm{SO}_8(K)^2 \mid \overline{y} = \Psi(\overline{x})\}$. *This group is isomorphic to* $\mathrm{Spin}_8(K)$.

6.3.3. *Zariski closure of* $R(B)$. Let $\widetilde{\mathrm{SO}}_8(K) = \mathrm{SO}_8(K) \times \{\pm 1\}$, with $\pi_0 : \widetilde{\mathrm{SO}}_8(K)$ the projection onto the second factor. Note that $\widetilde{\mathrm{OSP}}(\rho^i) \simeq \widetilde{\mathrm{SO}}_8(K)$. We let $\widetilde{\mathrm{PSO}}(K) = \mathrm{PSO}_8(K) \times \{\pm 1\}$. One clearly has $R(B) = \{(x,y) \in \widetilde{\mathrm{SO}}_8(K)^2 \mid \pi_0(x) = \pi_0(y)\}$. The surjective morphism $\epsilon \circ \pi : B \to \{\pm 1\}$ can thus be extended to a morphism $\tilde{\Gamma} \to \{\pm 1\}$, restriction of $\pi_0 \times \pi_0$ to $\tilde{\Gamma}$, $\tilde{\Gamma}$ designates the Zariski closure of $R(B)$. We thus have a short exact sequence $1 \to \Gamma \to \tilde{\Gamma} \to \{\pm 1\} \to 1$, which is split, by sending -1 to $(\rho^a(s), \rho^b(s))$ for some $s \in \mathcal{R}$, by the same argument as in proposition 6.5. This extension is not trivial, and moreover we have $Z(\tilde{\Gamma}) = Z(\Gamma)$. Indeed, by irreducibility of R_1, R_2 under the action of P, central elements in $\tilde{\Gamma} \subset \widetilde{\mathrm{SO}}_8 \times \widetilde{\mathrm{SO}}_8(K)$ have the form (λ, μ) for $\lambda, \mu \in K$, hence belong $Z(\widetilde{\mathrm{SO}}_8(K)^2) = Z(\Gamma)$, hence $Z(\tilde{\Gamma}) = Z(\Gamma)$. We thus proved the following.

Proposition 6.8. *Let* $\Gamma, \tilde{\Gamma}$ *denote the Zariski closures of* $R(B), R(P)$, *respectively. Then* $\tilde{\Gamma}$ *is a split extension of* $\{\pm 1\}$ *by* Γ *with* $Z(\tilde{\Gamma}) = Z(\Gamma)$.

Notice that this extension, which is uniquely up to isomorphism by the action of (s,s) on Γ, for $s \in \mathcal{R}$, is not isomorphic (as a group) to the usual $\mathrm{Pin}_8(K)$ extension, because the centers of $\mathrm{Pin}_8(K)$ and $\mathrm{Spin}_8(K)$ do not coincide.

6.4. **Proof of theorem 1 for** H_4**.** Theorem 1 of the introduction is known for $W \neq H_4$, as it is a consequence of theorem 2.21 proved in §5. We thus assume $W = H_4$. Let $\approx'$ be defined as the introduction, meaning $\rho^a \approx' \rho^b$ for the special representations above and $\rho^1 \approx \rho^2 \Rightarrow \rho^1 \approx' \rho^2$. Recall the morphism Φ from section 2. By lemma 6.6 is factorizes through

$$\Phi^+ : \mathcal{H}' \to \mathcal{I} \oplus \left(\bigoplus_{\rho \in (\mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref})/\approx} \mathcal{L}(\rho) \right)$$

and we only need to prove that Φ^+ is surjective. We know that $\mathcal{H}(\rho) \simeq \mathcal{L}(\rho)$ for all $\rho \notin \Lambda\mathrm{Ref}$, and one checks through the character table that, for $\rho \in \mathrm{Irr}(W)$, $\mathcal{H}(\rho) \simeq \mathfrak{so}_8(\Bbbk) \Leftrightarrow \rho \in \{\rho^a, \rho^b\}$. Then the arguments of proposition 2.23 and corollary 2.24 show that Φ^+ is surjective, which concludes the proof of theorem 1.

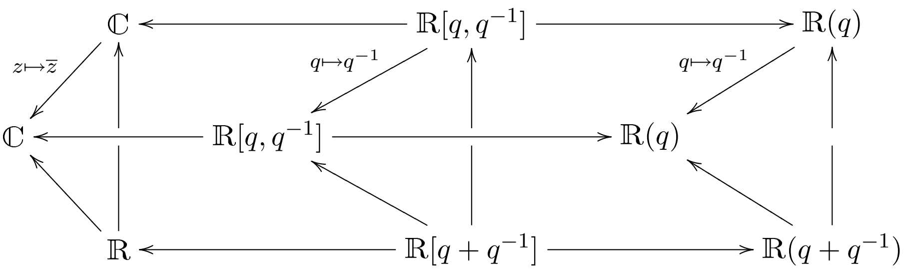
Finally, proposition 6.5 implies the following for $W \neq H_4$, the case $W = H_4$ being now straightforward.

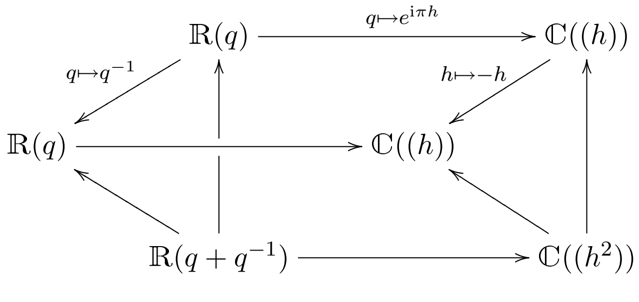
Theorem 6.9. *Assume that* W *is an irreducible complex reflection group which is* tackled. *The Zariski closures of the image of* P *and* B^2 *in* $H_W(q)^\times$ *coincide, they are connected and reductive, with Lie algebra* $\mathcal{H} \otimes K$. *They have index 2 in the Zariski closure of the image of* B, *and the corresponding extension is split.*

It can be noticed that the Zariski closure of (the image of) P is simply connected exactly for the groups W which do not admit $\rho \in \mathrm{Irr}(W) \setminus \Lambda\mathrm{Ref}$ with $\epsilon \hookrightarrow S^2\rho$. This is the case for all irreducible groups of rank 2 by using : lemma 2.11 , the fact that $G(e,1,2)$ as well as its reflection subgroups admit no irreducible representation of dimension greater than 2, and inspection of the character table of the groups G_{12}, G_{13}, G_{22} and $\mathfrak{S}_3$. Similarly, the groups $G(e,e,3)$ and $G(2e,e,3)$ have irreducible representations of dimension $1,2,3$ or 6. By lemma 7.4 below we cannot have $\epsilon \hookrightarrow S^2\rho$ with $\dim \rho = 6$ in rank 3, and the other representations belong to $\Lambda\mathrm{Ref}$ by lemma 2.18. Checking the exceptional groups and $\mathfrak{S}_4$ separately, it follows that the corresponding algebraic groups are simply connected if $\mathrm{rk}\, W \leq 3$, while many non-simply connected cases appear in rank 4 (e.g. F_4, G_{29}, H_4, G_{31}, etc.).

6.5. **Unitarisability in Coxeter types.** We assume that W is a Coxeter group, and for $q \in \mathbb{C}^{\times}$ we denote $\hat{H}(q)$ the specialized (complex) Iwahori-Hecke algebra associated to q. For q close enough to 1, $\hat{H}(q)$ is isomorphic to the group algebra $\mathbb{C}W$, and there is a natural 1-1 correspondance between its irreducible representations and the ones of W (we use [GP] as our main reference on these topics). We prove that, for q close enough to 1 these representations are unitarizable. This extends previous results in type A (see [Wz], and [Ma06]).

The main idea in the proof below is a combination of deformation and descent arguments along the following patterns.





Proposition 6.10. *For $q \in \mathbb{C}$ with $|q| = 1$ and close enough to 1, every representation of $\hat{H}(q)$ is unitarizable as a representation of B.*

Proof. We can assume that the representation considered here is irreducible. We let $Q_0 = \mathbb{R}(q)$ the field of rational fractions in one indeterminate over $\mathbb{R}$. By [GP] §9.3.8 this representation is a specialization of $R_0 : B \to \mathrm{GL}_N(A_0)$ where $A_0 = \mathbb{R}[q,q^{-1}]$ is the ring of Laurent polynomials, and the specialization $q = 1$ provides the corresponding representation of W. Since

W is a Coxeter group, up to conjugation by an element of $\mathrm{GL}_N(\mathbb{R})$ we can assume that this representation of W is orthogonal w.r.t. the standard scalar product of $\mathbb{R}^N$.

We let $\epsilon_0 \in \mathrm{Gal}(Q_0|\mathbb{R})$ defined by $\epsilon_0(q) = q^{-1}$ and $\epsilon \in \mathrm{Aut}(K_0)$ defined by $f(h) \mapsto f(-h)$. These automorphisms stabilized A_0 and A, respectively. The ring morphism $A_0 \hookrightarrow A$ defined by $q \mapsto \exp(\mathrm{i}\pi h)$ induces an embedding $Q_0 \hookrightarrow K_0$ equivariant under ϵ_0 and ϵ. The ring morphism is compatible with the reductions at $q = 1$ and $h = 0$, meaning that the following diagram is commutative.

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ {\scriptstyle q=1}\big\downarrow & & \big\downarrow{\scriptstyle h=0} \\ \mathbb{R} & \longrightarrow & \mathbb{C} \end{array}$$

By extension of scalars we deduce from R_0 a representation $K_0 \otimes R_0 : B \to \mathrm{GL}_N(K_0)$. Since it is a representation of the generic Hecke algebra over K_0, it is isomorphic to a monodromy representation $R : B \to \mathrm{GL}_N(A)$, that can be chosen with $R(B) \subset U_N^{\epsilon}(A)$, where $U_N^{\epsilon}(A)$ denotes the formal unitary group associated to ϵ, and $R_{h=0} = (R_0)_{q=1}$ (this follows from [Ma07d], prop. 2.4, since the reflections are both orthogonal and selfadjoint w.r.t. the W-invariant (standard) scalar product chosen here). It follows that there exists $P \in \mathrm{GL}_N(K_0)$ such that $R(g)P = PR_0(g)$ for all $g \in B$. Up to multiplication by some power of h, we can assume that P has coefficients in A and that its reduction $\overline{P} \in Mat_N(\mathbb{C})$ is nonzero. It follows that $\overline{P}$ is an intertwinner between the two absolutely irreducible representations $R_{h=0}$ and $R_{q=1}$. Since these two representations are actually equal, $\overline{P}$ is some scalar, that can be assumed to equal 1, up to multiplication by some real scalar, hence $P \in \mathrm{GL}_N(A)$.

Let $b_1, \dots, b_r$ a set of generators for B. The mapping

$$\Phi : J \mapsto \left(\epsilon_0({}^t R_0(b_i))J - JR_0(b_i)^{-1}\right)_{i=1..r}$$

from $Mat_N(Q_0)$ to $Mat_N(Q_0)^r$ is Q_0-linear. The dimension of its kernel is thus equal to the one of $K_0 \otimes \Phi : Mat_N(K_0) \to Mat_N(K_0)$. Considering the corresponding linear system $\epsilon_0({}^t R_0(b_i))J - JR_0(b_i)^{-1} = 0$ in $Mat_N(K_0)$ we see that it is in 1-1 correspondance with the space of B-module morphisms between $(K_0 \otimes R_0)^{\epsilon}$ and $K_0 \otimes R_0^*$, where $R_0 : b \to {}^t R(b^{-1})$ is the dual representation of R_0 and $(K_0 \otimes R_0)^{\epsilon}$ maps $b \mapsto \epsilon(R_0(b))$. These two representations being absolutely irreducible, by Schur lemma this space has dimension at most 1, and a corresponding nonzero J is necessarily invertible. Since R is isomorphic to R_0 over K_0, such a J exists. Indeed, from $R(b_i) = PR_0(b_i)P^{-1}$ and ${}^t\epsilon(R(b_i)) = R(b_i)^{-1}$, we get $\epsilon({}^t P^{-1})\epsilon({}^t R_0(b_i))\epsilon({}^t P) = PR_0(b_i)^{-1}P^{-1}$, meaning $\epsilon({}^t R_0(b_i))\epsilon({}^t P)P = \epsilon({}^t P)PR_0(b_i)^{-1}$, and this is the desired equation with $J = J' = \epsilon({}^t P)P$. Since $\overline{P} = 1$ we have $J'_{h=0} = 1$.

It follows that there exists a nonzero $J_0 \in M_N(Q_0)$ solving this linear system. Up to multiplication of J_0 by powers of $q - 1$, we can assume $J_0 \in Mat_N(Q_0) \cap Mat_N(A)$, and moreover $(J_0)_{h=0} \neq 0$. Since the reductions at $q = 1$ of R_0 and $\epsilon_0 \, {}^t R_0$ are (absolutely) irreducible and equal, we can actually assume $(J_0)_{h=0} = 1$, up to rescaling J_0 by $(J_0)_{h=0}^{-1} \in \mathbb{R}$.

On the other hand, there exists $\mu \in K^\times$ with $J' = \mu J_0$. Since J' and J_0 belong to $\mathrm{GL}_N(A)$ with $(J')_{h=0} = (J_0)_{h=0} = 1$, we get $\mu \in A$ and $\mu \equiv 1 \not\equiv 0$ modulo h. As a consequence we get that $1 + \epsilon(\mu)\mu^{-1} \in A$ is invertible. We let c denote its inverse. From $\epsilon(J') =^t J'$ we get $\epsilon(\mu)\epsilon(J_0) = \mu^t J_0$. Since $J_0 \neq 0$ we have $\epsilon(\mu)\mu^{-1} \in Q_0$ hence $c \in Q_0 \cap A$.

Let now $J = 2cJ_0 \in \mathrm{GL}_N(Q_0) \cap M_N(A)$. We check ${}^tJ = \epsilon_0(J)$, and $J_{h=0} = 2\frac{1}{2}(J_0)_{h=0} = 1$. It follows that J can be specialized in a neighbourhood of $q = 1$ and, when $|q| = 1$, it defines a unitary form which is positive definite when q is close enough to 1, which concludes the proof. □

6.6. **Explicit unitary models : the example of D_4.** We make the remark that the proof given above is constructive, in the sense that, starting from a given matrix model over $\mathbb{R}[q, q^{-1}]$, or even $\mathbb{R}(q)$, one can algorithmically derive a matrix J with coefficients in $\mathbb{R}(q)$ that affords a unitary structure for $|q| = 1$ and q close to 1. Indeed, the obtention of J_0 is merely a linear algebra matter from matrix models for R_0 (plus chasing $(q-1)$ denominators) ; moreover, since $\epsilon(\mu)\epsilon(J_0) = \mu J_0$ we get the value of $\epsilon(\mu)\mu^{-1}$ from any nonzero coefficient m_{ij} of J_0 as $\epsilon(\mu)\mu^{-1} = \epsilon_0(m_{ji})m_{ij}^{-1}$ and get c explicitely.

As an example, we carry out this procedure on the reflection representation of the Coxeter group D_4. A convenient matrix model is given by Hoefsmit matrices, with Artin generators $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ following the convention that $\sigma_1, \sigma_2, \sigma_4$ commute one to the other.

$$\sigma_1 \mapsto \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \frac{q-q^{-1}}{2} & -\frac{q+q^{-1}}{2} \\ 0 & 0 & -\frac{q+q^{-1}}{2} & \frac{q-q^{-1}}{2} \end{pmatrix} \sigma_2 \mapsto \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \frac{q-q^{-1}}{2} & \frac{q+q^{-1}}{2} \\ 0 & 0 & \frac{q+q^{-1}}{2} & \frac{q-q^{-1}}{2} \end{pmatrix}$$

$$\sigma_3 \mapsto \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \frac{q^2-1}{q^3+q} & \frac{2q}{1+q^2} & 0 \\ 0 & \frac{1+q^4}{q+q^3} & \frac{q^3-q}{1+q^2} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \sigma_4 \mapsto \begin{pmatrix} \frac{q^2-1}{q+q^5} & \frac{q+q^3}{1+q^4} & 0 & 0 \\ \frac{q^6+1}{q^5+q} & \frac{q^5-q^3}{q^4+1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Using the above procedure, we find for J the diagonal matrix $1, q^2 + 1 + q^{-2}, \frac{(q^2+q^{-2})(q^2+q^{-2}-1)}{2}, \frac{(q^2+q^{-2})(q^2+q^{-2}-1)}{2}$, which satisfies $\epsilon_0(J) =^t J$. Specializing at a complex number with $|q| = 1$ written $q = \exp(\mathrm{i}x)$ with $|x| \leq \frac{\pi}{2}$, we get that the form has signature $(4, 0)$ for $|x| < \frac{\pi}{6}$, $(1, 3)$ for $\frac{\pi}{6} < |x| < \frac{\pi}{4}$ and $(3, 1)$ for $\frac{\pi}{4} < |x|$.

6.7. **Topological closure for Coxeter groups.** We still assume that W is an irreductible Coxeter group. The representations of the Hecke algebras are actually defined over $\overline{\mathbb{Q}}_0[q, q^{-1}]$, where $\overline{\mathbb{Q}}_0 = \overline{\mathbb{Q}} \cap \mathbb{R}$. Like the definition over $\mathbb{R}[q, q^{-1}]$, this is a consequence of the existence of W-graphs for all Coxeter groups (see [GP]). Obviously, every $\rho \in \mathrm{Irr}(W)$ is defined over $\overline{\mathbb{Q}}_0$.

We now consider a transcendantal number $u \in \mathbb{C}$ with $|u| = 1$ and u close enough to 1, so that $\hat{R}$ is unitarizable by proposition 6.10, and investigate the topological closure of $\hat{R}(P)$. It is a compact (Lie) subgroup of U_N, hence lies inside $G(V_\rho) \cap U_N$, where $G(V_\rho)$ is the Zariski closure of $\hat{R}(P)$.

By irreducibility of the action of P, this group $G(V_\rho) \cap U_N$ is a maximal compact subgroup of $G(V_\rho)$; for, every intermediate compact subgroup should be contained in some unitary group, but the image of P acting irreductibly can be contained in only one such group. In particular, this is a real form of $G(V_\rho)$. The fact that it is the topological closure of $\hat{R}(P)$ is an immediate consequence of the following well-known lemma.

Lemma 6.11. *If $H_1 \subset H_2 \subset U_N$ is an inclusion of (compact) closed subgroups, then $H_1 = H_2$ if and only if they have the same Zariski closure in $\mathrm{GL}_N(\mathbb{C})$.*

Proof. If $H_1 \neq H_2$, then there exists a continuous $f : U_N \to \mathbb{C}$ with $f(H_1) = \{0\}$ and $f(gH_1) = \{1\}$ for some $g \in H_2 \setminus H_1$, that can be chosen H_1-invariant by averaging w.r.t. some Haar measure on H_1. We denote (z_{ij}) the complex coordinates of $Mat_N(\mathbb{C})$. By the Stone-Weierstrass theorem we get a polynomial P in the z_{ij} and $\overline{z}_{ij}$ with $|f - P| \leq 1/4$ and averaging P we get another polynomial $\tilde{P}$ which takes distinct constant values over H_1 and gH_1, so we can assume $\tilde{P}(H_1) = 0$ and $\tilde{P}(gH_1) = 1$.

Now the identity $\overline{M} = {}^tM^{-1}$ for $M \in U_N$ implies that $\tilde{P}$ coincides over U_N with some Q/det^r, where Q is a polynomial in the (z_{ij}) and det is the determinant. In particular, Q vanishes on H_1 but not on gH_1, a contradiction if H_1 and H_2 had the same Zariski closure.

□

We choose an isomorphism $\psi : \mathbb{C} \xrightarrow{\sim} K$ preserving $\overline{\mathbb{Q}}$ that maps our transcendantal number u to $e^{\mathrm{i}\pi h}$, by identifying both $\mathbb{C}$ and K to the algebraic closure of the field of rational fractions with coefficients in $\overline{\mathbb{Q}}$ over a continuous number of indeterminates, one of them being $u \in \mathbb{C}$ and $e^{\mathrm{i}\pi h} \in K$. Now $G(V_\rho)$ is actually defined over $\overline{\mathbb{Q}}_0(u)$ (by [Bor] prop 1.3 b) and R is isomorphic to the twisting of $\hat{R}$ by ψ. It follows that the Zariski closure of $R(P)$ is deduced from $G(V_\rho)$ by ψ followed by a conjugation in $\mathrm{GL}(V_\rho^h)$. In particular $G(V_\rho)$ is reductive and connected, and so are its maximal compact subgroups. It follows that the topological closure of $\hat{R}(P)$ is connected.

Let $\mathfrak{g}$ denote the (complex) Lie algebra of the Lie or algebraic group $G(V_\rho)$. Through ψ we have $\mathfrak{g} \otimes_{\mathbb{C},\psi} K \simeq \rho(\mathcal{H}) \otimes_{\mathbb{C}} K$, where $\otimes_{\mathbb{C},\psi}$ denotes the identification of $\mathbb{C}$ with K, so that $\mathbb{C} \otimes_{\mathbb{C},\psi} K$ is naturally isomorphic both to $\mathbb{C}$ and to K. Indeed, the latter Lie algebra $\rho(\mathcal{H}) \otimes K$ is the Lie algebra of the Zariski closure of $R(P)$ by the results of section 6.2, and R is isomorphic to the twisting of $\hat{R}$ by ψ. Let $\mathcal{H}^{\overline{\mathbb{Q}}_0} \subset \overline{\mathbb{Q}}_0 W$ denote the infinitesimal Hecke algebra with coefficients in $\overline{\mathbb{Q}}_0$. Then $\rho(\mathcal{H}) = \rho(\mathcal{H}^{\overline{\mathbb{Q}}_0}) \otimes_{\overline{\mathbb{Q}}_0} \mathbb{C}$, and, since ψ is the identity on $\overline{\mathbb{Q}}$,

$$\mathfrak{g} \otimes_{\mathbb{C},\psi} K \simeq (\rho(\mathcal{H}^{\overline{\mathbb{Q}}_0}) \otimes_{\overline{\mathbb{Q}}_0} \mathbb{C}) \otimes_{\mathbb{C},\psi} K.$$

which implies $\mathfrak{g} \simeq \rho(\mathcal{H}^{\overline{\mathbb{Q}}_0}) \otimes_{\overline{\mathbb{Q}}_0} \mathbb{C} \simeq \rho(\mathcal{H})$, because ψ is an isomorphism (or because $\mathbb{C}$ is algebraically closed). Since the same arguments work for B^2, we proved the following.

Theorem 6.12. *For $u \in \mathbb{C}^\times$ a transcendental number, $|u| = 1$ and u close to 1, if ρ is an irreducible real representation of (the irreducible finite Coxeter*

$p(\boldsymbol{\lambda})$	r	$\dim \boldsymbol{\lambda}$	λ^{i_1} or $(\lambda^{i_1})'$	λ^{i_2} or $(\lambda^{i_2})'$	λ^{i_3} or $(\lambda^{i_3})'$
1	$r \geq 1$	1	$[r]$		
	$2 \leq r \leq 9$	$r-1$	$[r-1,1]$		
	4	2	$[2,2]$		
	5	5	$[3,2]$		
	6	5	$[3,3]$		
	5	6	$[3,1,1]$		
2	$2 \leq r \leq 8$	r	$[r-1]$	$[1]$	
	4	6	$[2]$	$[2]$	
	5	8	$[3,1]$	$[1]$	
3	3	6	$[1]$	$[1]$	$[1]$

Table 6. Irreducible representations of $G(e,1,r)$ of dimension at most 8

group) W, then the closure of the image of P (or B^2) in the corresponding representation of $\widehat{H}(u)$ is connected and has for Lie algebra a compact real form of $\rho(\mathcal{H})$.

Finally, we recall from proposition 2.27 that a compact real form of $\mathcal{H}$ is given by the real Lie algebra $\mathcal{H}^c$ generated inside $\mathbb{C}W$ by the elements is, $s \in \mathcal{R}$. The compact real form of the theorem is thus isomorphic to $\rho(\mathcal{H}^c)$.

7. Technical results on representations of W

7.1. Small-dimensional representations.

Lemma 7.1. *The irreducible representations $\boldsymbol{\lambda}$ of $G(e,1,r)$ of dimension at most 8 satisfy $p(\boldsymbol{\lambda}) \leq 3$. The possible $\lambda_{i_k} \neq \emptyset$ for $1 \leq k \leq p(\boldsymbol{\lambda})$, with the i_k distinct indices are given by table 6.*

Proof. If $p(\boldsymbol{\lambda}) = 1$, according to the branching rule the dimension of $\boldsymbol{\lambda}$ is the same as the dimension of the symmetric group $\mathfrak{S}_{r+1}$ associated to λ. It is well-known that, besides 1-dimensional representations, $\mathfrak{S}_{r+1}$ admits no irreducible representations of dimension less than r, except when $r = 3$. The conclusion follows by case-by-case examination for $r \leq 8$.

If $p(\boldsymbol{\lambda}) = 2$, then $\dim(\boldsymbol{\lambda}) = \dim(\lambda, \mu)$, where (λ, μ) is a representation of $G(2,1,r)$, that is the Coxeter group of type B_r. Once again, it is sufficient to examine the case $r \leq 9$, and the conclusion follows by case-by-case examination.

Now assume $p(\boldsymbol{\lambda}) \geq 3$. We know $\dim(\boldsymbol{\lambda}) \geq p(\boldsymbol{\lambda})!$, hence $p(\boldsymbol{\lambda}) = 3$. Moreover the binary representations with $p(\boldsymbol{\lambda}) = 3$ have dimension $3! = 6$. If $\boldsymbol{\lambda}$ is not a binary representation, then $r \geq 4$ and its restriction to $G(e,1,4)$ contains a representation of the same dimension as $([2],[1],[1],\emptyset,\dots)$, whose dimension is 12. The conclusion follows. ☐

We will need the following result to check the inductive assumptions in case of multiplicity 2 components.

r	$\dim\rho$	e	$A(\boldsymbol{\lambda})$	$p(\boldsymbol{\lambda})$	λ^{i_1} or $(\lambda^{i_1})'$	λ^{i_2} or $(\lambda^{i_2})'$	λ^{i_3} or $(\lambda^{i_3})'$
3	2	$*$	1	1	$[2,1]$		
		$3\|e$	3	3	$[1]$	$[1]$	$[1]$
	3	$*$	1	2	$[2]$	$[1]$	
4	2	$*$	1	1	$[2,2]$		
	3	$*$	1	1	$[3,1]$		
		$2\|e$	2	2	$[2]$	$[2]$	

TABLE 7. Irreducible representations of $G(e,e,r)$ of dimension 2 or 3 for $r \geq 3$.

Proposition 7.2. *If $r \geq 5$ then the dimension of all irreducible representations of $G(e,e,r)$ differ from 2 and 3. If $r = 2$ all irreducible representations of $G(e,e,r)$ have dimension 1 or 2. If $r \in \{3,4\}$ and ρ is an irreducible representation of $G(e,e,r)$ of dimension 2 or 3 then ρ is a component of some of the representations $\boldsymbol{\lambda}$ of $G(e,1,r)$ listed in table 7.*

Proof. If $r = 2$ the group $G(e,e,r)$ is a dihedral group and the result is classical. Assume that $r > 2$ and $\dim\rho \in \{2,3\}$. If $A(\boldsymbol{\lambda}) = 1$ the result follows from table 6, hence we assume $A(\boldsymbol{\lambda}) \neq 1$.

From the inequality $\dim\rho \geq (p(\boldsymbol{\lambda})-1)!$ of lemma 4.3 we get $p(\boldsymbol{\lambda}) \leq 3$. Moreover $1 \neq A(\boldsymbol{\lambda})|p(\boldsymbol{\lambda})$ by lemma 4.2 hence $A(\boldsymbol{\lambda}) = p(\boldsymbol{\lambda}) \in \{2,3\}$. If $A(\boldsymbol{\lambda}) = p(\boldsymbol{\lambda}) < 3$ or $\dim\rho < 3$ then $\dim\boldsymbol{\lambda} = A(\boldsymbol{\lambda})\dim\rho \leq 8$ and the result follows by inspection of table 6. In case $A(\boldsymbol{\lambda}) = p(\boldsymbol{\lambda}) = 3$, then there exists i such that $\lambda^i \notin \{\emptyset, [1]\}$, otherwise $\dim\rho \geq p(\boldsymbol{\lambda})! = 6$ by lemma 4.3. It follows that $\boldsymbol{\lambda}$ is binary with $r = p(\boldsymbol{\lambda}) = 3$, which completes the proof.

□

7.2. **Proof of lemma 2.22.** We note the following facts. If $W = G(e,e,r)$ with $r \geq 3$, since W admits a single class of reflexions we have $\mathrm{X}(\rho) = \{\mathbb{1}\}$ as soon as $\dim\rho > 1$. The same holds for $W = G(2e,e,r)$ with $r \geq 3$ and $\dim\rho > 1$ as soon as the restriction of ρ to $W_0 = G(2e,2e,r)$ is *not* irreducible : we cannot have $\rho(s_i) = \pm 1$ since $\dim\rho > 1$, and $\rho(t^e) = \pm 1$ would contradict the irreducibility of ρ. Finally note that, if this restriction ρ_0 *is* irreducible (with $\dim\rho > 1$) then $\rho^* \otimes \epsilon \simeq \rho \otimes \eta$ for some $\eta \in \mathrm{X}(\rho)$ implies that $\rho_0^* \otimes \epsilon \simeq \rho_0$.

If $r = 2$, recall that all $\rho \in \mathrm{Irr}(W)$ have dimension 2, hence we do not have to consider this case in the sequel.

Lemma 7.3. *There exists $\rho \in \mathrm{Irr}(W)$ with $\dim\rho = 4$ and and $\eta \in \mathrm{X}(\rho)$ with $\epsilon \otimes \eta \hookrightarrow S^2\rho^*$ if and only if W is a Coxeter group of type F_4.*

Proof. We first check using the character tables in CHEVIE that $F_4 = G_{28}$ is the only exceptional type admitting such a representation. We then assume by contradiction that W has type $G(e,e,r)$, $r \geq 3$ and admits such a ρ, that is $\epsilon \hookrightarrow S^2\rho^*$. Let $\boldsymbol{\lambda}$ be an irreducible representation of $G(e,1,r)$ such that ρ embeds in the restriction of $\boldsymbol{\lambda}$. If $A(\boldsymbol{\lambda}) = 1$ then $\dim\boldsymbol{\lambda} = 4$ and we can use table 6 to check that there are no possibilities with $\boldsymbol{\lambda}^* \otimes \epsilon \simeq \boldsymbol{\lambda}$. It follows that $A(\boldsymbol{\lambda}) \neq 1$. By lemma 4.3 we have $4 \geq (p(\boldsymbol{\lambda})-1)!$ hence $p(\boldsymbol{\lambda}) \leq 3$. From $1 \neq A(\boldsymbol{\lambda})|p(\boldsymbol{\lambda}) \leq 3$ it follows that $A(\boldsymbol{\lambda}) = p(\boldsymbol{\lambda}) \in \{2,3\}$. If

$\boldsymbol{\lambda}$ was binary we would have $p(\boldsymbol{\lambda})! = \dim \boldsymbol{\lambda} = A(\boldsymbol{\lambda}) \dim \rho = p(\boldsymbol{\lambda}) \dim \rho$ hence $4 = \dim \rho = (p(\boldsymbol{\lambda}) - 1)!$, a contradiction. Then $\boldsymbol{\lambda}$ is not binary, $4 \geq p(\boldsymbol{\lambda})!$ by lemma 4.3 and $p(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda}) = 2$. But then $\dim \boldsymbol{\lambda} = 8$ and $A(\boldsymbol{\lambda}) = 2$, a case excluded by table 6.

Now we assume that W has type $G(2e, e, r)$ and that ρ embeds in some $\boldsymbol{\lambda} \in \mathrm{Irr}(G(2e, 1, r))$. If the restriction of ρ to $G(2e, 2e, r)$ is irreducible then we are reduced to the previous case. Otherwise it has two 2-dimensional components ρ^+, ρ^-. It follows that $2 | A(\boldsymbol{\lambda})$. Then $2 = \dim \rho^+ \geq (p(\boldsymbol{\lambda}) - 1)!$ implies $p(\boldsymbol{\lambda}) \leq 3$, and $2 | A(\boldsymbol{\lambda}) | p(\boldsymbol{\lambda})$ implies $A(\boldsymbol{\lambda}) = p(\boldsymbol{\lambda}) = 2$. Then $\dim \boldsymbol{\lambda} = 4$ contradicts $A(\boldsymbol{\lambda}) = 2$ by table 6, and this concludes the proof. □

Lemma 7.4. *Assume there exists* $\rho \in \mathrm{Irr}(W)$ *with* $\dim \rho = 6$ *and* $\eta \in \mathrm{X}(\rho)$ *with* $\epsilon \otimes \eta \hookrightarrow S^2\rho^*$. *Then* $\rho \in \Lambda\mathrm{Ref}$, *$W$ has rank 4, and ρ factorizes through a representation of either a Coxeter group of type* A_4, B_4, F_4, H_4, *or an exceptional group of type* G_{29}, G_{31}.

Proof. We first assume that $W = G(e, e, r)$ and take $\boldsymbol{\lambda}$ corresponding to ρ. We first exclude the case $A(\boldsymbol{\lambda}) \neq 1$, by contradiction. From $6 \geq (p(\boldsymbol{\lambda}) - 1)!$ we would get $p(\boldsymbol{\lambda}) \leq 4$. Then $p(\boldsymbol{\lambda}) = 4$ implies that $\boldsymbol{\lambda}$ is binary, $\dim \boldsymbol{\lambda} = 24$ and $A(\boldsymbol{\lambda}) = 4$. Then ρ factorizes through the representation $([1], [1], [1], [1])$ of $G(4, 4, 4)$, which we check to be of symplectic type. It follows that $p(\boldsymbol{\lambda}) \leq 3$. Since $1 \neq A(\boldsymbol{\lambda}) | p(\boldsymbol{\lambda})$ we have $p(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda}) \in \{2, 3\}$. Then $p(\boldsymbol{\lambda}) = 3$ implies $\dim \boldsymbol{\lambda} = 18$; since $\boldsymbol{\lambda}$ is not binary it follows that $\boldsymbol{\lambda}$ has dimension greater than $\dim([2], [2], [2]) = 36$, a contradiction. It follows that $p(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda}) = 2$ and $\dim \boldsymbol{\lambda} = 12$. Then $\boldsymbol{\lambda}$ has two non-empty parts of the same shape λ, and we can assume $\lambda \supset [2]$. If $\lambda = [2]$ then $\dim \boldsymbol{\lambda} = 6 < 12$, and otherwise we can assume either $\lambda \supset [2, 1]$ or $\lambda \supset [3]$. But $\dim([3], [3]) = 20 > 12$ and $\dim([2, 1], [2, 1]) = 84 > 12$, a contradiction.

We thus have $A(\boldsymbol{\lambda}) = 1$ and $\dim \boldsymbol{\lambda} = 6$. From table 6 the additional condition $\boldsymbol{\lambda}^* \otimes \epsilon \simeq \boldsymbol{\lambda}$ implies that either $\boldsymbol{\lambda} = (\lambda, \emptyset, \dots)$ with $\lambda = [3, 1, 1]$, in which case ρ factors through the alternating square of the reflection representation of $\mathfrak{S}_5$, or $p(\boldsymbol{\lambda}) = 2$, with three nonempty parts $\lambda_0 = \lambda_g = \lambda_{e-g} = [1]$ with $e \neq 3g$. We consider the explicit models (see e.g. [MM]) for this representation of $G(e, 1, 3)$, generated by t, s_1, s_2. For $\{i, j, k\} = \{1, 2, 3\}$ we denote e_{ijk} the tableau with (i, j, k) placed in position $(0, g, e - g)$. Then it is easily checked that $x = e_{312} \wedge e_{321} + e_{123} \wedge e_{132} - e_{213} \wedge e_{231}$ is fixed by t, and $s_i.x = -x$ for $i \in \{1, 2\}$. Since $G(e, e, 3)$ is generated by s_1^t, s_1, s_2 it follows that $\epsilon \hookrightarrow \Lambda^2\rho$, hence $\epsilon \not\hookrightarrow S^2\rho^*$.

There remains to consider the case $W = G(2e, e, r)$. If the restriction to $G(2e, 2e, r)$ is irreducible we are done, otherwise $\mathrm{X}(\rho) = \{\mathbb{1}\}$ hence $\rho^* \otimes \epsilon \simeq \rho$, and moreover it is the direct sum of ρ^+ and ρ^- of dimension 3. By table 7 we have $p(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda}) = 2$, and the condition $\rho^* \otimes \epsilon \simeq \rho$ implies that $\boldsymbol{\lambda}$ can be taken of the form $(\lambda_0, \emptyset, \dots, \lambda_{e-1}, \emptyset, \dots)$ with $\{\lambda_0, \lambda_{e-1}\} = \{[2], [1, 1]\}$. Then ρ factorizes through either representations $([2], [1, 1])$ or $([1, 1], [2])$ of the Coxeter group of type B_4, which are the alternating square of its reflection representation, and its tensor product by a multiplicative character, respectively. It follows that all these representations belong to $\Lambda\mathrm{Ref}$. The assertion on the exceptional groups is based on a case-by-case computer check. □

Finally we will prove that, for the groups $G(e,e,r)$ and $G(2e,e,r)$, the case $\epsilon \otimes \eta \hookrightarrow S^2\rho^*$. for some $\eta \in \mathrm{X}(\rho)$ with $\dim \rho = 8$ does not occur. We first need a lemma.

Lemma 7.5. *The 8-dimensional irreducible representations of $G(e,e,r)$ are restrictions of a representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ such that $A(\boldsymbol{\lambda}) = 1$. The 8-dimensional irreducible representations of $G(2e,e,r)$ are restrictions of a representation $\boldsymbol{\lambda}$ of $G(2e,1,r)$ such that $A(\boldsymbol{\lambda}) = 1$.*

Proof. Let ρ be a 8-dimensional irreducible representation of $G(e,e,r)$. There exists a well-defined irreducible representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ such that ρ embeds in the restriction of $\boldsymbol{\lambda}$. Assume by contradiction that $A(\boldsymbol{\lambda}) \neq 1$.

By lemma 4.3 we know that $8 = \dim \rho \geq (p(\boldsymbol{\lambda}) - 1)!$, hence $p(\boldsymbol{\lambda}) \leq 4$ and $p(\boldsymbol{\lambda}) \leq 3$ if $\dim \rho \neq 8$. First assume that $\boldsymbol{\lambda}$ is binary. If $p(\boldsymbol{\lambda}) = 4$, then $\dim(\boldsymbol{\lambda}) = 24$ and $A(\boldsymbol{\lambda})|4$. Since $A(\boldsymbol{\lambda}) \neq 1$ then $2|A(\boldsymbol{\lambda})$ and $\dim(\rho) \in \{6, 12\}$, which contradicts $\dim(\rho) = 8$.

We thus can assume that $\boldsymbol{\lambda}$ is not binary. Then there exists $\boldsymbol{\mu} \nearrow \boldsymbol{\lambda}$ with $p(\boldsymbol{\mu}) = p(\boldsymbol{\lambda})$, hence $\dim(\rho) \geq p(\boldsymbol{\lambda})!$ and $p(\boldsymbol{\lambda}) \leq 3$. The case $p(\boldsymbol{\lambda}) = 1$ is ruled out by $A(\boldsymbol{\lambda}) \neq 1$. If $p(\boldsymbol{\lambda}) = 3$ that is $A(\boldsymbol{\lambda}) = 3$, then $\dim(\rho) = 24$. Moreover, there exists i, j, k distincts and $\lambda \neq \emptyset$ such that $\lambda^i = \lambda^j = \lambda^k$. Since $\boldsymbol{\lambda}$ is not binary, we can assume $\lambda \supset [2]$. But $\dim([2],[2],[2]) = 36 \geq 24$, hence a contradiction. It follows that $p(\boldsymbol{\lambda}) = 2$, $A(\boldsymbol{\lambda}) = 2$ and $\dim(\boldsymbol{\lambda}) = 16$. Let i, j distincts and $\lambda \neq \emptyset$ such that $\lambda = \lambda^i = \lambda^j$. Since $([2],[2])$ has dimension 12 it follows that $\lambda \supset [2,1]$ hence $\dim(\boldsymbol{\lambda}) \geq \dim([2,1],[2,1]) = 84 > 16$, a contradiction.

Now let ρ be a irreducible representation of $G(2e,e,r)$, and $\boldsymbol{\lambda}$ a irreducible representation of $G(2e,1,r)$ of which it is a component. If its restriction to $G(2e,2e,r)$ is irreducible we are reduced to the former situation. Otherwise this restriction is the sum of two irreducible components ρ^+ and ρ^-, and $2|A(\boldsymbol{\lambda})$ hence $2|p(\boldsymbol{\lambda})$. Since $4 = \dim \rho^+ \geq (p(\boldsymbol{\lambda}) - 1)!$ we have $p(\boldsymbol{\lambda}) \leq 3$, hence $p(\boldsymbol{\lambda}) = 2$, $A(\boldsymbol{\lambda}) = 2$, $\dim \boldsymbol{\lambda} = 8 = \dim \rho$, which implies $A(\boldsymbol{\lambda}) = 1$. We thus get a contradiction and the conclusion.

□

Proposition 7.6. *Assume that W has type $G(e,e,r)$ or $G(2e,e,r)$, and let $\rho \in \mathrm{Irr}(W)$. If $\dim \rho = 8$, then $S^2\rho^*$ does not contain $\epsilon \otimes \eta$ for any $\eta \in \mathrm{X}(\rho)$.*

Proof. In both kind of types for W, such a representation ρ has to be the restriction of an 8-dimensional irreducible representation of some $G(d,1,r)$ by lemma 7.5. In the case of $G(2e,e,r)$, the restriction of ρ to $G(2e,2e,r)$ has to be irreducible by the same lemma and satisfy the same assumption, so we only have to deal with the $G(e,e,r)$ case, and we can assume by contradiction that $\epsilon \hookrightarrow S^2\rho^*$.

By the previous lemmas such a representation ρ would be the restriction of a 8-dimensional irreducible representation $\boldsymbol{\lambda}$ of $G(e,1,r)$ with $A(\boldsymbol{\lambda}) = 1$. In order that $\rho^* \otimes \epsilon \simeq \rho$ the set $\{\lambda^i \mid i \in [0, e-1]\}$ has to be stable under $\lambda \mapsto \lambda'$. The only possibility in table 6 is that $p(\boldsymbol{\lambda}) = 2$ and that there exists i, j such that $\lambda^i = [2,1]$ and $\lambda^j = [1]$. In particular $r = 4$.

We thus can assume $\lambda^0 = [2,1]$, and $\lambda^i = [1]$ with $i \neq 0$. Then $\rho^* \otimes \epsilon \simeq \rho$ implies $\lambda^{[-i]} = (\lambda^i)'$ hence e is even and $i = e/2$. From the matrix formulas (see e.g. [MM] formula 3.3) such a representation factorizes through

$G(e,e,4) \twoheadrightarrow G(2,2,4)$ and more precisely is induced by the restriction to the Coxeter group $D_4 = G(2,2,4)$ of $([2,1],[1])$. It is easily checked that ϵ embeds in the alternating square of this representation, not in the symmetric one. This concludes the proof. □

In order to conclude the proof of lemma 2.22, we then examine (using CHEVIE) the irreducible 8-dimensional representations of the exceptional groups, getting that only H_4 admits such a representation (actually two of them) with $\epsilon \otimes \eta \hookrightarrow S^2\rho^*$ for some $\eta \in \mathrm{X}(\rho)$.

Contents

References

[Bd] M. Benard, *Schur indices and splitting fields of the Unitary reflection groups* J. Algebra **38** (1976), 318–342.

[Bes] D. Bessis, *Sur le corps de définition d'un groupe de réflexions complexe*, Comm. Algebra **25** (1997), 2703–2716.

[Bor] A. Borel, *Linear algebraic groups*, 2nd edition, GTM 126, Springer 1991.

[Ar] S. Ariki, *Representation theory of a Hecke algebra for $G(r,p,n)$*, *J. Algebra* **177** (1995), 164–185.

[ArKo] S. Ariki, K. Koike, *A Hecke algebra of $\mathbb{Z}/r\mathbb{Z}\wr\mathfrak{S}_n$ and construction of its irreducible representations*, *Adv. Math.* **106** (1994), 216–243.

[Bo456] N. Bourbaki, *Groupes et algèbres de Lie, chapitres 4,5,6*, Hermann, 1968.

[Br] E. Brieskorn, *Sur les groupes de tresses [d'après V.I. Arnold]*, Séminaire Bourbaki 1971/1972, exposé 401, LNM 317, Springer-Verlag, 1973.

[BMR] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998) 127-190.

[FLW] M. Freedman, M. Larsen, Z. Wang, *The two-eigenvalue problem and density of Jones representation of braid groups*, Comm. Math. Phys. **228** (2002), 177–199.

[FH] W. Fulton, J. Harris, *Representation theory. A first course*, GTM 129, Springer, Berlin-Heidelberg-New York, 1991.

[GP] M. Geck, G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori-Hecke algebras*, London Math. Soc. Monographs new series 21, Clarendon Press, 2000.

[Ko] T. Kohno, *Série de Poincaré-Koszul associée aux groupes de tresses pures*, Invent. Math **82** (1985), 57–76.

[Ma01] I. Marin, *Représentations linéaires des tresses infinitésimales*, Thèse de l'université Paris XI-Orsay, 2001.

[Ma03a] I. Marin, *Quotients infinitésimaux du groupe de tresses*, Ann. Inst. Fourier (Grenoble) **53** (2003), 1323–1364.

[Ma03b] I. Marin, *Infinitesimal Hecke Algebras*, Comptes Rendus Mathématiques **337** Série I, 297–302 (2003).

[Ma04] I. Marin, *Irréductibilité générique des produits tensoriels de monodromies*, Bull. Soc. Math. Fr. **132**, 201–232 (2004).

[Ma05] I. Marin, *On the representation theory of braid groups*, preprint arxiv:RT/0502118 (v3), 2005.

[Ma06] I. Marin, *Caractères de rigidité du groupe de Grothendieck-Teichmüller*, Compos. Math. **142** (2006), 657–678.

[Ma07a] I. Marin, *L'algèbre de Lie des transpositions*, J. Algebra **310** (2007), 742–774.

[Ma07b] I. Marin, *Sur les représentations de Krammer génériques*, Ann. Inst. Fourier (Grenoble) **57**, 1883–1925 (2007).

[Ma07c] I. Marin, *Branching properties for the groups $G(de,e,r)$*, preprint arxiv:0711.1845 (v2).

[Ma07d] I. Marin, *Krammer representations for complex braid groups*, preprint arxiv:math/0711.3096.

[Ma08] I. Marin, *Group algebras of finite groups as Lie algebras*, preprint arxiv:math/0809.0074, to appear in Comm. Algebra.

[Ma09] I. Marin, *Reflection groups acting on their hyperplanes*, J. Algebra **322** (2009), 2848–2860.

[MM] I. Marin, J. Michel, *Automorphisms of complex reflection groups*, preprint arxiv:math/0701266.

[Sta] R. Stanley, *Relative invariants of finite groups generated by pseudoreflections*, J. Algebra **49** (1977), 134–148.

[Ste] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. A.M.S. **112** (1964) 392–400.

[Sp] T. Springer, *Linear algebraic groups, 2nd edition*, Birkhäuser, 1998.

[Wz] H. Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math. **92** (1988), 349–383.