HOOKS GENERATE THE REPRESENTATION RING OF THE SYMMETRIC GROUP

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Abstract. We prove that the representation ring of the symmetric group on n letters is generated by the exterior powers of its natural $(n-1)$-dimensional representation. The proof we give illustrates a strikingly simple formula due to Y. Dvir. We provide an application and investigate a possible generalization of this result to some other reflection groups.

Keywords. Symmetric group, representation rings.

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1. Introduction

We let $\mathfrak{S}_n$ denote the symmetric group on n letters, and $R(\mathfrak{S}_n)$ its ordinary representation ring, or equivalently the ring of its complex characters. It is a free $\mathbb{Z}$-module with basis $(V_\lambda)_{\lambda\vdash n}$ of irreductible characters classically indexed by the set of partitions $\lambda = [\lambda_1, \lambda_2, \dots]$ of $n = \lambda_1 + \lambda_2 + \dots$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$. As usual, we identify such a λ with a Young (or Ferrers) diagram, and we use the row-aligned, left-justified, top-to-bottom convention (e.g. the left-hand sides of figure 1 represent the partition $[3,2,1,1]$). The *size* n of the partition λ is denoted $|\lambda|$.

We refer to [FH] for classical facts about the correspondence between representations and partitions. The notation we use here is such that the partition $[n]$ is attached to the trivial representation $V_{[n]} = \mathbb{1}$, and the natural permutation representation $\mathfrak{S}_n < \mathrm{GL}_n(\mathbb{C})$ decomposes itself as $\mathbb{C}^n = \mathbb{1} + V$ with $V = V_{[n-1,1]}$. Among the classical results that can be found in [FH] we recall that the exterior powers $\Lambda^k V$ for $0 \leq k \leq n$ provide irreducible representations attached to the partitions $[n-k, 1^k]$. Such representations or the corresponding partitions are classically called *hooks*.

The purpose of this note is to prove the following.

Theorem 1.1. *For every $n \geq 1$, the representation ring $R(\mathfrak{S}_n)$ is generated by the hooks $\Lambda^k V, 0 \leq k \leq n-1$.*

Note that $\Lambda^{k+1}\mathbb{C}^n = \Lambda^{k+1}V \oplus \Lambda^k V$, hence the collection of the $\Lambda^k V$ and the collection of the $\Lambda^k \mathbb{C}^n$ span the same additive subgroup of $R(\mathfrak{S}_n)$. Another version of the same result is thus the following.

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Theorem 1.2. *For every $n \geq 1$, the representation ring $R(\mathfrak{S}_n)$ is generated by the representations $\Lambda^k\mathbb{C}^n, 0 \leq k \leq n-1$.*

This latter version can be compared with the similar classical result for $\mathrm{GL}_n(\mathbb{C})$, that its ring of rational representations is generated by the $\Lambda^k\mathbb{C}^n$ (which, in terms of characters, simply means that the symmetric polynomials are generated by the elementary symmetric ones – see e.g. [FH], (6.2) and appendix A).

It has been communicated to us by J.-Y. Thibon that, when translated in the language of symmetric functions, the theorems above are equivalent to the results of Butler and Boorman (see [Bu, Boo] and also [STW]). The main point of this note is thus to show how to derive this result from the strikingly simple formula of Dvir (see §3 below), and to explore natural generalizations.

It is indeed a remarkable fact that, while tensor product decompositions are very well-understood for the representations of reductive Lie groups, the ring structure of classical finite groups is often difficult to understand in terms of the natural indexing of their irreducible representations. Having a nice generating family for its representationg ring is typically one of the nice features of the symmetric group that one would like to generalize.

2. A filtration on $R(\mathfrak{S}_n)$

Let G be a finite group, V a faithful (finite-dimensional, complex, linear) representation of G and $\mathrm{Irr}(G)$ the set of all irreducible representations. Then the representation ring $R(G)$ is a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(G)$, and each $\rho \in \mathrm{Irr}(G)$ embeds into some $V^{\otimes r}$ for $r \in \mathbb{Z}_{\geq 0}$ (Burnside-Molien, see e.g. [FH] problem 2.37). The *level* (or *depth*) of $\rho \in \mathrm{Irr}(G)$ with respect to V is defined to be

$$N(\rho) = \min\{r \in \mathbb{Z}_{\geq 0} \mid \rho \hookrightarrow V^{\otimes r}\}$$

Obviously we have $N(\rho_1 \otimes \rho_2) \leq N(\rho_1) + N(\rho_2)$, $N(\mathbb{1}) = 0$. It follows that the subgroup $\mathcal{F}_r$ of $R(G)$ generated by the $\rho \in \mathrm{Irr}(G)$ with $N(\rho) \leq r$ defines a ring filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset$ of $R(G)$, hence a ring structure $(\mathrm{gr}\, R(G), \odot)$ on the graded ring

$$\mathrm{gr}\, R(G) = \bigoplus_{k=0}^{+\infty} (\mathcal{F}_k R(G))/(\mathcal{F}_{k-1} R(G))$$

with the convention $\mathcal{F}_{-1}R(G) = \{0\}$. Notice that $\mathrm{Irr}(G)$ provides a basis of $R(G)$ as a $\mathbb{Z}$-module.

We now let $G = \mathfrak{S}_n$. Considering $\mathfrak{S}_{n-1} < \mathfrak{S}_n$ through the natural embedding that leaves the n-th letter untouched, we let $\mathrm{Ind} : R(\mathfrak{S}_{n-1}) \to R(\mathfrak{S}_n)$ and $\mathrm{Res} : R(\mathfrak{S}_n) \to R(\mathfrak{S}_{n-1})$ denote the usual induction and restriction morphisms.

Recall that Res and Ind are easily described on Young diagrams by Young rule, as illustrated by figure 1. If λ is a Young diagram of size n, then $\mathrm{Res}\, V_\lambda$ is the sum (without multiplicities) in $R(\mathfrak{S}_{n-1})$ of the V_μ, with μ being deduced from λ by removing (respectively adding) one box. Similarly, if λ is a Young diagram of size $n-1$, then $\mathrm{Ind}\, V_\lambda$ is the sum (without multiplicities) in $R(\mathfrak{S}_n)$ of the V_μ, with μ being deduced from λ by adding one box.

The operator $\mathrm{Ind}\,\mathrm{Res}$ on Young diagrams then means summing all V_μ for μ a diagram deduced from λ by *moving* one box, and $\delta(\lambda)$ copies of V_λ where $\delta(\lambda) = \#\{i \mid \lambda_i \neq \lambda_{i+1}\}$ (see figure 2).

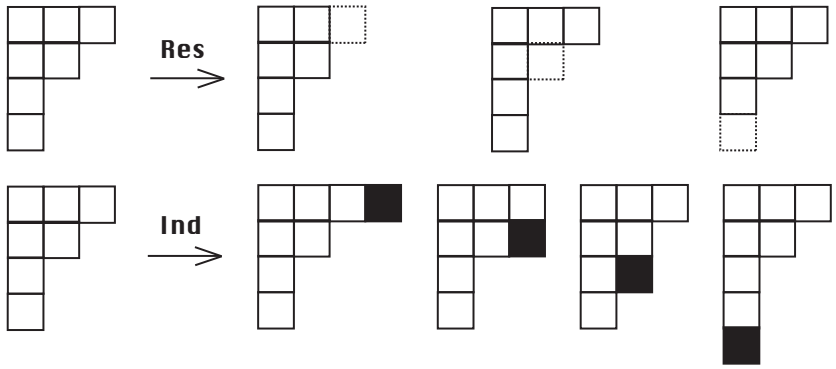

FIGURE 1. Restriction and induction on Young diagrams

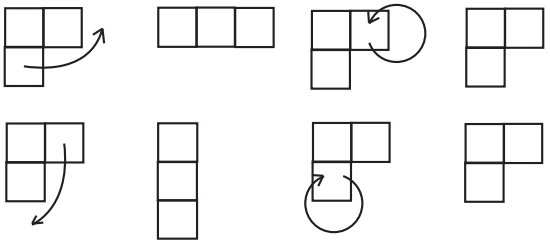

FIGURE 2. $\mathrm{Ind} \circ \mathrm{Res}\ [2,1]$

Let $V = V_{[n-1,1]}$. By the above Young rule, we have $\mathbb{C}^n = \mathrm{Ind}\,\mathbb{1} = \mathbb{1} + V$. Using the classical formula $U \otimes \mathrm{Ind}\, W \simeq \mathrm{Ind}((\mathrm{Res}\, U) \otimes W)$ we get, for all $U \in R(\mathfrak{S}_n)$,

$$U + U \otimes V = U \otimes (\mathbb{1} + V) = \mathrm{Ind}\,\mathrm{Res}\, U$$

i.e. $U \otimes V = (\mathrm{Ind}\,\mathrm{Res}\, U) - U$. Because of this, $N(\lambda) = N(V_\lambda)$ can be determined combinatorially. First note that, if $V_\lambda \hookrightarrow V^{\otimes(r-1)} \otimes V$, then $V_\lambda \hookrightarrow V_\mu \otimes V$ for some irreducible $V_\mu \hookrightarrow V^{\otimes(r-1)}$. An immediate consequence of the above remarks is thus that the number λ_1 of boxes in the first row for λ satisfies $\lambda_1 \geq \mu_1 - 1$. By induction on r this yields $r \geq n - \lambda_1$, hence $N(\lambda) \geq n - \lambda_1$. One then easily gets the following classical fact, for which we could not find an easy reference.

Proposition 2.1. *For all $\lambda \vdash n$, we have $N(\lambda) = n - \lambda_1$.*

Proof. The proof is by induction on $r = n - \lambda_1$, the case $r = 0$ being clear. Let $\lambda = [\lambda_1, \ldots, \lambda_s]$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, $n - \lambda_1 = r + 1$. Since $n - \lambda_1 > 0$ we have $s \geq 2$. We consider $\mu \vdash n$ defined by $\mu_1 = \lambda_1 + 1$, $\mu_i = \lambda_i$ for $1 < i < s$, and $\mu_s = \lambda_s - 1$. By the induction assumption, $N(\mu) = r$ and $V_\mu \hookrightarrow V^{\otimes r}$. One of the components of $V_\mu \otimes V$ is V_λ by the combinatorial rule, hence $V_\lambda \hookrightarrow V_\mu \otimes V \hookrightarrow V^{\otimes(r+1)}$ and the conclusion follows by induction. □

For a partition $\lambda = [\lambda_1, \lambda_2, \ldots]$ of n with $\lambda_i \geq \lambda_{i+1}$, we define the partition $\theta(\lambda) = [\lambda_2, \lambda_3, \ldots]$ of $n - \lambda_1$. In diagrammatic terms, $\theta(\lambda)$ is the diagram deduced from λ by *deletion*

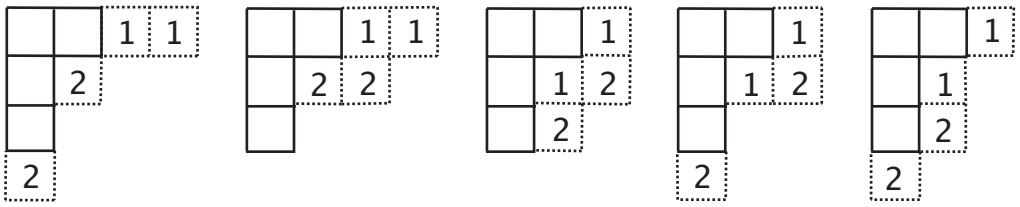

FIGURE 3. The $[2,2]$-expansions of $[2,1,1]$

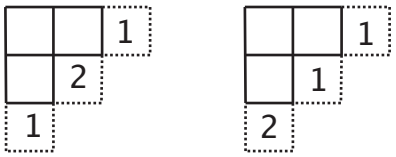

FIGURE 4. $L_{[2,1],[2,1],[3,2,1]} = 2$

of the first row (see figure 5). Proposition 2.1 can thus be reformulated as

$$|\theta(\lambda)| = N(\lambda).$$

3. Dvir's formula

For three partitions λ, μ, ν of arbitrary size, we let $L_{\lambda,\mu,\nu}$ denote the Littlewood-Richardson coefficient (see e.g. [FH]). A remarkable discovery of Y. Dvir is that the graded ring structure $(\mathrm{gr}\, R(\mathfrak{S}_n), \odot)$ is basically given by these coefficients.

We first recall how to compute $L_{\lambda,\mu,\nu}$ with $|\nu| = |\lambda| + |\mu|$ using the Littlewood-Richardson rule : $L_{\lambda,\mu,\nu}$ is the number of ways λ, as a Young diagram, can be expanded into ν by using a *μ-expansion*. Letting $\mu = [\mu_1, \dots, \mu_k]$, such a μ-expansion is obtained by first adding μ_1 boxes labelled by 1, then μ_2 boxes labelled by 2, and so on (that is, at the r-th step we add μ_r boxes labelled r to the preceedingly obtained diagram) so that

(1) at each step, one still has a Young diagram
(2) the labels strictly increase in each column
(3) when the labels are listed from right to left in each row and starting with the top row, we have the following property. For each $t \in [1, |\mu|]$, the following holds : each label p occurs at least as many times as the label $p+1$ (when it exists) in the first t entries.

As an example, see figure 3 for the list of the $[2,2]$-expansions of $[2,1,1]$ and figure 4 for the two expansions leading to $L_{[2,1],[2,1],[3,2,1]} = 2$. The reader can find in [FH] other examples and further details on this combinatorics.

For $\lambda, \mu, \nu \vdash n$, we let $C_{\lambda\mu\nu}$ denote the structure constants $V_\lambda \otimes V_\mu = \sum_\nu C_{\lambda\mu\nu} V_\nu$ of $R(\mathfrak{S}_n)$. These constants, whose study has been initiated by Murnaghan (1938), are notoriously complicated to understand.

For a partition $\lambda = [\lambda_1, \lambda_2, \dots]$ with $\lambda_i \geq \lambda_{i+1}$, of n, define the partition $\theta(\lambda) = [\lambda_2, \lambda_3, \dots]$ of $n - \lambda_1$, and let $d(\lambda) = |\theta(\lambda)| = \lambda_2 + \lambda_3 + \cdots = n - \lambda_1$. By proposition 2.1 above we have

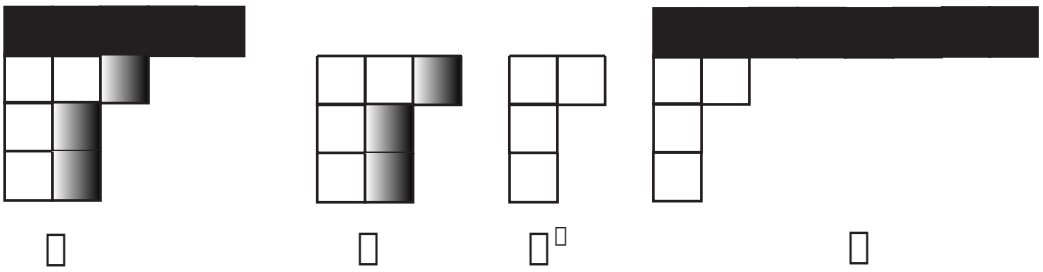

FIGURE 5. $\alpha = \theta(\lambda)$, α° and μ for $\lambda = [5,3,2,2]$

$d(\lambda) = N(\lambda) = \min\{r \geq 0 \mid V_\lambda \hookrightarrow V^{\otimes r}\}$. In particular $C_{\lambda,\mu,\nu} = 0$ whenever $d(\nu) > d(\lambda)+d(\mu)$. Dvir's formula can be stated as follows

Theorem 3.1. *(Dvir* [D]*, theorem 3.3) Let λ, μ, ν be partitions of n such that $d(\lambda) + d(\mu) = d(\nu)$. Then $C_{\lambda,\mu,\nu} = L_{\theta(\lambda),\theta(\mu),\theta(\nu)}$.*

In particular we get, inside $\operatorname{gr} R(\mathfrak{S}_n)$, the following formula :

$$V_\lambda \odot V_\mu = \sum_{d(\nu)=d(\lambda)+d(\mu)} L_{\theta(\lambda),\theta(\mu),\theta(\nu)} V_\nu.$$

4. The proof

The main theorem is then an immediate consequence of the following proposition. For the proof of this proposition, we will associate to a Young diagram $\alpha = [\alpha_1, \alpha_2, \dots]$ its interior α° defined by the partition $\alpha_i^\circ = \max(0, \alpha_i - 1)$, and its boundary $\partial\alpha$ is defined to be the ribbon made of the boxes in α which do not belong to α°. The size $|\partial\alpha|$ of $\partial\alpha$ (that is, its number of boxes) is clearly equal to the number of rows in α, or in other terms to the number of nonzero parts of the partition α.

Proposition 4.1. *The ring $(\operatorname{gr} R(\mathfrak{S}_n), \odot)$ is generated by the $\Lambda^k V$, $0 \leq k \leq n-1$.*

Proof. Recall that $\Lambda^k V = V_{[n-k,1^k]}$, and note that $\theta([n-k,1^k]) = [1^k]$. In particular $N(\Lambda^k V) = k$. We identify each V_λ with its image in $\operatorname{gr} R(\mathfrak{S}_n)$ and let Q denote the subring of $\operatorname{gr} R(\mathfrak{S}_n)$ generated by the $\Lambda^k V$. We prove that $V_\lambda \in Q$ for all partition λ of n ($\lambda \vdash n$), by induction on $d(\lambda) = |\theta(\lambda)|$. We have $d(\lambda) = 0 \Rightarrow \lambda = [n] \Rightarrow V_\lambda = \Lambda^0 V$ and $d(\lambda) = 1 \Rightarrow \lambda = [n-1,1] \Rightarrow V_\lambda = \Lambda^1 V$, hence $V_\lambda \in Q$ if $d(\lambda) \leq 1$. We thus assume $d(\lambda) \geq 2$ and that $V_\mu \in Q$ for all partitions μ with $d(\mu) < d(\lambda)$.

Letting $\alpha = \theta(\lambda)$ we use another induction on $|\alpha^\circ|$. Note that $|\alpha^\circ| \leq |\alpha|$, with equality only if $\alpha = \emptyset$. More generally, the case $|\alpha^\circ| = 0$ means that $V_\lambda = \Lambda^{|\partial\alpha|} V \in Q$, so we can assume $|\alpha^\circ| \geq 1$.

We let $r = |\partial\alpha| = |\alpha| - |\alpha^\circ|$. Since $d(\lambda) \geq 2$ we have $\theta(\lambda) \neq 0$ and in particular $r \geq 1$. Moreover $\lambda_1 = n - |\alpha| \geq \lambda_2$, hence $n - |\alpha^\circ| \geq \alpha_1 \geq \alpha_1^\circ$. We thus can introduce the partition $\mu = [n - |\alpha^\circ|, \alpha_1^\circ, \dots,]$ of n (see figure 5 for an example) and consider $M = V_\mu \odot \Lambda^r V \in \operatorname{gr} R(\mathfrak{S}_n)$. Since $|\alpha^\circ| < |\alpha|$ we have $d(\mu) < d(\lambda)$ hence $V_\mu \in Q$ by the first induction assumption so $M \in Q$. Let $\nu \vdash n$ such that M has nonzero coefficient on V_ν. We have $d(\nu) = d(\mu)+r = d(\lambda)$, hence $\nu_1 = n - |\alpha| = \lambda_1$, and this coefficient is $L_{\alpha^\circ,[1^r],\theta(\nu)}$ by Dvir formula.

By the Littlewood-Richardson rule, this coefficient $L_{\alpha^\circ,[1^r],\theta(\nu)}$ is the number of ways that one can add boxes labelled $1,\dots,r$ on the Young diagram of α° with at most one box on each row (with the graphic convention that α° has α_i° boxes on the i-th row), the labels increasing along the rows, and such that the augmented diagram corresponds to $\theta(\nu)$. We thus clearly have $L_{\alpha^\circ,[1^r],\alpha}=1$, this corresponding to adding a box marked i on the i-th row for each $1\leq i\leq r$. Moreover, if $L_{\alpha^\circ,[1^r],\theta(\nu)}$ is nonzero, then either $\theta(\nu)$ has (strictly) more nonzero parts than α, which means that one box has been added to the empty $(r+1)$-st row, and in that case we know that $V_\nu\in Q$ by the second induction hypothesis (as this means $|\partial\theta(\nu)|>r=|\partial\alpha|$, hence $|\theta(\nu)^\circ|<|\alpha^\circ|$ since $|\alpha|=|\theta(\nu)|$); or, the r boxes have been added to the first row, which implies $\theta(\nu)=\alpha$ hence $\nu=\lambda$. We thus get $M\equiv V_\lambda$ modulo Q, $V_\lambda\in Q$ and the conclusion follows by induction. □

A careful look at the above proof shows that we proved a more technical but also more precise result. For $\lambda,\mu\in\mathfrak{S}_n$, we define $\lambda\prec\mu$ if either $N(\lambda)<N(\mu)$, or $N(\lambda)=N(\mu)$ and $|\theta(\lambda)^\circ|<|\theta(\mu)^\circ|$, and we denote by R_λ (resp. $\overline{R}_\lambda$) the $\mathbb{Z}$-submodule of $R(\mathfrak{S}_n)$ (resp. $\operatorname{gr}R(\mathfrak{S}_n)$) spanned by the $\kappa\in\operatorname{Irr}(\mathfrak{S}_n)$ with $\kappa\prec\lambda$. The above proof actually shows the following.

Proposition 4.2. *For every $\lambda\in\operatorname{Irr}(\mathfrak{S}_n)\setminus\{\mathbb{1}\}$, there exists $\hat{\lambda}\in\operatorname{Irr}(\mathfrak{S}_n)$ with $\hat{\lambda}\prec\lambda$ and $k\in\mathbb{Z}_{\geq 0}$ such that $\hat{\lambda}\odot\Lambda^kV\in\lambda+\overline{R}_\lambda$.*

Since $\mathcal{F}_{N(\kappa)-1}(\mathfrak{S}_n)\subset R_\kappa$ this immediately implies

Corollary 4.3. *For every $\lambda\in\operatorname{Irr}(\mathfrak{S}_n)\setminus\{\mathbb{1}\}$, there exists $\hat{\lambda}\in\operatorname{Irr}(\mathfrak{S}_n)$ with $\hat{\lambda}\prec\lambda$ and $k\in\mathbb{Z}_{\geq 0}$ such that $\hat{\lambda}\otimes\Lambda^kV\in\lambda+R_\lambda$.*

5. An application

One can use this result to give a proof of the well-known fact that all complex linear representations of the symmetric group can actually be realized over $\mathbb{Q}$. We first recall the following lemma.

Lemma 5.1. *Let G be a finite group, $\Bbbk$ a number field, $\rho:G\to\mathrm{GL}_N(\Bbbk)$ a linear representation of G defined over $\Bbbk$, and $\rho_{\mathbb{C}}:G\to\mathrm{GL}_N(\mathbb{C})$ its complexification. If φ is an irreducible subrepresentation of $\rho_{\mathbb{C}}$ occuring with multiplicity one whose character takes values in $\Bbbk$, then φ can be realized over $\Bbbk$.*

Proof. This is an immediate consequence of the fact that the projection on the φ-isotopic component of $\rho_{\mathbb{C}}$ is given by $\frac{\dim\varphi}{|G|}\sum_{g\in G}\overline{\chi(g)}\rho(g)$ (see e.g. [FH] (2.32)), which is an endomorphism of $\Bbbk^N$ under our assumptions. □

We now can deduce the following well-known result.

Theorem 5.2. *Every complex linear representation of $\mathfrak{S}_n$ can be realized over $\mathbb{Q}$.*

Proof. We use first that the natural permutation module $\mathbb{C}^n$ is obviously realizable over $\mathbb{Q}$, and that $\mathbb{C}^n=\mathbb{1}+V$. This implies that the character associated to V is defined over $\mathbb{Q}$, hence V can be realized over $\mathbb{Q}$ by lemma 5.1 (or, directly, V can be identified to the rational subspace $\{(x_1,\dots,x_n)\in\mathbb{Q}^n\mid x_1+\cdots+x_n=0\}$). It follows that all the Λ^rV can be realized over $\mathbb{Q}$.

Since $\operatorname{Irr}(\mathfrak{S}_n)$ is clearly a well-founded set under $\prec$, with minimal element $\mathbb{1}$, one can now use this relation to prove our statement by induction.

Let $\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$. Corollary 4.3 implies that there exists $\hat{\lambda} \prec \lambda$ and $k \in \mathbb{Z}_{\geq 0}$ such that $M = \hat{\lambda} \otimes \Lambda^k$, which is realizable over $\mathbb{Q}$ by our induction assumption, contains λ with multiplicity 1, and has the property that the quotient representation M/λ is also realizable over $\mathbb{Q}$ by the same induction assumption. This proves that the character of λ takes values in $\mathbb{Q}$, and then that λ is realizable over $\mathbb{Q}$ by lemma 5.1. This concludes the proof.

□

6. Generalization attempts

The symmetric group is an irreducible complex (pseudo-)reflection group. Recall that such a group is a finite subgroup W of $\mathrm{GL}(V)$ for V some finite-dimensional complex vector space acted upon irreductibly by W, with W generated by its reflections, namely elements of $\mathrm{GL}(V)$ which fix a hyperplane. The dimension of V is called the *rank* of W.

For such a group, it is a classical result of Steinberg that the representations $\Lambda^k V$ are irreducible (see e.g. [Bou] ch. 5 §2 exercice 2), and are thus natural generalization of *hooks*.

Among other similarities, theorem 5.2 admits a natural generalization to these groups. Indeed, it can first be shown that the representation V can be realized over its character field $\Bbbk$ (i.e. the number field generated by the values taken by its character), sometimes called its *field of definition*. Moreover, it is a theorem of M. Benard that *every* representation of W can be realized over $\mathbb{Q}$ (see [Bena], and also [Bes], [MM] for other proofs), thus providing a complete generalization of theorem 5.2. We now investigate to what extent theorem 1.1 could be generalized.

The irreducible complex reflection groups have been classified by Shephard and Todd (see [ST]). There is an infinite series $G(de, e, r)$ depending on three integral parameters d, e, r, plus 34 exceptions $G_4, \ldots, G_{37}$. For the representation theory of the $G(de, e, r)$ we refer to [AR].

Note that, for a given group with known character table, it is easy to check by computer whether a given subset $\mathcal{B}$ of $\mathrm{Irr}(W)$ generates $R(W)$. Indeed, the ring $R(W) = \mathbb{Z}\,\mathrm{Irr}(W)$ is a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(W)$; assume we are given a subset $\mathcal{B} \subset \mathrm{Irr}(W)$ with $\mathbb{1} \in \mathcal{B}$, and let A denote the subring of $R(W)$ generated by $\mathcal{B}$. The embedding $R(W) \subset \mathrm{End}_{\mathbb{Z}} R(W) \simeq \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}\,\mathrm{Irr}(W))$ identifies A with the minimal $\mathbb{Z}$-submodule of $\mathbb{Z}\,\mathrm{Irr}(W)$ containing $\mathbb{1}_W$ which is stable under multiplication by $\mathcal{B}$. This identifies $a \in A$ with $a.\mathbb{1} \in \mathbb{Z}\,\mathrm{Irr}(W)$. Starting with the $\mathbb{Z}$-module $A_0 = \mathbb{Z}\mathbb{1}$ of rank 1, multiplication by the elements of $\mathcal{B}$ iteratively provides a sequence of submodules $A_0 \subset A_1 \subset \ldots$ which eventually stops at $A_\infty = A$ by noetherianity of the $\mathbb{Z}$-module $R(W)$.

If W has rank 2, we are able to prove case-by-case the following.

Proposition 6.1. *If W is an irreducible complex reflection group of rank 2, then $R(W)$ is generated by V and the 1-dimensional representations.*

Proof. The case of exceptional reflection groups is checked by computer, using the algorithm above. The non-exceptional ones are the $G(de, e, 2)$, so we assume $W = G(de, e, 2)$. The irreducible representations of W have dimension at most 2. The ones of dimension 2 can be extended to $G(de, 1, 2)$, so we can assume without loss of generality that $e = 1$. The group W is generated by $t = \mathrm{diag}(1, \zeta)$ with $\zeta = \exp(2i\pi/d)$ and s the permutation matrix $(1\ 2)$. Its two-dimensional representations are indexed by couples (i, j) with $0 \leq i < j < d$. We extend this notation to $i, j \in \mathbb{Z}$ with $j \not\equiv i \mod d$ by taking representatives modulo d and letting

$(i,j)=(j,i)$. A matrix model for the images of t and s in the representation $(r,r+k)$ is

$$t\mapsto\begin{pmatrix}\zeta^r & 0\\ 0 & \zeta^{r+k}\end{pmatrix}\quad s\mapsto\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

In particular, $V=(0,1)$. From these explicit models it is straightforward to check that $(0,1)\otimes(0,1)$ is the sum of $(0,2)$ and 1-dimensional representations, and that $(0,1)\otimes(0,k)=(0,k+1)+(1,k)$. Then we consider the 1-dimensional representation $\chi_1:t\mapsto\zeta,s\mapsto 1$. It is clear that $(i,j)\otimes\chi_1=(i+1,j+1)$. Letting Q denote the subring of $R(W)$ generated by V and the 1-dimensional representations, through tensoring by χ_1 is it enough to show that $(0,k)\in Q$ for all $1\leq k\leq d$. By definition $(0,1)\in Q$, tensoring by $(0,1)$ yields $(0,2)\in Q$, and finally $(0,1)\otimes(0,k)=(0,k+1)+\chi_1\otimes(0,k-1)$ proves the result by induction on k. □

Among the higher rank exceptional groups, we check by computer that the union of the Λ^kV and the one-dimensional representations generates $R(W)$ exactly for the groups $G_{23}=H_3, G_{24}, G_{25}, G_{26}, G_{30}=H_4, G_{33}, G_{35}=E_6$ (but not E_7 nor E_8 !).

In the more classical case of the Coxeter groups W of type B_n and D_n, it is easily checked that the subring generated by the Λ^kV has not full rank in $R(W)$ (for $n\geq 4$). It is thus natural to consider the non-faithful reflection representations U of dimension $n-1$ of these groups, which correspond to $([n-1,1],\emptyset)$ and $\{[n-1,1],\emptyset\}$ in the usual parametrizations of their irreducible representations (see [GP]). These are deduced from $V_{[n-1,1]}\in\mathrm{Irr}(\mathfrak{S}_n)$ through a natural morphism $W\twoheadrightarrow\mathfrak{S}_n$. A computer check for small values of n motivates the following conjecture.

Conjecture 6.2. *For W a Coxeter group of type B_n or D_{2n+1}, $R(W)$ is generated by the $\Lambda^kV,\Lambda^kU,k\geq 0$.*

The proof of such a conjecture would probably involve an understanding of the structure constants in $R(W)$ comparable to Dvir's formula for $\mathfrak{S}_n$. Unfortunately, the combinatorial study of the representation ring of these more general Coxeter groups seems to be only at the beginning.

For a group of type D_{2n}, it can be checked that the subring generated by such elements has smaller rank already for D_4. This is a general phenomenon, as can be seen in the following way. Recall that a group W of type D_n is an index 2 subgroup of a Coxeter group $\widetilde{W}$ of type B_n. By Clifford theory, an irreducible representations of $\widetilde{W}$ parametrized by (λ,μ) with $|\lambda|+|\mu|=n$ restricts either to an irreducible representation $\{\lambda,\mu\}$ of W, precisely in the case $\lambda\neq\mu$, or, in the case $\lambda=\mu$, to a direct sum of two irreducibles usually denoted λ^+ and λ^-. Note that such $\lambda^\pm$ exist if and only if n is even.

Choosing some $s\in\widetilde{W}\setminus W$ and letting $\mathrm{Ad}\,s:x\mapsto sxs^{-1}$ be the automorphism of W induced by s, the map $\rho\mapsto\rho\circ\mathrm{Ad}\,s$ induces a $\mathbb{Z}$-linear involution η of $R(W)$ which fixes the $\{\lambda,\mu\}$ and maps $\lambda^\pm$ to $\lambda^\mp$. Letting $R(W)^\eta$ denote the invariant subspace, we have $R(W)^\eta=R(W)$ if and only if n is odd. Clearly the Λ^kV and Λ^kU are always fixed by η, and this explains why the subring they generate cannot be $R(W)$ when $R(W)^\eta\neq R(W)$. We do not have any serious guess for a natural generating set in these cases.

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