PROOF OF THE BMR CONJECTURE FOR G_{20} AND G_{21}

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Abstract. We prove two new cases of the Broué-Malle-Rouquier freeness conjecture for the Hecke algebras associated to complex reflection groups, using methods inspired by non-commutative Gröbner bases. These two cases are the complex reflection groups of rank 2 called G_{20} and G_{21} in the Shephard and Todd classification. This reduces the number of remaining unproven cases to 3.

1. Introduction

Two decades ago, M. Broué, G. Malle and R. Rouquier conjectured in [3] that the generalized Hecke algebras that they attached to an arbitrary complex reflection group satisfy the crucial structural property of the ordinary (Iwahori-)Hecke algebras attached to any finite Coxeter group, namely that they are free modules of rank equal to the order of the group. This is known as the BMR *freeness conjecture*, and it can be easily reduced to the case where the complex reflection group W is irreducible. We refer to [13] for a general exposition of this conjecture and standard results about it.

The Shephard-Todd classification of irreducible complex reflection groups defines an infinite family $G(de, e, n)$ of such groups, for which the conjecture was already known to hold by work of Ariki and Ariki-Koike (see [1, 2]), and a long list of exceptional groups. Subsequent works have proved it for most of the exceptional groups, notably all the ones of rank at least 3 (see [13, 11, 14]), and most of the ones of rank 2 (see [4, 6, 7]). In rank 2, the 5 remaining ones are named, in Shephard-Todd notation, G_{17}, G_{18}, G_{19}, G_{20} and G_{21}. Here we prove the cases of G_{20} and G_{21}, by a method of a different nature. This reduces the list of remaining cases to the groups G_{17}, G_{18} and G_{19}, for which it appears difficult to apply readily the methods of this paper.

In section 2 we recall the main definitions, and prove a technical property that will enable us to work over rings of definitions which are polynomial rings, instead of the usual Laurent polynomial rings. In section 3 we explain the general method : how we find a potential basis for the Hecke algebras and how we find a list of rewriting rules. The first idea is to build on some *heuristic* input coming from a software for computing (non-commutative) Gröbner bases over a field (here GBNP) applied to *specializations* of our Hecke algebras, in order to get a list of leading terms that should be simplifiable using the defining relations. Secondly, to write down an (easy) algorithm that deduces a list of rewriting rules for the leading terms we already have, as a linear combination of *positive* words, from some rewriting rules with *signed* words, that is words in the generators *and their inverses*. Finally, to check by standard algorithms that this rewriting system provides a spanning set of the right size.

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Of course, there is one crucial piece missing in this description, which is the main mathematical content : a list of convenient rewriting rules with signed words. We provide the one we found ('by hand') in sections 4 and 5 for the groups G_{20} and G_{21}, respectively. Applying these rules then yields a convenient basis for these algebras, which has the nice property to originate from positive elements of the corresponding braid groups (which happen to be dihedral Artin groups in these two cases). This proves the BMR freeness conjecture for G_{20} and G_{21}.

The GAP4 programs used for G_{21} can be found on my webpage `http://www.lamfa.u-picardie.fr/marin/G20G21code-en.html`.

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2. Definitions and preliminaries

Let W be a finite complex (pseudo-)reflection group. We let B denote the braid group of W, as defined in [3] §2 B, and recall that a (pseudo-)reflection $s \in W$ is called distinguished if its only nontrivial eigenvalue is $\exp(2\pi \mathrm{i}/o(s))$, where $\mathrm{i} \in \mathbb{C}$ is the chosen square root of -1 and $o(s)$ denotes the order of $s \in W$.

We let $R = \mathbb{Z}[a_{s,i}, a_{s,0}^{-1}, 0 \leq i \leq o(s)-1]$ where s runs over the distinguished reflections in W and $o(s)$ is the order of s in W, with the convention $a_{s,i} = a_{s',i}$ if s, s' are conjugates in W. For the standard notion of a braided reflection associated to s we refer to [3], where they are described as 'generators-of-the-monodromy' around the divisors of the orbit space. The definition of the Hecke algebra associated to W reads as follows.

Definition 2.1. *The generic Hecke algebra is the quotient of the group algebra RB by the relations $\sigma^{o(s)} - a_{s,o(s)-1}\sigma^{o(s)-1} - \cdots - a_{s,0} = 0$ for each braided reflection σ associated to s.*

Actually, it is enough to choose one such relation per conjugacy class of distinguished reflection, as all the corresponding braided reflections are conjugates in B. Although we are not going to use this result in our proof, we mention that it was already known by work of Etingof and Rains (see [9]) that the Hecke algebras of the groups considered here are modules of finite type. Our main result can now be stated as follows.

Theorem 2.2. *When W is a complex reflection group of Shephard-Todd type G_{20} or G_{21}, then the generic Hecke algebra of W is a free R-module of rank $|W|$.*

Let $R_0 = \mathbb{Z}[b_{s,i}, 1 \leq i \leq o(s)]$ where s runs over the distinguished reflections, with the convention $b_{s,i} = b_{s',i}$ if s, s' are conjugates in W, and define H_0 as the quotient of $R_0 B$ by the relations

$$\sigma^{o(s)} - b_{s,o(s)-1}\sigma^{o(s)-1} - \cdots - b_{s,1}\sigma - 1 = 0$$

for each braided reflection σ associated to s. Again, it is enough to choose one such relation per conjugacy class of distinguished reflection. We let H denote the usual Hecke algebra, defined over R, as in definition 2.1.

The next proposition is useful in order to reduce the number of parameters involved in the computations, and in order to replace Laurent polynomials with ordinary polynomials.

Proposition 2.3.

(i) *H_0 is spanned by $|W|$ elements as a R_0-module iff it is a free R_0-module of rank $|W|$.*
(ii) *H is a free R-module of rank $|W|$ iff H_0 is a free R_0-module of rank $|W|$.*

Proof. The proof of (i) is the same as the one of [13], proposition 2.4. We prove (ii). We have a ring morphism $\phi_1 : R \to R_0$ defined by $a_{s,i} \mapsto b_{s,i}$ if $i \geq 1$, $a_{s,0} \mapsto 1$, for which $H_0 = H \otimes_{\phi_1} R_0$. Therefore, if H is a free R-module of rank $|W|$, we get the $H_0 \simeq R^{|W|} \otimes_{\phi_1} R_0 \simeq R_0^{|W|}$ is also free of rank $|W|$. We prove the converse. Assume that H_0 is R_0-free of rank $|W|$. Let $A = \mathbb{Z}[x_s, x_s^{-1}]$ where s runs among the distinguished reflections of W with $x_s = x_{s'}$ if s, s' are conjugates in W. We have an injective ring morphism $R \to A \otimes_{\mathbb{Z}} R_0$ defined by $a_{s,0} \mapsto x_s^{o(s)} = x_s^{o(s)} \otimes 1$, and $a_{s,i} \mapsto b_{s,i} x_s^{o(s)-i} = x_s^{o(s)-i} \otimes b_{s,i}$ for $i \geq 1$. We first note that $A \otimes R_0$ is a free R-module of finite rank, since it is easily checked that

$$A \otimes R_0 = \bigoplus_{s \in \mathcal{S}} \bigoplus_{0 \leq i < o(s)} x_s^i R$$

where $\mathcal{S}$ is a system of representatives of the conjugacy classes of distinguished reflections.

We denote $\check{H}_0$ the quotient of the group algebra $(A \otimes_{\mathbb{Z}} R_0)B$ of B over $A \otimes_{\mathbb{Z}} R_0$ by the relations $(x_s\sigma)^{o(s)} - b_{s,o(s)-1} x_s (x_s\sigma)^{o(s)-1} - \cdots - b_{s,1} x_s^{o(s)-1}(x_s\sigma) - x_s^{o(s)} = 0$ for each braided reflection σ associated to s. We consider the composite map

$$AB \xrightarrow[\Delta]{} (AB) \otimes_A (AB) \xrightarrow[\mathrm{Id} \otimes Ab]{} (AB) \otimes_A (AB^{ab}) \xrightarrow[\mathrm{Id} \otimes (s \mapsto x_s)]{} (AB) \otimes_A A \xrightarrow[\simeq]{} AB$$

where Δ is the usual coproduct of the Hopf algebra AB, $Ab : B \to B^{ab}$ the abelianization morphism and, by abuse of notations, the associated linear map $AB \to AB^{ab}$, and '$s \mapsto x_s$' denotes the map $B^{ab} \to A$ defined as follows. It is known (see e.g. [3]) that B^{ab} is a free $\mathbb{Z}$-module admitting a natural basis indexed by the conjugacy classes of distinguished reflections. The map is defined by mapping the basis element associated to (a conjugacy class of) distinguished reflection s to the scalar $x_s \in A$.

The composite map is easily checked to be an A-algebra isomorphism. Its natural extension $(A \otimes R_0)B \to (A \otimes R_0)B$ induces an isomorphism $\check{H}_0 = H \otimes_R (A \otimes_{\mathbb{Z}} R_0)$.

Now, if H_0 is R_0-free of rank $|W|$, then $\check{H}_0 = H_0 \otimes_{R_0} A$ is $A \otimes R_0$-free of rank $|W|$. Since $A \otimes R_0$ is a free R-module of finite rank, this implies that $\check{H}_0$ is a free R-module of finite rank, and also that, since $\check{H}_0 = H \otimes_R (A \otimes_{\mathbb{Z}} R_0)$, that the R-module H is a direct factor of $\check{H}_0$. Therefore H is projective as a R-module and this implies that H is free of rank $|W|$ by [13], proposition 2.5.

□

The groups we hare interested in are the ones denoted G_{20} and G_{21} in the Shephard-Todd notation. They admit presentations symbolized by the following diagrams

that is $G_{20} = \langle s_1, s_2 \mid s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2,\ s_1^3 = s_2^3 = 1 \rangle$ and $G_{21} = \langle s_1, s_2 \mid (s_1 s_2)^5 = (s_2 s_1)^5,\ s_1^2 = s_2^3 = 1 \rangle$, respectively. In these presentations, s_1, s_2 are distinguished reflections, and every distinguished reflection is a conjugate of one of them. Moreover, s_1 and s_2 are conjugates in G_{20}, as is readily deduced from the presentation itself. The corresponding braid groups admit the same presentations, with the order relations removed (see [3]).

We use the above proposition to define the Hecke algebras of G_{20} and G_{21} over R_0, where $R_0 = \mathbb{Z}[a, b]$ for G_{20} and $R_0 = \mathbb{Z}[a, b, q]$ for G_{21}, with relations

G_{20} :	$s_1^3 = as_1^2 + bs_1 + 1$	G_{21} :	$s_1^2 = qs_1 + 1$
	$s_2^3 = as_2^2 + bs_2 + 1$		$s_2^3 = as_2^2 + bs_2 + 1$

In the subsequent section we prove these Hecke algebras are spanned by the 'right' number of elements, and this proves theorem 2.2 by proposition 2.3.

3. General method

In this section we describe the general method we used to prove the conjecture in these cases. It proceeds in several steps.

(i) Heuristics/Experimentation
(ii) Incremental determination of computational rules
(iii) Right multiplication table

3.1. **Heuristics/Experimentation.** The first element is of heuristic nature, provided by a software able to compute non-commutative Gröbner basis for finitely presented associative $\mathbb{Q}$-algebras. We used the GAP4 package GBNP (see [8]) with the standard ('`deglex`') ordering for monomials, taking as input the presentations of [3], where we specialized the Hecke algebras at more or less random parameters. For G_{20} and G_{21} it finished in reasonable time for all the specializations we tried, while for G_{18} and G_{19} it was not able to complete the computation after several months of running time, except for the simple case of the group algebra specialization, that is the presentation of W viewed as a presentation of the Hecke algebra at very special parameters. For all the groups of the so-called icosahedral series of complex reflection groups of rank 2, numeroted from G_{16} to G_{22}, GBNP nevertheless finds a Gröbner basis of the rational group algebra of W.

It turns out that most if not all the specializations we tried for G_{20} and G_{21} (including the group algebra specialization) provided the same number of elements for the Gröbner basis. As an indication of the complexity of this heuristic data, we provide table 1, were $\#W$ is the order of W and $\#gb$ is the number of elements in the Gröbner basis. The groups whose name appears in bold fonts are the ones for which the BMR freeness conjecture is now proved, after work of Chavli for G_{16} (see [4, 5]), of Marin-Pfeiffer for G_{22} (see [14]), and by the present work for G_{20} and G_{21}.

The output of GBNP we are interested in is the collection $\mathcal{G}$ of leading monomials of the Gröbner basis. In case we had computed the Gröbner basis for several specializations this collection turned out to be independent of the specialization. From this one computes easily the set $\mathcal{B}$ of all words avoiding the patterns which belong to $\mathcal{G}$. As expected, it has cardinality $|W|$ and provides for these specializations a basis of the Hecke algebra.

We provide an example of our use of the GBNP package of GAP4. It concerns the rational specialisation of the Hecke algebra for G_{20} at $a = -2$ and $b = -3$. The following computation lasted less than 10 minutes on a (not particularly fast) laptop.

```
gap> LoadPackage("GBNP");;
????????????????????????????????????????????????????????????????????????????
Loading  GBNP 1.0.3 (Non-commutative Gröbner bases)
by A.M. Cohen (http://www.win.tue.nl/~amc) and
   J.W. Knopper (J.W.Knopper@tue.nl).
Homepage: http://mathdox.org/products/gbnp/
????????????????????????????????????????????????????????????????????????????
gap> g1 := [[[1,2,1,2,1],[2, 1, 2, 1, 2 ]],[1,-1]];;
gap> g2 := [[[1,1,1],[1,1],[1],[]],[1,2,3,-1]];
[gap> g3 := [[[2,2,2],[2,2],[2],[]],[1,2,3,-1]];; KI := [g1,g2,g3];;
```

W	$\#W$	$\#gb$	W	$\#W$	$\#gb$
$\mathbf{G_{16}}$	600	44	$\mathbf{G_{20}}$	360	36
G_{17}	1200	49	$\mathbf{G_{21}}$	720	30
G_{18}	1800	138	$\mathbf{G_{22}}$	240	66
G_{19}	3600	558			

TABLE 1. Summary of GBNP's heuristics for the icosahedral series

Num.	Word	Num.	Word	Num.	Word
1	111	13	2112122121	25	2112112212211
2	222	14	212212211	26	2112122122122
3	21212	15	211221122	27	2112211211221
4	2112121	16	221122112	28	2112212212212
5	2121122	17	2112112211	29	2122122122122
6	22122121	18	221121121	30	2212212212212
7	2211212	19	21221121122	31	21121121121121
8	21211211	20	21122121121	32	21121121121122
9	212112122	21	22112112212	33	21121121122122
10	221211212	22	211211211212	34	21122122112112
11	21221122	23	211211212212	35	211211221221221
12	22112212	24	211212211211	36	2112112112212112

TABLE 2. Dominant terms of the Gröbner basis for G_{20}

```
gap> GI := Grobner(KI);; time;
464612
Length(GI); GG := List(GI,y->y[1][1]);;
gap> Maximum(List(GI,y->Length(y[2])));
193
```

The variable `GG` is what we are interested in. It provides the list $\mathcal{G}$ of table 2. In the case of G_{21} the list is given in table 5. The runtime in case of G_{21} is about 4 hours on a standard PC.

We insist that this part (that uses GBNP) is used for *heuristics* only. In particular, it does not play a role in the logical part of the proof, that could start now from just providing tables 2 and 5 pretending they come out of nowhere.

A last piece of information that we would like to mention is provided by the following command

```
gap> Maximum(List(GI,y->Length(y[2])));
193
```

It says that, in the Gröbner basis provided by GBNP, already in this specialized case, the linear combinations involved there can have as many as 193 terms. Therefore the rewriting rules corresponding to this Gröbner basis, which involves only positive words in the generators, may certainly not be found by hand, already in the case of G_{20}.

3.2. **Incremental determination of computational rules.** It so happens that all defining relations are included in the Gröbner bases provided by GBNP. We view these as the first step in the construction of an *ordered list* $\mathcal{L}$ of rewriting rules of the form $w \rightsquigarrow c_w$ where $w \in \mathcal{G}$ and c_w is a R_0-linear combination of elements of $\mathcal{B}$, with the property that the equality $w = c_w$ holds inside the Hecke algebra H_0. More precisely, the defining relations of the braid groups of the form $b_1 = b_2$ are included under the form $b_1 \rightsquigarrow b_2$ for $b_1 > b_2$. One checks that $b_1 \in \mathcal{G}$ and $b_2 \in \mathcal{B}$ in all cases. The defining relations the form $\sigma^m = b_{s,m-1}\sigma^{m-1} + \cdots + b_{s,1}\sigma + 1$, are also included under the form $\sigma^m \rightsquigarrow b_{s,m-1}\sigma^{m-1} + \cdots + b_{s,1}\sigma + 1$. We denote $\mathcal{L}_0$ the ordered list of leading terms $w \in \mathcal{G}$ of the rules in $\mathcal{L}$.

The incremental process aims at enlarging $\mathcal{L}$ so that $\mathcal{L}$ contains at the end as many elements as $\mathcal{G}$, with the set of elements inside $\mathcal{L}_0$ being equal to $\mathcal{G}$.

The way we enlarge $\mathcal{L}$ uses two algorithms.

The first one is an algorithm for computing a given word as a R_0-linear combination of words, using as additional input an ordered list $\mathcal{L}$ of rewriting rules of the form $v \rightsquigarrow c_v$, where the c_v are linear combinations of *signed* words (that is a word in the generators and their inverses), while the list $\mathcal{L}_0 = (v)_{(v \mapsto c_v) \in \mathcal{L}}$ is formed of *positive words.* The output is either a linear combination of positive words, or `fail`. It is a straightforward rewriting algorithm.

Algorithm 1

- Input : a word w in the generators and their inverses together with the list $\mathcal{L}$
- If w contains the inverse of a generator, replace w by a linear combination $\sum_{m \in E} \lambda_m m$, $\lambda_m \in R_0$ of *positive* words, by applying the rewriting rules $\sigma^{-1} \rightsquigarrow \sigma^{m-1} - b_{s,m-2}\sigma^{m-1} - \cdots - b_{s,1}$ as many times as needed, and apply the present algorithm to these words m (and the same list $\mathcal{L}$). This provides a collection $(r(m))_{m \in E}$ of linear combinations of (positive) words. Then return $\sum_m \lambda_m r(m)$.
- If $w \in \mathcal{B}$, then return w.
- If not, then look for the first element in $\mathcal{L}_0$ which appear as a subword in w. If there is none, return `fail`. If there is one v, with $w = avb$, then replace it with the linear combination $ac_v b$, where $v \rightsquigarrow c_v$ belongs to $\mathcal{L}$. Write the linear combination $ac_v b$ as $\sum_{m \in E} \lambda_m m$, $\lambda_m \in R_0$, and apply the algorithm to each (signed) monomial m (and the same list $\mathcal{L}$). It provides a collection $(r(m))_{m \in E}$ of linear combinations of (positive) words. Then return $\sum_m \lambda_m r(m)$.

Note that the list $\mathcal{L}$ given as input is not changed by algorithm 1. It is clear that, if the algorithm 1 terminates for a given word w, producing a R_0-linear combination b_w, then the equality $w = b_w$ holds inside H_0. Adding more elements in $\mathcal{L}$ will not change the result if the input is one for which the algorithm already terminated, but instead potentially increases the number of words for which it does provide a result.

We denote $\mathcal{G} = (w_1, w_2, \dots)$. Our strategy is then to establish a number of equalities inside H_0 of the form $w_i = b_{w_i}$, where $w_i \in \mathcal{G}$ and b_{w_i} is a linear combination of words with possibly negative powers. This part does *not* follows from an algorithm. In the case of G_{20} we will establish these equalities completely by hand, while for G_{21} we will use a linear combination of handwritten computations and automatic expansions (see section 5).

Once this list is built, it is not suitable for our previous algorithm because the b_w are linear combinations of *signed* words, while positive words are needed in order to apply algorithm 1. We build a suitable list $\mathcal{L}$ incrementally as follows. We first define $\mathcal{L}$ to originally contain the first rewriting rules $w_1 \rightsquigarrow c_{w_1}, \dots, w_{n_0} \rightsquigarrow c_{w_{n_0}}$ originating from the defining relations.

By definition the $c_{w_i}, i \leq n_0$ are linear combinations of positive words. We then complete $\mathcal{L}$ incrementally as follows.

Algorithm 2

- If $\mathcal{L} = (w_1 \rightsquigarrow c_{w_1}, \dots, w_n \rightsquigarrow c_{w_n})$ with n smaller than the length of $\mathcal{G}$, then write $b_{w_{n+1}}$ as $\sum_{m \in E} \lambda_m m$ with $\lambda_m \in R_0$ and E a collection of signed words. Apply **algorithm 1** to each $m \in E$ and get a linear combination $r(m)$ of positive words. Set $c_{w_{n+1}} = \sum_{m \in E} \lambda_m r(m)$, and add to $\mathcal{L}$ the rule $w_{n+1} \rightsquigarrow c_{w_{n+1}}$.
- If the length of $\mathcal{L}$ is equal to the length of $\mathcal{G}$, then return $\mathcal{L}$. Otherwise start again **algorithm 2** with the new $\mathcal{L}$ as input.

For the first group (G_{20}) we are interested in, we managed to produce a convenient list of rewriting rules $w_i \rightsquigarrow b_{w_i}$ completely by hand (see section 4). For the group G_{21} the production of this list had to be partly automatized, too (see section 5). Note that this algorithm looks really dependent on the *ordering* of the list $b_{w_1}, b_{w_2}, \dots$. It is actually so : we checked that interchanging two lines in the list of rewriting rules provided below sometimes makes this algorithm fail. As an indication of running time, algorithm 2 for G_{21} and the list given below lasted about 5 minutes on a standard laptop.

3.3. **Right multiplication table.** Completing the (right)multiplication table is then merely a way to check that H_0 is indeed spanned by the elements of $\mathcal{B}$. It is sufficient to calculate, using algorithm 1, each word ws where $w \in \mathcal{B}$ and s a generator, as a R_0-linear combination of the words in $\mathcal{B}$. The fact that these algorithms complete provides a proof of the conjecture, because we already checked that $\mathcal{B}$ has cardinality $\#W$. It also provides a concrete way to calculate in these algebras. In the case of G_{21} the computation of the whole table lasted only a few minutes.

4. Rules for G_{20}

We first provide the list $w \rightsquigarrow b_w$ of (signed) rewriting rules, and subsequently justify that they indeed hold inside H_0. We use for compactness the notation $1 = s_1$, $2 = s_2$, $\bar{1} = s_1^{-1}$, $\bar{2} = s_2^{-1}$, and $\emptyset$ denotes the emptyword.

(1)	111	$\rightsquigarrow$	$a.11 + b.1 + \emptyset$
(2)	222	$\rightsquigarrow$	$a.22 + b.2 + \emptyset$
(3)	21212	$\rightsquigarrow$	12121
(4)	2112121	$\rightsquigarrow$	1212112
(5)	2121122	$\rightsquigarrow$	$1122121 + a.212112 - a.122121 - b.22121 + b.21211$
(6)	22122121	$\rightsquigarrow$	$12122122 + a.2122121 - a.1212212 + b.122121 - b.121221$
(7)	2211212	$\rightsquigarrow$	$1212211 + a.211212 - a.121221 - b.12122 + b.11212$
(8)	21211211	$\rightsquigarrow$	$11211212 + a.2121121 - a.1211212 + b.212112 - b.211212$
(9)	212112122	$\rightsquigarrow$	$112122121 + a.21211212 - a.12122121 - b.2122121 + b.2121121$
(10)	221211212	$\rightsquigarrow$	$121221211 + a.21211212 - a.12122121 - b.1212212 + b.1211212$

(11) $21221122 \rightsquigarrow a.2121122 + b.211122 + a.21\bar{2}122 + b.212 + a.21\bar{2}\bar{1}2 + b.21\bar{2}\bar{1} + \bar{1}\bar{2}\bar{1}21$

(12) $22112212 \rightsquigarrow a.2112212 + b.112212 + a.\bar{2}12212 + b.212 + a.\bar{2}\bar{1}212 + b.\emptyset + 12\bar{1}\bar{2}\bar{1}$

(13) $2112122121 \rightsquigarrow a.2212122121\bar{2} + b.221212121\bar{2} + 22121121\bar{2}$

(14) $212212211 \rightsquigarrow a.21221221 + b.2122122 + a.212212\bar{1} + b.2122 + 212\bar{1}\bar{2}\bar{1}212$

(15) $211221122 \rightsquigarrow a.21122112 + b.2112211 + a.211221\bar{2} + b.2112 + a.2112\bar{1}\bar{2} + b.21\bar{2}$
$+a.21\bar{2}\bar{1}\bar{2} + b.\bar{1}\bar{2} + \bar{1}\bar{2}\bar{1}\bar{2}1$

(16) $221122112 \rightsquigarrow a.21122112 + b.1122112 + a.\bar{2}122112 + b.2112 + a.\bar{2}\bar{1}2112$
$+b.\bar{2}12 + a.\bar{2}\bar{1}\bar{2}12 + b.\bar{2}\bar{1} + 1\bar{2}\bar{1}\bar{2}\bar{1}$

(17) $2112112211 \rightsquigarrow a.211212211 + b.21122211 + a.2112\bar{1}211 + b.21121$
$+a.2112\bar{1}\bar{2}1 + b.2112\bar{1}\bar{2} + 21\bar{2}\bar{1}\bar{2}12$

(18) $221121121 \rightsquigarrow a.21121121 + b.1121121 + a.\bar{2}121121 + b.1121 + 121\bar{2}\bar{1}\bar{2}121$

(19) $21221121122 \rightsquigarrow a.2122112122 + b.212211222 + a.2122112\bar{1}2$
$+b.2122112\bar{1} + 212\bar{1}\bar{2}\bar{1}21121$

(20) $21122121121 \rightsquigarrow a.2122121121 + b.222121121 + a.2\bar{1}2121121$
$+b.221121 + 2212212\bar{1}\bar{2}$

(21) $22112112212 \rightsquigarrow a.2211212212 + b.221122212 + a.22112\bar{1}212$
$+b.221122 + 22112212\bar{1}\bar{2}\bar{1}$

(22) $211211211212 \rightsquigarrow 2112212122212\bar{1}$

(23) $211211212212 \rightsquigarrow 2112121212\bar{1}212$

(24) $211212211211 \rightsquigarrow 21\bar{2}12121211211$

(25) $2112112212211 \rightsquigarrow a.211211221211 + b.21121122111 + a.211211221\bar{2}1$
$+b.211211221\bar{2} + 21\bar{2}\bar{1}2112212$

(26) $2112122122122 \rightsquigarrow 2121212\bar{1}2122122$

(27) $2112211211221 \rightsquigarrow a.212211211221 + b.22211211221 + a.2\bar{1}211211221$
$+b.21211221 + a.2\bar{1}\bar{2}1211221 + b.21221 + 2211221\bar{2}\bar{1}\bar{2}$

(28) $2112212212212 \rightsquigarrow a.212212212212 + b.22212212212 + a.2\bar{1}212212212$
$+b.22212212 + 2212\bar{1}\bar{2}\bar{1}212212$

(29) $2122122122122 \rightsquigarrow \bar{1}21212\bar{1}2122122122$

(30) $2212212212212 \rightsquigarrow 2212212212\bar{1}21212\bar{1}$

(31) $21121121121121 \rightsquigarrow 21121121121\bar{2}12121\bar{2}$

(32) $21121121121122 \rightsquigarrow a.2112112112122 + b.211211211222$
$+a.2112112112\bar{1}2 + b.2112112112\bar{1} + 21121121\bar{2}\bar{1}\bar{2}121$

(33) $21121121122122 \rightsquigarrow 21\bar{2}12121\bar{2}121122122$

(34) $21122122112112 \rightsquigarrow 2112\bar{1}21212\bar{1}2112112$

(35) $211211221221221 \rightsquigarrow 21\bar{2}12121\bar{2}1221221221$

(36) $2112112112212112 \rightsquigarrow 211211211212121\bar{2}12$

We now justify each one of the above rules. We first notice that the braid relation may appear under the following different guises

$$21212 = 12121, \bar{2}\bar{1}\bar{2}12 = 12\bar{1}\bar{2}\bar{1}, 21\bar{2}\bar{1}\bar{2} = \bar{1}\bar{2}\bar{1}21, \bar{2}\bar{1}\bar{2}\bar{1}2 = 1\bar{2}\bar{1}\bar{2}\bar{1}, \bar{1}\bar{2}\bar{1}\bar{2}1 = 2\bar{1}\bar{2}\bar{1}\bar{2}, \bar{1}\bar{2}121 = 212\bar{1}\bar{2}$$

and the the other relations imply $\bar{1} = 11 - a.1 - b.\emptyset$, $\bar{2} = 22 - a.2 - b.\emptyset$. In order to clarify the computations, we underline the subwords to which one of these defining relations or their obvious variants are applied.

(i) defining relation.
(ii) defining relation.

(iii) defining relation.

(iv) follows from $21\underline{12121} = \underline{21212}12 = 1212112$.

(v) we have

$$\begin{aligned}
2121122 &= \bar{1}\underline{12121}122 \\
&= \bar{1}2\underline{12121}22 \\
&= \bar{1}22121\underline{222} \\
&= a.\bar{1}2212122 + b.\bar{1}221212 + \bar{1}22121 \\
&= a.\bar{1}\underline{21212}12 + b.\bar{1}\underline{21212}1 + (11 - a.1 - b.\emptyset)22121 \\
&= a.\bar{1}1212112 + b.\bar{1}121211 + 1122121 - a.122121 - b.22121 \\
&= a.212112 + b.21211 + 1122121 - a.122121 - b.22121
\end{aligned}$$

(vi) We have $22122121 = 2212\underline{21212}\bar{2} = 2\underline{21212}121\bar{2} = \underline{21212}1121\bar{2} = 1212\underline{111}21\bar{2}$ hence

$$\begin{aligned}
22122121 &= 1212(a.11 + b.1 + \emptyset)21\bar{2} \\
&= a.\underline{12121}121\bar{2} + b.12\underline{12121}\bar{2} + 121221\bar{2} \\
&= a.212\underline{12121}\bar{2} + b.1221212\bar{2} + 121221\bar{2} \\
&= a.21221212\bar{2} + b.122121 + 121221(22 - a.2 - b\emptyset) \\
&= a.2122121 + b.122121 + 12122122 - a.1212212 - b.121221
\end{aligned}$$

(vii) We have $2211212 = 221\underline{12121}\bar{1} = 22\underline{12121}2\bar{1} = \underline{222}12122\bar{1}$ hence

$$\begin{aligned}
2211212 &= (a.22 + b.2 + \emptyset)12122\bar{1} \\
&= a.2\underline{21212}2\bar{1} + b.\underline{21212}2\bar{1} + 12122\bar{1} \\
&= a.21\underline{21212}\bar{1} + b.1\underline{21212}\bar{1} + 1212211 - a.121221 - b.12122 \\
&= a.2112121\bar{1} + b.112121\bar{1} + 1212211 - a.121221 - b.12122 \\
&= a.211212 + b.11212 + 1212211 - a.121221 - b.12122
\end{aligned}$$

(viii) We have $21211211 = \bar{1}\underline{12121}1211 = \bar{1}21\underline{21212}11 = \bar{1}211212\underline{111}$ hence

$$\begin{aligned}
21211211 &= \bar{1}211212(a.11 + b.1 + \emptyset) \\
&= a\bar{1}21\underline{12121}1 + b\bar{1}21\underline{12121} + \bar{1}211212 \\
&= a\bar{1}\underline{21212}121 + b\bar{1}\underline{21212}12 + (11 - a.1 - b.\emptyset)211212 \\
&= a\bar{1}12121121 + b\bar{1}1212112 + 11211212 - a.1211212 - b.211212 \\
&= a2121121 + b212112 + 11211212 - a.1211212 - b.211212
\end{aligned}$$

(ix) Similarly, we have $212112122 = \bar{1}\underline{12121}12122 = \bar{1}212\underline{12121}22 = \bar{1}2122121\underline{222}$ hence

$$\begin{aligned}
212112122 &= \bar{1}2122121(a.22 + b.2 + \emptyset) \\
&= a.\bar{1}212\underline{21212}2 + b.\bar{1}212\underline{21212} + \bar{1}2122121 \\
&= a.\bar{1}\underline{21212}1212 + b.\bar{1}\underline{21212}121 + (11 - a.1 - b)2122121 \\
&= a.\bar{1}121211212 + b.\bar{1}12121121 + 112122121 - a.12122121 - b.2122121 \\
&= a.21211212 + b.2121121 + 112122121 - a.12122121 - b.2122121
\end{aligned}$$

(x) Finally we have $221211212 = 22121\underline{12121}\bar{1} = 22\underline{12121}212\bar{1} = \underline{222}1212212\bar{1}$ hence

$$\begin{aligned}
221211212 &= (a.22 + b.2 + \emptyset)1212212\bar{1} \\
&= a.2\underline{21212}212\bar{1} + b.\underline{21212}212\bar{1} + 1212212\bar{1} \\
&= a.2121\underline{21212}\bar{1} + b.121\underline{21212}\bar{1} + 1212212(11 - a.1 - b) \\
&= a.212112121\bar{1} + b.12112121\bar{1} + 121221211 - a.12122121 - b.1212212 \\
&= a.21211212 + b.1211212 + 121221211 - a.12122121 - b.1212212
\end{aligned}$$

(xi) We have $21\underline{22}1122 = a.2121122 + b.211122 + 21\bar{2}1122$ and $21\bar{2}\underline{11}22 = a.21\bar{2}122 + b.21\bar{2}22 + 21\bar{2}\bar{1}22 = a.21\bar{2}122 + b.212 + 21\bar{2}\bar{1}22$. Then, $21\bar{2}\bar{1}\underline{22} = a.21\bar{2}\bar{1}2 + b.21\bar{2}\bar{1} + 21\bar{2}\bar{1}\bar{2}$. Since $21\bar{2}\bar{1}\bar{2} = \bar{1}\bar{2}\bar{1}21$ this establishes the rule.

(xii) We have $\underline{22}112212 = a.2112212 + b.112212 + \bar{2}112212$, then $\bar{2}\underline{11}2212 = a.\bar{2}12212 + b.\bar{2}2212 + \bar{2}\bar{1}2212$ and $\bar{2}\bar{1}2212 = a.\bar{2}\bar{1}212 + b\bar{2}\bar{1}12 + \bar{2}\bar{1}\bar{2}12$. Finally, $\bar{2}\bar{1}\bar{2}12 = 12\bar{1}\bar{2}\bar{1}$ and this establishes the rule.

(xiii) We have $2112122121 = 211212\underline{21212}\bar{2} = 21\underline{12121}2121\bar{2} = 2\underline{12121}22121\bar{2}$ which is equal to $22121\underline{222}121\bar{2} = a.22121221212\bar{2} + b.2212121212\bar{2} + 22121121\bar{2}$

(xiv) We have $2122122\underline{11} = a.21221221 + b.2122122 + 2122122\bar{1}$, and $21221\underline{22}\bar{1} = a.212212\bar{1} + b.21221\bar{1} + 21221\bar{2}\bar{1}$. Now, $21221\bar{2}\bar{1} = 212\underline{21\bar{2}\bar{1}\bar{2}}2 = 212\bar{1}\bar{2}\bar{1}212$ and this establishes the rule.

(xv) We expand 2(11)(22)(11)(22) by using four times the relation $x^2 = a.x + b + x^{-1}$ for $x \in \{1,2\}$ at the four places between parenthesis we get $211221122 = a.21122112 + b.2112211 + a.211221\bar{2} + b.2112 + a.2112\bar{1}\bar{2} + b.21\bar{2} + a.21\bar{2}\bar{1}\bar{2} + b.\bar{1}\bar{2} + 2\bar{1}\bar{2}\bar{1}\bar{2}$. Now $\bar{1}\bar{2}\bar{1}\bar{2}1 = 2\bar{1}\bar{2}\bar{1}\bar{2}$ and this establishes the rule.

(xvi) Rule #16 is similar to rule #15 : we expand (22)(11)(22)(11)2 and use $1\bar{2}\bar{1}\bar{2}\bar{1} = \bar{2}\bar{1}\bar{2}\bar{1}2$.

(xvii) Rule #17 is similar to rules #15 and #16 : expand 2112(11)(22)(11) and use $2112\bar{1}\bar{2}\bar{1} = 211\underline{2\bar{1}\bar{2}\bar{1}\bar{2}}2 = 211\bar{1}\bar{2}\bar{1}\bar{2}12 = 21\bar{2}\bar{1}\bar{2}12$.

(xviii) By expanding (22)(11)21121 we get $221121121 = a.21121121 + b.1121121 + a.\bar{2}121121 + b.1121 + \bar{2}\bar{1}21121$. Since $\bar{2}\bar{1}21121 = 1\underline{\bar{1}\bar{2}\bar{1}21}121 = 121\bar{2}\bar{1}\bar{2}121$ this establishes the rule.

(xix) By expanding 2122112(11)(22) we get $21221121122 = a.2122112122 + b.212211222 + a.2122112\bar{1}2 + b.2122112\bar{1} + 2122112\bar{1}\bar{2}$ and $2122112\bar{1}\bar{2} = 21221\underline{12\bar{1}\bar{2}\bar{1}}1 = 212\underline{21\bar{2}\bar{1}\bar{2}}121 = 212\bar{1}\bar{2}\bar{1}21121$ establishes the rule.

(xx) By expanding 2(11)(22)121121 we get $21122121121 = a.2122121121 + b.222121121 + a.2\bar{1}2121121 + b.221121 + 2\bar{1}\bar{2}121121$ and $2\underline{\bar{1}\bar{2}121}121 = 2212\underline{\bar{1}\bar{2}121} = 2212212\bar{1}\bar{2}$ which establishes the rule.

(xxi) By expanding 22112(11)(22)12 we get $22112112212 = a.2211212212 + b.221122212 + a.22112\bar{1}212 + b.221122 + 22112\bar{1}\bar{2}12$ and $22112\bar{1}\bar{2}12 = 22112\underline{\bar{1}2121}\bar{1} = 2211221 2\bar{1}\bar{2}\bar{1}$ establishes the rule.

(xxii) We have $211211211212 = 21121121\underline{12121}\bar{1} = 21121\underline{12121}212\bar{1} = 2112\underline{12121}2212\bar{1} = 2112212122212\bar{1}$ and this establishes the rule.

(xxiii) We have $211211212212 = 21121\underline{12121}\bar{1}212 = 2112121212\bar{1}212$ and this establishes the rule.

(xxiv) We have $211212211211 = 21\bar{2}\underline{21212}211211 = 21\bar{2}12121211211$ and this establishes the rule.

(xxv) We expand 211211221(22)(11) and get

$$2112112212211 = a.211211221211 + b.21121122111 + a.21121122\bar{2}1 + b.211211221\bar{2} + 21121122\bar{2}\bar{1}$$

and since $21121122\bar{2}\bar{1} = 2112112\underline{21\bar{2}\bar{1}\bar{2}}2 = 21121\underline{12\bar{1}\bar{2}\bar{1}}212 = 211\underline{21\bar{2}\bar{1}\bar{2}}12212 = 211\bar{1}\bar{2}\bar{1}2112212 = 21\bar{2}\bar{1}2112212$ this establishes the rule.

(xxvi) We have $2112122122122 = 21\underline{12121}\bar{1}2122122 = 212121 2\bar{1}2122122$ and this establishes the rule.

(xxvii) By expanding 2(11)(22)(11)211221 we get $2112211211221 = a.212211211221 + b.22211211221 + a.2\bar{1}211211221 + b.21211221 + a.2\bar{1}\bar{2}1211221 + b.21221 + 2\bar{1}\bar{2}\bar{1}211221$ and $2\underline{\bar{1}\bar{2}\bar{1}21}1221 = 221\underline{\bar{2}\bar{1}\bar{2}12}21 = 22112\underline{\bar{1}\bar{2}\bar{1}21} = 2211221\bar{2}\bar{1}\bar{2}$ and this establishes the rule.

(xxviii) By expanding 2(11)(22)12212212 we get

$$\begin{aligned} 2112212212212 &= a.212212212212 + b.22212212212 + a.2\bar{1}212212212 \\ &\quad + b.22212212 + 2\bar{1}\bar{2}12212212 \end{aligned}$$

and $2\bar{1}\bar{2}12212212 = 2\underline{\bar{1}\bar{2}121}\bar{1}212212 = 2212\bar{1}\bar{2}\bar{1}212212$ and this establishes the rule.

(xxix) We have $2122122122122 = \bar{1}\underline{12121}\bar{1}2122122122 = \bar{1}21212\bar{1}2122122122$ and this establishes the rule.

(xxx) We have $2212212212212 = 2212212212\bar{1}\underline{12121}\bar{1} = 2212212212\bar{1}21212\bar{1}$ and this establishes the rule.

(xxxi) We have $21121121121121 = 2112112112\bar{2}\underline{21212}\bar{2} = 2112112112\bar{2}12121\bar{2}$ and this establishes the rule.

(xxxii) By expanding 2112112112(11)(22) we get $21121121121122 = a.2112112112122 + b.211211211222 + a.2112112112\bar{1}2 + b.2112112112\bar{1} + 2112112112\bar{1}\bar{2}$, and $2112112112\bar{1}\bar{2} = 21121121\underline{12\bar{1}\bar{2}\bar{1}}1 = 21121121\bar{2}\bar{1}\bar{2}121$, which establishes the rule.

(xxxiii) We have $21121121122122 = 21\bar{2}\underline{21212}\bar{2}121122122 = 21\bar{2}12121\bar{2}121122122$ which establishes the rule.

(xxxiv) We have $21122122112112 = 2112\bar{1}12121\bar{1}2112112 = 2112\bar{1}21212\bar{1}2112112$ which establishes the rule.

(xxxv) We have $211211221221221 = 21\bar{2}21212\bar{2}1221221221 = 21\bar{2}12121\bar{2}1221221221$ which establishes the rule.

(xxxvi) We have $2112112112212112 = 2112112112\underline{21212}\bar{2}12 = 211211211212121\bar{2}12$ which establishes the rule.

5. Rules for G_{21}

5.1. **Semi-manual procedures.** Recall that the Hecke algebra in this case is defined over $R_0 = \mathbb{Z}[a, b, q]$. Let Y be the alphabet $\{1, 2, \bar{1}, \bar{2}\}$, $M(Y)$ the free monoid over Y, and $F(Y) \subset M(Y)$ the subset of *freely reduced words*, that is the set of natural representatives of the free group on $\{1, 2\}$ viewed as a quotient of $M(Y)$. We denote $M^+(Y) = M(\{1, 2\}) \subset M(Y)$ the submonoid of *positive words*. We let red : $M(Y) \to F(Y)$ denote the usual reduction procedure, and red : $R_0M(Y) \to R_0F(Y)$ its natural linear extension, where we let $R_0M(Y)$ denote the monoid algebra over R_0 and $R_0F(Y)$ the (free) submodule spanned by $F(Y)$.

A more complicated procedure which we describe now is what we call *expansion*. By convention we set $\bar{\bar{y}} = y$ for all $y \in \{1, 2\}$. For $I \subset \mathbb{N}^*$, let us first define the *I-inversion* map $\mathrm{inv}_I : M(Y) \to M(Y)$ as follows. If $y = y_1y_2y_3 \dots y_n \in M(Y)$ is a word in n letters, with $y_k \in Y$, $\mathrm{inv}_I(y) = y' = y'_1y'_2y'_3 \dots y'_n \in M(Y)$ is defined by $y'_k = \bar{y}_k$ if $k \in I$, $y'_k = y_k$ if $k \notin I$. We now define the partially defined *expansion* map $\exp_I : M(Y) \dashrightarrow M(Y)$ with respect to I by induction on the cardinality of I. If $I = \emptyset$, then $\exp_\emptyset$ is the identity map. If not, let $i_0 = \min(I)$, and let J such that $I = J \sqcup \{i_0\}$. If $y = y_1y_2y_3 \dots y_n \in M(Y)$ is a word in n letters, with $y_k \in Y$, then $\exp_I(y)$ is defined if $n \geq i_0$ and if

- either $y_{i_0} = 1$, in which case $\exp_I(y) = q.y' + z$ with $y' = y'_1y'_2 \dots y'_{n-1}$ where $y'_k = y_k$ for $k < i_0$, $y'_k = y_{k+1}$ for $k \geq i_0$, and $z = \exp_{\check{J}}(\mathrm{inv}_{\{i_0\}}(y))$ with $\check{J} = J$, if it is defined.
- either $y_{i_0} = y_{i_0} + 1 = 2$, in which case $\exp_I(y) = a.y' + b.y'' + z$ with
  - $y'' = y''_1y''_2 \dots y''_{n-2}$ where $y''_k = y_k$ for $k < i_0$, $y''_k = y_{k+2}$ for $k \geq i_0$
  - $y' = y'_1y'_2 \dots y'_{n-1}$ where $y'_k = y_k$ for $k < i_0$, $y'_{i_0} = y_{i_0} = 2$, $y'_k = y_{k+1}$ for $k \geq i_0 + 1$.
  - $z = \exp_{\check{J}}(\mathrm{inv}_{\{i_0\}}(y''))$ with $\check{J} = J - 1 = \{x - 1; x \in J\}$, if it is defined.

I	$\mathrm{head}_I(1221)$	$\exp_I(1221)$
$\emptyset$	1221	1221
$\{1\}$	$\bar{1}221$	$q.221 + \bar{1}221$
$\{2\}$	$1\bar{2}1$	$a.121 + b.11 + 1\bar{2}1$
$\{1,2\}$	$\bar{1}\bar{2}1$	$q.221 + \exp_{\{2\}}(\bar{1}221)$ $= q.221 + a.\bar{1}21 + b.\bar{1}1 + \bar{1}\bar{2}1$
$\{4\}$	$122\bar{1}$	$q.122 + 122\bar{1}$
$\{1,4\}$	$\bar{1}22\bar{1}$	$q.221 + \exp_{\{4\}}(\bar{1}221)$ $= q.221 + q.\bar{1}22 + \bar{1}22\bar{1}$
$\{2,4\}$	$1\bar{2}\bar{1}$	$a.121 + b.11 + \exp_{\{3\}}(1\bar{2}1)$ $= a.121 + b.11 + q.1\bar{2} + 1\bar{2}\bar{1}$
$\{1,2,4\}$	$\bar{1}\bar{2}\bar{1}$	$q.221 + \exp_{\{2,4\}}(\bar{1}221)$ $= q.221 + a.\bar{1}21 + b.\bar{1}1 + \exp_{\{3\}}(\bar{1}\bar{2}1)$ $= q.221 + a.\bar{1}21 + b.\bar{1}1 + q.\bar{1}\bar{2} + \bar{1}\bar{2}\bar{1}$

TABLE 3. Example : partial expansions of the word 1221.

It is easily checked that, when defined, $\exp_I(y)$ can be written as $\exp_I(y) = \mathrm{tail}_I(y) + \mathrm{head}_I(y)$ with $\mathrm{head}_I(y) \in M(Y)$ being characterized, with the above notations, by $\mathrm{head}_{\{i_0\}\sqcup J}(y) = \mathrm{head}_{\tilde{J}}(z)$, and $\mathrm{head}_{\emptyset}(y) = y$.

In table 3 we provide as an example all the partial expansions of the word $w = 1221$. Of course, the definitions of $\exp_I$ and red are made in such a way that the images of any two words of $M(Y)$ under one of these maps yield the same element under the natural algebra morphism $R_0 M(Y) \to H_0$.

5.2. **Rules.** We can now give the set of rules for G_{21}, the justification that they correspond to genuine relations inside its Hecke algebra easily relying on the properties of $\exp_I$ and tail_I established in the above section. Indeed, most of them are of the form $w \rightsquigarrow \mathrm{red}(\mathrm{tail}_I(a*w*b) + w')$, where $*$ denotes the concatenation of two words, $a, b \in M(Y)$ satisfy $\mathrm{red}(a) = \mathrm{red}(b) = \emptyset$, and w' is a word deduced from $\mathrm{head}_I(a*w*b)$ using braid relations. Some more complicated ones have the form $\mathrm{red}(\mathrm{tail}_I(a*w*b) + \mathrm{tail}_J(w') + w'')$ where w' is deduced from $\mathrm{head}_I(a*w*b)$ and w'' is deduced from $\mathrm{tail}_J(w')$, etc.

(1)	11	$\rightsquigarrow$	$(q).1 + \emptyset$
(2)	222	$\rightsquigarrow$	$a.22 + b.2 + \emptyset$
(3)	2121212121	$\rightsquigarrow$	1212121212
(4)	21212121221	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{11}(\bar{1}1 * w) + \bar{1}\bar{2}1212121211)$
(5)	221221212121	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{1,3,4}(w) + 1212121\bar{2}\bar{1}\bar{2})$
(6)	212121221221	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{7,9,10,12}(212121221221) + \overline{1212}121212)$
(7)	2121221221221	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{5,7,8,10,11,13}(2121221221221) + \overline{121212}1212)$
(8)	2212212212121	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{1,3,4,6,7}(2212212212121) + 12121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2})$
(9)	212122121221221	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{10,12,13,15}(w * \bar{2}\bar{1}12) + \bar{1}\bar{2}\bar{1}\bar{2}\bar{1}2121121212)$
(10)	22121221212121	$\rightsquigarrow$	$\mathrm{red}(\mathrm{tail}_{5,6,8}(w * 2\bar{2}) + 22122121212\bar{1}\bar{2}\bar{1}\bar{2})$

(11) $21212122121221 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{11,13}(\bar{1}\bar{2}21 * w) + \mathrm{tail}_{12,14,15}(w') + w'')$
$w' = \bar{1}\bar{2}\bar{1}\bar{2}1212121221221$
$w'' = \bar{1}\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}12121211$

(12) $21221221221221 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{3,5,6,8,9,11,12,14}(w) + \bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}12)$

(13) $22122122122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{1,3,4,6,7,9,10,12}(w) + 12\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2})$

(14) $2121212212121221 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{14,16}(w * \bar{2}\bar{1}12) + 2121212\bar{1}\bar{2}\bar{1}\bar{2}12121212)$

(15) $221221212212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{8,9}(w * 212\bar{2}\bar{1}\bar{2}) + 221221221212121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2})$

(16) $2212121221212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{7,8}(w * 2\bar{2}) + 22121221212121\bar{2}\bar{1}\bar{2})$

(17) $2212212212122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{3,5,6,8,9}(1\bar{1} * w) + 12121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}2121)$

(18) $22122121212212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{3,5,6}(1\bar{1} * w) + 1212121\bar{2}\bar{1}\bar{2}\bar{1}212121)$

(19) $212122122121221221 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{12,13,15,16,18}(w * \bar{2}\bar{1}12) + \bar{1}\bar{2}\bar{1}\bar{2}121221221212)$

(20) $221221212212212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{8,9,11,12}(w * 2\bar{2}) + 2212212212121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2})$

(21) $2212122121212212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{5,6}(\hat{w}) + 221221212121\bar{2}\bar{1}\bar{2}\bar{1}212121)$
$\hat{w} = 221212212121212\bar{2}\bar{1}212121$

(22) $2212212212122122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{3,5,6,8,9}(1\bar{1} * w) + 12121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}2122121)$

(23) $22121221221212122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{5,6,8,9}(w) + \mathrm{tail}_{5,7,8}(w')) + w''$
$w' = 12\bar{2}\bar{1}2212212121\bar{2}\bar{1}\bar{2}\bar{1}2121$
$w'' = 1212121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{1}2121$

(24) $22122121221212212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{13,14}(w * 212\bar{2}\bar{1}\bar{2}) + 22122121221221212121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2})$

(25) $221221212212212122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{7,9,10,12}(121\bar{1}\bar{2}\bar{1} * w) + 121212212212\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}2121)$

(26) $2121221212212122121221 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{20,22}(w * \bar{2}\bar{1}\bar{2}\bar{1}1212) + 21212\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}21212212112121212)$

(27) $2212122122121221212122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{5,6,8,9}(\hat{w}) + 2212212121121221212\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2})$
$\hat{w} = 2212122122121212\bar{2}\bar{1}212122121$

(28) $2212212122121221212122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{7,9,10}(121\bar{1}\bar{2}\bar{1} * w) + 12121211212\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}212122121)$

(29) $2212122121221212212121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{15,16}(w * 212\bar{2}\bar{1}\bar{2}) + \mathrm{tail}_{10,11,13,14}(w') + w'')$
$w' = 2212122121221221212121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}$
$w'' = 22121221221212121\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}$

(30) $2212122121221212122121 \rightsquigarrow \mathrm{red}(\mathrm{tail}_{14,15}(\hat{w}) + \mathrm{tail}_{5,6,8,9}(w') + \mathrm{tail}_{5,7,8}(w'') + w''')$
$\hat{w} = 22121221221\bar{1}\bar{2}1221212122121$
$w' = 22121221221212121\bar{2}\bar{1}\bar{2}\bar{1}2121$
$w'' = 12\bar{2}\bar{1}2212212121\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{1}2121$
$w''' = 1212121\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{1}2121$

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∅	112122	2112212	21221121	211212212	2112112122	12211211221	121122122122	2112112112212
1	112211	2121121	21221211	211221121	2112112212	12212211211	121221221221	2112112122112
2	112212	2122112	21221221	211221211	2112122112	12212212212	122122112112	2112112212112
11	121121	2122121	22112112	211221221	2112122122	21121121121	122122122122	2112112212212
12	121122	2122122	22112211	212211211	2112211211	21121121122	211211211211	2112212211211
21	121211	2211211	22121121	212212112	2112212112	21121121221	211211211221	11211211211211
22	121221	2211221	22122112	212212212	2112212211	21121122121	211211212211	11211211211221
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121	122121	2212211	112112112	221221121	2122112112	21121221121	211211221221	11211211221211
122	122122	2212212	112112122	221221221	2122121121	21121221221	211212212212	11211211221221
211	211211	11211211	112112211	1121121121	2122122122	21122112112	211221121122	11211212212212
212	211212	11211212	112112212	1121121122	2211211221	21122122112	211221221121	11211221121122
221	211221	11211221	112121121	1121121221	2212211211	21122122122	211221221221	11211221221121
1121	212112	11212112	112122112	1121122112	2212212212	21221221221	212212212212	11211221221221
1122	212211	11212211	112122121	1121122121	11211211211	22122112112	221221221221	11212212212212
1211	212212	11212212	112122122	1121122122	11211211212	22122122122	1121121121121	11221221221221
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112112	2112122	21122122	211211221	1221221221	12122121121	121122112112	1221221221221	1211211211221211
112121	2112211	21211212	211212211	2112112112	12122122122	121122122112	2112112112112	11211211211221211

TABLE 4. Basis for G_{20}

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Num.	Word	Num.	Word
1	11	16	2212121221212121
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3	2121212121	18	22122121212212121
4	21212121221	19	212122122121221221
5	221221212121	20	221221212212212121
6	212121221221	21	2212122121212212121
7	2121221221221	22	2212212212122122121
8	2212212212121	23	22121221221212122121
9	212122121221221	24	22122121221212212121
10	22121221212121	25	221221212212212122121
11	21212122121221	26	2121221212212122121221
12	21221221221221	27	22121221221212212122121
13	22122122122121	28	22122121221212212122121
14	2121212212121221	29	2212122121221212212121
15	221221212212121	30	2212122121221212122121

Table 5. Dominant terms of the Gröbner basis for G_{21}

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