TECHNICAL NOTE
Numerical characterization of the regularity loss of minimal surfaces

M. Hamouda[*] ,[2]
and Hervé V.J. Le Meur[†]

November 17, 2006

Abstract. – In this Technical Note, we numerically study the regularity loss of the solutions of non-parametric minimal surfaces with non-zero boundary conditions. As expected from theoretical results when parts of the boundaries have non-positive mean curvature, the solutions may or may not be regular close to the boundary. From approximate solutions, we look for a process that will say whether the *exact* solution is regular or not. We elaborate this process inside an astroid to get a threshold value of the constraint. More theoretical results are also given on the approximation by a regularized solution. Last, we check this process in the catenoid.

[*]The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405, USA.

[†]Laboratoire d'analyse numérique et EDP, CNRS and Université Paris-Sud, Bât. 425, 91.405 Orsay Cedex, France. Herve.LeMeur@math.u-psud.fr

Key Words. Boundary layers, minimal surfaces, non-regular solutions, singular perturbations.

1 Introduction

Almost two centuries and a half after Euler's works on the catenoid, the topic of minimal surfaces is still active. For instance the Mathematical Sciences Research Institute (MSRI) hosted the Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces. The nature of the meeting made it possible to give a panoramic view of this subject. An edited book was published in 2005 [1].

The problem of non-parametric minimal surfaces consists in finding a graph function u solution of the following minimization problem :

$$\min_{u|_\Gamma=\Phi} \int_\Omega \sqrt{1+|\nabla u|^2}\, dx, \tag{1}$$

for u defined inside $\Omega \subset \mathbb{R}^N$, Γ being the frontier of Ω and Φ a given L^∞ function.

Many works were concerned with the existence of a strong solution for (1) in the case of boundaries of non-negative curvature. See for instance [2], ... among many others. Since the 1970s, some papers proved the existence of weak solutions of (1) called "generalized solutions" satisfying either $u|_{\Gamma'} = \Phi$ and $\partial u/\partial n(\Gamma') < \infty$ or $u|_{\Gamma'} \neq \Phi$ and $\partial u/\partial n(\Gamma') = \infty$, for $\Gamma' \subset \Gamma$. A "generalized solution" is defined as the limit of a regularized solution to (2). Serrin [3] proved that if the domain's boundary Γ is not of non-negative mean curvature, the generalized solution may be non-regular ($u \neq \Phi$ at the frontier) even if Φ is $\mathcal{C}^\infty$. So the generalized solution develops a "vertical branch" and its normal derivative becomes infinite near the boundary in the region where $u = \Phi$ is not satisfied.

Here, we are interested in the way one may know from computed minimal surfaces whether the *exact* (generalized) solution is regular or not. In this technical note, we justify a numerical process to characterize, from the computed minimal surface, whether the exact (and unknown) solution is regular ($u = \Phi$ and $\partial u/\partial n|_\Gamma < \infty$) or non-regular ($u \neq \Phi$ and $\partial u/\partial n|_\Gamma = \infty$). Such a process could be useful even outside optimization.

One of the author has studied in [4] the non-parametric catenoid for which he exhibited an explicit value of the height above which the solution is regular and below which it is non-regular. We will check our process on this case.

In the Section 2 of the Note, we validate the numerical code and set up a process for a minimal surface inside an astroid with boundary conditions depending on a real parameter K. We justify the existence of a threshold value of K above which the solution is non-regular. This numerical study is completed by some theoretical insights into the behavior of the regularized solutions when $\varepsilon \to 0$ (the boundary layer behavior). In Section 3, we apply the preceding method to the study on the catenoid. We conclude in Section 4.

2 Investigations in the astroid

Theoretical papers mentionned above force us to take a domain whose boundary is of non-negative curvature. A simple one is the interior of an astroid ($x(\theta) = 4\cos^3\theta, y(\theta) = 4\sin^3\theta$). If K is a positive real, the boundary condition Φ is either $+K$ on the part Γ_1 of the frontier where $xy > 0$ or $-K$ on the part Γ_2 of the frontier where $xy > 0$. Because of the symmetries we expect the regularity loss to appear at $I(\sqrt{2}, \sqrt{2}) \in \Gamma_1$ where we will measure the normal slope.

We need to validate the program that will compute the approximate optimal solutions, then the process that will tell us whether the exact solution is regular or not.

2.1 The resolution

Because of the symmetries, we meshed a fourth of the astroid with `emc2` ([5]). Our meshes are exponentially refined close to the point $I(\sqrt{2}, \sqrt{2})$ with 20, 50 and 80 nodes on $\Omega \bigcap (x = y)$ for mesh1, mesh2 and mesh3 respectively.

We discretized functions with P_1 finite elements and wrote a program of evaluation of the functional. As it only depends on the gradient and is highly non-linear, we created two more arrays that stored the derivatives of the local basis functions. They enable to speed up the evaluation.

The constraint is a simple Dirichlet condition $u|_{\partial\Omega} = \Phi$ depending only on a positive K.

We plug all this in `DONLP2` (see [6, 7]) which is a free optimization code written either in F77, F90 or C. It is available on the internet. It uses a slightly modified version of the Pantoja-Mayne update for the Hessian of the Lagrangian, variable dual scaling and an improved Armijo-type stepsize algorithm.

By using two very different initial try, we could compute two solutions for $K = 10$ where normal slopes at $I(\sqrt{2}, \sqrt{2})$ differ only of 0.03 %. So the program does not depend on the initial try.

Similarly, we checked that the result on our best mesh3 is 0.3% from the result on mesh2. Thanks to other tests, we could validate our program.

2.2 Threshold value in the astroid

Theoretical results propose two ideas to prove the regularity loss : either $u|_\Gamma \neq \Phi$ or $\frac{\partial u}{\partial n}|_\Gamma = +\infty$ at I.

So as to catch the condition $u|_\Gamma \neq \Phi$ of the exact solution, we used in a first time extrapolations from the *computed* one.

It happens that the computed solution extrapolates (piecewise either cubic Hermite or cubic spline) almost exactly to K. Only its difference with $K \mapsto K$ seemed to show a kind of difference at about 3. Since the flag is unclear, we abandonned this idea.

In a second time, still trying to catch the condition $u|_\Gamma \neq \Phi$, we solved the optimization for K and $K' = K+1$. From the computed solutions, we drew $|U_{K+1} - U_K|_{L^2}/|U_K|_{L^2}$ as a function of K. Yet, no abrupt change appeared. So we abandonned this idea.

In a third time, although measuring a slope is less accurate (and more sensitive), we tried to catch the equivalent condition $\partial u/\partial n|_\Gamma = +\infty$ at I by computing the normal slope of the *computed* solution. As it can be expected at a finite and fixed space step, the infinite is too close to 1. Even though the slope at $K = 10$ is about 240, this is not a clear infinite. Yet, it appears on Figure 1 that this slope depends drastically on the space step. This remark led us to the next process.

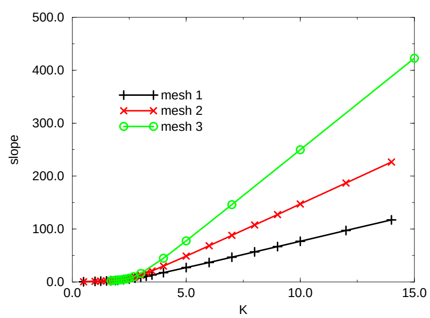


Figure 1: Slopes versus K

In a fourth time, we tried to catch the condition $\partial u/\partial n|_\Gamma = +\infty$ at I by non-convergence. In other words, if the normal slope increases with the local space step, we may guess that this is a sign of non-convergence and so of local non-regularity of the *exact* solution. The results for our three meshes are drawn on Figure 1. These results enable us to claim first that this process gives a reasonable answer and next that there is very likely a threshold value K_0 between 2 and 4 such that if $K > K_0$ the exact solution to (1) in an astroid is no more regular close to I.

Zooming in the range $K \in [0.1, 3]$ shows little discrepancy between the results on the three meshes. Zooming in the range $K \in [2, 8]$ is less acurate than making a linear regression from Figure 1 in the straight parts of the curves (almost exactly linear). All these lines cross at the same point whose abscissa is 3 ± 0.2.

An other numerical post-treatment after the resolution would be to draw the difference of the normal slope on our two best meshes. It provides the same value.

2.3 Dependence of the boundary layer on the regularization

An asymptotic analysis enables us to guess the order of the boundary layer from the optimal solution to the regularized problem :

$$\min_{u|_{\Gamma_1}=K,\, u|_{\Gamma_2}=-K} \int_\Omega \varepsilon|\nabla u|^2/2 + \sqrt{1+|\nabla u|^2}\,dx. \qquad (2)$$

First we need some preliminary results. Let the new coordinates ; $z = (x+y)/\sqrt{2}$, $t = (x-y)/\sqrt{2}$ and $v(z,t) = u^\varepsilon(x,y)$. We may state the following theorem.

THEOREM 2.1 *The solution $v(z,t)$ of the regularized equation (2) inside an astroid has a trace along the axis of symmetry $t = 0$ that satisfies for all ε :*

$$\varepsilon\Big(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial t^2}\Big)\left(1+(\frac{\partial v}{\partial z})^2\right)^{3/2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial t^2}\Big(1+(\frac{\partial v}{\partial z})^2\Big) = 0. \quad (3)$$

Moreover, at the point $I(\sqrt{2},\sqrt{2})$, any solution of (2) is odd in t and satisfies :

$$\frac{\partial^2 v}{\partial t^2} = -1/6\frac{\partial v}{\partial z}. \tag{4}$$

Sketch of the proof

The proof of (3) is straightforward from the Euler's equations associated to (1).

Concerning the proof of (4), it suffices to make an expansion of the function $v(z,t)$ restricted to Γ_1 where it is constant, and use the fact that odd in t derivatives of v are zero along the segment $t = 0$ by symmetry. An expansion in $\theta - \pi/4$ of the solution leads to (4). By pursuing the expansion, one might even prove that at $(z = 2, t = 0)$;

$$54\frac{\partial^3 v}{\partial z \partial t^2} + 9/2\frac{\partial^2 v}{\partial z^2} + 54\frac{\partial^4 v}{\partial t^4} - 30\frac{\partial^2 v}{\partial t^2} - 13/4\frac{\partial v}{\partial z} = 0.$$

$\square$

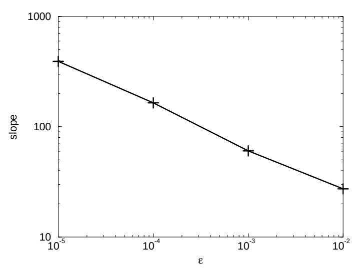
Let us assume the boundary layer is of size ε^α. Then if we let $z = \varepsilon^\alpha Z$ and use this in (4), one sees that the second derivative in t close to I will be negligible with respect to $\partial^2 v/\partial z^2$ which is of order $\varepsilon^{-2\alpha}$. We find back a well-known result that in a boundary layer the most important variation holds in the normal direction (z direction in our case). But this negligibility is limited as $\partial^2 v/\partial t^2 = -\varepsilon^{-\alpha}/6\ \partial v/\partial Z$ and $\partial^2 v/\partial z^2 = \varepsilon^{-2\alpha}\partial^2 v/\partial Z^2$.

Then the leading terms in the equation (3) are given by

$$\varepsilon^{1-5\alpha}\frac{\partial^2 v}{\partial Z^2}\left(\frac{\partial v}{\partial Z}\right)^3 + \varepsilon^{-2\alpha}\frac{\partial^2 v}{\partial Z^2} + \varepsilon^{-2\alpha}\frac{\partial^2 v}{\partial t^2}(\frac{\partial v}{\partial Z})^2 = 0, \quad \text{at} \quad t = 0, \tag{5}$$

sufficiently close to I (to have (4)). One must notice that the equation (4) makes the third term dominate the second one. By homogeneity we obtain $1 - 5\alpha = -3\alpha$ or $\alpha = 1/2$. Had we neglected all the t derivatives as usual in boundary layers' literature, we would have been led to a different result. Of course here we just intend to get an order of magnitude of the boundary layer and a complete study of this problem is postponed to a subsequent work.

Now, we are going to numerically check the value found by our simple asymptotic analysis. For this purpose, we look for a discretized solution to (2), extract the slope at I and draw its variation with the regularizing parameter ε in the left part of Figure 2. This slope is finite because of the regularizing parameter ε in (2). The computations are done on mesh 3 and K is set equal to $20 > K_0$. The normal slope as a function of ε varies approximatively as $\varepsilon^{-\alpha}$ with $\alpha \simeq 0.4$ for mesh 3 (astroid). This is compatible with what we expected from our asymptotic analysis ($\alpha = 0.5$).



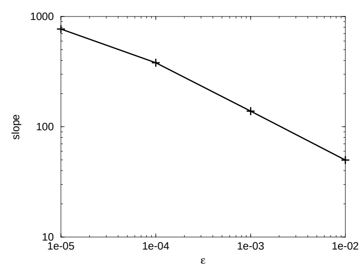


Figure 2: Dependence on ε of the normal slope in the astroid (left-mesh 3) and the catenoid (right-mesh 4)

3 The case of the catenoid

In order to check our numerical process, we need a test case where the exact value of the threshold can be known exactly. This is the case for the (non-parametric) catenoid. As it is proved in [4], the exact value of K_0 is $K_0 = R_1 \ln\left((R_2 + \sqrt{R_2^2 - R_1^2})/R_1\right)$

For $R_1 = 1, R_2 = 2$, it is about 1.32.

In order to find the catenoid, we meshed non-radially a fourth of an annulus between $R = 1$ and $R = 2$ and solved (1) with DONLP2 (cf. [6], [7]). Then we read the normal gradient along a section radially uniformly meshed with 20, 30, 50 or 100 points. The results are reported on Figure 3.

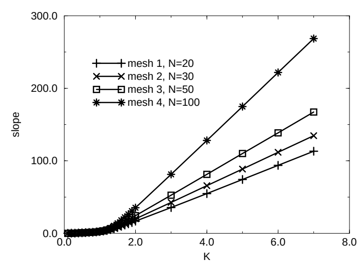


Figure 3: Normal gradient on the interior circle (catenoid)

When one draws the four straight lines for each mesh after the non-convergence, there is, like in the astroid case, only one intersection. Its abscissa is $K \simeq 1.3$. This numerical value is sufficiently close to the exact one (about 1.32) to give confidence. *A posteriori*, it gives one more argument for the study in the astroid.

Like in the astroid case, one may draw the difference of the computed slopes as a function of K for the two best meshes. Such a curve drives us to predict a value $K \simeq 1.4$ which is reasonable too.

Let us notice that even if the meshes had increasing number of points along one section, the rest of the meshes was not equally refined. It proves that the accuracy in the vicinity of the point of regularity loss is more important than elsewhere.

We also drew in Figure 2-right the slope of the regularized catenoid as a function of the regularizing parameter ε. While the theoretical value of the slope is 1/3 (see [4]) we found a reasonable value of 0.38.

4 Conclusion

Our numerical study enables us to claim that there exists no regular solution to the problem (1) in an astroid for K above a certain value K_0. This is due to the curvature of the boundary. An estimate for this threshold in an astroid is $K_0 = 3 \pm 0.2$. We also described both theoretically and numerically how the regularized solution converges to the generalized one. This measures the boundary layer behavior.

All these computations are checked in the case of the catenoid where one of the authors has computed the exact threshold K_0 in [4].

Moreover, it appeared that some more "natural" approches were less efficient in catching the regularity loss than our numeric-based approch. Even outside the field of optimization, a numerical attempt to characterize the regularity loss could use the numerical process given in this article.

References

[1] D. Hoffman, Editor, *Global theory of minimal surfaces.* American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2005.

[2] J. Douglas, *Solution of the Problem of Plateau.* Trans. Amer. Math. Soc. 33, 263-321, 1931.

[3] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables,* Philos. Trans. Roy. Soc. London, Ser. A **264** (1969), 413-496.

[4] M. Hamouda, *Asymptotic analysis for the regularized minimal surface problem in the radially symmetric case,* Asymptotic Analysis **32** (2002), no. 2, 107–130.

[5] P.L. Georges http://www-rocq.inria.fr/modulef/Doc/GB/Guide3-14/welcome.html.

[6] P. Spellucci, *An SQP method for general nonlinear programs using only equality constrained subproblems.* Math. Programming 82 (1998), no. 3, Ser. A, 413–448

[7] P. Spellucci, *A new technique for inconsistent QP problems in the SQP method.* Math. Meth. of Oper. Res. 47 (1998) 355-400 (Physica Verlag Heidelberg, Germany