BRANCHING PROPERTIES FOR THE GROUPS $G(de, e, r)$

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Abstract. We study general properties of the restriction of the representations of the finite complex reflection groupes $G(de, e, r+1)$ to their maximal parabolic subgroups of type $G(de, e, r)$, and focus notably on the multiplicity of components. In combinatorial terms, this amounts to the following question : which symmetries arise or disappear when one changes (exactly) one pearl in a combinatorial necklace ?

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1. Introduction

1.1. **Motivations.** It is well-known that, for irreducible, classical Coxeter groups of type A_{n+1}, B_{n+1} and D_{n+1}, the restriction of irreducible representations to their natural maximal parabolic subgroups of type A_n, B_n and D_n is multiplicity free. This is a useful, although mysterious, classical fact, which is easily proved once we know it for the symmetric groups, as B_n is a wreath product and D_n is a subgroup of index 2 of B_n. This generalizes to the following also classical fact :

Fact 1. *If W is a finite irreducible Coxeter group, it admits a maximal parabolic subgroup W_v such that the restriction to W_v of any irreducible representation of W is multiplicity free, except if W has type E_8 or H_4.*

In case W has type E_8 or H_4, there are a number of irreducible representation whose restriction to maximal parabolic subgroups of types E_7 and H_3 have irreducible components with multiplicity 2. This is the worst case scenario, so the above observation can be refined :

Fact 2. *If W is a finite irreducible Coxeter group, it admits a maximal parabolic subgroup W_v such that the restriction to W_v of any irreducible representation of W contains multiplicities of order at most 2, and is even multiplicity free, except if W has type E_8 or H_4.*

A first goal of this note is to prove a analogous result for the more general setting of irreducible (finite) complex pseudo-reflection groups. Recall that such groups belong to either a finite set of 34 exceptions or to an infinite family with three integer parameters $G(de, e, r)$. In this family, two families can be thought of as generalisations of Coxeter groups. The first one is when $e = 1$: the group $G(d, 1, r)$ is a wreath product that generalizes $B_n = G(2, 1, n)$. The second one is for $d = 1$: the groups $G(e, e, r)$ generalize both $D_n = G(2, 2, n)$ and the

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dihedral groups $I_2(e) = G(e,e,2)$. Another noticeable fact, which generalizes the relation between D_n and B_n, is that $G(de,e,r)$ is a normal subgroup of index e of $G(de,1,r)$ with cyclic quotient.

It follows that the classical case-by-case approaches to the representation of complex reflection groups and their cyclotomic Hecke algebras usually starts with the wreath products $G(d,1,r)$ and then uses an avatar of Clifford theory to deal with the more general groups $G(de,e,r)$ (see e.g [RR, MM]). This approach is however not always satisfactory. To understand this, we can remember that many results about Coxeter groups and root systems are simpler to prove and/or state for groups of type ADE, which have a single conjugacy class of reflections, and then extended or generalized to the other cases, including types B. The analagous approach to complex reflection groups would be to deal first with the groups which have a single class of reflections, and these groups are the groups $G(e,e,r)$. In particular, in order to generalize the above facts the crucial case concerns the groups $G(e,e,r)$.

1.2. **Main results.** To make the next statements precise, we need to recall some terminology about finite complex (pseudo-)reflection groups. Let V be a finite-dimensional complex vector space. A pseudo-reflection of V is an element $s \in \mathrm{GL}(V)$ of finite order such that $\mathrm{Ker}\,(s-1)$ is an hyperplane of V. A finite subgroup W of $\mathrm{GL}(V)$ is called a *reflection group* if it is generated by pseudo-reflections. It is called irreducible if its action on V is irreducible. A *reflection subgroup* of W is a subgroup of W generated by pseudo-reflections. A *maximal parabolic subgroup* of W is the subgroup W_v of the elements of W which stabilize some given $v \in V \setminus \{0\}$. It is a classical result due to Steinberg that W_v is a reflection subgroup of W, generated by the pseudo-reflections of W which stabilize v.

Recall that a matrix is called monomial if it admits exactly one non-zero entry in each row and in each column. Let $d,e,r \geq 1$ be integers. The group $G(de,e,r+1)$ is the subgroup of $\mathrm{GL}_{r+1}(\mathbb{C})$ of the monomial matrices with non-zero entries in μ_{de} such that the product of these entries lies in μ_e. The maximal parabolic subgroup of elements leaving the $(r+1)$-th coordinate unchanged can obviously be identified with the reflection group $G(de,e,r)$. We refer to [Ar, AK] or [MM] for a general account on these groups. It is known and easily checked that they are irreducible, provided $de \neq 1$ and $(d,e) \neq (1,2)$.

We will then prove the following

Theorem 1. *The induction table between the group $G(de,e,r+1)$ and their maximal parabolic subgroups of type $G(de,e,r)$ contains multiplicities of order at most 2.*

Moreover, these multiplicities appear in a systematic way that we describe. A consequence is the following.

Theorem 2. *Any irreducible complex reflection group W admits a maximal parabolic subgroup W_v such that the restriction to W_v of any irreducible representation of W contains multiplicities of order at most 2, except if W has type G_{22}, G_{27}. In these cases, W admits a maximal parabolic subgroup for which the multiplicities have order at most 3.*

To deduce this result from the former one, we only need to check it for the exceptional complex reflection groups which are not Coxeter groups. We used computer means, namely the GAP package CHEVIE. In Table 1 we list all these complex reflection groups, giving a set of generators for a maximal parabolic subgroup satisfying our conditions, where the names of the generators follow the conventions of the tables in [BMR]. In all cases there exists such a subgroup which can be generated by a subset of the usual generators, which makes things easier to describe. In the case of G_{22} and G_{27} we checked that no other maximal parabolic subgroup behaves in a nicer way.

In the case of the groups $G(de, e, r + 1)$, and in order to be more specific about which representations of $G(de, e, r)$ occur with multiplicity 2 in the restriction of an irreducible representation of $G(de, e, r + 1)$, we get several other results to understand the " square of inclusions "

$$\begin{array}{ccc} G(de,1,r) & \longrightarrow & G(de,1,r+1) \\ \uparrow & & \uparrow \\ G(de,e,r) & \longrightarrow & G(de,e,r+1) \end{array}$$

in representation-theoretic terms. These technical results are listed and proved in section 3.

1.3. **Representations and necklaces.** In order to prove these results for the groups $G(de, e, r)$, we translate the questions in terms of combinatorial data, which are called *necklaces*. In general, a necklace is a function from a group Γ, usually assumed to be cyclic, to some set of ornaments, that can be called pearls or colours – and is considered modulo the Γ-action. It is now well-known that representations of $G(de, e, r)$ are naturally indexed by such objects (see e.g. [HR]). It turns out that understanding the branching problem involves the following strange problem : what happens when one changes (exactly) one pearl in a necklace ?

We did not find occurences of this problem in the literature. Because we found it interesting in its own right, we tried to solve it in some generality. As a consequence, the reader interested in the proofs of the statements in section 3 may prefer to read before that the sections 4, 6 and 7, which deal with necklaces in general, as a whole. Section 2 deals with a simple general result that we use in section 3 but which does not involve necklaces. Section 5 contains preliminary lemmas about cyclic groups which are used in section 6 and 7.

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2. A general symmetry breaking result

The aim of this section is to prove the following result, which we may have independant interest, and that we view as a combinatorial symmetry breaking result. For a group G acting on a set X, and $x \in X$, we let $G_x \subset G$ denote the stabilizer of x.

Proposition 2.1. *Let X be a set, G a group which does not contain a free product, and X^G denote the set of functions from G to X, endowed with the natural action of G. Let $\alpha, \beta \in X^G$ such that there exists a unique $g_0 \in G$ with $\alpha(g_0) \neq \beta(g_0)$. Then $G_\alpha \neq \{1\} \Rightarrow G_\beta = \{1\}$.*

First note that the condition on G is optimal. Indeed, we can construct a counterexample whenever G contains a non-trivial free product $F = A * B$. Let $X = G$ endowed by the left-multiplication G-action, and pick $a \neq b$ in $X = G$. We denote e the neutral element of G. We let $\alpha(e) = a$, $\beta(e) = b$. Assume $w \in F \setminus \{e\}$. In the decomposition of w in the free product $A * B$, if the rightmost syllabon lies in A we let $\alpha(w) = \beta(w) = a$, and otherwise $\alpha(w) = \beta(w) = b$. Finally, let e.g. $\alpha(w) = \beta(w) = a$ for all $w \in G \setminus F$. It is easily checked that $G_\alpha \supset A$ and $G_\beta \supset B$, although there exists a unique $x = g_0 = e \in X$ such that $\alpha(x) \neq \beta(x)$.

We remark that the condition on G is closely related to the condition of not having a free subgroup of rank 2, but is not equivalent to it. Indeed, recall that a nontrivial free product $A * B$ contains the commutator subgroup (A, B) which is a free group on the set $\{aba^{-1}b^{-1} \mid a \in A \setminus \{e\}, b \in B \setminus \{e\}\}$ (see e.g. [Ro] §6.2 exercise 7). In particular, any nontrivial free product contains a free subgroup of rank 2, except for the infinite dihedral group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}$ which does not. For groups satisfying the Tits alternative, this condition can thus be translated as G being virtually solvable but not containing any infinite dihedral group.

To prove this result in its full generality, we first need a criterium for a group to be a free product, which we did not find in the literature and which has a somewhat different flavour than the more common pingpong lemma.

Proposition 2.2. *Let G be a group acting freely on a set X, $e \in G$ its neutral element, and $g_0 \in X$. Let A, B be two subgroups generating G, $K, L \subset X$ such that $K \cap L = \{g_0\}$, $K^* = K \setminus \{g_0\}$, $L^* = L \setminus \{g_0\}$. If $AK \subset K$, $BL \subset L$, $AL^* \subset L^*$ and $AK^* \subset K^*$ then $A \cap B = \{e\}$ and $G = A * B$.*

Proof. We let $A^* = A \setminus \{e\}$, $B^* = B \setminus \{e\}$. We have $A^* g_0 \subset K^*$, $B^* g_0 \subset L^*$ hence $A^* \cap B^* = \emptyset$ that is $A \cap B = \{e\}$.

Let $\varphi : A * B \twoheadrightarrow G$ be the natural morphism. We want to show that φ is injective. For this we define a map $l : A * B \to \mathbb{N} \cup \{+\infty\}$. Let $w \in A * B$. If w can be written as $X_s Y_s X_{s-1} Y_{s-1} \dots X_1 Y_1$ for some $s \geq 1$, with $X_i \in A^*$ for $i < s$, $Y_i \in B^*$ for $i \leq s$, and $X_s \in A$, we let $l(w) = s$; similarly $l(w) = s$ if $w = Y_s X_s Y_{s-1} X_{s-1} \dots Y_1 X_1$ for some $s \geq 1$, with $X_i \in A^*$ for $i \leq s$, $Y_i \in B^*$ for $i < s$, and $Y_s \in B$; finally $l(e) = +\infty$. By the existence and uniqueness of such decompositions in a free product, l is well-defined.

Assume by contradiction that $\operatorname{Ker} \varphi \neq \{e\}$. Then $s = \min l(\operatorname{Ker} \varphi) \in \mathbb{Z}_{>0}$ is reached for some $w_0 \in \operatorname{Ker} \varphi \setminus \{e\}$. Up to interchanging A and B we may assume $w_0 = X_s Y_s \dots X_1 Y_1$ with $X_i \in A^*$ for $i < s$, $Y_i \in B^*$ for $i \leq s$, and $X_s \in A$.

We let $\overline{w_0} = \varphi(w_0)$, $x_i = \varphi(X_i)$, $y_i = \varphi(Y_i)$. Note that $s \geq 2$ otherwise $y_1 = x_1^{-1} \in A \cap B = \{e\}$ contradicting $y_1 \neq e$. We have $x_s y_s \dots x_1 y_1 . g_0$ hence $y_s x_{s-1} y_{s-1} \dots x_1 y_1 . g_0 = x_s^{-1} . g_0$. If $x_s = e$, it follows that $y_s x_{s-1} y_{s-1} \dots x_1 y_1 = e$ hence $x_{s-1} y_{s-1} \dots x_1 (y_1 y_s^{-1}) = e$; then $w_1 = X_{s-1} Y_{s-1} \dots X_1 (Y_1 Y_s^{-1}) \in \operatorname{Ker} \varphi \setminus \{e\}$, and $l(w_1) = s - 1 < s$, a contradiction.

We thus have $x_s \in A^*$, hence $x_s^{-1} . g_0 \in K^*$. We prove by induction that the element $y_r x_{r-1} y_{r-1} \dots x_1 y_1 . g_0$ lies in L^* for $1 \leq r \leq s$. The case $r = 1$ is a consequence of $B^* g_0 \subset B^* L \subset L^*$. Assuming the assertion proved for r, if $r+1 \leq s$ we let $u = y_{r-1} x_{r-2} y_{r-2} \dots x_1 y_1 . g_0 \in L^*$. Since $A^* L^* \subset L^*$ we have $x_{r-1} . u \in L^*$, and $y_r x_{r-1} . u \in L$ since $BL \subset L$. If $y_r x_{r-1} . u = g_0$ we would have $y_r x_{r-1} y_{r-1} \dots x_1 y_1 = e$ hence $x_{r-1} y_{r-1} \dots x_1 (y_1 y_r^{-1})$ contradicting once again the minimality of s. It follows that $y_r x_{r-1} . u \in L^*$ and we conclude by induction.

In particular, for $r = s$ we proved that $x_s^{-1} . g_0 \in K^* \cap L^* = \emptyset$, a contradiction. It follows that φ is injective and $G = A * B$.

□

We can now prove the main result of this section.

Proof. By contradiction we assume $G_\alpha \neq \{1\}$ and $G_\beta \neq \{1\}$. Let $a = \alpha(g_0)$, $b = \beta(g_0)$, $K = \alpha^{-1}(\{a\})$, $K^* = \beta^{-1}(\{a\}) = K \setminus \{g_0\}$ and similarly $L = \beta^{-1}(\{b\})$, $L^* = \alpha^{-1}(\{b\}) = L \setminus \{g_0\}$. It is clear that $G_\alpha K \subset K$, $G_\beta L \subset L$ and $K \cap L = \{g_0\}$.

We claim that $G_\beta K^* \subset K^*$. Indeed, let $u \neq g_0$ in K and $g \in G_\beta$. Then $\alpha(u) = \beta(u)$ because $u \neq g_0$ and $\beta(u) = \beta(gu)$ because $g \in G_\beta$. Now, if $gu \notin K$ then $gu \neq g_0$ hence $\beta(gu) = \alpha(gu)$ whence $\alpha(gu) = \alpha(u)$ and $gu \in K$, a contradiction. It follows that $gu \in K$. Moreover, $gu = g_0$ would imply that $u = g^{-1} g_0$ would satisfy both $\beta(u) = \beta(g^{-1} g_0) = \beta(g_0)$, since

$g \in G_\beta$, and $\beta(u) = \alpha(u)$ because $u \neq g_0$, hence $\beta(u) = \alpha(g_0)$, contradicting $\alpha(g_0) \neq \beta(g_0)$. The claim follows.

In the same way, $G_\alpha L^* \subset L^*$. By the criterium above it follows that the subgroup of G generated by G_α and G_β is the free product of both, contradicting the assumption on G. $\square$

This result will be applied here only for a commutative group G, in which case the proof does not need the criterium above. Indeed, taking $g \in G_\alpha \setminus \{1\}$ and $g' \in G_\beta \setminus \{1\}$, we have $g'gg_0 \in K^*$ and $gg'g_0 \in L^*$, a contradiction since $gg' = g'g$ and $K^* \cap L^* = \emptyset$.

3. Representations and necklaces

For $m \geq 2$, $d, e \geq 1$ such that $m = de$ we let

$$G_m = \bigsqcup_{r=0}^{\infty} \operatorname{Irr} G(m, 1, r), \qquad G_{d,e} = \bigsqcup_{r=0}^{\infty} \operatorname{Irr} G(de, e, r)$$

and let $L : G_m \to \mathbb{N} = \mathbb{Z}_{\geq 0}$ be the map $\rho \in \operatorname{Irr} G(m, 1, r) \mapsto r$. Similarly and by abuse of notation we also denote $L : G_{d,e} \to \mathbb{N}$ the map $\rho \mapsto r$.

Let $\Gamma = \mathbb{Z}/m\mathbb{Z}$, $\Gamma' = d\Gamma$, $E_m = \{X \to Y\}$ where $X = \Gamma$, viewed as a simply transitive Γ-set, and Y is the set of all partitions. There is a natural action (on the left) of Γ on E_m, given by $(\gamma.c)(x) = c(\gamma^{-1}.x)$. For $c \in E_m$ we let $\operatorname{Aut}(c) \subset \Gamma$ denote the stabilizer of c in Γ. There is a natural encoding of G_m by m-tuples of partitions (see e.g. [Ze]), hence a natural bijective map $\Phi : G_m \to E_m$. We have a natural map $L : E_m \to \mathbb{N}$ defined by $L(c) = \sum_{x \in X} |c(x)|$ where $|\lambda|$ denotes the size of the partition λ. This abuse of notation is justified by $\Phi \circ L = L$.

Let t be a generator of $G(m, 1, 1) \simeq \mathbb{Z}/m\mathbb{Z}$. There are natural inclusions $G(m, 1, r) \subset G(m, 1, r + 1)$ hence $t \in G(m, 1, r)$ for all $r \geq 1$. We let $t' = t^d$. The image of t' generates the cyclic quotient $G(de, 1, r)/G(de, e, r) \simeq \mathbb{Z}/e\mathbb{Z}$. Let $\zeta \in \mathbb{C}^\times$ be primitive e-th root of unity. There exists a well-defined character $\epsilon : G(de, 1, r) \to \mathbb{C}^\times$ with kernel $G(de, e, r)$ such that $\epsilon(t') = \zeta$. It is a classical fact (see e.g. [HR]) that ζ can be chosen such that, for all $\rho \in G_m$, we have $\Phi(\rho \otimes \epsilon) = \overline{d}.\Phi(\rho)$, where $\overline{d} \in \Gamma = \mathbb{Z}/m\mathbb{Z}$ is the residue of d modulo m. We let $r = L(\rho)$.

We refer e.g. to the chapter 6 of [Is] for a general account on Clifford theory. It says that, for $\rho_1, \rho_2 \in G_m$ with $L(\rho_i) = r$, the restrictions to $G(de, e, r)$ of ρ_1 and ρ_2 are isomorphic iff $\rho_2 \simeq \rho_1 \otimes \epsilon^n$ for some $n \in \mathbb{N}$, that is if $\Phi(\rho_1)$ and $\Phi(\rho_2)$ lies in the same Γ'-orbit. On the other hand, if $\rho \in G_{d,e}$ there exists $\tilde{\rho} \in G_m$ such that ρ embeds in the restriction of $\tilde{\rho}$, and two such $\tilde{\rho}$ are conjugated by some power of t' ; in particular they have the same restriction to $G(de, e, r)$ and, denoting $\overline{x}$ the image of $x \in E_m$ in E_m/Γ', it follows that there exists a well-defined map $\overline{\Phi} : G_{d,e} \to E_m/\Gamma'$ which sends ρ to $\overline{\Phi(\tilde{\rho})}$. Moreover, the preimage of $\overline{c} \in E_m/\Gamma'$ by $\overline{\Phi}$ has $\#\{\tilde{\rho} \otimes \epsilon^n \mid n \in \mathbb{N}\}$ elements, that is $\#\operatorname{Aut}_{\Gamma'}(c)$ elements, where $\operatorname{Aut}_{\Gamma'}(c) = \operatorname{Aut}(c) \cap \Gamma'$.

The set Y of partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ is naturally endowed with a size function $\lambda \mapsto |\lambda| = \sum \lambda_i$ and of the following usual binary relations : (non-)equality, a total (lexicographic) ordering $\leq$, and the relation $\lambda \nearrow \mu$, common in the combinatorial representation theory of the symmetric groups, which means $\forall i \; \lambda_i \leq \mu_i$ and $|\mu| = |\lambda| + 1$. In particular $\lambda \nearrow \mu$ implies $\lambda < \mu$.

The set E_m inherits from these the following binary relations :

- $\alpha \perp \beta$ if there exists exactly one $x \in X$ such that $\alpha(x) \neq \beta(x)$;
- $\alpha < \beta$ if $\forall x \; \alpha(x) \leq \beta(x)$ and $\alpha \perp \beta$;
- $\alpha \nearrow \beta$ if $\alpha \perp \beta$ and $\exists x \in X \quad \alpha(x) \nearrow \beta(x)$.

Note that these relations are listed from the coarser to the thiner, that the first one is symmetric and that $<$ is not a strict ordering. In general, for an arbitrary Γ-set X and $E = \{X \to Y\}$, with Γ acting freely on X, the set of necklaces E/Γ will be said to have *ordered pearls* if Y is given a total ordering, and *rough pearls* otherwise. The corresponding combinatorics is dealt with in section 7 for the former case, in section 6 for the latter. The relation $\perp$ is always available, while the relation $<$ needs ordered pearls.

Let $W_r = G(de, e, r)$, $\widetilde{W_r} = G(de, 1, r)$. A combinatorial description of the branching rule for the pair $(\widetilde{W_r}, \widetilde{W_{r+1}})$ is

$$\mathrm{Res}_{\widetilde{W_r}}^{\widetilde{W_{r+1}}} \rho = \bigoplus_{\Phi(\psi) \nearrow \Phi(\rho)} \psi = \bigoplus_{\alpha \nearrow \Phi(\rho)} \Phi^{-1}(\alpha) \tag{1}$$

for $\rho \in \mathrm{Irr}(\widetilde{W_{r+1}})$ (see [Ze, p. 104].) By Clifford theory and the discussion above, a combinatorial description of the branching rule for the pairs $(W_r, \widetilde{W_r})$ is given by

$$\mathrm{Res}_{W_r}^{\widetilde{W_r}} \rho = \bigoplus_{\varphi \in \overline{\Phi}^{-1}(\overline{\Phi(\rho)})} \varphi \tag{2}$$

We say that a representation $\rho \in \mathrm{Irr}(W_r)$ *extends to* $\widetilde{W_r}$ if there exists $\tilde{\rho} \in \mathrm{Irr}(\widetilde{W_r})$ such that $\rho = \mathrm{Res}_{W_r}^{\widetilde{W_r}} \tilde{\rho}$.

Proposition 3.1. *If $\rho_1 \in \mathrm{Irr}(W_{r+1})$ does not extend to $\widetilde{W_{r+1}}$ then any $\rho_2 \in \mathrm{Irr}(W_r)$ such that $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2) \neq 0$ extends to $\widetilde{W_r}$. Conversely, if for $\rho_1 \in \mathrm{Irr}(W_{r+1})$ there exists some $\rho_2 \in \mathrm{Irr}(W_r)$ not extending to $\widetilde{W_r}$ such that ρ_2 is an irreducible component of $\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1$, then ρ_1 extends to $\widetilde{W_{r+1}}$.*

Proof. Let $\tilde{\rho}_1 \in \mathrm{Irr}(\widetilde{W_{r+1}})$ such that ρ_1 is an irreducible component of $\mathrm{Res}_{W_{r+1}}^{\widetilde{W_{r+1}}} \tilde{\rho}_1$ and $c_1 = \Phi(\tilde{\rho}_1)$. If $\rho_2 \in \mathrm{Irr}(W_r)$ is such that $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2) \neq 0$, with $\overline{\Phi}(\rho_2) = \overline{c}_2$ for some $c_2 \in E_m$, then

$$0 \neq (\mathrm{Res}_{W_r}^{\widetilde{W_{r+1}}} \tilde{\rho}_1 | \rho_2) = (\mathrm{Res}_{W_r}^{\widetilde{W_r}} \mathrm{Res}_{\widetilde{W_r}}^{\widetilde{W_{r+1}}} \tilde{\rho}_1 | \rho_2) = (\mathrm{Res}_{\widetilde{W_r}}^{\widetilde{W_{r+1}}} \tilde{\rho}_1 | \mathrm{Ind}_{W_r}^{\widetilde{W_r}} \rho_2)$$

meaning that we can choose $c_2 \in E_m$ such that $c_2 \nearrow c_1$. The first assumption states $\mathrm{Aut}_{\Gamma'}(c_1) \neq 1$ hence $\mathrm{Aut}(c_1) \neq 1$. But then $c_2 \nearrow c_1$ implies $\mathrm{Aut}(c_2) = 1$ by proposition 2.1 whence $\mathrm{Aut}_{\Gamma'}(c_2) = 1$ and $\rho_2 = \mathrm{Res}_{W_r}^{\widetilde{W_r}} \Phi^{-1}(c_2)$.

The converse assumption states $\mathrm{Aut}_{\Gamma'}(c_2) \neq 1$ hence $\mathrm{Aut}(c_1) \neq 1$. But then $c_2 \nearrow c_1$ hence $c_1 \perp c_2$ and $\mathrm{Aut}(c_1) = 1$ by proposition 2.1 whence $\mathrm{Aut}_{\Gamma'}(c_1) = 1$ and $\rho_1 = \mathrm{Res}_{W_r}^{\widetilde{W_r}} \Phi^{-1}(c_1)$. ☐

Note that $\mathrm{Res}_{W_r}^{W_{r+1}} \mathrm{Res}_{W_{r+1}}^{\widetilde{W_{r+1}}} = \mathrm{Res}_{W_r}^{\widetilde{W_{r+1}}} = \mathrm{Res}_{W_r}^{\widetilde{W_r}} \mathrm{Res}_{\widetilde{W_r}}^{\widetilde{W_{r+1}}}$ hence (1) and (2) imply

$$\mathrm{Res}_{W_r}^{W_{r+1}} \mathrm{Res}_{W_{r+1}}^{\widetilde{W_{r+1}}} \tilde{\rho} = \bigoplus_{\alpha \nearrow \Phi(\tilde{\rho})} \mathrm{Res}_{W_r}^{\widetilde{W_r}} \Phi^{-1}(\alpha) = \bigoplus_{\alpha \nearrow \Phi(\rho)} \bigoplus_{\varphi \in \overline{\Phi}^{-1}(\overline{\alpha})} \varphi \tag{3}$$

Let $\tilde{\rho}_1 \in \mathrm{Irr}(\widetilde{W_{r+1}})$. Then, for all $\rho_2 \in \mathrm{Irr}(W_r)$,

$$(\mathrm{Res}_{W_r}^{\widetilde{W_{r+1}}} \tilde{\rho}_1 | \rho_2) = \sum_{\alpha \nearrow \Phi(\tilde{\rho}_1)} \sum_{\varphi \in \overline{\Phi}^{-1}(\overline{\alpha})} (\varphi | \rho_2) = \sum_{\alpha \nearrow \Phi(\tilde{\rho}_1)} (\alpha \in \overline{\Phi}(\rho_2)) = \#\{\alpha \in \overline{\Phi}(\rho_2) | \alpha \nearrow \Phi(\tilde{\rho}_1)\}$$

In particular, if $\rho_1 = \mathrm{Res}_{W_{r+1}}^{\widetilde{W}_{r+1}} \tilde{\rho}_1$ is irreducible, we have

$$(\mathrm{Res}_{W_r}^{W_{r+1}} \tilde{\rho}_1 | \rho_2) = \#\{\alpha \in \overline{\Phi}(\rho_2) \mid \alpha \nearrow \Phi(\tilde{\rho}_1)\}.$$

Otherwise, by the previous proposition we know that $\rho_2 = \mathrm{Res}_{W_r}^{\widetilde{W}_r} \tilde{\rho}_2$ for some irreducible $\tilde{\rho}_2 \in \mathrm{Irr}(\widetilde{W}_r)$. We let $c_2 = \Phi(\tilde{\rho}_2)$. Let $\tilde{\rho}_1 \in \mathrm{Irr}(\widetilde{W}_{r+1})$ such that ρ_1 is an irreducible component of $\mathrm{Res}_{W_{r+1}}^{\widetilde{W}_{r+1}} \tilde{\rho}_1$ and $c_1 = \Phi(\tilde{\rho}_1)$. We have

$$\mathrm{Res}_{W_{r+1}}^{\widetilde{W}_{r+1}} \tilde{\rho}_1 = \rho_1^{(1)} + \cdots + \rho_1^{(s)}$$

with $\rho_1^{(1)} = \rho_1$, $s = \#\mathrm{Aut}_{\Gamma'}(c_1)$. For $x \in \widetilde{W}_{r+1}$ we let $\mathrm{Ad}(x) : \widetilde{W}_{r+1} \to \widetilde{W}_{r+1}$ denote the conjugation $g \mapsto xgx^{-1}$, or more properly the automorphism it induces on the normal subgroup W_{r+1}. For all $1 \leq i,j \leq s$, there exists u such that $\rho_1^{(j)} = \rho_1^{(i)} \circ \mathrm{Ad}(t^u)$. On the other hand, $\rho_2 = \rho_2 \circ \mathrm{Ad}(t)$ hence

$$(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1^{(j)} | \rho_2) = (\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1^{(i)} \circ \mathrm{Ad}(t^u) | \rho_2) = (\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1^{(i)} | \rho_2 \circ \mathrm{Ad}(t^{-u})) = (\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1^{(i)} | \rho_2).$$

It follows that $(\mathrm{Res}_{W_r}^{\widetilde{W}_{r+1}} \tilde{\rho}_1 | \rho_2) = s(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2)$. We thus proved the following.

Proposition 3.2. *If $\rho_1 \in \mathrm{Irr}(W_{r+1})$, $\rho_2 \in \mathrm{Irr}(W_r)$ with $\overline{\Phi}(\rho_1) = \overline{c_1}$, then*

$$\left(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2\right) = \frac{\#\{\alpha \in \overline{\Phi}(\rho_2) \mid \alpha \nearrow c_1\}}{\#\mathrm{Aut}_{\Gamma'}(c_1)}$$

We are now ready to prove the main theorem, using combinatorial results to be proved in the sequel.

Theorem 3.3. *Let $\rho_1 \in \mathrm{Irr}(W_{r+1})$, $\rho_2 \in \mathrm{Irr}(W_r)$. Then $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2) \leq 2$. Moreover, if $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2) = 2$ then ρ_1 extends to $\widetilde{W}_{r+1}$.*

Proof. Let $c_1 \in E_m$ chosen such that $\overline{c_1} = \overline{\Phi}(\rho_1)$. If $\mathrm{Aut}_{\Gamma'}(c_1) = 1$, we have to prove $\#\{\alpha \in \overline{\Phi}(\rho_2) \mid \alpha \nearrow c_1\} \leq 2$, which is a consequence of proposition 6.2. We thus assume $\#\mathrm{Aut}_{\Gamma'}(c_1) \neq 1$. Let $\{\alpha_1, \ldots, \alpha_r\} = \{\alpha \in \overline{\Phi}(\rho_2) \mid \alpha \nearrow c_1\}$. Since $\overline{\Phi}(\rho_2)$ is a Γ'-orbit we have well-defined and distincts $\gamma_i \in \Gamma' \setminus \{1\}$ for $2 \leq i \leq r$ such that $\alpha_i = \gamma_i.\alpha_1$. By lemma 6.1, for all i we have $\gamma_i \in \mathrm{Aut}(c_1)$ hence $\gamma_i \in \mathrm{Aut}_{\Gamma'}(c_1) = \mathrm{Aut}(c_1) \cap \Gamma'$. Thus $\#\mathrm{Aut}_{\Gamma'}(c_1) \geq \#\{\alpha \in \overline{\Phi}(\rho_2) \mid \alpha \nearrow c_1\}$ and $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho_1 | \rho_2) \leq 1$. □

Proposition 3.4. *Let $\rho \in \mathrm{Irr}(W_{r+1})$. If $\rho_1, \ldots, \rho_s \in \mathrm{Irr}(W_r)$ do not extend to $\widetilde{W}_r$ and satisfy that, for all i, ρ_i is an irreducible component of $\mathrm{Res}_{W_r}^{W_{r+1}} \rho$, then for all i we have $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho | \rho_i) = 1$ and there exists $\tilde{\rho}_0 \in \mathrm{Irr}(\widetilde{W}_r)$ such that each ρ_i is an irreducible component of $\mathrm{Res}_{W_r}^{\widetilde{W}_r} \tilde{\rho}_0$. In particular, for all i,j there exists $g \in \widetilde{W}_r$ such that $\rho_j \simeq \rho_i \circ \mathrm{Ad}(g)$.*

Proof. Let $c \in E_m$ such that $\overline{c} = \overline{\Phi}(\rho)$. The statement is void if $s \leq 1$, hence we assume $s \geq 2$. For $i \in [1,s]$, by proposition 3.2, the fact that $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho | \rho_i) \neq 0$ implies the existence of $\alpha_i \in \overline{\Phi}(\rho_i)$ such that $\alpha_i \nearrow c$. Moreover, we have $\mathrm{Aut}_{\Gamma'}(\alpha_i) \neq 1$ hence $\mathrm{Aut}(\alpha_i) \neq 1$. Then proposition 7.1 implies $\alpha_1 = \cdots = \alpha_s$, which proves the existence of $\tilde{\rho}_0 = \Phi^{-1}(\alpha_1)$. Moreover, $\mathrm{Aut}(\alpha_i) \neq 1$ implies $\mathrm{Aut}(c) = 1$ by proposition 2.1. Then proposition 3.2 states $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho | \rho_i) = \{\alpha \in \overline{\Phi}(\rho_i) \mid \alpha \nearrow c\}$, hence $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho | \rho_i) = 1$ again by proposition 7.1. The final assertion is an immediate consequence of Clifford theory. □

We may wonder for which $\rho \in \mathrm{Irr}(W_{r+1})$ there exists several $\rho_1, \dots, \rho_s \in \mathrm{Irr}(W_r)$ such that $(\mathrm{Res}_{W_r}^{W_{r+1}} \rho | \rho_i) = 2$. By theorem 3.3, ρ extends to some $\tilde{\rho} \in \mathrm{Irr}(\widetilde{W}_{r+1})$ and by proposition 3.4 each ρ_i extends to some $\tilde{\rho}_i \in \mathrm{Irr}(\widetilde{W}_r)$. Let $\alpha_i \in \Phi(\tilde{\rho}_i)$, $c = \Phi(\tilde{\rho})$. We have $\mathrm{Aut}_{\Gamma'}(\alpha_i) = \mathrm{Aut}_{\Gamma'}(c) = 1$, $\alpha_i \nearrow c$, and there exists $\gamma_i \in \Gamma' \setminus \{e\}$ such that $\gamma_i.\alpha_i \nearrow c$ with $r \geq 2$. Then the possible shapes of c are implicitely given by proposition 7.3.

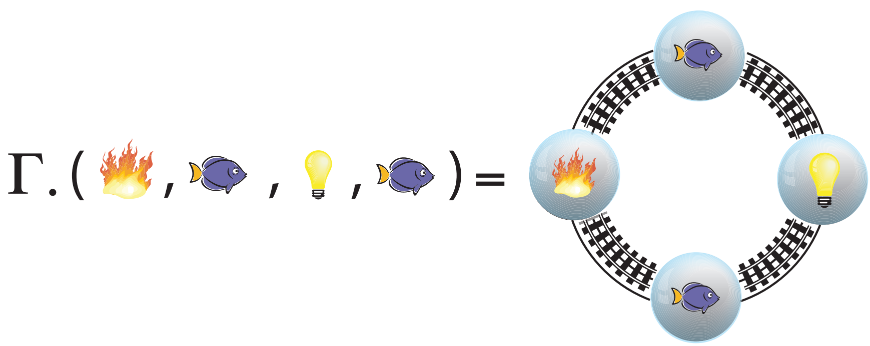
They are most easily described when $d = 1$, that is $\Gamma = \Gamma'$ and $W = G(e, e, r+1)$. Choose some nontrivial subgroup $\Gamma_0 = m\Gamma$ of Γ, and define $c : \Gamma \to Y$ as follows. Subdivide $\Gamma = \Gamma_0 \sqcup (x_1 + \Gamma_0) \sqcup \dots \sqcup (x_{m-1} + \Gamma_0)$ in Γ_0-cosets, pick one $\lambda_i \in Y$ for each $i \in [1, m-1]$ and put $c(x) = \lambda_i$ for $x \in x_i + \Gamma_0$; choose $\lambda_0, \mu_0 \in Y$ such that $\mu_0 \nearrow \lambda_0 \in Y$ and define $c(0) = \mu_0$, $c(x) = \lambda_0$ for $x \in \Gamma_0 \setminus \{0\}$. If $u = e/m = |\Gamma_0|$, it is easily checked that $\mathrm{Aut}(c) = 1$ and that the restriction of $\rho = \mathrm{Res}_{W_{r+1}}^{\widetilde{W}_{r+1}} \Phi^{-1}(c)$ to W_r admits at least $\llcorner \frac{u-1}{2} \lrcorner$ components of multiplicity 2. Proposition 7.3 and its corollary state that all $\rho \in \mathrm{Irr}(W)$ whose restriction to W_r contains at least 2 components with multiplicity 2 are obtained in this way, with $u \geq 5$.

4. A basic lemma on necklaces

Here we let X be a finite set acted upon *freely* by a group Γ, Y be a set containing at least 2 elements, and $E = \{X \to Y\}$ be the set of maps from X to Y. There is a natural action of Γ on E. We recall that for $\alpha, c \in E$ the notation $\alpha \perp c$ means that there exists exactly one $x \in X$ such that $\alpha(x) \neq c(x)$, and we denote $\mathrm{Aut}(c) \subset \Gamma$ the stabilizer of c under the action of Γ.

When Γ is cyclic of order n and $\gamma \in \Gamma$, we also introduce the following notation. Let $v \in X$ and $u \in \Gamma.v$. If $r \in [0, n-1]$ is defined by $u = \gamma^r.v$, then we let $[v, u]_\gamma = \{\gamma^k.v \mid k \in [0, r]\}$. The companion notations $]v, u]_\gamma$, $[v, u[_\gamma$, $]v, u[_\gamma$ are self-explaining.

The set Y has to be thought of as a set of (types of) pearls, each decorated by some ornament ; accordingly, when Γ is cyclic, E/Γ can be thought of as a necklace made out of the pearls in Y. We illustrate this by the picture below in the case $\Gamma = X = \mathbb{Z}/4\mathbb{Z}$; then E is the set of 4-tuples of elements (pearls) in Y, and orbits in E/Γ can be illustrated as follows.



In order to clearly distinguish elements of the Γ-set X from elements of the set Y we use ornamental symbols $\spadesuit, \heartsuit, \diamondsuit, \clubsuit$ for elements of Y in the proofs.

The following technical lemma is basic for our purposes.

Lemma 4.1. *Assume that Γ is cyclic with generator γ and acts freely on X. Let $c \in E$. The following are equivalent*

(i) $\exists \alpha, \beta \in E$ *such that* $\alpha \neq \beta$, $\alpha \perp c$, $\beta \perp c$ *and* $\beta = \gamma.\alpha$
(ii) $\exists \mathcal{O} \in X/\Gamma$ *such that*
  (a) c *is constant on each* $P \neq \mathcal{O}$ *in* X/Γ.
  (b) *there exists* $u, v \in \mathcal{O}$ *such that* $u \neq v$ *and* c *is constant on both* $[v, u]_\gamma$ *and its complement in* $\mathcal{O}$.

Under these assumptions, we have $\gamma \in \mathrm{Aut}(c) \Leftrightarrow |c(\mathcal{O})| = 1 \Leftrightarrow v = \gamma.u$. *Moreover,* u *and* v *are characterized in* X *by* $\alpha(u) \neq c(u)$ *and* $\beta(v) \neq c(v)$. *Finally we have* $|\alpha(\mathcal{O})| = |\beta(\mathcal{O})| = 2$.

Proof. *(ii)* $\Rightarrow$ *(i)*. (b) implies $|c(\mathcal{O})| \leq 2$. We let $\clubsuit \in Y$ such that $c(\mathcal{O}) = \{c(u), \clubsuit\}$ if $|c(\mathcal{O})| = 2$, and let $\clubsuit$ be an arbitrarily chosen element of $Y \setminus c(\mathcal{O})$ otherwise, using $|Y| \geq 2$. We have to define α and β fulfilling *(i)*. We define $\alpha(u) = \clubsuit$, $\alpha(x) = c(x)$ for $x \neq u$ and $\beta(v) = \clubsuit$, $\beta(x) = c(x)$ for $x \neq v$. We have $\alpha(u) \neq c(u)$ hence $\alpha(u) \neq \beta(u) = c(u)$ because $u \neq v$. It follows that $\alpha \neq \beta$, and clearly $\alpha \perp c$, $\beta \perp c$. It remains to show that $\beta = \gamma.\alpha$.

Let $x \in X$. If $\Gamma.x \neq \mathcal{O}$ then $c(\gamma.x) = c(x)$ by (a) hence $\beta(\gamma.x) = c(\gamma.x) = c(x) = \alpha(x)$ since $x, \gamma.x \in \Gamma.x$ and $u, v \notin \Gamma.x$. Now (b) tells us that β equals $c(u)$ on $[u, \gamma^{-1}.v]_\gamma$, $\clubsuit$ on its complement in $\mathcal{O}$, and α equals $c(u)$ on $[\gamma.u, v]_\gamma = \gamma.[u, \gamma^{-1}.v]_\gamma$, $\clubsuit$ on its complement. It follows that $\beta(\gamma.x) = \alpha(x)$ also for $x \in \mathcal{O}$, hence $\beta = \gamma.\alpha$.

(i) $\Rightarrow$ *(ii)*. Since $\alpha \perp c$ and $\beta \perp c$ there exist well-defined $u, v \in X$ such that $\alpha(u) \neq c(u)$ and $\beta(v) \neq c(v)$. Let $\clubsuit = \alpha(u)$ and $e = |c^{-1}(\clubsuit)|$. If $u = v$, $\alpha \neq \beta$ implies $\beta(u) \neq \clubsuit$ hence $|\beta^{-1}(\clubsuit)| = e$, but $|\beta^{-1}(\clubsuit)| = |\alpha^{-1}(\clubsuit)| = e + 1$, since $\beta = \gamma.\alpha$. It follows that $u \neq v$ and $|\beta^{-1}(\clubsuit)| = e + 1$, hence $\beta(v) = \clubsuit$. Similarly, considering $\spadesuit = c(u) \neq \clubsuit$ and $f = |c^{-1}(\spadesuit)|$, we get $c(u) = \beta(u) = \alpha(v) = c(v) = \spadesuit$, since $f - 1 = |\alpha^{-1}(\spadesuit)| = |\beta^{-1}(\spadesuit)|$.

We let $\mathcal{O} = \Gamma.u \in X/\Gamma$, and $n = |\Gamma|$. We will prove first that $v \in \mathcal{O}$, and then prove (b). Note that (b) is equivalent to

$$\beta(y) = \begin{cases} c(u) & \text{if } y \in [u, \gamma^{-1}.v]_\gamma \\ \clubsuit & \text{otherwise.} \end{cases} \quad \alpha(y) = \begin{cases} c(u) & \text{if } y \in [\gamma.u, v]_\gamma \\ \clubsuit & \text{otherwise.} \end{cases}$$

and that these two equalities are deduced one from the other through $\beta = \gamma.\alpha$.

We assume by contradiction that $v \notin \mathcal{O}$. Then by induction we have $\beta(\gamma^r.u) = \alpha(u)$ for all $r \in [1, n]$. Indeed, $\beta = \gamma.\alpha$ proves the case $r = 1$, and also implies $\beta(\gamma^{r+1}.u) = \alpha(\gamma^r.u)$; then $r < n$ implies $\gamma^r.u \neq u$ hence $\alpha(\gamma^r.u) = c(\gamma^r.u)$; $v \notin \mathcal{O}$ implies $c(\gamma^r.u) = \beta(\gamma^r.u)$, which equals $\alpha(u)$ by the induction hypothesis. This yields the contradiction $c(u) = \beta(u) = \beta(\gamma^n.u) = \alpha(u)$.

We thus know $v \in \mathcal{O}$, hence $u = \gamma^m.v$ for some $m \in [1, n-1]$. For $0 \leq r \leq m - 1$ we have $\gamma^{r+1}.v \neq v$ and $\gamma^r.v \neq u$ hence $c(\gamma^{r+1}.v) = \beta(\gamma^{r+1}.v) = \alpha(\gamma^r.v) = c(\gamma^r.v) = c(\gamma^r.v)$ meaning that c is constant on $[v, u]_\gamma$. Similarly, if $v = \gamma^m.u$ for some $m \in [1, n-1]$, then for $1 \leq r \leq m - 2$ we have $\gamma^r.u \neq u$ and $\gamma^{r+1}.u \neq v$, whence $c(\gamma^{r+1}.u) = \beta(\gamma^{r+1}.u) = \alpha(\gamma^r.u) = c(\gamma^r.u)$ and c is constant on the complement of $[v, u]_\gamma$ in $\mathcal{O}$, which proves (b).

Let now $P \in X/\Gamma$ with $P \neq \mathcal{O}$. Since $u, v \notin P$ and $\beta = \gamma.\alpha$ we have $c(\gamma.x) = \beta(\gamma.x) = \alpha(x) = c(x)$ for all $x \in P$, which proves (a).

The proof that $\gamma \in \mathrm{Aut}(c) \Leftrightarrow |c(\mathcal{O})| = 1 \Leftrightarrow v = \gamma.u$ is straightforward. Finally, we show that $|\alpha(\mathcal{O})| = |\beta(\mathcal{O})| = 2$. If $|c(\mathcal{O})| = 1$ we have $|\alpha(\mathcal{O})| = |\beta(\mathcal{O})| = |\{\beta(v), c(v)\}| = 2$. We thus can assume $|c(\mathcal{O})| = 2$, which implies $v \neq \gamma.u$. Assume by contradiction that $\beta(v) = \heartsuit \notin c(\mathcal{O})$. We have $\alpha(u) = \beta(v) = \heartsuit$ and, for $x \in \mathcal{O}$, $x = v \Leftrightarrow \beta(x) = \heartsuit$ and

$x = u \Leftrightarrow \alpha(x) = \heartsuit$. Then $\beta(\gamma.u) = \alpha(u) = \heartsuit$ implies $\gamma.u = v$ which has been excluded. Thus $\beta(v) \in c(\mathcal{O})$ and $|\alpha(\mathcal{O})| = |\beta(\mathcal{O})| = |c(\mathcal{O})| = 2$.

□

5. Preliminaries on cyclic groups

Lemma 5.1. *Let Γ be a cyclic group acting freely and transitively on a finite set X. Let Γ_1, Γ_2 be subgroups of Γ such that $\Gamma = \Gamma_1\Gamma_2$. For all $(P,Q) \in X/\Gamma_1 \times X/\Gamma_2$ we have $P \cap Q \neq \emptyset$.*

Proof. Let $n = |\Gamma|$ and $n_i = |\Gamma_i|$ for $i = 1, 2$. Since Γ is cyclic we have $|\Gamma_1 \cap \Gamma_2| = \gcd(n_1, n_2)$ and the assumption $\Gamma = \Gamma_1\Gamma_2$ means $|\Gamma| = \mathrm{lcm}(n_1, n_2)$. If $(P,Q) \in X/\Gamma_1 \times X/\Gamma_2$ satisfies $P \cap Q \neq \emptyset$, then $\Gamma_1 \cap \Gamma_2$ acts freely on $P \cap Q$. Moreover, if $x, y \in P \cap Q$, we know that there exists $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ such that $y = \gamma_1.x = \gamma_2.x$, hence $\gamma_2^{-1}\gamma_1.x = x$ and $\gamma_1 = \gamma_2 \in \Gamma_1 \cap \Gamma_2$ because Γ acts freely on X. It follows that $\Gamma_1 \cap \Gamma_2$ acts freely and transitively on $P \cap Q$ hence $|P \cap Q| = |\Gamma_1 \cap \Gamma_2| = \gcd(n_1, n_2)$. Now let $P \in X/\Gamma_1$. It is the disjoint union of the $P \cap Q$ for $Q \in X/\Gamma_2$, hence

$$n_1 = |P| = \sum_{Q \in X/\Gamma_2} |P \cap Q| \leqslant \gcd(n_1,n_2)|X/\Gamma_2| \leqslant \frac{\gcd(n_1,n_2)\mathrm{lcm}(n_1,n_2)}{n_2} = n_1$$

and $\sum_{Q \in X/\Gamma_2} |P \cap Q| = \gcd(n_1, n_2)|X/\Gamma_2|$ hence $|P \cap Q| \neq 0$ for all $Q \in X/\Gamma_2$. □

We define Aff_n to be the group of bijective affine functions from $\mathbb{Z}/n\mathbb{Z}$ to itself :

$$\mathrm{Aff}_n = \{\varphi \in \mathrm{Bij}(\mathbb{Z}/n\mathbb{Z}) \mid \exists \alpha, \beta \in \mathbb{Z}/n\mathbb{Z} \;\; \forall x \in \mathbb{Z}/n\mathbb{Z} \;\; \varphi(x) = \alpha + \beta x\}.$$

We will use the following lemma.

Lemma 5.2. *Let $n \geq 3$, $0 \leq m \leq n-2$ and $I_m = \{\overline{0}, \overline{1}, \dots, \overline{m}\} \subset \mathbb{Z}/n\mathbb{Z}$. Let $\varphi \in \mathrm{Aff}_n$ such that $\varphi(I_m) \subset I_m$. Then :*

1. *If $1 \leq m \leq n-3$ then $\varphi = \mathrm{Id}$ or $\forall x \in \mathbb{Z}/n\mathbb{Z} \;\; \varphi(x) = m - x$.*
2. *If $m = 0$ there exists $r \in [0, n-1]$ with $\gcd(r,n) = 1$ such that $\forall x \in \mathbb{Z}/n\mathbb{Z} \;\; \varphi(x) = rx$.*
3. *If $m = n-2$ there exists $r \in [0, n-1]$ with $\gcd(r,n) = 1$ such that $\forall x \in \mathbb{Z}/n\mathbb{Z} \;\; \varphi(x) = r - 1 + rx$.*

Proof. Since φ is injective we know that $\varphi(I_m) = I_m$. Let $a \in [0, n-1]$ and $r \in (\mathbb{Z}/n\mathbb{Z})^\times$ such that $\varphi(x) = \overline{a} + rx$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. If $m = 0$ then $\varphi(\overline{0}) = \overline{0}$ hence $a = 0$ and the conclusion follows. If $m = n-2$ then $\{-\overline{1}\} = (\mathbb{Z}/n\mathbb{Z}) \setminus I_m$ hence $\varphi(-\overline{1}) = -\overline{1}$ that is $\overline{a} = r - \overline{1}$ and the conclusion follows.

We thus can restrict ourselves to assumption (1). Assume for now that $m < n - m$. Let $\Delta : I_m \times I_m \to \mathbb{Z}/n\mathbb{Z}$ be defined by $\Delta(x, y) = x - y$. The set $\Delta(I_m \times I_m) = \{-\overline{m}, \dots, -\overline{1}, \overline{0}, \overline{1}, \dots, \overline{m}\}$ has cardinality $2m+1 \leq n$. Let Φ be the restriction of $\varphi \times \varphi$ to $I_m \times I_m$. This is a bijection of $I_m \times I_m$. We have $|\Delta^{-1}(y)| = m$ if and only if $y \in \{-\overline{1}, \overline{1}\}$. Since $|(\Delta \circ \Phi)^{-1}(\overline{1})| = |\Delta^{-1}(\overline{1})|$ by bijectivity of Φ and $(\Delta \circ \Phi)^{-1}(\overline{1}) = \Delta^{-1}(r^{-1})$ by direct calculation, it follows that $r \in \{-\overline{1}, \overline{1}\}$. If $r = \overline{1}$ then $\varphi(x) = \overline{a} + x$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. Consider in that case the iterated maps φ^j of φ for $j \in \mathbb{N}$. These induce bijections of I_m. If $a \neq 0$ there would exist $j \in \mathbb{Z}_{>0}$ such that $ja > m$ and $(j-1)a \leq m$. But $\overline{ja} = \varphi^j(\overline{0}) \in I_m$ hence $ja \geq n$ and $a = ja - (j-1)a > n - m > m$ by assumption, a contradiction since $a \in [0, m]$. It follows that $\varphi = \mathrm{Id}$. If $r = -\overline{1}$, we introduce $\psi_m \in \mathrm{Aff}_n$ defined by $\psi_m(x) = \overline{m} - x$. Then $\psi_m \circ \varphi \in \mathrm{Aff}_n$ sends I_m into itself and $\psi_m \circ \varphi(x) = \overline{m} - \overline{a} + x$, hence $\psi_m \circ \varphi = \mathrm{Id}$ by the above discussion, $\overline{m} = \overline{a}$ and $a = m$ hence $\varphi = \psi_m$ since $\psi_m^2 = \mathrm{Id}$.

Now assume $m \geq n-m$. Let $S \in \mathrm{Aff}_n$ defined by $S(x) = -\overline{1} - x$ for all $x \in \mathbb{Z}/n\mathbb{Z}$, and $I'_m = \{\overline{m+1}, \dots, \overline{n-1}\}$. We have $\varphi(I'_m) = I'_m$. Let $\varphi' = S \circ \varphi \circ S \in \mathrm{Aff}_n$. We have $S(I_m) = I'_{n-m-2}$, $S(I'_m) = I_{n-m-2}$ hence $\varphi'(I_{n-m-2}) = I_{n-m-2}$. Moreover $1 \leq m \leq n-3$ implies $1 \leq n-m-2 \leq n-3$. It follows that $\varphi' \in \{\mathrm{Id}, \psi_{n-m-2}\}$ since $n-m-2 < n-m \leq m < m+2 = n-(n-m-2)$ and thus $\varphi \in \{\mathrm{Id}, \psi_m\}$. □

6. Necklaces with rough pearls

In this section we deal with the case where the set Y has no additional structure. We recall that Γ is cyclic and acts freely on the finite set X.

Lemma 6.1. *Let $c \in E$ such that $\mathrm{Aut}(c) \neq 1$. If there exists $\alpha \in E$ such that $\alpha \perp c$ and $\gamma \in \Gamma$ such that $\gamma.\alpha \perp c$ then $\gamma \in \mathrm{Aut}(c)$.*

Proof. Let $\beta = \gamma.\alpha$. By assumption there exists $\delta \in \mathrm{Aut}(c) \setminus \{1\}$. We assume by contradiction that $\gamma \notin \mathrm{Aut}(c)$. In particular $\gamma \neq 1$ and $\alpha \neq \beta$. Let $\Gamma_0 =< \gamma >$, $\Delta_0 =< \delta >$ and $\Gamma' =< \Gamma_0, \delta_0 >$.

Lemma 4.1 applied to $\Gamma_0 =< \gamma >$ defines an orbit $\mathcal{O}$ in X/Γ_0, and $u, v \in \mathcal{O}$ such that c is constant on each orbit in X/Γ_0 which is distinct from $\mathcal{O}$, and c is constant both on $[v,u]_\gamma$ and on its complement in $\mathcal{O}$. Also recall that u, v are characterized in X by $\alpha(u) \neq (u)$ and $\beta(v) \neq c(v)$. Since $\gamma \notin \mathrm{Aut}(c)$ we have $|c(\mathcal{O})| = 2$ and $v \neq \gamma.u$.

We let $X' = \Gamma'.v$. Obviously $\mathcal{O} \subset X'$.

Since $\Gamma' = \Gamma_0\Delta_0$ acts freely and transitively on X' we get by lemma 5.1 that $P \cap Q \neq \emptyset$ for all $(P,Q) \in X'/\Gamma_0 \times X'/\Delta_0$. Since $\delta \in \mathrm{Aut}(c)$ the map c is constant on each $Q \in X'/\Delta_0$, hence induces a map $\overline{c} : X'/\Delta_0 \to Y$. If there were $P \neq \mathcal{O}$ in X'/Γ_0 then c would be constant on P, hence $\overline{c}$ and c would be constant. This is a contradiction because c is not constant on $\mathcal{O} \subset X'$. It follows that $X'/\Gamma_0 = \{\mathcal{O}\}$ and $\Gamma' = \Gamma_0$. In particular $\delta \in \Gamma_0$ and $\delta = \gamma^r$, for some $r \in [2, n-2]$ since $\gamma \notin \Delta_0 \subset \mathrm{Aut}(c)$.

Since $X' = \mathcal{O}$ with $|c(\mathcal{O})| = 2$ and c is constant on each $Q \in \mathcal{O}/\Delta_0$, we know that $[v,u]_\gamma$ is a union of Δ_0-orbits, hence is δ-stable. Let n denote the order of γ. We identify $\mathcal{O}$ with $[0, n-1]$, v with 0, u with $m \in [1, n-2]$, γ to $\overline{1} \in \mathbb{Z}/n\mathbb{Z}$. Since $\delta.v \in [v,u]$ and $r \in [2, n-1]$ we have $r \leq m$. Let $w = \gamma^{-1}.v$, identified with $n-1$. We have $c(w) \notin c([v,u]_\gamma)$ hence $\delta.w \notin [v,u]_\gamma$. On the other hand $\delta.w$ is identified with $r-1 \geq 0$, but $r \in [2, n-2]$ implies $r-1 \geq 0$ and $r \leq m$ implies $r-1 \leq m$. It follows that $\delta.w \in [v,u]_\gamma$, a contradiction.

□

Proposition 6.2. *Let $\alpha_1, \alpha_2, \beta, c \in E$ such that $\alpha_1 \perp c$, $\alpha_2 \perp c$, $\beta \perp c$, $\beta = \gamma_1.\alpha_1 = \gamma_2.\alpha_2$ with $\gamma_1, \gamma_2 \in \Gamma \setminus \mathrm{Aut}(c)$. Then $\alpha_1 = \alpha_2$.*

Proof. We assume by contradiction that α_1, α_2 and β are all distinct. Let $\Gamma_i =< \gamma_i >$. Since $\gamma_1, \gamma_2 \notin \mathrm{Aut}(c)$, lemma 4.1 provides two special orbits $\mathcal{O}_1, \mathcal{O}_2$ and $u_1, u_2, v_1, v_2 \in X$ with $u_1 \neq u_2$. Since v_1, v_2 are characterized in X by $\beta(v_i) \neq c(v_i)$ we have $v_1 = v_2 = v$.

We first rule out the possibility that $\Gamma_1 = \Gamma_2$. In that case, let $\Gamma_0 = \Gamma_1 = \Gamma_2$. We have $\gamma_2 = \gamma_1^r$ for some r prime to $n = |\Gamma_0|$, and $\mathcal{O}_1 = \mathcal{O}_2 = \Gamma_0.v = \mathcal{O}$ can be identified with $\mathbb{Z}/n\mathbb{Z}$, v with $\overline{0}$, γ_1 with $\overline{1} \in \mathbb{Z}/n\mathbb{Z}$. Let $\varphi : x \mapsto rx$ in Aff_n. We have $[v,u_1]_{\gamma_1} = [v,u_2]_{\gamma_2}$, which means that φ preserves $I_m \subset \mathbb{Z}/n\mathbb{Z}$ where u_1 is identified with $\overline{m}$ for some $m \in [1, n-2]$ (recall that $|c(\mathcal{O})| = 2$ hence $[v,u_1]_{\gamma_1} \neq \mathcal{O}$). Since in addition $\varphi(\overline{0}) = \overline{0}$, lemma 5.2 implies $\varphi = \mathrm{Id}$ meaning $\gamma_1 = \gamma_2$ and $\alpha_1 = \alpha_2$, a contradiction.

Let $\Gamma' = \Gamma_1\Gamma_2$, $X' = \Gamma'.v$ and c' the restriction of c to X'. We have $\mathcal{O}_1, \mathcal{O}_2 \subset X'$. Assume that $X' \neq \mathcal{O}_1$ and $X' \neq \mathcal{O}_2$, or equivalently $\Gamma' \neq \Gamma_1$, $\Gamma' \neq \Gamma_2$. Then there exists $P \neq \mathcal{O}_1$ in

X'/Γ_1 and $Q \neq \mathcal{O}_2$ in X'/Γ_2. Since P intersects each element of X'/Γ_2 and Q intersects each element of X'/Γ_1 by lemma 5.1, we get that c' is constant on $X' \setminus \mathcal{O}_1 \cap \mathcal{O}_2$. Let $\heartsuit \in Y$ be the value it takes. Since $\gamma_i \notin \mathrm{Aut}(c)$ we have $|c(\mathcal{O}_i)| = 2$. We know that $c(X' \setminus \mathcal{O}_1) = \{\heartsuit\}$. On the other hand, $\mathcal{O}_1 \cap \mathcal{O}_2 \subsetneq \mathcal{O}_1$ otherwise $\mathcal{O}_1 \subset \mathcal{O}_2$, in particular $\gamma_1.v \in \mathcal{O}_2$ and $\gamma_1 \in \Gamma_2$ by the freeness assumption, hence $\Gamma_1 \subset \Gamma_2$ contradicting $\Gamma_2 \neq \Gamma'$. It follows that $\heartsuit \in c(\mathcal{O}_1)$ and $c(X') = c(\mathcal{O}_1) = c(\mathcal{O}_2) = \{\heartsuit, \clubsuit\}$ for some $\clubsuit \neq \heartsuit$. We claim that there exists only one $x \in X'$ such that $c(x) = \clubsuit$. By contradiction assume otherwise. These elements belong to $\mathcal{O}_1$, hence by lemma 4.1 they belong either to $[v, u_1]_{\gamma_1}$ or to its complement $]u_1, v[_{\gamma_1}$. If there are at least two of them, we then have some $x \in \mathcal{O}_1$ such that $c(\gamma_1.x) = c(x) = \clubsuit$. But then $x, \gamma_1.x \in \mathcal{O}_1 \cap \mathcal{O}_2 \subset \mathcal{O}_2$ hence $\gamma_1 \in \Gamma_2$ by the freeness assumption and $\Gamma_1 \subset \Gamma_2$, a contradiction. Let then x denote the only element in X' satisfying $c(x) = \clubsuit$. Since $c(v) = c(u_1)$ and $v \neq u_1$ we have $c(v) = c(u_1) = \heartsuit$. Likewise, $c(u_2) = \heartsuit$. This implies $\gamma_1.x = v$ and $\gamma_2.x = v$, hence $\gamma_1 = \gamma_2$, a contradiction.

It follows that $\Gamma_1 \subset \Gamma_2$ or $\Gamma_2 \subset \Gamma_1$. By symmetry we may assume $\Gamma_1 \subset \Gamma_2$, that is $< \gamma_1 > = < \gamma_2^r >$ for some $r \in [1, n-1]$ dividing the order n of Γ_2, and $r \geq 2$ since $\Gamma_1 \neq \Gamma_2$ (of course we do not necessarily have $\gamma_1 = \gamma_2^r$). We identify $\mathcal{O}_2$ and Γ_2 with $\mathbb{Z}/n\mathbb{Z}$, v with $\overline{0}$, γ_2 with $\overline{1}$. Then $u_1 = \overline{m_1 r}$ for some $1 \leq m_1 < n/r$ since $u_1 \neq v$. Similarly $u_2 = \overline{m_2}$ for some $m_2 \in [1, n[$. Let $P = \overline{1} + \overline{r}\mathbb{Z} \subset \mathcal{O}_2$. We have $P \cap \mathcal{O}_1 = \emptyset$ since $r \geq 2$, hence c is constant on P. Let $c(v) = \spadesuit$, $c(\mathcal{O}_2) = \{\spadesuit, \clubsuit\}$. Since $c([v, u_2]_{\gamma_2}) = \{\spadesuit\}$, $[v, u_2]_{\gamma_2}$ has been identified with $[0, m_2]$, and the class of $1 \in [0, m_2]$ belongs to P, we get $c(P) = \{\spadesuit\}$.

On the other hand, $c(\mathcal{O}_1) = \{\spadesuit, \clubsuit\}$ since $\gamma_1 \notin \mathrm{Aut}(c)$, hence there exists $k \in [1, \frac{n}{r}[$ such that $c(n - kr) = \clubsuit$. It follows that $n - kr \in]u_2, v[_{\gamma_2}$ and $c(x) = \clubsuit$ for all $x \in [n - kr, n-1]$. In particular $c([n-r, n-1]) = \{\clubsuit\}$ hence $[n-r, n-1] \cap P = \emptyset$. But $\overline{n} \notin P$ hence $P \cap \overline{[n-r, n]} = \emptyset$, a contradiction since $P = 1 + \overline{r}\mathbb{Z}$. $\square$

7. Necklaces with ordered pearls

We assume here that Y is endowed with a *total* ordering $\leq$. This enables one to introduce the following relation on E : we denote $\alpha < \beta$ if $\alpha \perp \beta$ and $\alpha(x) \leq \beta(x)$ for all $x \in X$. In terms of pearls, we can imagine that the elements of Y are greyscales, and that α is deduced from β by fading one pearl. If $\alpha < \beta$ we call α a *child* of β. Recall that $\mathrm{Aut}(\alpha)$ denotes the stabilizer of α in Γ. We say that two children $\alpha_1 \neq \alpha_2$ of c are *twins* if there exists $\gamma \notin \mathrm{Aut}(c)$ such that $\alpha_2 = \gamma.\alpha_1$. Figure 4 provides, in the case $\Gamma = X = \mathbb{Z}/4\mathbb{Z}$ and $Y = \{\text{black}, \text{white}\}$ with black < white, an example of a necklace with 4 children, forming two pairs of twins.

We assume again that Γ is cyclic and acts freely on the finite set X. By proposition 6.2 above, we know that triplets do not occur.

7.1. At most one child admits symmetries.

Proposition 7.1. *Let* $\alpha_1, \alpha_2, c \in E$ *such that* $\alpha_1 < c$ *and* $\alpha_2 < c$. *If* $\mathrm{Aut}(\alpha_1) \neq 1$ *and* $\mathrm{Aut}(\alpha_2) \neq 1$ *then* $\alpha_1 = \alpha_2$.

Proof. We argue by contradiction, assuming $\alpha_1 \neq \alpha_2$. Let $\Gamma_1 = \mathrm{Aut}(\alpha_1)$, $\Gamma_2 = \mathrm{Aut}(\alpha_2)$ and $x_1, x_2 \in X$ such that $\alpha_i(x_i) < c(x_i)$ for $i \in \{1, 2\}$.

As a first step, we prove that this implies $x_1 \neq x_2$. Assume to the contrary that $x_1 = x_2 = x_0$. An element $g \in \Gamma_1 \cap \Gamma_2 \setminus \{e\}$ would yield $\alpha_1(x_0) = \alpha_1(g.x_0) = c(g.x_0) = \alpha_2(g.x_0) = \alpha_2(x_0)$ hence $\alpha_1 = \alpha_2$, a contradiction. Let then $g_i \in \Gamma_i \setminus \{e\}$ for $i \in \{1, 2\}$. If $g_2 g_1.x_0 = x_0$ then $g_2 = g_1^{-1} \in \Gamma_1 \cap \Gamma_2 \setminus \{e\}$ which has been ruled out. Thus $\alpha_2(g_2 g_1.x_0) = c(g_2 g_1.x_0)$ and

$$c(g_2 g_1.x_0) = \alpha_1(g_2 g_1.x_0) = \alpha_1(g_1 g_2.x_0) = \alpha_1(g_2.x_0) = c(g_2.x_0) = \alpha_2(g_2.x_0) = \alpha_2(x_0)$$

and also $\alpha_2(g_2g_1.x_0) = \alpha_2(g_1.x_0) = c(g_1.x_0) = \alpha_1(g_1.x_0) = \alpha_1(x_0)$ hence $\alpha_1 = \alpha_2$, a contradiction.

We thus proved $x_1 \neq x_2$. As a second step, we prove $\Gamma_1 \cap \Gamma_2 = \{e\}$, by contradiction. Assume we have $g \in \Gamma_1 \cap \Gamma_2$ with $g \neq e$, and recall $x_1 \neq x_2$. If $x_2 \neq g.x_1$ then, on the one hand we have $\alpha_2(g.x_1) = c(g.x_1) = \alpha_1(g.x_1) = \alpha_1(x_1)$, and on the other hand we have $\alpha_2(g.x_1) = \alpha_2(x_1) = c(x_1)$ since $x_2 \neq x_1$. We thus get $c(x_1) = \alpha_1(x_1)$, a contradiction. It follows that $x_2 = g.x_1$. This implies $|\Gamma_1 \cap \Gamma_2| = 2$ by freeness of the Γ-action, hence $g = g^{-1}$ and $x_1 = g.x_2$. But then follows the following contradiction :

$$\begin{cases} c(x_1) &= \alpha_2(x_1) &= \alpha_2(g.x_2) &= \alpha_2(x_2) &< c(x_2) \\ c(x_2) &= \alpha_1(x_2) &= \alpha_1(g.x_1) &= \alpha_1(x_1) &< c(x_1). \end{cases}$$

As a consequence we get that, for all $(g_1, g_2) \in \Gamma_1 \times \Gamma_2$ with $g_1, g_2 \neq e$, we have

$$|\{x_1, g_1.x_1, g_2.x_1, g_1g_2.x_1\}| = 4 \text{ and } |\{x_2, g_1.x_2, g_2.x_2, g_1g_2.x_2\}| = 4.$$

As a third step we prove that $x_2 \notin \Gamma_2.x_1$ and $x_1 \notin \Gamma_1.x_2$. By symmetry considerations it is sufficient to show that $x_2 \notin \Gamma_2.x_1$. We argue by contradiction, assuming $x_2 = g_2.x_1$ with $g_2 \in \Gamma_2$. Since $x_1 \neq x_2$ we know that $g_2 \neq e$. Moreover, this also implies $c(x_1) = \alpha_2(x_1) = \alpha_2(g_2.x_1) = \alpha_2(x_2) < c(x_2)$ and $c(x_2) = \alpha_1(x_2) = \alpha_1(g_1.x_2) = \alpha_1(g_1g_2.x_1)$ for all $g_1 \in \Gamma_1$. By assumption we can choose $g_1 \in \Gamma_1$ with $g_1 \neq e$. Since $\Gamma_1 \cap \Gamma_2 = \{e\}$ we know that $g_1 \notin \Gamma_2$ hence $g_1g_2.x_1 \neq x_1$ and $c(x_2) = \alpha_1(g_1g_2.x_1) = c(g_1g_2.x_1)$. Moreover $g_1 \neq e$ and $x_2 = g_2.x_1$ hence $g_1g_2.x_1 \neq x_2$. It follows that $c(x_2) = c(g_1g_2.x_1) = \alpha_2(g_1g_2.x_1) = \alpha_2(g_2g_1.x_1) = \alpha_2(g_1.x_1)$. We have $g_1.x_1 \neq x_2 = g_2.x_1$ since $g_1 \notin \Gamma_2$, hence $c(x_2) = \alpha_2(g_1.x_1) = c(g_1.x_1) = \alpha_1(g_1.x_1) = \alpha_1(x_1) < c(x_1)$, contradicting $c(x_1) < c(x_2)$.

As a fourth step we prove that there exists $(g_1, g_2) \in \Gamma_1 \times \Gamma_2$ such that $g_1, g_2 \neq e$ and $x_2 \neq g_1g_2.x_1$. We argue by contradiction. Let $g_2 \in \Gamma_2$ with $g_2 \neq e$. If, for all $g_1 \in \Gamma_1 \setminus \{e\}$, we have $g_1g_2.x_1 = x_2$ then $|\Gamma_1| = 2$ by freeness of the Γ-action. Similarly, we get $|\Gamma_2| = 2$. Since Γ is cyclic, $|\Gamma_1| = |\Gamma_2|$ implies $\Gamma_1 = \Gamma_2$ contradicting $\Gamma_1 \cap \Gamma_2 = \{e\}$.

We can now conclude the proof. Let $(g_1, g_2) \in \Gamma_1 \times \Gamma_2$ such that $g_1, g_2 \neq e$ and $x_2 \neq g_1g_2.x_1$. Then $\alpha_1(x_1) = \alpha_1(g_1.x_1) = c(g_1.x_1)$. Moreover

$$\begin{array}{rcll} c(g_1.x_1) &=& \alpha_2(g_1.x_1) & \text{since } x_2 \notin \Gamma_1.x_1 \Leftrightarrow x_1 \notin \Gamma_1.x_2 \\ &=& \alpha_2(g_2g_1.x_1) & = \alpha_2(g_1g_2.x_1) \\ &=& c(g_1g_2.x_1) & \text{since } x_2 \neq g_1g_2.x_1 \\ &=& \alpha_1(g_1g_2.x_1) & \text{since } \Gamma_1 \cap \Gamma_2 = \{e\} \Rightarrow g_1g_2.x_1 \neq x_1 \\ &=& \alpha_1(g_2.x_1) & = c(g_2.x_1) \text{ since } g_2 \neq e \\ &=& \alpha_2(g_2.x_1) & \text{since } x_2 \notin \Gamma_2.x_1 \\ &=& \alpha_2(x_1) & = c(x_1) \text{ since } x_1 \neq x_2. \end{array}$$

It follows that $\alpha_1(x_1) = c(x_1)$, contradicting $\alpha_1(x_1) \neq c(x_1)$. □

Figure 1 illustrates the necessity of considering necklaces with ordered pearls, and figure 2 shows that the assertion is false if Γ is not cyclic. However, the reader can check that the proof provided here works for Γ a (finite) commutative group with at most one subgroup of order 2.

7.2. **How many twins can one have ?** Our goal is to study which necklaces appear in pairs while fading one pearl in a given necklace c.

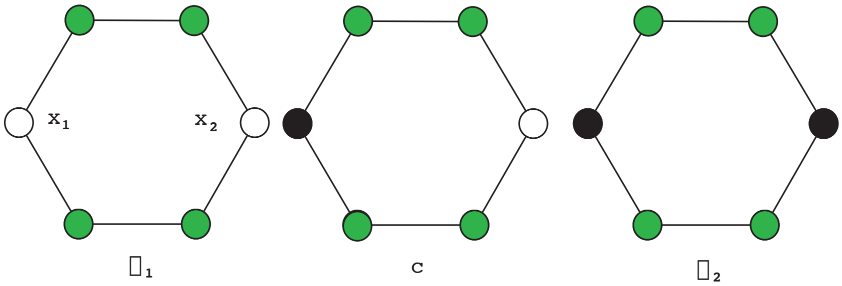


FIGURE 1. Necessity of assumption $\alpha < c$ instead of $\alpha \perp c$

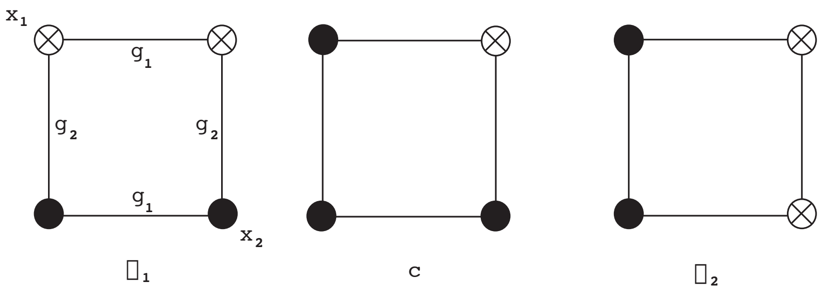


FIGURE 2. $\Gamma =< g_1, g_2 \mid g_1^2 = g_2^2 = e >$ and $\otimes < \bullet$

Lemma 7.2. *Let* $\alpha_1, \beta_1, \alpha_2, \beta_2 \in E$ *such that* $\alpha_1 < c, \beta_1 < c, \alpha_2 < c, \beta_2 < c$, $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 4$, $\beta_1 = \gamma_1.\alpha_1$, $\beta_2 = \gamma_2.\alpha_2$ *with* $< \gamma_2 > \subset < \gamma_1 >$, $\gamma_1, \gamma_2 \notin \mathrm{Aut}(c)$. *Then there exists exactly one* $\mathcal{O} \in X/ < \gamma_1 >$ *such that*

$$\left| c_{|\mathcal{O}}^{-1} (\max c(\mathcal{O})) \right| = |\mathcal{O}| - 1$$

and, for all $P \in X/ < \gamma_1 >$, $P \neq \mathcal{O} \Rightarrow |c(P)| = 1$.

Proof. Let $\Gamma_i =< \gamma_i >$ for $i = 1, 2$ and $n = |\Gamma_1|$. We apply lemma 4.1 to α_1 and $\beta_1 = \gamma_1.\alpha_1$. This provides an orbit $\mathcal{O} = \mathcal{O}_1$ in X/Γ_1, as well as elements $u_1, v_1 \in \mathcal{O}_1$ being characterized by $\alpha_1(u_1) \neq c(u_1)$ and $\beta(v_1) \neq c(v_1)$. Since $\alpha_1 < c$ and $\beta_1 < c$ we get $\alpha_1(u_1) < c(u_1)$ and $\beta_1(v_1) < c(v_1)$.

Moreover, since $\gamma_1 \notin \mathrm{Aut}(c_1)$ we have $|c(\mathcal{O})| = 2$, so we let $c(\mathcal{O}) = \{\clubsuit, \spadesuit\}$ with $c(v_1) = \clubsuit \neq \spadesuit$. For $P \in X/\Gamma_1$ with $P \neq \mathcal{O}$ we have by the lemma $|c(P)| = 1$.

We identify Γ_1 and $\mathcal{O}$ to $\mathbb{Z}/n\mathbb{Z}$, γ_1 to $\overline{1}$ and v_1 to $\overline{0}$. Thus u_1 is identified to some $\overline{m} \in \mathbb{Z}/n\mathbb{Z}$ for some $m \in [0, n-1]$. Since $u_1 \neq v_1$ we have $m \geq 1$. We have also $m \leq n-2$, as $\overline{m} = -\overline{1}$ would mean $v_1 = \gamma_1.u_1$ hence $\gamma_1 \in \mathrm{Aut}(c)$ by lemma 4.1.

We finally have $|\beta_1(\mathcal{O})| = 2$ and

$$\beta_1(\mathcal{O}) = \{\beta_1(\overline{m}), \beta_1(\overline{0}), \beta_1(-\overline{1})\} = \{c(\overline{m}), \beta_1(\overline{0}), c(-\overline{1})\} = \{\clubsuit, \beta_1(v_1), \spadesuit\}.$$

Since $\beta_1(\overline{0}) < c(\overline{0}) = \clubsuit$ it follows that $\beta_1(\overline{0}) = \spadesuit$ and $\spadesuit < \clubsuit$, that is $\clubsuit = \max c(\mathcal{O})$.

We need to show that $|c_{|\mathcal{O}}^{-1}(\clubsuit)| = |\mathcal{O}| - 1$. Since, by lemma 4.1, $c(\overline{y}) = c(\overline{0}) = \clubsuit$ for $y \in [0, m]$ and $c(\overline{y}) = \spadesuit$ for $y \in [m+1, n-1]$, this amounts to saying that $m = n-2$, as $|\mathcal{O}| = |\Gamma_1| = n$.

We now apply lemma 4.1 to α_2 and $\beta_2 = \gamma_2.\alpha_2$. It provides a Γ_2-orbit $\mathcal{O}_2$ such that $|c(Q)| = 1$ for all $Q \in X/\Gamma_2$ with $Q \neq \mathcal{O}_2$, and $u_2, v_2 \in \mathcal{O}_2$. Moreover, the assumption $\Gamma_2 \subset \Gamma_1$ implies that this Γ_2-orbit $\mathcal{O}_2$ is included in some Γ_1-orbit P. But since $\gamma_2 \notin \mathrm{Aut}(c)$, by lemma 4.1 we know that $|c(\mathcal{O}_2)| \geq 2$, hence $|c(P)| \geq 2$. It follows that $P = \mathcal{O}$ and $\mathcal{O}_2 \subset \mathcal{O}$. Through our identification of $\mathcal{O}$ to $\mathbb{Z}/n\mathbb{Z}$, v_2 is then identified with $\overline{a}$ for some $a \in [0, n-1]$.

Assume first that $\Gamma_2 \neq \Gamma_1$. Then Γ_2 is generated by γ_1^r for some $r \geq 2$ dividing n. Since $|c(\mathcal{O}_2)| = 2$ and $\mathcal{O}_2 \subset \mathcal{O}$ we have $c(\mathcal{O}_2) = c(\mathcal{O})$. Moreover, $\beta_2 < c$ and $\beta_2(v_2) \neq c(v_2)$ implies $\beta_2(v_2) < c(v_2)$. But $\beta_2(\mathcal{O}) = c(\mathcal{O})$ hence $c(\overline{a}) = c(v_2) = \max c(\mathcal{O}) = \clubsuit$.

It follows that $\overline{a} \in [v_1, u_1]_{\gamma_1} = [\overline{0}, \overline{m}]_{\overline{1}}$, that is $a \in [0, m]$, and, since $m \geq 1$, there exists $b \in [0, n-1]$ such that $\overline{b} \in \{\overline{a-1}, \overline{a+1}\}$ and $c(\overline{b}) = \clubsuit$. Since $\Gamma_1 \neq \Gamma_2$ and $\overline{a} \in \mathcal{O}_2$ we have $\overline{b} \notin \mathcal{O}_2$ (otherwise $\overline{b} - \overline{a} \in \{\overline{1}, -\overline{1}\}$ would be a generator of Γ_1 belonging to Γ_2). Moreover c is constant on every Γ_2-orbit different from $\mathcal{O}_2$ hence, for all $x \in \mathbb{Z}$, the congruence $x \equiv b \bmod r$ implies $c(\overline{x}) = \clubsuit$. In particular there exists $x, x+r \in [0, n-1]$ such that $c(\overline{x}) = c(\overline{x+r}) = \clubsuit$. Since the set of elements $z \in [0, n-1]$ for which $c(\overline{z}) = \clubsuit$ is an interval (namely $[0, m]$), we get that $c(\overline{z}) = \clubsuit$ for all $z \in [x, x+r]$. Now every Γ_2-orbit in $\mathcal{O}$ intersects $[\overline{x}, \overline{x+r}]_{\overline{1}}$ hence $c(P) = \{\clubsuit\}$ for all $P \in \mathcal{O}/\Gamma_2$ with $P \neq \mathcal{O}_2$. It follows that $c(\mathcal{O} \setminus \mathcal{O}_2) = \{\clubsuit\}$. Let $\overline{x}, \overline{y} \in \mathcal{O}_2$ such that $c(\overline{x}) = c(\overline{y}) = \spadesuit$ with $x, y \in [0, n-1]$. If $x \neq y$ we may assume $x < y$ hence $m < x < y \leq n-1$ and $c(\overline{y-1}) = \spadesuit$ contradicting $\overline{y-1} \notin \mathcal{O}_2$ (again because $\Gamma_2 \neq \Gamma_1$). It follows that $x = y$, that is there is only one $x \in \mathcal{O}_2$ such that $c(x) = \spadesuit$, hence $|c_{|\mathcal{O}}^{-1}(\clubsuit)| = |\mathcal{O}| - 1$.

Now assume that $\Gamma_2 = \Gamma_1$, and let $r \in [0, n-1]$ with $\gcd(r, n) = 1$ such that $\gamma_2 = \gamma_1^r$. Since $\Gamma_2 = \Gamma_1$ and $\mathcal{O}_2 \subset \mathcal{O}_1 = \mathcal{O}$ we have $\mathcal{O}_2 = \mathcal{O}$. In particular $c(v_2) = c(\overline{a}) = \max c(\mathcal{O}) = \clubsuit$ and $|[v_2, u_2]_{\gamma_2}| = |[v_1, u_1]_{\gamma_1}| = m+1$. Let $\varphi : x \mapsto \overline{a} + rx \in \mathrm{Aff}_n$ (see section 5 for the definition of Aff_n). Since $\gamma_2 = \gamma_1^r$ we have $[v_2, u_2]_{\gamma_2} = \varphi([v_1, u_1]_{\gamma_1}) = \{\varphi(\overline{0}), \dots, \varphi(\overline{m})\}$. On the other hand $[v_2, u_2]_{\gamma_2} = \{x \in \mathcal{O} \mid c(x) = \clubsuit\} = \{\overline{0}, \dots, \overline{m}\}$. If $m < n-2$ lemma 5.2 implies that either $\varphi = \mathrm{Id}$ or $\forall x \ \varphi(x) = \overline{m} - x$. The first case implies $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ and the second one implies $\alpha_1 = \beta_2, \alpha_2 = \beta_1$. Both thus yield a contradiction, hence $m = n-2$ and $|c_{|\mathcal{O}}^{-1}(\clubsuit)| = |\mathcal{O}| - 1$. □

The above lemma is a special case of the following proposition, and will be used in its proof.

Proposition 7.3. *Let $\alpha_1, \beta_1, \alpha_2, \beta_2 \in E$ with $\alpha_1 < c$, $\beta_1 < c$, $\alpha_2 < c$, $\beta_2 < c$ and such that $|\{\alpha_1, \beta_1, \alpha_2, \beta_2\}| = 4$, $\beta_1 = \gamma_1.\alpha_1$, $\beta_2 = \gamma_2.\alpha_2$ with $\gamma_1, \gamma_2 \notin \mathrm{Aut}(c)$. Let $\Gamma' =< \gamma_1, \gamma_2 >$. Then there exists exactly one $\mathcal{O} \in X/\Gamma'$ such that*

$$\left| c_{|\mathcal{O}}^{-1} \left(\max c(\mathcal{O}) \right) \right| = |\mathcal{O}| - 1$$

and, for all $P \in X/\Gamma'$, $P \neq \mathcal{O} \Rightarrow |c(P)| = 1$. Moreover $c(x) = \alpha_i(x) = \beta_i(x)$ for all $x \notin \mathcal{O}$.

Proof. Let $\Gamma_i =< \gamma_i >$. Lemma 4.1 provides $v_1, u_1, v_2, u_2, \mathcal{O}_1, \mathcal{O}_2$. We let $X' = \Gamma'.v_1$, α_i', β_i', c' the restriction of α_i, β_i, c to X' and we denote by $\mathrm{Aut}(c')$ the stabilizer of c' under the action of Γ'. Since $\mathcal{O}_1 \subset X'$, we know that c is constant on the Γ_1-orbits not included in X', therefore $\gamma_1 \notin \mathrm{Aut}(c) \Rightarrow \gamma_1 \notin \mathrm{Aut}(c')$. Moreover $\mathcal{O}_1 \subset X'$ implies $\alpha_1' \perp c$, $\beta_1' \perp c$, $\beta_1' = \gamma_1.\alpha_1'$, whence $\mathrm{Aut}(c') = \{e\}$ by lemma 6.1. We show that $\mathcal{O}_2 \subset X'$. From $\beta_2' = \gamma_2.\alpha_2'$ and $\gamma_2 \neq 1$ we deduce $\alpha_2' \neq c'$, because otherwise we would have $\beta_2' = \gamma_2.c'$ hence

- either $\beta'_2 = c'$ and $\gamma_2 \in \mathrm{Aut}(c') \setminus \{1\}$, contradicting $\mathrm{Aut}(c') = \{1\}$,
- or there exists exactly one $x \in X'$ such that $\beta'_2(x) = c'(x)$ contradicting $\forall y \in Y \ |(\beta'_2)^{-1}(y)| = |c'^{-1}(y)|$ since $\beta'_2 = \gamma_2.c'$.

Likewise, we have $\beta'_2 \neq c'$ hence $\alpha'_2 \perp c'$, $\beta'_2 \perp c'$. Since $\beta'_2 = \gamma_2.\alpha'_2$ and $\gamma_2 \notin \mathrm{Aut}(c') = \{1\}$, lemma 4.1 implies that X' contains some Γ_2-orbit on which c takes two distinct values, hence $\mathcal{O}_2 \subset X'$.

If $\Gamma_1 \subset \Gamma_2$ or $\Gamma_2 \subset \Gamma_1$, that is $\Gamma' = \Gamma_1$ or $\Gamma' = \Gamma_2$, lemma 7.2 gives the conclusion so from now on we exclude these cases. This assumption implies in particular that there exists $P \in X'/\Gamma_1$, $Q \in X'/\Gamma_2$ with $P \neq \mathcal{O}_1$, $Q \neq \mathcal{O}_2$. Since $\Gamma' = \Gamma_1\Gamma_2$ acts freely and transitively on X', lemma 5.1 implies that c is constant on both $X' \setminus \mathcal{O}_1$ and $X' \setminus \mathcal{O}_2$. Now $\Gamma_1 \not\subset \Gamma_2$ and $\Gamma_2 \not\subset \Gamma_1$ implies that $\Gamma_1 \cup \Gamma_2$ is not a subgroup of Γ' hence $\Gamma_1 \cup \Gamma_2 \neq \Gamma'$. By lemma 5.1 there exists $v_0 \in \mathcal{O}_1 \cap \mathcal{O}_2$ hence $\mathcal{O}_1 \cup \mathcal{O}_2 = (\Gamma_1 \cup \Gamma_2).v_0$ and $|\mathcal{O}_1 \cup \mathcal{O}_2| = |\Gamma_1 \cup \Gamma_2| < |\Gamma'| = |X'$. It follows that $X' \neq \mathcal{O}_1 \cup \mathcal{O}_2$ therefore c is also constant on $X' \setminus (\mathcal{O}_1 \cap \mathcal{O}_2)$. We let $\clubsuit \in Y$ denote this value taken by c. Since $Q \cap \mathcal{O}_1 \neq \emptyset$ we have $\clubsuit \in c(\mathcal{O}_1)$ and similarly $P \cap \mathcal{O}_2 \neq \emptyset \Rightarrow \clubsuit \in c(\mathcal{O}_2)$. Let $\spadesuit \in Y$ such that $c(\mathcal{O}_1) = \{\clubsuit, \spadesuit\}$. We have $\clubsuit \neq \spadesuit$ since $|c(\mathcal{O}_1)| = 2$. Then $c(\mathcal{O}_1) = c(X') = c(\mathcal{O}_2)$ since $|c(\mathcal{O}_2)| = 2$.

If c takes twice the value $\spadesuit$ on $\mathcal{O}_1$, by lemma 4.1 there exists $x \in \mathcal{O}_1$ such that $c(x) = c(\gamma.x) = \spadesuit$, hence $x, \gamma_1.x \in \mathcal{O}_1 \cap \mathcal{O}_2$ and $\gamma_1 \in \Gamma_1 \cap \Gamma_2$ since Γ_1, Γ_2 act freely transitively on $\mathcal{O}_1 \cap \mathcal{O}_2$. In particular $\Gamma_1 \subset \Gamma_2$, contradicting our assumption.

Since c takes the value $\clubsuit$ on all the others Γ_1-orbits, it follows that there exists a unique $x \in X'$ such that $c(x) = \spadesuit$, and $]u_1, v_1[_{\gamma_1} =]u_2, v_2[_{\gamma_2} = \{x\}$. In particular $v_1 = \gamma_1.x$ and $v_2 = \gamma_2.x$. As in the proof of lemma 7.2, the existence of $\alpha'_2 < c$ implies $\clubsuit > \spadesuit$. Letting $\mathcal{O} = X'$ we then have $|c_{|\mathcal{O}}^{-1}(\max c(\mathcal{O}))| = |\mathcal{O}| - 1$. Let now $R \in X/\Gamma'$ with $R \neq \mathcal{O}$. Since $\mathcal{O}_1 \subset \mathcal{O}$ and $\mathcal{O}_2 \subset \mathcal{O}$ we know that c is constant on each Γ_1-orbit and each Γ_2-orbit in R. We deduce from lemma 5.1 that $|c(R)| = 1$ and the conclusion. □

Note that, for this proposition, we really need to put a total order on Y. Figure 3 shows a simple necklace with four of its children which gives a counterexample to the proposition if $<$ is replaced by the weaker relation $\perp$. Figures 4 and 5 show typical examples of necklaces (with ordered pearls, where black is smaller than white) having several twins.

Corollary 7.4. *Under the assumptions of proposition 7.3, we have* $|\mathcal{O}| \geq 5$.

Proof. Let $x \in \mathcal{O}$ such that $c(x) = \min c(\mathcal{O}) = \spadesuit$ and $\clubsuit = \max c(\mathcal{O})$. We have $\alpha_i(x) = c(x)$ because otherwise $c(x) \notin \alpha_i(\mathcal{O})$. Since $\beta_i = \gamma_i.\alpha_i$ and $\beta_i < c$ this implies $\beta_i(x) = c(x)$ hence $c(x) \in \beta_i(\mathcal{O}) = \alpha_i(\mathcal{O})$, a contradiction. Similarly, $\beta_i(x) = c(x)$.

We also have, for $y \in \mathcal{O} \setminus \{x\}$, $\beta_i(y) \neq c(y) \Rightarrow \beta_i(y) = c(x)$. Indeed, $\beta_i(\gamma_i.x) = \alpha_i(x) = c(x) \neq c(\gamma_i.x)$ by $\gamma_i \neq 1$ and the proposition. Since $\beta_i < c$ this implies $\gamma_i.x = y$ and $\beta_i(y) = c(x)$. Similarly, $\alpha_i(y) \neq c(y) \Rightarrow \alpha_i(y) = c(x)$. In particular, $|\mathcal{O} \setminus \{x\}| \geq |\{\alpha_1, \alpha_2, \beta_1, \beta_2\}| = 4$ and the conclusion. □

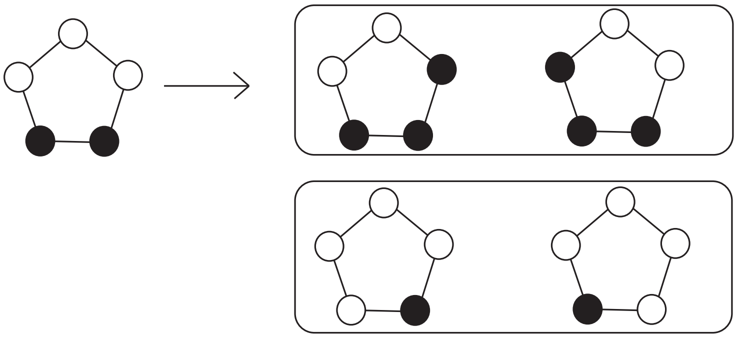


FIGURE 3. Necklaces with unordered pearls

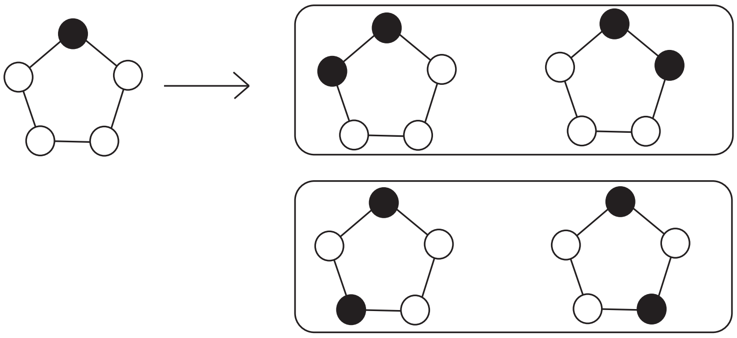


FIGURE 4. Necklace providing several twins (1)

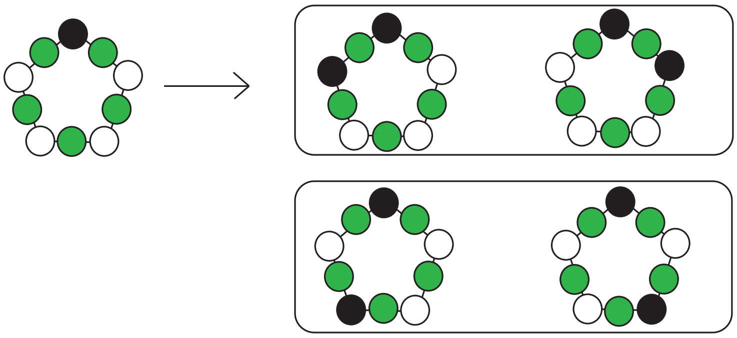


FIGURE 5. Necklace providing several twins (2)

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Table 1. Table for exceptional groups

Group	Multiplicities	Generators of parabolic
G_4	1	$< s >$
G_5	1	$< s >$
G_6	1	$< t >$
G_7	1	$< t >$
G_8	1	$< s >$
G_9	1	$< t >$
G_{10}	1	$< t >$
G_{11}	1	$< u >$
G_{12}	2	$< s >$
G_{13}	2	$< s >$
G_{14}	2	$< s >$
G_{15}	2	$< s >$
G_{16}	2	$< s >$
G_{17}	2	$< t >$
G_{18}	2	$< s >$
G_{19}	2	$< t >$
G_{20}	2	$< s >$
G_{21}	2	$< t >$
G_{22}	3	$< s >$
G_{24}	2	$< s,t >$
G_{25}	1	$< s,t >$
G_{26}	1	$< s,t >$
G_{27}	3	$< t,u >$
G_{29}	2	$< s,t,u >$
G_{31}	2	$< s,t,u,v >$
G_{32}	2	$< s,t,u >$
G_{33}	2	$< s,t,u,w >$
G_{34}	2	$< s,t,u,v,w >$

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