THE THEORETICAL ASPECTS OF THE ORR-SOMMERFELD METHODS. THE CASE OF TWO VISCOELASTIC FLUIDS.

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Abstract In this article, conceived for physicists and mathematicians, we describe various Orr-Sommerfeld Equations (OSE) and stress their differences, both in modeling, justification and in the results. These equations are derived from the Poiseuille flow of two viscoelastic or Newtonian fluids. The literature proposes a link between computation and experiment which is modeled by two different equations. We reinvestigate it and stress a hidden assumption. Then, we study extensively the long wave asymptotic stability of the flow of two viscoelastic fluids and exhibit a formula for characterization of loss of stability in a new case. Some waves are found through an OSE and cannot be found through the other. We give their growth rate implicitely for some of them. Last, we prove a theorem that says whether such a wave could be unstable or not.

KEY WORDS: Orr-Sommerfeld, Stability, Visco-Elastic Fluids, Poiseuille flow

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1. Introduction

The stability of various experimental problems can be investigated through the stability of mathematical models. Yet, they are often nonlinear and the study of their stability is very difficult. A weak definition of stability involves the stability of the linearized model and a good literature has been devoted to it ([6], ...) and to the links between linear and nonlinear stability ([12], ...).

Once the linearized equations are derived, the Orr-Sommerfeld Equations (OSE) are often used to test linear stability. They are derived after testing for various exponentials. These equations correspond to different experiments, and different geometries. The (existing) justification for derivation is scarcely presented. The most used OSE corresponds to an experiment rarely performed. So when calculations are associated to experiments, one chooses sometimes the "wrong" OSE.

So, we can wonder to what extent the various OSEs will yield the same unstable modes and also the same stable ones. The answer is clear for intermediate waves and some OSEs, but is not investigated for short waves. This article deals with these questions in the general case of two ViscoElastic Fluids (VEF).

After submission, our attention was drawn to absolute and convective instabilities. These notions seem to give a more precise prediction of instability (see [11] and [4]). We have not investigated them in this article.

In section 2, we present the various OSEs, recall the mathematical justification for their derivation, stress their possible differences and discuss the modeling and aims of these equations. We end with some comments on the literature on this topic. Section 3 is devoted to the study of the long wave asymptotic study of two VEF. We stress a condition needed to ensure the equivalence of two OSE and give new results more accurate than those of [14]. Finally, section 4 gives the proof of the main theorem that states the differences between two OSEs for the case of this specific flow.

2. Different ways of deriving different equations

2.1 The physical experiment

It appears, through the literature, that two main types of experiments could be used to test the stability of the flow of two fluids under infinitesimal perturbance.

In the first one, one tests the *amplification in Space* (S). Namely, a ribbon is kept vibrating at the entrance of a box where enter the two fluids. The stationary interface is said to be stable if, when perturbed (see Figure 1a), its amplitude remains bounded up to the exit which is supposed sufficiently far. If the amplitude of the variation of the interface grows with x, the flow is said to be unstable (see Figure 1b). So, the characterization of instability is manageable.

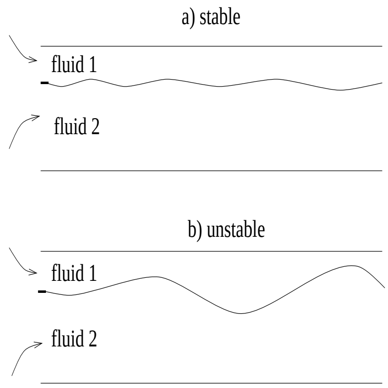

Figure 1: Amplification in space

In the second one, one tests the *amplification in Time* (T). The source of the instability cannot be at any entry point, but in the initial condition. For that reason, one starts with an initial interface having a sine shape with a sufficiently small amplitude and lets the system evolve. This shape is said to be unstable if the amplitude grows with time (see Figure 2). The major con of this experiment is that it is very difficult to design an experimental apparatus enabling such a fine control of the initial shape of the interface and this instability is almost never investigated.

2.2 Deriving the Orr-Sommerfeld Equations

As we are mainly interested in the theoretical aspects linked to the Orr-Sommerfeld Equations (OSE) for the flow of two VEF, we will start from dimensionless equations in dimensionless geometries. The details of how to derive these equations can be found in [13] for example but are meaningless here.

First, let us state part of the geometry, bounded between $\varepsilon_2(=-\varepsilon)$ and $\varepsilon_1(=1)$ in y. The stationary interface of the Poiseuille flow of the two fluids is at $y=0$ (see Figure 3).

Then, let us state the VEF equations. The parameters, velocities $(\underline{u})$, pressure (p), and extra stress $(\underline{\underline{\tau}})$ are indiced by $k=1,2$ which denotes the index of the fluid k that flows in Ω_k (see Figure 3). Vectors are underlined once while matrices are underlined twice. For the present paper, we will focus our attention on the classical Oldroyd-B model for which the constitutive law is written:

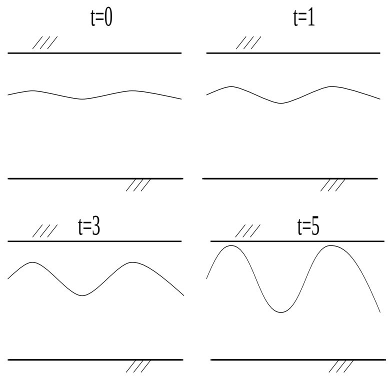

Figure 2: Amplification in time

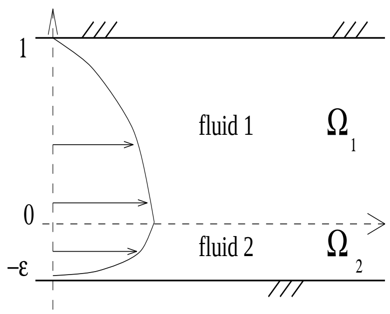

Figure 3: Overall geometry

(1)
$$\underline{\underline{\tau}}_k + \mathrm{We}_k \frac{\mathcal{D}\underline{\underline{\tau}}_k}{\mathcal{D}t} = 2\alpha_k\ m_k \underline{\underline{D}}[u_k],$$

where We_k is the Weissenberg number, α_k is the polymeric viscosity, m_k is the total viscosity and $\dfrac{\mathcal{D}.}{\mathcal{D}t}$ is the Oldroyd derivative :

$$\frac{\mathcal{D}\underline{\underline{\tau}}}{\mathcal{D}t} = \frac{\partial\underline{\underline{\tau}}}{\partial t} + \underline{u}.\nabla\underline{\underline{\tau}} - \underline{\underline{\nabla u}}\ \underline{\underline{\tau}} - \underline{\underline{\tau}}\ \underline{\underline{\nabla u}}^T.$$

The conservation of momentum and incompressibility are more usual:

(2)
$$\begin{aligned} r_k \mathrm{Re}\left(\frac{\partial \underline{u}_k}{\partial t} + (\underline{u}_k.\ \nabla)\ \underline{u}_k\right) + \nabla p_k -\ & m_k\ (1-\alpha_k)\ \Delta\underline{u}_k\ = \\ & \mathrm{div}\ \underline{\underline{\tau}}_{p,k} + r_k \mathrm{Re}\ F^{-2}\underline{k}, \\ \nabla.\underline{u}_k = 0, & \end{aligned}$$

where Re is the Reynolds number (common to the two fluids), r_k is the dimensionless density, F is the Froude number and $\underline{k}$ is the unit vector vertical. Non-dimensionalizing is done with respect to the quantities of fluid 1, so that $m_1 = 1, r_1 = 1, \varepsilon_1 = 1$. We close the equations with classical boundary and interface conditions, including the effect of surface tension (measured by the coefficient S) and with $[\![.]\!] = (.)_1 - (.)_2$ the jump at the interface :

$$[\![-p\underline{\underline{I}} + 2m(1-\alpha)(\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}^T)/2 + \underline{\underline{\tau}}]\!].\underline{n} = -2H\ S\ \underline{n}\ \text{ and }\ [\![\underline{u}]\!].\underline{n} = 0,$$

where $\underline{n}$ is a unit vector normal to the interface. The goal of the Orr-Sommerfeld methods is to study stability under infinitesimal perturbations. So, we restrict our attention to equations (1-2) linearized around the basic Poiseuille state (see [14]). This linearization being an easy calculation, we will not reproduce it here. Suffice it to say that the linear equations have the shape :

$$P(\partial_x, \partial_y)U = \partial_t Q(\partial_x, \partial_y)U, \tag{3}$$

where ∂ are derivation operators, P and Q are matrices applied to U, vector of the unknown fields of the fluids and the height of the interface.

As the equations are linear, a principle of superposition applies and so, one may assume that the unknown fields are waves. At that point of derivation, we meet three possibilities.

A first one consists in looking for the waves that can be represented as varying in x and t as

$$U_0 \exp(\alpha x - ict)), \tag{4}$$

where $\alpha \in \mathbb{C}$, $c \in \mathbb{R}$ and U_0 is the constant amplitude (see [9] and [20]). An other possibility is to look for waves like

$$U_0 \exp(iq(x - ct)), \tag{5}$$

where $q \in \mathbb{R}$ and $c \in \mathbb{C}$ (see for instance [7], [21], [8]). Its meaning of a wave with constant velocity appears clearly.

An other way is to introduce fields varying as :

$$U_0 \exp(iqx + st), \tag{6}$$

where $q \in \mathbb{R}$ and $s \in \mathbb{C}$ and U_0 is the amplitude (see [16], [1] for instance).

In both assumptions, it seems natural to consider that we are looking for "perturbation quantities that have an exponential time and periodic spatial dependence" (as is said for instance in [22] p. 318). We will discuss this point of view in subsection 2.4.

Whether one chooses one representation or another, one is led to solve the Orr-Sommerfeld Equations (OSEs) respectively for (4), (5) and (6) :

$$\begin{aligned}
&(7) \qquad & P(\alpha, \partial_y)U_0(y) &= -ic \quad Q(\alpha, \partial_y)U_0(y), \\
&(8) \qquad & P(iq, \partial_y)U_0(y) &= -iqc \quad Q(iq, \partial_y)U_0(y), \\
&(9) \qquad & P(iq, \partial_y)U_0(y) &= s \quad Q(iq, \partial_y)U_0(y).
\end{aligned}$$

These are, somehow, eigenvalue problems. As can be seen from these three systems, the equivalence of their solutions is not totally obvious and will be discussed in detail hereafter. The point would be here to make sure that these methods will detect the same unstable flows.

2.3 Another way to derive some OSE

In the precedent derivation, we linearized around a basic flow and also around its geometry (see [19] for justification) in a preliminary phase. As a consequence, these new equations are applied in a straight geometry.

As a consequence, if the problem is amplification in time (cf. Figure 2), the geometry is such that one may make a Fourier transform in x (the domain is infinite in x and the transform is onto). In the same way, one may make a Laplace transform in time (onto) and be led to the equation (9) rigorously.

In a very similar way, if the experiment deals with amplification in space (cf. Figure 1), the geometry enables a Laplace transform in x and a Fourier transform in t. This proves (7) in this configuration.

At that level, an assumption must be clearly stated. We could derive (7) or (9) rigorously because we assumed implicitely, that

$$\text{(A1)}\left\{\begin{array}{c}\text{The initial fields in Laplace/Fourier transforms are zero, or}\\ \text{one order of magnitude less than the linear fields.}\end{array}\right.$$

2.4 Preliminary discussion

It can be seen from the mathematical derivation, that (A1) is assumed in deriving (9) or (7). This condition is often satisfied in experiments, but rarely stressed.

It is also obvious that the two equations correspond to different experiments. Very few are the articles that stress this difference. Yet, we can quote [22] (p.325) who uses [9] (commented hereafter) to transform time instability into space instability.

Also it can be seen that no evidence can be given of the possibility to derive OSE rigorously in more complex geometries (4:1 contraction or entry flow for instance) because of the non-locality of the transforms. If such a generalization gave good results, it would mean that one may localize the Fourier transform, which is a non-obvious assumption.

The first positive result is that one may prove the derivation of (7) or (9), from the linear equations, under the assumption of the geometry (Figure 1 or 2 respectively). So, these equations appear to model very different experiments. Space instability is modeled by (7) while time instability by (9).

Moreover, as one knows that Fourier and Laplace transforms are onto, it can be claimed that solving (7) or (9) is equivalent to solving (3) in the right geometry and in a very general functional space (say L^2). As a consequence, any discrepancy between an experiment and the corresponding OSE cannot come from the assumption that "disturbances were periodic" in space or time (cf. [22] p. 318 or [20] p. 212) because this assumption appears only in an unclear derivation using exponentials and disappears when the Laplace and Fourier transforms are used.

Mathematical modeling also justifies that, if the spectrum of (7) or (9) is discrete, the back transformed function will be exponentially increasing or decreasing in the variable associated to the Laplace variable and sine-varying in the Fourier variable. This behavior is experimentally found in most cases. Yet some polynomial increases are reported in [23] and could be attributed to the continuous spectrum.

One of the clearest conclusions of the preceeding is that experimentations of amplification in x (most common) cannot be, *a priori* investigated rigorously only through (9) (see [9] though and comments below).

Last, (8) and (9) would be the same if one might set (both s and c are complex here):

$$s = -iqc.$$

In the case of intermediate waves, it is clearly possible and was argued already in [20]. Concerning the longwave studies, it will be possible if and only if the 0th order term of the expansion of s in powers of q is zero ($s = s^0 + q\, s^1 + q^2\, s^2 ...$):

$$\text{(A2)} \quad s^0 = 0.$$

The sequel will be devoted to discussing this assumption (A2).

2.5 Comments on the literature

In [9], M. Gaster investigated some relations between (7) and (9). To that purpose, he denoted the general disturbance as varying in exp $(i(\alpha x - \beta t))$ with $\alpha = \alpha_r + i\alpha_i, \beta = \beta_r + i\beta_i$ complex and $(\alpha_r, \alpha_i, \beta_r, \beta_i)$ real. Then, time amplification (T) was characterized by $\beta_i \neq 0$ and $\alpha_i = 0$ while space amplification (S) by $\alpha_i \neq 0$ and $\beta_i = 0$. Gaster assumed that there exists an analytic function $\beta(\alpha)$ between (T) and (S) states, that α_r is constant and also :

1. $\dfrac{\partial \beta_i}{\partial \alpha_r}$ and $\dfrac{\partial \alpha_i}{\partial \beta_r}$ small (of order 10^{-3}),
2. α_i small (of order 10^{-3}).

These assumptions are experimentally argued in [18]. Then, through the use of the Cauchy-Riemann equations, he proved that :

$$(10) \qquad \begin{aligned} \alpha_r(T) &= \alpha_r(S), \\ \beta_r(T) &= \beta_r(S), \\ \frac{\beta_i(T)}{\alpha_i(S)} &= -\frac{\partial \beta_r}{\partial \alpha_r}. \end{aligned}$$

The Gaster relation (10) enables to link time amplification (T) and space amplification (S). Yet, one may pursue the proof and integrate the Cauchy-Riemann relation (5) of [9] ($\partial \beta_r / \partial \alpha_r = \partial \beta_i / \partial \alpha_i$ because $\beta(\alpha)$ is analytic) in α_r between (S) and (T) states. So we have :

$$\beta_i(T) - \beta_i(S) = \beta_i(T) = \int_S^T \frac{\partial \beta_r}{\partial \alpha_i}\, d\alpha_r.$$

But since α_r is assumed to be constant, and as $\partial \beta_r / \partial \alpha_i$ is analytic, the integral must vanish exactly and $\beta_i(T)$ be identically zero (no amplification in time). This is a contradiction.

If we still assume analyticity, one may find a more straightforward justification, as (10) is a mere Finite Difference approximation of one of the Cauchy-Riemann relations :

$$\frac{\partial \beta_r}{\partial \alpha_r} = \frac{\partial \beta_i}{\partial \alpha_i} = \frac{\beta_i(T) - \beta_i(S)}{\alpha_i(T) - \alpha_i(S)} = -\frac{\beta_i(T)}{\alpha_i(S)},$$

and does not require so many assumptions rarely satisfied. With this relation, and as one may measure $\partial \beta_r / \partial \alpha_r$ (through a simple FD approximation), we can relate the growth rate of the theoretical amplification in time (T) to the one of the experimental amplification in space (S).

In [3] Brevdo already stressed that the assumption " that there exists a continuous contour (the contour of integration), connecting state (T) with state (S)" is only an assumption. Still in [3], the author proves even more. With an exotic but authorized dispersion relation, he proves that there may not exist any contour of integration if α_r is constant.

The main hypothesis in what precedes is that the function $\beta(\alpha)$ should be analytic. We do not see any justification for this. For instance, let us look for affine functions $\underline{\beta} = \underline{\underline{A}}(\underline{\alpha} - \underline{\alpha(S)}) + \underline{\beta(S)}$ that would interpolate the values of $\beta(\alpha)$ at $\alpha(S)$ and $\alpha(T)$. Then, for all C, one may find a matrix $\underline{\underline{A}}$ for which $\partial\beta_r/\partial\alpha_r = C\partial\beta_i/\partial\alpha_i$! So even the Cauchy-Riemann relations do not seem physically meaningfull as analytic functions are not the only one in nature and $\beta(\alpha)$ could be anything if any ...

Moreover, normal modes are quite criticized since the introduction of absolute and convective instabilities in fluids by Monkewitz and Huerre (see [10], [11]). For instance, Brevdo and Bridges [5] give examples where normal modes for a local analysis give false results on model equations and a good review is in [17].

So the justification of these equations seems very weak. Yet it gives good results (see [21], [22]) but also bad results predicted and partially explained (see [4] for instance).

To summarize, we have no definitive justification for the Gaster relation that appears to have been already criticized. Moreover, there are interesting tracks in the literature on other tools than Gaster relations and normal modes that we did not investigate (absolute and convective instabilities [11], [17] and pseudo-spectra [15]).

After having revisited the links between (7) and (9), we come back to the longwave analysis so as to show some differences between (8) and (9) and pave the way for the main Theorem 2.

3. Long wave stability of two VEF

3.1 Conditional equivalence of two OSE

Starting from the whole system algebraically denoted by (9), one has to expand all the amplitudes of fields and functions : $X(q) = X^0 + qX^1 + q^2X^2 + ...$, where the superscript on X^i denotes the order of the coefficient in the expansion. We also expand $s = s^0 + q\,s^1 + q^2\,s^2 ...$ and assume $s^0 \neq 0$. We denote $\underline{u} = (u, v)$ and $\underline{\underline{\tau}} = \begin{pmatrix} \sigma & \tau \\ \tau & \gamma \end{pmatrix}$. The subscript 0 denotes the basic flow around which we linearize (Poiseuille flow found in [14]). The other notations were given in section 2 except for the Froude number $F = U^*/\sqrt{gd_1}$ (with characteristic dimensioned quantities and $g = 9.8$ in standard units) that measures the effect of gravity. So we are led to :

$$
(11) \left\{ \begin{array}{l}
r_k \mathrm{Re}\left(s^0 \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} + \begin{pmatrix} u_0' v^0 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ p^{0'} \end{pmatrix} = \begin{pmatrix} (1-\alpha_k) m_k u^{0''} + \tau^{0'} \\ (1-\alpha_k) m_k v^{0''} + \gamma^{0'} \end{pmatrix}, \\
v^{0'} = 0, \\
\sigma^0 (1 + \mathrm{We}_k s^0) + \mathrm{We}_k \left(\sigma_0' v^0 - 2u_0' \tau^0 - 2\tau_0 u^{0'}\right) = 0, \\
\tau^0 (1 + \mathrm{We}_k s^0) + \mathrm{We}_k \left(\tau_0' v^0 - u_0' \gamma^0\right) = \alpha_k m_k u^{0'}, \\
\gamma^0 (1 + \mathrm{We}_k s^0) = 2\alpha_k m_k v^{0'} (= 0), \\
v^0(0) = s^0 h^0, u_k^0(\varepsilon_k) = 0 = v_k^0(\varepsilon_k), \\
[\![u_0']\!] h^0 + [\![u^0]\!] = 0, [\![v^0]\!] = 0, \\
[\![(1-\alpha_k) m_k (u^{0'} + u_0'' h^0) + \alpha_k m_k u_0'' h^0 + \tau^0]\!] = 0, \\
[\![-p^0 + 2(1-\alpha_k) m_k v^{0'} + \gamma^0]\!] = -[\![r_k]\!] \mathrm{Re} F^{-2} h^0.
\end{array} \right.
$$

It is easily seen that the zeroth order vertical velocity (v^0) is zero. If (A2) is false ($s^0 \neq 0$), then one deduces $h^0 = 0$ (so the perturbation is rather volumic and not carried by an interfacial mode).

If $s^0 = -1/\mathrm{We}_k$ (stable wave) for both k, then one may see that there exist eigenvectors for any σ^0. Moreover, the set of eigenvectors is such that

1. if $1-2\alpha_k = 0$ for one k, then the components of fluid k (other than σ^0_k) in the vector X^0 are zero.

2. if $1-2\alpha_k \neq 0$ for both k, let :

$$\omega_k = \sqrt{\frac{r_k \mathrm{Re}\ s^0}{(1-2\alpha_k)m_k}},$$

then the system reads, for both k :

$$\text{(12)} \qquad \begin{cases} u_k^{0\,\prime\prime}(y) = \omega_k^2 u_k^0(y), \\ [\![u^0]\!] = 0, \\ [\![(1-2\alpha_k)m_k u_k^{0\,\prime}]\!] = 0, \\ u_1^0(1) = 0 = u_2^0(-\varepsilon), \end{cases}$$

and has non-zero solutions if and only if (ω_k is given above in the present case) :

$$\text{(13)} \qquad \frac{\mathrm{th}\ (\omega_k \varepsilon_k)}{\mathrm{th}\ (\omega_{k'} \varepsilon_{k'})} = \frac{(1-2\alpha_k)m_k\omega_k}{(1-2\alpha_{k'})m_{k'}\omega_{k'}} \quad \text{for } k \neq k' \in \{1,2\},$$

where k, k' are supposed to be different in $\{1,2\}$ and such that $s^0 = -1/\mathrm{We}_k = -1/\mathrm{We}_{k'}$.

The most general case requires the following definition :

Definition 1 : *For any $k \in \{1,2\}$, we define :*

$$\text{(14)} \qquad X_k = \left| \begin{array}{ll} \dfrac{\alpha_k m_k}{1+\mathrm{We}_k s^0} & \textit{if } 1 + We_k s^0 \neq 0 \\ -\alpha_k m_k & \textit{if } 1 + We_k s^0 = 0 \end{array} \right. ,$$

and

$$\text{(15)} \qquad \omega_k = \sqrt{\frac{r_k \mathrm{Re}\ s^0}{(1-\alpha_k)m_k + X_k}} \quad \textit{if } (1-\alpha_k)m_k + X_k \neq 0.$$

With that definition, straightforward investigations that mimic the particular case studied above ($s^0 = -1/\mathrm{We}_k$ for both k) prove the following theorem :

Theorem 1. *Let $s^0 \neq 0, X_k, \omega_k$ defined in* **Definition 1**.
If $(1-\alpha_k)m_k + X_k = 0$ there is no non-trivial solution to (11) except if $s^0 = -1/We_k = -1/We_{k'}$.
Else if $s^0 = -1/We_k$ for a $k \in \{1,2\}$, there is a non-trivial eigenvector such that $\sigma_k^0 \neq 0$.
If $s^0 \neq -1/We_k$ for any $k \in \{1,2\}$, there will be an eigenvector to (11) if and only if its growth rate s^0 satisfies :

$$\text{(16)} \qquad \frac{\mathrm{th}\ (\omega_k \varepsilon_k)}{\mathrm{th}\ (\omega_{k'} \varepsilon_{k'})} = \frac{((1-\alpha_k)m_k + X_k)\,\omega_k(s^0)}{((1-\alpha_{k'})m_{k'} + X_{k'})\,\omega_{k'}(s^0)}$$

for k, k' different in $\{1,2\}$.

It appears that the modes corresponding to these eigenvectors would not be found through (8) (they satisfy $s^0 \neq 0$) and appear in a precise investigation of (9) in the long wave limit. There are still only two cases that enable such mode. The first ($s^0 = -1/\mathrm{We}_k$) gives a stable mode while the second (see (16)) will be investigated in section 4 to see whether there could exist an s^0 of positive real part, solution to (16).

3.2 Characterization of the stability of long waves

We have seen, in the last subsection, that the assumption $s^0 \neq 0$ may lead to one stable mode more and to a possibly unstable one. Yet, for the present subsection, we will assume (A2) and so $s = -iqc$. By this, we intend also to cope with the literature. Here we will study, thoroughly and carefully, the asymptotic stability and we will neglect the effect of the gravity by assuming $F = \infty$.

When one writes the full system (11) with $s = -iqc$, one sees that the zeroth order of v and γ are zero. So we set $v = iqw$ and $\gamma = iq\delta$ and the new system is written ($s = -iqc$) with dimensionless surface tension S :

$$(17)\left\{\begin{array}{ll}
r_k \mathrm{Re}\left((-c+u_0(y))\begin{pmatrix} iqu \\ -q^2 w\end{pmatrix} + \begin{pmatrix} iqu_0' w \\ 0\end{pmatrix}\right) + \begin{pmatrix} iqp \\ p'\end{pmatrix} = & \\
\qquad (1-\alpha_k)m_k\begin{pmatrix} u''-q^2u \\ iq(w''-q^2w)\end{pmatrix} + \begin{pmatrix} iq\sigma+\tau' \\ iq\tau + iq\delta'\end{pmatrix} & \\
u + w' = 0 & \\
\sigma + \mathrm{We}_k\left(iq(-c+u_0)\sigma + iqw\sigma_0' - 2u_0'\tau - 2iq\sigma_0 u - 2\tau_0 u'\right) = 2iq\alpha_k m_k u & \\
\tau + \mathrm{We}_k iq\left((-c+u_0)\tau + \tau_0' w - u_0'\delta - iq\sigma_0 w\right) = \alpha_k m_k(u' - q^2 w) & \\
\delta + \mathrm{We}_k\left((-c+u_0)iq\delta - 2iq\tau_0 w\right) = 2\alpha_k m_k w' & \\
w = (-c+u_0(0))h & \text{at } y=0 \\
u_k(\varepsilon_k) = 0 = w_k(\varepsilon_k) & \text{with } \varepsilon_1 = 1\ \varepsilon_2 = -\varepsilon \\
[\![u_0']\!]h + [\![u]\!] = 0 & \text{and } [\![w]\!] = 0 \\
[\![(1-\alpha_k)m_k(u'-q^2w) - iq\sigma_0 h + \tau]\!] = 0 & \\
[\![-p + 2iq(1-\alpha_k)m_k w' + iq\delta]\!] = q^2 S h. &
\end{array}\right.$$

We expand the fields $X = X^0 + (iq)X^1 + (iq)^2X^2 + \ldots$, where X is any field $u_k, w_k, \sigma_k, \tau_k, \delta_k, p_k, h$ or c, and keep the notations of subsection 3.1. If, classicaly, we keep the first order term, we split the discussion in two cases.

1. $[\![u_0']\!] = 0$
   To compute the first relevant terms in c^i, we have two sub-cases :

   (a) $h^0 \neq 0 (\Leftrightarrow c^0 = 1)$. Then we have to write down the system at order 1. Then, two sub-sub-cases are to be dealt with :

      i. If $[\![\sigma_0]\!] = 0$ we have to write the system till order 3, and we get the effect of the surface tension. After tedious but straightforward calculations, we find :

$$(18)\qquad c^3 = -S\frac{\varepsilon^3(2m_2+3\varepsilon^3+5\varepsilon)}{3(m_2^2+4m_2\varepsilon+6m_2\varepsilon^2+4m_2\varepsilon^3+\varepsilon^4)}. \qquad c^0 = 1, c^1 = 0 = c^2$$

      So, the flow will be stable if and only if the real part of the growth rate is negative $\Re(iq((iq)^3c^3)) = \Re(q^4c^3) < 0$. As we see on (18), this is always true (but at order 3 !).

ii. If $[\![\sigma_0]\!] \neq 0$, after some calculations, one finds the amplitude of oscillations of the interface :

$$h^0 = -a\frac{m_2^2 + 4m_2\varepsilon + 6m_2\varepsilon^2 + 4m_2\varepsilon^3 + \varepsilon^4}{6m_2\varepsilon(1+\varepsilon)[\![\sigma_0]\!]},$$

with a a free parameter (the equations are linear). Then, we find :

$$c^1 = -\frac{\varepsilon^2[\![\sigma_0]\!](m_2 - \varepsilon^2)}{2(m_2^2 + 4m_2\varepsilon + 6m_2\varepsilon^2 + 4m_2\varepsilon^3 + \varepsilon^4)}. \tag{19}$$

Yet, $[\![u_0']\!] = \dfrac{(m_2-\varepsilon^2)(m_2-1)}{\varepsilon(1+\varepsilon)m_2} = 0$ in the present case. So, unless we are still at the frontier between stability and unstability ($m_2 = \varepsilon^2$ and $c^1 = 0$), we have $m_2 = 1$ (same viscosity) and (19) transforms into a condition which only involves the jump of elasticity. The flow will be stable if and only if :

$$(1-\varepsilon^2)[\![\alpha_k \mathrm{We}_k]\!] > 0. \tag{20}$$

(b) if $h^0 = 0$, all the components vanish.

2. $[\![u_0']\!] \neq 0$ (general case) : The calculation is tedious, but thanks to a software of symbolic calculation (we used MAPLE), we got the formula which has already been published in [14].

The formula for the general case is already published in [14]. The particular case Re$= 0$, We$\neq 0$, $m_2 = 1$ is envisaged in [14] and is similar to our 1,(a) ii. But our present formula (20) is more general as it does not depend on Re and the physical meaning appears much more easily than in the function J_5 from [14]. Why this particular case $[\![u_0']\!] = 0$ and $[\![\sigma_0]\!] \neq 0$ (1,(a) ii) should it be interesting ? It happens that one may experimentally act on the viscosity through temperature and fix it up to some percents. So, reaching the "particular case" can be experimentally achieved. In the very precise case of the stability of two VEF, one would have to make the viscosities equal and to act on the jump of the normal stress of the basic flow to cross the line of marginal asymptotic long wave stability given by (20) to test the model (and the process of linearization).

After having studied the precise long wave stability and stated Theorem 1, we come back to the difference between (8) and (9) to state the main Theorem 2.

4. Quasi equivalence of two OSE

The present section deals with the alledged equivalence of (9) and (8). We proved above that it is equivalent to the fact that $s(q)$ should be zero and derivable at $q = 0$ (assumption $A2$). We already proved that this is false by exhibiting waves that do not satisfy this condition, but these waves were stable.

It remains to see whether (9) could give rise to *unstable* waves, not found by (8). Their growth rate is given by Theorem 1 and their existence is denied by the following theorem :

Theorem 2 *Let $k \in \{1,2\}$, the parameters $\alpha_k, We_k, Re, r_k, m_k$ be real nonnegative, $\alpha_k \in [0,1], X_k$ and ω_k defined by (14-15), $\varepsilon_1 = 1, m_1 = 1, r_1 = 1, \varepsilon_2 = -\varepsilon < 0$. There is no s^0 complex with nonnegative real part such that (16) be satisfied.*

Notice that if any of the ω_k, for $k \in \{1,2\}$, is not defined, there is no s^0 solution.

Corollary 3 *In the long wave asymptotic stability study of the flow of two VEF, (8) and (9) give the same unstable modes, although they give different stable waves.*

Remark : The case $\alpha_k = 0 = \alpha_{k'}$ gives the case of two Newtonian fluids.

Proof of Theorem 2.

We need to define two sets depending on β, R, θ and $\hat{\beta}, \hat{R}$ (cf. Figure 4):

$$\begin{aligned} \Omega_{\beta,R,\theta} &= \{z = re^{i\phi}/\beta < r < R, -\theta < \phi < \theta\} \\ O_{\hat{\beta},\hat{R}} &= \{z = re^{i\phi}/\hat{\beta} < r < \hat{R}, -\pi/2 < \phi < \pi/2\}. \end{aligned}$$

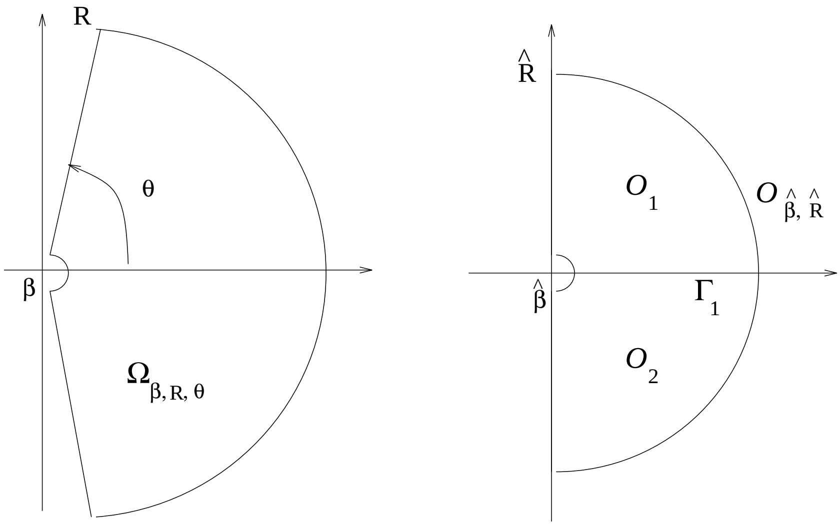

Figure 4: Complex domains

First, we check that ω_k, given by (15), is well defined for $s^0 \in O_{\hat{\beta},\hat{R}}$. After this easy examination, we prove the following lemma (easy too) :

Lemma 4 *Let $(\hat{\beta}, \hat{R})$ be given with $0 < \hat{\beta} < \hat{R}$. Then there exists (β, R, θ) in $\mathbb{R}^{+*} \times \mathbb{R}^{+*} \times [0, \pi/2[$ with $\beta < R$ such that $\omega_k(s^0) \in \Omega_{\beta,R,\theta}$ for all $s^0 \in O_{\hat{\beta},\hat{R}}$.*

This lemma ensures that for any $s^0 \in O_{\hat{\beta},\hat{R}}$, there will be no pole for the hyperbolic tangent th. Then we see easily that the function

$$f(s^0) = r_k \text{th } (\omega_{k'}(s^0)\varepsilon_{k'})\omega_{k'} - r_{k'}\text{th } (\omega_k(s^0)\varepsilon_k)\omega_k(s^0),$$

is analytic on $O_{\hat{\beta},\hat{R}}$. We define O_1 as the $y > 0$ part of $O_{\hat{\beta},\hat{R}}$, O_2 as the $y < 0$ part of $O_{\hat{\beta},\hat{R}}$ and Γ_1 as the $y = 0$ part (see Figure 4). T. Bousch [2] suggested us to look for the sufficient condition $(k' = 1, k = 2)$:

$$\begin{cases} (21) \qquad \text{Im } s^0 > 0 \Rightarrow \text{Im } f(s^0) > 0 \text{ in } O_1, \\ (22) \qquad f \text{ is nonzero on } \Gamma_1, \\ (23) \qquad \text{Im } s^0 < 0 \Rightarrow \text{Im } f(s^0) < 0 \text{ in } O_2, \end{cases}$$

where Im denotes the imaginary part of a complex. It appears that conditions (21) and (23) have the prose to be additive. Then, it is sufficient that f satisfies (22), and the function $s^0 \mapsto \text{th } (\omega_2(s^0)\varepsilon_2)\omega_2(s^0)\varepsilon$ satisfy the conditions (21) and (23) to retrieve the property for f. We stress that $\varepsilon_2 = -\varepsilon < 0$.

As the real part of s^0 is nonnegative and $s^0 \neq 0$, it suffices to prove :

$$\text{Im } (s^0) > 0 \Rightarrow \exists \varepsilon_0 / \left\{ \begin{array}{l} \text{Im } (\omega_k) > 0 \\ \Re(\omega_k) \geq \varepsilon_0 > 0 \end{array} \right. \tag{24}$$

and

$$\left\{ \begin{array}{l} \text{Im } (\omega) > 0 \\ \Re(\omega) \geq \varepsilon_0 > 0 \end{array} \right. \Rightarrow \text{Im } (\omega \text{th } (\omega)) > 0, \tag{25}$$

to have (21). To prove (24), one studies the sign of the imaginary part of ω_k and Lemma 4 gives the property on the real part of ω_k thanks to a compactness argument. A carefull study of $\omega \mapsto \omega \text{th } \omega$ proves (25). The proof about O_2 is very similar. Finally, a simple but tedious calculation on Γ_1 enables to prove (22). This completes the proof. □

5. Conclusion

In this article, we described the two types of instabilities for the 1D flow of two VEF. Then, rigorous derivation of the corresponding OSE enabled us to stress the applicability of the various OSE to experiments. Also, we discussed the Gaster relation that links two of them ((7) and (9)) and proved that it is not justified although it is commonly used in comparisons between experiments and computations. We gave some hints of deeper understandings through absolute and convective instabilities [11] [17], but also through pseudo-spectra [15].

We proved that the OSE (9) under long wave perturbations could trigger stable modes not found by (8). On the opposite, we proved that detection of unstable modes by (8) or (9) in the long wave limit are equivalent. Moreover, we gave a precise discussion of the study of long wave asymptotic stability of two VEF that might be of interest for experiments.

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