Derivation of a Viscous KP Equation Including Surface Tension, and Related Equations

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Abstract

The aim of this article is to derive surface wave models in the presence of surface tension and viscosity. Using the Navier-Stokes equations with a free surface, flat bottom and surface tension, we derive the viscous 2D Boussinesq system with a weak transverse variation. The assumed scale of transverse variation is larger than the one along the main propagation direction. This Boussinesq system is proved to be consistent with the Navier-Stokes equations. This system is only an intermediate result that enables us to derive the Kadomtsev-Petviashvili (KP) equation which is a 2D generalization of the KdV equation. In addition, we get the 1D KdV equation, and lastly the Boussinesq equation. All these equations are derived for general initial conditions either slipping (Euler's fluid) or sticking (Navier-Stokes fluid) with a given profile in the boundary layer different from the Euler's one. We discuss whether the Euler's initial condition is physical.

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List of Figures

1 Introduction

1.1 Motivation

Understanding the evolution of water waves is a longstanding problem. For instance, 130 years ago, Stokes discussed a wave (Stokes (1880)) which was named after him. This wave was still being discussed more than one century later (Hasimoto and Ono (1972)) !

One of the striking phenomena is the existence of surface waves that travel almost without shape modification. Such a behavior is an impetus to investigate the dynamics of these waves.

The first motivation is to understand how they may disperse and yet remain without shape modification for some time. It was discovered that there is such a regime in which nonlinearity compensates dispersion (with no dissipation).

Beyond mathematical and industrial considerations, mathematicians and physicists define the water waves as the waves at the free surface of a fluid flowing in a channel of (often) flat bottom, without meniscus (appart in Mei and Liu (1973)). Since the Euler or Navier-Stokes equations are rather prone to faithfully model such a fluid, one could be satisfied with either of these models. Yet, the Direct Numerical Simulation is too expensive and so, asymptotic models are required. In such models, one assumes a regime of small dimensionless parameters and makes expansions of the equations to derive a simplified model with less fields and less dimensions.

1.2 Literature

The techniques are very different depending on whether one chooses Euler or Navier-Stokes equations as a model for the fluid.

On the one hand, for an inviscid flow (Euler), one assumes very often irrotationality. Then one may use a potential function and a Dirichlet to Neumann operator in the Zakharov-Craig-Sulem formulation. Numerous articles use this formulation and a review of the mathematical proofs has been published in Lannes (2013). Some authors used a velocity-pressure formulation, but they still assumed irrotationnality (Iguchi (2006)). The derivation of Kadomtsev-Petviashvili (KP) equation assumes a predominant propagation direction along x and a weakly transverse propagation along y. It was initially done for an inviscid fluid in Kadomtsev and Petviashvili (1970). This approximation was justified in Lannes (2013) (subsection 7.2). In Lannes (2003), the author proved that the sum of one wave propagating to the right and one to the left, both obeying a KP equation, is consistent (when ε tends to zero) on $[0, T_0/\varepsilon]$ with the Boussinesq system. A precise definition of consistency may be found in Lannes (2013) (Def. 5.1). Roughly, a function u solution of $Au = 0$ is said to be consistent with the operator B on an interval I_ε up to $\mathrm{O}(\varepsilon^k)$ if $Bu = \mathrm{O}(\varepsilon^k)$ in the suitable function space and on I_ε.

A tough difficulty is that two different scales of the transverse velocity are choosen in the literature. The classical choice drives to write $\sqrt{\varepsilon}\partial_y\Psi$ for the vertical velocity. The (implicit) assumption that $\partial_y\Psi = \mathrm{O}(1)$ enables then to get rid of some terms with a coefficient $\varepsilon^{3/2}$ (because $\sqrt{\varepsilon}\mathrm{O}(\varepsilon^{3/2}) = \mathrm{O}(\varepsilon^2)$). But these terms remain in the more general case where one replaces $\sqrt{\varepsilon}\partial_y\Psi$ by v. This (implicit) assumption is discussed in Remark 3 of Lannes and Saut (2006) and here in Remark 3. It leads to different Boussinesq systems and KP equations.

In the seminal article Alvarez-Samaniego and Lannes (2008), the authors did justify all the classical models such as the shallow-water equations, the Boussinesq system, the KP equation with a variable bottom, thanks to a nonlocal energy adapted to the equations.

In Ming et al. (2012a), the authors proved that a dimensionless water wave system in an infinite strip under the influence of gravity and surface tension has a unique solution on $[0, T/\varepsilon]$. More precisely, if the initial solution is sufficiently regular and small ($\mathrm{O}(\sqrt{\varepsilon})$), there exists a solution, on this time interval $[0, T/\varepsilon]$, that will remain small ($\mathrm{O}(\sqrt{\varepsilon})$). In Ming et al. (2012b), the same authors proved that, on the same time interval, these solutions can be accurately approximated by sums of solutions of two decoupled KP equations. They improved the results of Lannes and Saut (2006) by taking surface tension and a variable bottom into consideration.

The Boussinesq *equation* is a generalization of the KdV equation for waves moving both to the right and to the left. So as to derive it, we start from the Boussinesq system. Instead of changing to a frame moving to the right, as is done for KdV, we remain in the same coordinate system and derive higher order equation (perturbed wave equation). This was done in Johnson (1997) (p. 216-219) for an inviscid fluid.

On the other hand, when one assumes the fluid to be viscous, one may not use the potential function. In-between, a recent article (Castro and Lannes (2015)) generalized the potential Zakharov-Craig-Sulem formulation to the rotational case (but not viscous). Anyway, the number of boundary conditions changes when viscosity is taken into account. Viscosity has been considered for water waves since 1895 (Boussinesq (1895)) even though scarcely. The dynamic of viscous water waves on finite depth was investigated more recently in Kakutani and Matsuuchi (1975). The authors derived the viscous KdV equation from unspecified initial conditions (and so of Euler type) but made the error of using a Fourier transform in time while the problem is of Cauchy type. Such a solution is initially slipping while the Navier-Stokes equation prohibits such a behavior. The plausibility of such non-well prepared (slipping instead of sticking) initial conditions is discussed in remark 5 of Le Meur (2015) and remark 4 below.

The authors of Liu and Orfila (2004), and the coauthors of P.L.-F. Liu in subsequent articles, investigated the viscous Boussinessq system with slipping initial condition, and validated their KdV equation by some experiments. For instance they experimentally proved the reverse flow in the boundary layer, which is predicted by theory.

Although it was not done in Sobolev spaces, the derivation of the viscous Boussinesq system (1D and isotropic 2D) and 1D KdV equation with a non-slipping initial condition was done in Le Meur (2015) without surface tension. It was partially the goal of Le Meur (2015) not to rely on the irrotationnality assumption so as to derive the Boussinesq system and KdV equation for a viscous fluid. This could be achieved thanks to the fact that one of the equations contains $u_z = \mathrm{O}(\varepsilon)$ which is the zeroth order of the irrotationnality assumption. In Le Meur (2015), a more detailed bibliography was given and the reason why all the preceding articles did not derive the same KdV equation was explained.

In this article, we intend to improve some results concerning inviscid fluids (KP, Boussinesq equation, surface tension) by taking viscosity and surface tension into consideration.

The outline of this article is as follows. In section 2, we present the geometry and the equations and state the viscous Boussinesq system in 2D with surface tension and a predominant propagation direction. This result is needed to get the new results on the viscous 1D KdV equation in section 3 (see (3.2)), the viscous KP equation (2D generalization of KdV with a weak transverse propagation) in section 4 (see (4.3)), and the viscous Boussinesq equation in section 5 (see (5.3) and (5.4)). Finally, we justify that our Boussinesq system is consistent with the Navier-Stokes equations in Section 6. All these derivations are done for a viscous fluid with surface tension and non-slipping initial conditions (viscous initial condition).

2 2D geometry with viscosity and surface tension

Below, we define the geometry, write the equations, and then make these equations dimensionless (subsection 2.1). These equations model a propagation predominantly in the x-direction. This will enable us to make a linear analysis with the dispersion relation and an asymptotic of the phase velocity in subsection 2.2. A discussion of the relevance of viscosity and surface tension will be given in subsection 2.3. We will then use the results of Le Meur (2015), where the 1D viscous isotropic Boussinesq system and KdV equation are derived (with no surface tension), so as to state the weakly transverse 2D viscous Boussinesq system (subsection 2.4). Our present derivation is new because it includes surface tension and a propagation mainly along x and weakly along y. Since the already published results are very close, some proofs are omitted.

2.1 Navier-Stokes equations

Let $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$ be the velocity of a fluid in a 3-D domain $\tilde{\Omega} = \{(\tilde{x}, \tilde{y}, \tilde{z}) \ / \ (\tilde{x}, \tilde{y}) \in \mathbb{R}^2, \ \tilde{z} \in (-h, \tilde{\eta}(\tilde{x}, \tilde{y}, \tilde{t}))\}$. So we assume the bottom is flat and the free surface is characterized by $\tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{y}, \tilde{t})$ with $\tilde{\eta}(\tilde{x}, \tilde{y}, \tilde{t}) > -h$ (the bottom does not get dry). Let $\tilde{p}$ be the pressure and $\tilde{\mathbf{D}}[\tilde{\mathbf{u}}]$ the symmetric part of the velocity gradient. The dimensionless domain is illustrated in Fig. 1. We also denote ρ the density of the fluid, ν the viscosity of the fluid, g the

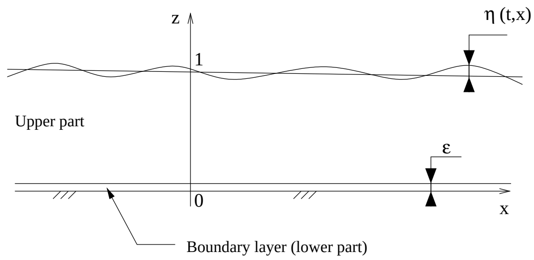

Figure 1: The dimensionless domain

gravitational acceleration, $\mathbf{k}$ the unit vertical vector, $\mathbf{n}$ the outward unit normal to the upper frontier of $\tilde{\Omega}$, σ the surface tension coefficient, and $\tilde{p}_{atm}$ the atmospheric pressure. The original system reads:

$$\left\{ \begin{array}{ll} \rho \left(\dfrac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\mathbf{u}}.\tilde{\nabla}\tilde{\mathbf{u}} \right) - \nu \tilde{\Delta}\tilde{\mathbf{u}} + \tilde{\nabla}\tilde{p} = -\rho g \mathbf{k} & \text{in } \tilde{\Omega}, \\ \widetilde{\mathrm{div}} \ \tilde{\mathbf{u}} = 0 & \text{in } \tilde{\Omega}, \\ \left(-\tilde{p}\mathbf{I} + 2\nu \tilde{\mathbf{D}}[\tilde{\mathbf{u}}] \right) .\mathbf{n} = -\tilde{p}_{atm}\mathbf{n} + \sigma \kappa \mathbf{n} & \text{on } \tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t}), \\ \tilde{\eta}_{\tilde{t}} + \tilde{u}\tilde{\eta}_{\tilde{x}} + \tilde{v}\tilde{\eta}_{\tilde{y}} - \tilde{w} = 0 & \text{on } \tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t}), \\ \tilde{\mathbf{u}} = 0 & \text{on } \tilde{z} = -h, \end{array} \right. \tag{2.1}$$

where we write the second order tensors and the vectors with bold letters. The differentiated functions are denoted either $\partial f/\partial x$, f_x, or $\partial_x f$. The surface tension term includes the radius of curvature κ given by geometrical computations:

$$\kappa = \frac{(1+\tilde{\eta}_{\tilde{y}}^2)\tilde{\eta}_{\tilde{x}\tilde{x}} + (1+\tilde{\eta}_{\tilde{x}}^2)\tilde{\eta}_{\tilde{y}\tilde{y}} - 2\tilde{\eta}_{\tilde{x}}\tilde{\eta}_{\tilde{y}}\tilde{\eta}_{\tilde{x}\tilde{y}}}{(1+\tilde{\eta}_{\tilde{x}}^2+\tilde{\eta}_{\tilde{y}}^2)^{3/2}}.$$

Of course, we need to add an initial condition and conditions at infinity.

So as to get dimensionless fields and variables, we need to choose a characteristic horizontal length l which is the wavelength along the propagation direction, a characteristic vertical length h which is the water's height, and the amplitude A of the propagating perturbation. Moreover, let U, V, W, P be the characteristic horizontal velocity along x, horizontal velocity along y, vertical velocity and pressure respectively. Since the zeroth order equation is a wave equation of dispersion relation $\omega = \sqrt{gh}k_x(1+k_y^2/(2k_x^2)) + o(k_y^2)$, one may infer that a good scaling for the y direction is if the first extra term is of the same order of magnitude as ε. So $k_y = \mathrm{O}(\varepsilon^{1/2})$ and the interesting scale for $\tilde{y}$ is $\mathrm{O}(\varepsilon^{-1/2})$ more than for the x direction. We may then define:

$$\begin{gathered} c_0 = \sqrt{gh},\ \varepsilon = \frac{A}{h},\ \beta = \frac{h^2}{l^2},\ U = \varepsilon c_0, V = \varepsilon c_0, \\ W = \sqrt{\varepsilon} c_0,\ P = \rho g A,\ \mathrm{Re} = \frac{\rho c_0 h}{\nu}, \mathrm{Bo} = \frac{\rho g l^2}{\sigma}, \end{gathered}$$

where c_0 is the phase velocity. Then, one may make the fields dimensionless and unscaled:

$$\tilde{u} = Uu,\ \tilde{v} = Vv,\ \tilde{w} = Ww,\ \tilde{p} = \tilde{p}_{atm} - \rho g \tilde{z} + Pp,\ \tilde{\eta} = A\eta,$$

and the variables too:

$$\tilde{x} = lx,\ \tilde{y} = \varepsilon^{-1/2} l y,\ \tilde{z} = h(z-1),\ \tilde{t} = t\, l/c_0. \tag{2.2}$$

We also make the Boussinesq approximation (β and ε of the same order of magnitude) and even $\beta = \varepsilon$. Notice that we take the scale used by Lannes and Saut (2006) ($V = \varepsilon c_0$) and not the one taken by Johnson (1997) ($V = \varepsilon^{3/2} c_0$). The difference will be highlighted in Remark 3.

With these definitions, the new system with the new fields, variables and outward unit normal, still denoted $\mathbf{n}$, writes in the new domain $\Omega_t = \{(x, y, z), (x, y) \in \mathbb{R}^2,\ z \in$

$(0, 1 + \varepsilon\eta(x, y, t))\}$ (see Figure 1):

$$\begin{cases} u_t + \varepsilon u u_x + \varepsilon^{3/2} v u_y + w u_z - \dfrac{\sqrt{\varepsilon}}{\mathrm{Re}}\left(u_{xx} + \varepsilon u_{yy} + \frac{u_{zz}}{\varepsilon}\right) + p_x = 0 & \text{in } \Omega_t, \\ v_t + \varepsilon u v_x + \varepsilon^{3/2} v v_y + w v_z - \dfrac{\sqrt{\varepsilon}}{\mathrm{Re}}\left(v_{xx} + \varepsilon v_{yy} + \frac{v_{zz}}{\varepsilon}\right) + \dfrac{p_y}{\sqrt{\varepsilon}} = 0 & \text{in } \Omega_t, \\ w_t + \varepsilon u w_x + \varepsilon^{3/2} v w_y + w w_z - \dfrac{\sqrt{\varepsilon}}{\mathrm{Re}}(w_{xx} + \varepsilon w_{yy} + \frac{w_{zz}}{\varepsilon}) + p_z = 0 & \text{in } \Omega_t, \\ u_x + \sqrt{\varepsilon} v_y + w_z/\varepsilon = 0 & \text{in } \Omega_t, \\ (\eta - p)\mathbf{n} + \frac{1}{\mathrm{Re}} \begin{pmatrix} 2\sqrt{\varepsilon} u_x & (\varepsilon u_y + \sqrt{\varepsilon} v_x) & u_z + w_x \\ (\varepsilon u_y + \sqrt{\varepsilon} v_x) & 2\varepsilon v_y & (v_z + \sqrt{\varepsilon} w_y) \\ u_z + w_x & (v_z + \sqrt{\varepsilon} w_y) & 2\dfrac{w_z}{\sqrt{\varepsilon}} \end{pmatrix} .\mathbf{n} & \\ \qquad = \dfrac{1}{\mathrm{Bo}} \dfrac{(1 + \varepsilon^4 \eta_y^2)\eta_{xx} + \varepsilon(1 + \varepsilon^3 \eta_x^2)\eta_{yy} - 2\varepsilon^4 \eta_x \eta_y \eta_{xy}}{(1 + \varepsilon^3 \eta_x^2 + \varepsilon^4 \eta_y^2)^{3/2}} .\mathbf{n} & \text{on } z = 1 + \varepsilon\eta, \\ \eta_t + \varepsilon u \eta_x + \varepsilon^{3/2} v \eta_y - w/\varepsilon = 0 & \text{on } z = 1 + \varepsilon\eta, \\ \mathbf{u} = 0 & \text{on } z = 0. \end{cases} \tag{2.3}$$

Simple computations give the outward non-unit normal $\mathbf{n} = (-\varepsilon\eta_x, -\varepsilon\eta_y, 1)$.

2.2 Linear theory

The whole subsection below is a straightforward modification of the case of a viscous fluid without surface tension, fully studied in Le Meur (2015).

2.2.1 Dispersion relation

We are looking for small fields. So we linearize the system (2.3). We get, after simplification of the dynamic condition:

$$\begin{cases} u_t - \frac{\sqrt{\varepsilon}}{\mathrm{Re}}(u_{xx} + \varepsilon u_{yy} + u_{zz}/\varepsilon) + p_x = 0 & \text{in } \mathbb{R}^2 \times [0,1], \\ v_t - \dfrac{\sqrt{\varepsilon}}{\mathrm{Re}}(v_{xx} + \varepsilon v_{yy} + v_{zz}/\varepsilon) + \sqrt{\varepsilon} p_y = 0 & \text{in } \mathbb{R}^2 \times [0,1], \\ w_t - \dfrac{\sqrt{\varepsilon}}{\mathrm{Re}}(w_{xx} + \varepsilon w_{yy} + w_{zz}/\varepsilon) + p_z = 0 & \text{in } \mathbb{R}^2 \times [0,1], \\ u_x + \sqrt{\varepsilon} v_y + w_z/\varepsilon = 0 & \text{in } \mathbb{R}^2 \times [0,1], \\ (u_z + w_x)/\mathrm{Re} = 0 & \text{on } z = 1, \\ (v_z + \sqrt{\varepsilon} w_y)/\mathrm{Re} = 0 & \text{on } z = 1, \\ (\eta - p) + \dfrac{2w_z}{\mathrm{Re}\sqrt{\varepsilon}} = \dfrac{1}{\mathrm{Bo}}(\eta_{xx} + \varepsilon\eta_{yy}) & \text{on } z = 1, \\ \eta_t - w/\varepsilon = 0 & \text{on } z = 1, \\ \mathbf{u} = 0 & \text{on } z = 0. \end{cases} \tag{2.4}$$

In order to eliminate the pressure, we differentiate $(2.4)_1$ with respect to z and $(2.4)_3$ with respect to x and compute their difference so as to simplify p_{xz}. Symmetrically, we differentiate $(2.4)_2$ with respect to z and $(2.4)_3$ with respect to y and compute their difference (with a $\sqrt{\varepsilon}$ coefficient). Pressure is then eliminated in two equations on three:

$$u_{zt} - w_{xt} - \frac{\sqrt{\varepsilon}}{\mathrm{Re}}\left((u_{xxz} - w_{xxx}) + \varepsilon(u_{yyz} - w_{yyx}) + (u_{zzz} - w_{xzz})/\varepsilon\right) = 0,$$
$$v_{zt} - \sqrt{\varepsilon} w_{yt} - \frac{\sqrt{\varepsilon}}{\mathrm{Re}}\left((v_{xxz} - \sqrt{\varepsilon} w_{xxy}) + \varepsilon(v_{yyz} - \sqrt{\varepsilon} w_{yyy}) + (v_{zzz} - \sqrt{\varepsilon} w_{yzz})/\varepsilon\right) = 0.$$

So as to eliminate u and v thanks to the incompressibility $(2.4)_4$ from the previous system, we differentiate the first equation with respect to x and the second with respect to y. After some simplifications, a good combination of the two gives (thanks to $(2.4)_4$):

$$(\partial_z^2 + \varepsilon\partial_x^2 + \varepsilon^2\partial_y^2)(-\mathrm{Re}\sqrt{\varepsilon}\partial_t + (\partial_z^2 + \varepsilon\partial_x^2 + \varepsilon^2\partial_y^2))w = 0. \tag{2.5}$$

Let w be of the form $\mathcal{A}(z)\exp i(k_x x + k_y y - \omega t)$ with non-negative k_x, k_y and a (complex) pulsation ω. We can define a complex parameter with non-negative real part, similar to the one used by Kakutani and Matsuuchi (1975):

$$\mu^2 = \varepsilon k^2 - i\omega\mathrm{Re}\sqrt{\varepsilon}, \tag{2.6}$$

where $k^2 = k_x^2 + \varepsilon k_y^2$. Thanks to this notation, the solutions of (2.5) are such that

$$\begin{aligned}\mathcal{A}(z) = C_1\cosh\sqrt{\varepsilon}k(z-1) + C_2\sinh\sqrt{\varepsilon}k(z-1)\\ + C_3\cosh\mu(z-1) + C_4\sinh\mu(z-1).\end{aligned} \tag{2.7}$$

The surface tension did not appear yet because it is only in the boundary conditions.

Up to now we have eliminated u, v and p only in the volumic equations. We still have to use the boundary conditions of (2.4) to find the conditions on the remaining field w.

Easy computations on the boundary conditions of (2.4), similar to the ones done in Le Meur (2015), give:

$$\begin{aligned}&w_z(0) = 0,\\ &w(0) = 0,\\ &w_{zz}(1) - \varepsilon w_{xx}(1) - \varepsilon^2 w_{yy}(1) = 0,\\ &w_{xx} + \varepsilon w_{yy}(1) - w_{ztt}(1) + \tfrac{3\sqrt{\varepsilon}}{\mathrm{Re}}(w_{xxzt} + \varepsilon w_{yyzt}) + \tfrac{1}{\mathrm{Re}\sqrt{\varepsilon}}w_{zzzt}(1) = \tfrac{1}{\mathrm{Bo}}(\partial_x^2 + \varepsilon\partial_y^2)^2 w.\end{aligned} \tag{2.8}$$

The solutions (2.7) satisfies a homogeneous linear system in the constants C_1, C_2, C_3, C_4. Its matrix is:

$$\begin{pmatrix} \sqrt{\varepsilon}k\sinh(\sqrt{\varepsilon}k) & -\sqrt{\varepsilon}k\cosh(\sqrt{\varepsilon}k) & \mu\sinh\mu & -\mu\cosh\mu \\ \cosh(\sqrt{\varepsilon}k) & -\sinh(\sqrt{\varepsilon}k) & \cosh\mu & -\sinh\mu \\ 2k^2\varepsilon & 0 & \mu^2 + \varepsilon k^2 & 0 \\ -k^2 - \frac{k^4}{\mathrm{Bo}} & \sqrt{\varepsilon}\omega^2 k + \frac{2i\varepsilon\omega k^3}{\mathrm{Re}} & -k^2 - \frac{k^4}{\mathrm{Bo}} & \frac{2\mu\sqrt{\varepsilon}i\omega k^2}{\mathrm{Re}} \end{pmatrix}, \tag{2.9}$$

where $k^2 = k_x^2 + \varepsilon k_y^2$. It suffices to compute its determinant to get the dispersion relation:

$$\begin{aligned}4\varepsilon k^2\mu(\varepsilon k^2 + \mu^2) + 4\mu k^3\varepsilon^{3/2}(\mu\sinh(k\sqrt{\varepsilon})\sinh\mu - k\sqrt{\varepsilon}\cosh(k\sqrt{\varepsilon})\cosh\mu)\\ - (\varepsilon k^2 + \mu^2)^2(\mu\cosh(k\sqrt{\varepsilon})\cosh\mu - k\sqrt{\varepsilon}\sinh(k\sqrt{\varepsilon})\sinh\mu)\\ - (k + k^3/\mathrm{Bo})\sqrt{\varepsilon}\mathrm{Re}^2(\mu\sinh(k\sqrt{\varepsilon})\cosh\mu - k\sqrt{\varepsilon}\cosh(k\sqrt{\varepsilon})\sinh\mu) = 0.\end{aligned} \tag{2.10}$$

This relation is identical to the one of Le Meur (2015) appart from the surface tension term and the new definition of k for the 3D geometry ($k^2 = k_x^2 + \varepsilon k_y^2$). Our process of non-dimensionnalizing makes a difference between x and z. This is why Kakutani and Matsuuchi (1975) have k terms, and we have $k\sqrt{\varepsilon}$ terms instead. Here, in addition, a difference is made between x and y

2.2.2 Asymptotic of the phase velocity (very large Re)

In this subsection, we state the following Proposition concerning the phase velocity (where $k = \sqrt{k_x^2 + k_y^2}$):

Proposition 1. *Under the assumptions*

$$\begin{array}{ll} k\sqrt{\varepsilon}\,\mathrm{Re}\, c \to +\infty, & \mathrm{Re} \to +\infty, \\ k = \mathrm{O}(1), & c = \mathrm{O}(1)\ \ (\textit{and } c \textit{ bounded away from } 0), \\ \varepsilon \to 0, & \mathrm{Bo} \textit{ bounded away from } 0, \end{array}$$

if there exists a complex phase velocity $c = \omega/k$ solution of (2.10), then it is such that:

$$c = \sqrt{\left(1 + \frac{k^2}{\mathrm{Bo}}\right)\frac{\tanh(k\sqrt{\varepsilon})}{k\sqrt{\varepsilon}}} - \frac{e^{i\pi/4}\,\mathrm{Re}^{-1/2}(k\sqrt{\varepsilon})^{1/4}}{2\tanh^{3/4}(k\sqrt{\varepsilon})} + o(\varepsilon^{-1/4}\mathrm{Re}^{-1/2}). \tag{2.11}$$

Moreover, the decay rate in our finite-depth geometry does not depend on the surface tension but only on the viscosity. It is:

$$Im(\omega) = Im(kc) = \frac{-1}{2\sqrt{2}}\frac{k^{5/4}\varepsilon^{1/8}}{\sqrt{\mathrm{Re}}\tanh^{3/4}(k\sqrt{\varepsilon})} + o(\varepsilon^{-1/4}\mathrm{Re}^{-1/2}). \tag{2.12}$$

Boussinesq and Lamb already published a decay rate, in infinite depth. It was different from ours. The reason is that our geometry is finite while their's is infinite. As a consequence, one should come back before the non-dimensionnalizing process to have common equations.

Appart from the surface tension terms and the change of notation of k, the proof is not different from the one of Proposition 1 in Le Meur (2015).

To what extent is surface tension relevant ?

2.3 Relevance of surface tension and viscosity

As was explained in the previous subsection, surface tension yields only a real term in the phase velocity. So it does not give dissipation, but only dispersion. On the contrary, viscosity influences both the real and the imaginary part of the phase velocity. Although they do not act on the same part of the spectrum, we will try to compare these two effects.

We choose two different fluids (water and mercury) in order to discuss the relevance of surface tension and viscosity. Their physical parameters are listed in Table 1.

	$\sigma(Nm^{-1})$	$\rho(\mathrm{kg}\,m^{-3})$	$\nu(\mathrm{Pa}\,s)$	$1/\mathrm{Re}$	$1/\mathrm{Bo}$
water 20^oC	$7.3\,10^{-2}$	10^3	10^{-3}	$3.2\,10^{-7}/h^{3/2}$	$7.4\,10^{-6}/l^2$
mercury 20^oC	$4.4\,10^{-1}$	$1.3\,10^4$	$1.56\,10^{-3}$	$3.8\,10^{-8}/h^{3/2}$	$3.5\,10^{-6}/l^2$

Table 1: Physical parameters for two fluids

Typical values of σ, ρ, ν, and the Re and Bo associated numbers are given in Table 1. Depending on h or l, the Re or Bo dimensionless parameters are greater or less than 1.

The critical values for each of them are given in Table 2. One must be cautious with this because the $1/\mathrm{Bo}$ is before a k^2 while $\mathrm{Re}^{-1/2}$ is before a $k^{-1/2}$ (for long waves). It is well-known that surface tension regularizes short waves that are not in the asymptotic considered here.

	$h_{crit\text{–}\mathrm{Re}}$	$l_{crit\text{–}\mathrm{Bo}}$
water 20^oC	$4.6\,10^{-5}$	$2.7\,10^{-3}$
mercury 20^oC	$1.1\,10^{-5}$	$1.9\,10^{-3}$

Table 2: Critical values of length (in m) for which $1/\mathrm{Re}$ or $1/\mathrm{Bo}$ is 1.

It appears that viscosity is not so relevant as surface tension. Furthermore, so as to compare $1/\mathrm{Bo}$ and ε, and under the Boussinesq approximation, one may compute the ratio

$$\frac{1}{\mathrm{Bo}}/\varepsilon = \frac{\sigma}{\rho g h^2}.$$

The critical value of h, for which this ratio is 1, is $h_{app} = \sqrt{\sigma/(\rho g)}$. This other critical value is either $2.7\,mm$ for water or $1.9\,mm$ for mercury. In other words if $h > h_{app}$, then the surface tension term would be irrelevant and even less important than gravity measured by ε in the equation. In most current experiments, h is more than $0.1m$ and so, surface tension should (globally) not be taken into account. As D. Lannes says: "for coastal waves of characteristic length $L_x = 10m$, capillary effects represent only 0.0003% of the gravity effects" (Lannes (2013) section 9.1.2).

Surface tension should be irrelevant for flows with h_0 more than $1\,cm$. Yet, even if the initial flow satisfies this, it may or may not remain true. For instance, in case of flows with ripples created by the wind or in case of wave breaking, very short wavelengths appear. See section 9.1.2 of Lannes (2013) for some references.

In the case of a flow in a shallow channel (some mm height), and for mercury, a depression solitary wave was predicted (as early as the initial article Korteweg and de Vries (1895)) and observed in Falcon et al. (2002). In this experiment, $A = 0.064\,mm$, $h = 2.12\,mm$. As a consequence, Re=2547 and viscosity should have been taken into account.

In the following, we assume $1/\mathrm{Bo}$ between $\mathrm{O}(\varepsilon)$ ($h \sim 1\,mm$) and $\mathrm{O}(\varepsilon^2)$ ($h^2/l \sim 2\,mm$).

2.4 Viscous Boussinesq System

Let us first discuss the regime binding $\varepsilon, \mathrm{Re}, \mathrm{Bo}$.

The complex phase velocity (2.11) contains gravitational and viscous terms that we want to compare so as to find the regime at which their variations are of the same order of magnitude. The first term is the gravitational term ($\sqrt{(1+k^2/\mathrm{Bo})\tanh{(k\sqrt{\varepsilon})}/(k\sqrt{\varepsilon})}$) which may be expanded when ε tends to zero: $1 - k^2\varepsilon(1-3/(\varepsilon\mathrm{Bo}))/6 + \mathrm{O}(\varepsilon^2) + \mathrm{O}(\mathrm{Bo}^{-2})$. The second is purely viscous and can be expanded: $-\sqrt{2}(1+i)(4\sqrt{k})^{-1}(\mathrm{Re}\sqrt{\varepsilon})^{-1/2} + o(\mathrm{Re}\sqrt{\varepsilon})^{-1/2})$. So if $1/\mathrm{Bo} = \mathrm{O}(\varepsilon)$ or less but $1/(\varepsilon\mathrm{Bo}) \not\sim 1/3$, the variations of c on the gravitational and on the viscous effects are of the same order of magnitude when $\varepsilon(1-3/(\varepsilon\mathrm{Bo}))$ and $(\mathrm{Re}\sqrt{\varepsilon})^{-1/2}$ are of the same order. In this regime of very large Re, studied hereafter, the dependence of Re on ε is such that (to simplify, we assume here

$1/\mathrm{Bo} = o(\varepsilon)$):

$$\mathrm{Re} \sim \varepsilon^{-5/2}. \tag{2.13}$$

This would be wrong if $1/(\varepsilon\mathrm{Bo}) \sim 1/3$ as is well-known for non-viscous fluids. In that case, we would need to go further in the expansion. But since it was assumed above that $1/\mathrm{Bo}$ is between $\mathrm{O}(\varepsilon^2)$ and $\mathrm{O}(\varepsilon)$, we are driven to choose the regime (2.13) and exclude $1/(\varepsilon\mathrm{Bo}) \sim 1/3$.

Our main purpose here is to state an asymptotic system of reduced size from the global Navier-Stokes equations in the whole *moving* domain with surface tension and a predominant propagation direction.

Proposition 2. *Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Let $\eta(\mathbf{x}, t)$ be the free boundary's height. Let $(u^{b,0}(\mathbf{x}, \gamma), v^{b,0}(\mathbf{x}, \gamma))$ for $\gamma \in (0, +\infty)$ (resp. $(u^{u,0}(\mathbf{x}, z), v^{u,0}(\mathbf{x}, z))$ for $z \in (0, 1+\varepsilon\eta(\mathbf{x}, t))$) be the initial horizontal velocity in the boundary layer (resp. in the upper part of the domain). Let $f(\mathbf{x}, \gamma) := u^{b,0}(\mathbf{x}, \gamma) - u^{u,0}(\mathbf{x}, z = 0)$ and $g(\mathbf{x}, \gamma) := v^{b,0}(\mathbf{x}, \gamma) - v^{u,0}(\mathbf{x}, z = 0)$. If for any $\mathbf{x} \in \mathbb{R}^2$, $\gamma \mapsto f$ and $\gamma \mapsto g$ are uniformly continuous in γ and $f_x, g_y \in L^1_\gamma(\mathbb{R}^+)$, then the solution of the Navier-Stokes equation with this given initial condition satisfies the weakly transverse Boussinesq system:*

$$\begin{aligned}
&u_t + \eta_x - \frac{\eta_{xxx}}{\mathrm{Bo}} + \varepsilon u u_x + \varepsilon^{3/2} v u_y - \varepsilon \eta_{xtt}\frac{(z^2-1)}{2} = \mathrm{O}(\varepsilon^2) + \mathrm{O}(\varepsilon/\mathrm{Bo}),\\
&v_t + \sqrt{\varepsilon}\eta_y - \frac{\sqrt{\varepsilon}\eta_{xxy}}{\mathrm{Bo}} + \varepsilon u v_x + \varepsilon^{3/2} v v_y - \varepsilon \eta_{ytt}\frac{(z^2-1)}{2} = \mathrm{O}(\varepsilon^2) + \mathrm{O}(\varepsilon/\mathrm{Bo}),\\
&\eta_t + u_x(\mathbf{x}, z, t) - \frac{\varepsilon}{2}\eta_{xxt}(z^2 - \tfrac{1}{3}) + \varepsilon(u\eta)_x + \sqrt{\varepsilon} v_y + \varepsilon^{3/2}(v\eta)_y - \frac{\varepsilon}{\sqrt{\pi R}}(u_x + \sqrt{\varepsilon} v_y) * \frac{1}{\sqrt{t}}\\
&\quad + \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} \mathrm{div} \begin{pmatrix} (u^{b,0}(\mathbf{x}, \gamma'') - u^{u,0}(\mathbf{x}, z = 0)) \\ \sqrt{\varepsilon}(v^{b,0}(\mathbf{x}, \gamma'') - v^{u,0}(\mathbf{x}, z = 0)) \end{pmatrix} \int_{\gamma'=0}^{\sqrt{\frac{R}{4t}}\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma''\\
&\quad = O(\varepsilon^2) + \mathrm{O}(\varepsilon/\mathrm{Bo}),
\end{aligned} \tag{2.14}$$

where the convolution, denoted with $*$, is in time, the parameters ε, and R have already been defined and $z \in (0, 1 + \varepsilon\eta(\mathbf{x}, t))$.

With the change of notation $k^2 = k_x^2 + k_y^2$, the whole proof is the very same as in Le Meur (2015) and is omitted.

Had we proceeded in the same way as Kakutani and Matsuuchi (1975), we would have distinguished two subdomains: the upper part ($z > \varepsilon$) where viscosity can be neglected, and the lower part ($0 < z < \varepsilon$) which is a boundary layer at the bottom and where viscosity must be taken into account. The resolution in each part would have enabled to match the solution on the common boundary. Instead, we assume overlapping domains.

***Remark* 3.** *As is stressed at the non-dimensionnalizing step, there are two choices for the scale of the transverse horizontal velocity. Either one takes $V = \varepsilon^{3/2} c_0$ (see the book Johnson (1997) for instance or many others), or one uses $V = \varepsilon c_0$ (see the article Lannes and Saut (2006) for instance). Firstly one must notice that since this field is not eliminated (as w is), its scale is relevant. Moreover the latter scaling is more general than the former one because it does not assume v to be $\mathrm{O}(\sqrt{\varepsilon})$. The main difference is that $v_{Lannes-Saut} = \sqrt{\varepsilon} v_{Johnson}$, where one implicitly assumes that the fields $v_{Lannes-Saut}$ and $v_{Johnson}$ are $\mathrm{O}(1)$. To be clear, Lannes-Saut's choice amounts to assume $v = \mathrm{O}(1)$*

although it can be written $\sqrt{\varepsilon}\partial_y\psi$ *! If we plug this* $v = \sqrt{\varepsilon}\partial_y\psi$ *in (2.14), we get the same transverse Boussinesq system as Johnson (1997) (p. 210) because* $\varepsilon^{3/2}v_{Lannes-Saut}$ *transforms into* $\varepsilon^2 v_{Johnson}$ *and goes to* $\mathrm{O}(\varepsilon^2)$. *So the non-dimensionnalizing process of Lannes-Saut is weaker than the one done by many other authors.*

Which of these two assumption is right ? The article Lannes and Saut (2006) gives one argument by exhibiting a v *solution to the linearized system (in their Remark 3) that depends on a free function* f*:*

$$u = 0,\ v = f'(y - \sqrt{\varepsilon}t) - f'(y + \sqrt{\varepsilon}t), \eta = f'(y - \sqrt{\varepsilon}t) + f'(y + \sqrt{\varepsilon}t).$$

It is of size $\mathrm{O}(1)$ *and not* $\mathrm{O}(\sqrt{\varepsilon})$ *and so the assumption* $v = \sqrt{\varepsilon}\partial_y\psi = O(\sqrt{\varepsilon})$ *is wrong. This could also be tested numerically.*

We want to notice here that the representation of the velocity with a potential might have induced the idea that if the potential is physical, then the deduced field (the velocity) is such that $(u, v) = (\partial_x\phi, \sqrt{\varepsilon}\partial_y\phi)$ *and so, that* $v = \mathrm{O}(\sqrt{\varepsilon})$. *Had we started from the velocity pressure formulation, this assumption would not be natural. Yet, such an attempt of explanation does not match reality. The book Johnson (1997) uses the velocity pressure representation as we do and yet assumes* $v = \mathrm{O}(\sqrt{\varepsilon})$, *while Lannes and Saut (2006) use the potential representation and do not make this assumption !*

***Remark* 4.** *It was already noticed in Le Meur (2015) that the double integral term in (2.14) is new and surprising because of its dependence on the initial condition. If we assume an initial flow of Euler type (so that* $u^{b,0} - u^{u,0} = 0$*) or slipping, the double integral is zero. But is it physical ? In other words, does an initial inviscid flow in the boundary layer (where Navier-Stokes applies) establishes (as a Navier-Stokes flow) fast or not ? Can we assume an initial Euler flow and a Navier-Stokes evolution PDE without loss of generality ?*

We claim that the answer is negative for at least two reasons. On the one hand, the characteristic time for the viscous effects to appear is roughly $T_{NSE} = \rho h_0^2/\nu$ *or* $T_{NSE} = \rho l^2/\nu$ *(since* $\rho\partial\tilde{u}/\partial\tilde{t} \simeq \nu\Delta\tilde{u}$*). Then, its ratio with the characteristic time of the inviscid gravity flow (*l/c_0*) is either* $Re\ \sqrt{\varepsilon} = \varepsilon^{-2}$ *or* $\mathrm{Re}/\sqrt{\varepsilon} = \varepsilon^{-3}$ *respectively. Whatever the chosen characteristic time, this ratio is large and the flow in the boundary layer does not establish fast enough. In other words, the flow in the boundary layer establishes from inviscid to viscous, but not so fast as the surface wave travels.*

On the other hand, as is argued in Le Meur (2015), the value of this integral for a typical exponential flow in the boundary layer can be computed. It tends to zero only like $1/\sqrt{t}$ *as is classical for Navier-Stokes flows. As a consequence, one may assume that it goes from slipping (Euler) to exponential (boundary layer) within the time* $(1/\sqrt{t})$ *which is large with respect to the characteristic time of the flow.*

Lastly, as will be seen below, when one makes the KdV change of variable, one looks at large time behavior, and this term vanishes as will be argued. But it remains in the Boussinesq regime.

3 Viscous KdV with surface tension

Starting from (2.14), one may assume the fields do not depend on y, and $v = 0$. So the flow is purely one dimensional (one direction). One may see that the zeroth order of this Boussinesq system is the wave equation. Then one knows that there are two waves propagating in each direction. If we look only for the waves that propagate to the right, one may make a change of variables suggested by the zeroth order equation:

$$\xi = x - t, \ \tau = \varepsilon t.$$

Every term can easily be converted in these new coordinates appart from the convolution (one derivative and a half integration), the new surface tension term and the double integral. The first was treated in a very clean way in Le Meur (2015). The second is new, but very simple. The third one was treated in Le Meur (2015), but its treatment is improved below. After the KdV change of variable, this term writes:

$$+\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(\xi+\frac{\tau}{\varepsilon},\gamma'')-u_x^{u,0}(\xi+\frac{\tau}{\varepsilon},z=0)\right)\times\int_{\gamma'=0}^{\sqrt{\frac{R\varepsilon}{4\tau}}\gamma''}e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' \tag{3.1}$$

The article Le Meur (2015) argued that if the initial horizontal velocity is localized and τ not too small, then $u^{b,0}(\xi+\tau/\varepsilon,\gamma'')-u^{u,0}(\xi+\tau/\varepsilon,z=0)$ will be negligible in comparison with ε. More precisely, if $\xi\mapsto u^{b,0}(\xi,\gamma'')-u^{u,0}(\xi,z=0)$ tends to zero at least as ξ^{-1} when ξ tends to $+\infty$, then the term (3.1) is at least $\mathrm{O}(\varepsilon^2)$.

One may add one more argument if τ is still not too small. The upper bound of the inner integral contains a $\sqrt{\varepsilon}$. So either γ'' is small with respect to $1/\sqrt{\varepsilon}$ and then the most inner integral is small, or γ'' is large (larger than $1/\sqrt{\varepsilon}$) and then, the $u^{b,0}(\xi+\tau/\varepsilon,\gamma'')-u^{u,0}(\xi+\tau/\varepsilon,z=0)$ is small uniformly in $\xi+\tau/\varepsilon$ because of the matching condition on u^0 between the boundary layer and the upper part (at large γ''). So (3.1) is small for two different reasons and we can state the viscous KdV equation with surface tension in the following Proposition.

Proposition 5. *If the initial flow is localized, the KdV change of variables applied to the 1D version of the system (2.14) leads to*

$$2\tilde{\eta}_\tau+3\tilde{\eta}\tilde{\eta}_\xi+\left(\frac{1}{3}-\frac{1}{\varepsilon\mathrm{Bo}}\right)\tilde{\eta}_{\xi\xi\xi}-\frac{1}{\sqrt{\pi R}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\tilde{\eta}_\xi(\xi+\xi',\tau)}{\sqrt{\xi'}}\mathrm{d}\xi'=o(1), \tag{3.2}$$

for not too small times τ, where we set Re$= R\,\varepsilon^{-5/2}$.

We do not replace the upper bound of the convolution (τ/ε) by $+\infty$ for two reasons. On the one hand we do not know how fast this integral converges when τ/ε tends to $+\infty$. On the other hand the τ term reminds us that this integral on ξ' is a mixture of time and space.

The assumption of not too small times τ is relevant since we applied the change of variables so as to look for large times in the KdV regime. This enabled us to eliminate some terms. Mainly the one depending on the initial conditions in the boundary layer. But if we look back at small times (in τ), we may not drop these terms. The behavior in small times cannot be recovered from the KdV equation after the change of variables.

4 Viscous KP equation with surface tension

Below, we derive the viscous KP equation, then discuss the zero-mass (in ξ) constraint.

4.1 Derivation

We recall that we set Re$= R\,\varepsilon^{-5/2}$, and $1/\mathrm{Bo} = \mathrm{O}(\varepsilon)$ but $1/(\varepsilon\mathrm{Bo}) \neq 1/3 + o(1)$. With these assumptions, the change of variables

$$\xi = x - t,\ y = y,\ \tau = \varepsilon t,$$

and the assumption that the initial flow is localized, the system (2.14) writes (since $\varepsilon/\mathrm{Bo} = \mathrm{O}(\varepsilon^2)$):

$$\begin{aligned}
-u_\xi(\xi, y, z, \tau) + \eta_\xi + \varepsilon(u_\tau + uu_\xi - \eta_{\xi\xi\xi}(z^2-1)/2 - \frac{\eta_{\xi\xi\xi}}{\varepsilon\mathrm{Bo}} + \sqrt{\varepsilon}vu_y) &= \mathrm{O}(\varepsilon^2),\\
-v_\xi + \sqrt{\varepsilon}\eta_y + \varepsilon(v_\tau + uv_\xi - \sqrt{\varepsilon}\eta_{y\xi\xi}(z^2-1)/2 - \frac{\eta_{\xi\xi y}}{\sqrt{\varepsilon}\mathrm{Bo}} + \sqrt{\varepsilon}vv_y) &= \mathrm{O}(\varepsilon^2),\\
-\eta_\xi + u_\xi + \varepsilon\left(\eta_\tau + \eta_{\xi\xi\xi}(z^2-1/3)/2 + (u\eta)_\xi + \sqrt{\varepsilon}(v\eta)_y\right) + \sqrt{\varepsilon}v_y &\\
-\frac{\varepsilon}{\sqrt{\pi R}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{u_\xi(\xi+\xi', y, z, \tau) + \sqrt{\varepsilon}v_y(\xi+\xi', y, z, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi' &= \mathrm{O}(\varepsilon^2).
\end{aligned} \tag{4.1}$$

In the above system, we did not write the double integral term from (3.1). Indeed, it was justified in the previous section that this term can be neglected. Moreover, one might prove that the transverse velocity is such that $v(z) = v(z') + \mathrm{O}(\varepsilon)$ in a similar way to Lemma 11 of Le Meur (2015) (roughly, one differentiate $(2.14)_2$ with respect to z, then integrate with respect to time t). So as to go further, since, at the first order $u_\xi = \eta_\xi$, one may justify that $u_\tau = \eta_\tau + o(1)$ like in the inviscid case, and $u = \eta + o(1)$. One may then add the first and third equation:

$$\begin{aligned}
2\varepsilon\eta_\tau + 3\varepsilon\eta\eta_\xi + \left(\frac{\varepsilon}{3} - \frac{1}{\mathrm{Bo}}\right)\eta_{\xi\xi\xi} + \varepsilon^{3/2}vu_y + \varepsilon^{3/2}(v\eta)_y + \sqrt{\varepsilon}v_y&\\
-\frac{\varepsilon}{\sqrt{\pi R}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\eta_\xi(\xi+\xi', y, \tau) + \sqrt{\varepsilon}v_y(\xi+\xi', y, z, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi' &= \mathrm{O}(\varepsilon^2).
\end{aligned} \tag{4.2}$$

In order to eliminate v, one must differentiate with respect to ξ to use the corresponding $(4.1)_2$. Our choice of non-dimensionalization forces us to manage extra terms, but we get the classical KP equation with a viscous term:

$$\begin{aligned}
\left(2\eta_\tau + 3\eta\eta_\xi + \left(\frac{1}{3} - \frac{1}{\varepsilon\mathrm{Bo}}\right)\eta_{\xi\xi\xi}\right)_\xi + \eta_{yy} - \frac{1}{\sqrt{\pi R}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\eta_{\xi\xi}(\xi+\xi', y, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi'&\\
= -\sqrt{\varepsilon}\left(2v\eta_{\xi y} + v_y\eta_\xi + v_{\tau y}\right) + \mathrm{O}(\varepsilon) + \mathrm{O}(1/\mathrm{Bo}),&
\end{aligned}$$

where v satisfies $(4.1)_2$. This completes the proof of the following Proposition.

Proposition 6. *If the initial flow is localized, in the Boussinesq approximation and if* $1/\mathrm{Bo} = \mathrm{O}(\varepsilon)$ *but* $1/\mathrm{Bo} \neq \varepsilon/3 + o(\varepsilon)$, *the surface waves of a viscous fluid propagating predominantly along* x *and weakly along* y *satisfy the viscous KP equation:*

$$\left(2\eta_\tau + 3\eta\eta_\xi + \left(\frac{1}{3} - \frac{1}{\varepsilon \mathrm{Bo}}\right)\eta_{\xi\xi\xi}\right)_\xi + \eta_{yy} - \frac{1}{\sqrt{\pi}R}\int_{\xi'=0}^{\tau/\varepsilon} \frac{\eta_{\xi\xi}(\xi+\xi', y, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi' = \mathrm{O}(\sqrt{\varepsilon}), \quad (4.3)$$

or the equivalent mixed system

$$\begin{aligned} 2\eta_\tau + 3\eta\eta_\xi + \left(\frac{1}{3} - \frac{1}{\varepsilon \mathrm{Bo}}\right)\eta_{\xi\xi\xi} - \frac{1}{\sqrt{\pi}R}\int_{\xi'=0}^{\tau/\varepsilon} \frac{\eta_{\xi}(\xi+\xi', y, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi' + v_y &= \mathrm{O}(\sqrt{\varepsilon}), \\ -v_\xi + \eta_y &= \mathrm{O}(\sqrt{\varepsilon}). \end{aligned} \quad (4.4)$$

One must keep in mind that our choice of scale triggers the $\sqrt{\varepsilon}$ (even inside the convolution) in (4.1). This choice is already discussed in Remark 3. Had we made the same choice of scale for v as Johnson (1997) and many others, we would have replaced v by $\sqrt{\varepsilon}v_J$ in (4.1). Then, the equivalent (4.2) could be simplified by ε and it would write :

$$2\eta_\tau + 3\eta\eta_\xi + \left(\frac{1}{3} - \frac{1}{\varepsilon \mathrm{Bo}}\right)\eta_{\xi\xi\xi} + v_{Jy} - \frac{1}{\sqrt{\pi}R}\int_{\xi'=0}^{\tau/\varepsilon} \frac{\eta_{\xi}(\xi+\xi', y, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi' = \mathrm{O}(\varepsilon),$$

and v_J would satisfy $(4.1)_2$ modified:

$$-v_{J\xi} + \eta_y + \varepsilon\left(v_{J\tau} + uv_{J\xi} - \eta_{y\xi\xi}(z^2-1)/2 - \frac{\eta_{\xi\xi y}}{\varepsilon \mathrm{Bo}}\right) = \mathrm{O}(\varepsilon^{3/2}).$$

Oddly, this would lead to the same KP equation as (4.3) but up to the order $\mathrm{O}(\varepsilon)$ and not $\mathrm{O}(\sqrt{\varepsilon})$. During the proof, we exhibit the $\mathrm{O}(\sqrt{\varepsilon})$ terms which would be $\mathrm{O}(\varepsilon)$ if $v = \sqrt{\varepsilon}v_J$ although they are not kept in the Proposition. So there is no contradiction and the accuracy of KP is tied to the property/assumption that $v = \mathrm{O}(\sqrt{\varepsilon})$ or not.

The viscous KP equation had already been found in Molinet (1996).

4.2 The zero-mass constraint

As is very well discussed in Molinet et al. (2007), the usual KP equation is often written with an operator $\partial_\xi^{-1}\partial_y^2$. Yet such an operator assumes the solution is differentiated (in ξ) from a function that tends to 0 when $\xi \to \pm\infty$. It is proved in Molinet et al. (2007) that although it is not obvious, this assumption is right for inviscid fluids. Their proof is done in two steps.

In the first step, the linear KP equation is solved with inverse Fourier transforms. For more general equations, but only dispersive, like $u_t - Lu_\xi + \partial_\xi^{-1}\partial_{y\,y}u = 0$ (the symbol of L is $\varepsilon \mid X \mid^\alpha$ for $\alpha > 1/2$), the fundamental solution writes

$$G(t, \xi, y) = \mathcal{F}^{-1}_{(X,y')\to(\xi,y)}\left[e^{it(\varepsilon X|X|^\alpha - y'^2/X)}\right],$$

where $\varepsilon = \pm 1$ depends on the KPI or KPII equation (or on the sign of $1/3 - 1/(\varepsilon \mathrm{Bo})$). For the usual KPI and KPII, $\alpha = 2$.

Thanks to the Lebesgue's Dominated convergence theorem and inventive changes of variables, it is proved that the fundamental solution is regular, and is differentiated from a more regular function A (which is $\mathcal{C}(\mathbb{R}^2)\bigcap L^\infty(\mathbb{R}^2)\bigcap \mathcal{C}^1_\xi(\mathbb{R}^2)$) such that $A * u_0 \to 0$ when $\mid \xi \mid \to +\infty$.

In a second step, the nonlinearity is treated through the Duhamel formulation. The solution is then the convolution of the group of the linear equation with the nonlinear term $u\,u_\xi$ and all what is already proved for the linear equation extends to the nonlinear one.

One may wonder whether such an interesting result can be extended to the viscous case. Indeed, the proof is even simpler and the details are left to the interested reader. The Fourier transform of the dissipative term writes

$$\begin{aligned}
\tfrac{-1}{\sqrt{\pi R}}\mathcal{F}_{\xi\to X}\left(\int_0^{+\infty}\frac{\eta_\xi(\xi+\xi',y,\tau)}{\sqrt{\xi'}}\,\mathrm{d}\xi'\right) &= \tfrac{-1}{\sqrt{2\pi^2 R}}\int_{\mathbb{R}_\xi}\int_{\xi'=0}^{+\infty}\frac{\eta_\xi(\xi+\xi')}{\sqrt{\xi'}}e^{-i\xi X}\,\mathrm{d}\xi'\mathrm{d}\xi \\
&= -\tfrac{i\,X}{\sqrt{\pi R}}\hat{\eta}(X)\int_{\xi'=0}^{+\infty}\frac{e^{i\xi' X}}{\sqrt{\mid \xi'\mid}}\mathrm{d}\xi'.
\end{aligned}$$

By a simple change of variable, one may see that the last integral is indeed only a function of the sign of X (denoted $\mathrm{sgn}(X)$) times $1/\sqrt{\mid X\mid}$. This function of $\mathrm{sgn}(X)$ can be computed :

$$\int_{\xi=0}^{+\infty}\frac{e^{i\,\mathrm{sgn}(X)\xi}}{\sqrt{\mid\xi\mid}}\mathrm{d}\xi = \sqrt{\frac{\pi}{2}}(1+i\,\mathrm{sgn}(X)),$$

and the Fourier corresponding term reads:

$$-i\,\hat{\eta}(X)\,\frac{X}{\sqrt{2R\mid X\mid}}(1+i\,\mathrm{sgn}(X)).$$

As a consequence, one could study the fundamental solution

$$G^{NS}(t,\xi,y) = \mathcal{F}^{-1}_{(X,y')\to(\xi,y)}\left[e^{it\left(\varepsilon X|X|^\alpha - y'^2/X + (1+i\,\mathrm{sgn}(X))X(2R|X|)^{-1/2}\right)}\right].$$

The only real part (non-dispersive) inside the exponential is $it\times i\,\mathrm{sgn}(X)X(2R|X|)^{-1/2} = -t\mid X\mid^{1/2}(2R)^{-1/2}$. Its sign is compatible with the dissipation and ensures convergence of all the integrals. So one do not even need to use the Lebesgue dominated convergence theorem and the proof is simpler than in Molinet et al. (2007).

5 Viscous Boussinesq equation with surface tension

The Boussinesq equation is a second order equation that takes into account waves going both to the right and to the left. Since the derivation is straightforward, we only sketch it, following Johnson (1997) (p. 216-219) and state the Proposition 7. The book Johnson (1997) uses a different scaling, but only along the transverse direction which is not used here.

Starting from the Boussinesq system (2.14) written only in 1D (with x, z, t and not x, y, z, t), one may differentiate $(2.14)_3$ with respect to t and $(2.14)_1$ with respect to x. The difference of these two equations writes after some easy computations:

$$\eta_{tt} - \eta_{xx} - \varepsilon\left(u^2 + \frac{\eta^2}{2}\right)_{xx} - \frac{\varepsilon}{3}\eta_{xxtt} + \frac{\eta_{xxxx}}{\mathrm{Bo}} + \frac{\varepsilon}{\sqrt{\pi R}}\eta_{tt} * \frac{1}{\sqrt{t}}$$
$$+ \frac{\varepsilon}{\sqrt{\pi R}}\mathrm{p.v.}\frac{1}{\sqrt{t}}\left[u_x^{u,0}(x, z=0) - \int_{\gamma'=0}^{+\infty} u_{x\gamma}^{b,0}(x,\gamma')\, e^{-\frac{R}{4t}\gamma'^2}\mathrm{d}\gamma'\right] = \mathrm{O}(\varepsilon^2), \quad (5.1)$$

where the convolution is in time and p.v. denotes the principal value as defined in the theory of distributions. The p.v. stems from an integration by parts in which one has a boundary integral

$$\frac{\varepsilon}{\sqrt{\pi R t}}\left[f_x^0(x,\gamma')e^{-\frac{R\gamma'^2}{4t}}\right]_{\gamma'=0}^{+\infty}.$$

The $u^{u,0}$ and $u^{b,0}$ are the initial values of u^u and u^b. They must be provided. Appart from the two viscous terms and the term generated by surface tension, this equation is identical to (3.41) of Johnson (1997). Since we concentrate on $x = \mathrm{O}(1)$, we may write $u = \int_{-\infty}^{x} u_x = -\int_{-\infty}^{x} \eta_t(x', t)\mathrm{d}x' + \mathrm{O}(\varepsilon)$. The above equation is then the Eulerian form of the Boussinesq equation rewritten in (5.3).

Let us now come back to the Lagrangian coordinates and change of fields. In Johnson (1997), the author proposes $X = x + \varepsilon\int_{-\infty}^{x}\eta(x',t)\mathrm{d}x'$ (p. 218), but we consider the following change of variable ($x \to X$) and of field ($\eta \to H$) easier to justify:

$$\begin{aligned} X &= x - \varepsilon\textstyle\int_0^t u(x,t')\mathrm{d}t' \\ H(X,t) &= H(x - \varepsilon\textstyle\int_0^t u(x,t')\mathrm{d}t', t) = \eta(x,t) - \varepsilon\eta^2(x,t). \end{aligned} \quad (5.2)$$

The function H is assumed to be defined on X in the initial domain. Using these definitions, it is straightforward to rewrite (5.1) under the form of a Lagrangian Boussinesq equation with surface tension and viscosity taken into account. Only the term containing the initial horizontal velocities needs some insight. It is the last one in (5.1) and both $u^{u,0}(x, z = 0)$ and $u_{x\gamma}^{b,0}(x,\gamma')$ can be considered as depending on x or on X since an ε appears in front of the whole term and the difference between x and X is $\mathrm{O}(\varepsilon)$. The following Proposition states the two forms of the Boussinesq equation in 1D.

Proposition 7. *Let a flow of a viscous fluid with surface tension in a 2D channel with a flat bottom like the one depicted in Figure 1. Let $u^{u,0}(x, z = 0)$ denote the trace (at the bottom of the upper part) of the initial horizontal velocity. Let $u^{b,0}(x,\gamma)$ denote the initial horizontal velocity in the whole boundary layer. We assume the Boussinesq approximation and $1/\mathrm{Bo} = \mathrm{O}(\varepsilon)$. Under these assumptions, the surface waves obey the viscous Boussinesq equation in the Eulerian form:*

$$\eta_{tt} - \eta_{xx} - \varepsilon\left(u^2 + \eta^2/2\right)_{xx} - \left(\frac{\varepsilon}{3} - \frac{1}{\mathrm{Bo}}\right)\eta_{xxxx} + \frac{\varepsilon}{\sqrt{\pi R}}\eta_{tt} * \frac{1}{\sqrt{t}}$$
$$+ \frac{\varepsilon}{\sqrt{\pi R}}\mathrm{p.v.}\frac{1}{\sqrt{t}}\left(-(\eta_t)_{t=0} - \int_{\gamma'=0}^{+\infty} u_{x\gamma}^{b,0}(x,\gamma')\, e^{-\frac{R\gamma'^2}{4t}}\mathrm{d}\gamma'\right) = \mathrm{O}(\varepsilon^2), \quad (5.3)$$

where the convolution is in time and p.v. *denotes the principal value as defined in the theory of distributions. The Lagrangian form (with (5.2)) writes:*

$$H_{tt}(X,t) - H_{XX} - \frac{3}{2}\varepsilon(H^2)_{XX} - \left(\frac{\varepsilon}{3} - \frac{1}{\mathrm{Bo}}\right) H_{XXXX} + \frac{\varepsilon}{\sqrt{\pi R}} H_{tt} * \frac{1}{\sqrt{t}}$$
$$+ \frac{\varepsilon}{\sqrt{\pi R}} \mathrm{p.v.}\frac{1}{\sqrt{t}} \left[-(H_t(X,t))_{t=0} - \int_{\gamma'=0}^{+\infty} u_{x\gamma}^{b,0}(x,\gamma')\, e^{-\frac{R}{4t}\gamma'^2} \mathrm{d}\gamma' \right] = \mathrm{O}(\varepsilon^2), \quad (5.4)$$

where the initial velocities defined on x can be evaluated either on x or on X.

Notice that the formulae (5.3, 5.4) also have a formulation in terms of $u^{u,0}(x, z = 0) - u^{b,0}(x,\gamma)$ which is physically meaningful.

6 Is the Boussinesq system consistent with Navier-Stokes ?

The weakly transverse Boussinesq system is stated in Proposition 2. We intend to justify here that any of its solution is consistent with the initial Navier-Stokes equations (2.3).

What is consistency ? In Le Meur (2015), the author justifies, but does not prove in any functionnal space, that if there is a solution to Navier-Stokes equations, with a free boundary and a flat bottom, then, under the Bousinesq approximation, the solution satisfies the viscous isotropic Boussinesq system. Such a necessary result deserves to be completed by a sufficient one which is consistency. This is the goal of the present section.

Notice that the consistency as defined in Lannes (2013) (Definition 5.1) is much more rigorous. While we use only expansions, Lannes asked whether a solution to the asymptotic model (assuming it exists) in the asymptotic regime, satisfied the initial full model on a convenient time-interval and in convenient norms, up to a given power of the small parameter. Lannes proved in Lannes (2003) that KP solution (with his scaling) is consistent with the Boussinesq system.

We state below the following Proposition in the 2D case but prove it in 1D.

Proposition 8. *Let $u^u(\mathbf{x}, z, t)$ be the horizontal velocity in the upper part of a flow in a geometry as in Figure 1, and $\eta(\mathbf{x}, t)$ be the height of the free boundary above the fluid, for $\mathbf{x} = (x, y) \in \mathbb{R}^2, z \in (\varepsilon, 1 + \varepsilon\eta(\mathbf{x}, t))$ and nonnegative time t. Let $(u^{u,0}(\mathbf{x}, z), v^{u,0}(\mathbf{x}, z))$ denote the initial horizontal velocity in the upper part (respectively $(u^{b,0}(\mathbf{x},\gamma), v^{b,0}(\mathbf{x},\gamma))$ in the boundary layer with $\gamma = z/\varepsilon$).*

If (u^u, v^u, η) satisfies the non-isotropic Boussinesq system (2.14), then there exist fields in the upper part (u^u, v^u, w^u, η) (resp. (u^b, v^b, w^b, η)) satisfying the Navier-Stokes equations (2.3) in the upper part (resp. the boundary layer) of the domain and the interface continuity at least as $\mathrm{O}(\varepsilon)$.

Proof. We start from the final Boussinesq model (2.14) written in 1D:

$$
\begin{aligned}
u^u_t+\eta_x-\eta_{xxx}/\mathrm{Bo}+\varepsilon u^u u^u_x-\varepsilon\eta_{xtt}(z^2-1)/2 &= \mathrm{O}(\varepsilon^2)+\mathrm{O}(\varepsilon/\mathrm{Bo}),\\
\eta_t+u^u_x(x,z,t)-\frac{\varepsilon}{2}\eta_{xxt}(z^2-\tfrac{1}{3})+\varepsilon(u^u\eta)_x-\frac{\varepsilon}{\sqrt{\pi R}}u^u_x*\frac{1}{\sqrt{t}}&\\
+\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u^{b,0}_x(x,\gamma'')-u^{u,0}_x(x,z=0)\right)\int_{\gamma'=0}^{\sqrt{\frac{R}{4t}}\gamma''}e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' &= \mathrm{O}(\varepsilon^2)+\mathrm{O}(\varepsilon/\mathrm{Bo}),
\end{aligned}
\tag{6.1}
$$

where u^u is the horizontal velocity in the upper part ($z\in(\varepsilon,1+\varepsilon\eta(x,t))$), $\eta(x,t)$ is the free boundary's height, $u^{u,0}(x,z)$ (resp. $u^{b,0}(x,\gamma)$) is the initial horizontal velocity in the upper part (resp. in the boundary layer). In the boundary layer, we set $z=\varepsilon\gamma$.

First, one must justify that $\eta_{tt}=\eta_{xx}+\mathrm{O}(\varepsilon)$, which is easy from the zeroth order of (6.1). Then, Equation $(6.1)_1$ enables to prove the following Lemma.

Lemma 9. *Any localized solution of (6.1) is such that*

$$
\begin{aligned}
\textstyle\int_0^1 u = &\ u(x,z,t)-\varepsilon\eta_{xt}(z^2-1/3)/2+\mathrm{O}(\varepsilon^2),\\
u(x,0,t)= &\ u(x,z,t)-\varepsilon\eta_{xt}z^2/2+\mathrm{O}(\varepsilon^2),\\
u(x,1,t)= &\ u(x,z,t)+\varepsilon\eta_{xt}(1-z^2)/2+\mathrm{O}(\varepsilon^2).
\end{aligned}
$$

The way to prove this Lemma is identical to the proof of Lemma 11 of Le Meur (2015). One differentiates $(6.1)_1$ with respect to z, integrate with respect to time t, prove that the constants of integration vanish if the wave is localized, and use that $\eta_{tt}=\eta_{xx}+\mathrm{O}(\varepsilon)$ to get $u_z=\varepsilon\eta_{xt}z+\mathrm{O}(\varepsilon^2)$. Completing the proof is then easy. We assume here that the $\mathrm{O}(\varepsilon^2)$ terms are L^1 integrable, which would be obvious if we exhibited them as higher order derivatives of known fields u,η.

Thanks to the previous Lemma, if we define

$$
\begin{aligned}
p^u(x,z,t)= &\ (\eta-\eta_{xx}/\mathrm{Bo})-\varepsilon\eta_{tt}(z-1)+\varepsilon\textstyle\int_1^z\int_1^{z'}u^u_{xt}(x,z'',t)\mathrm{d}z''\,\mathrm{d}z'\\
w^u(x,z,t)= &\ \varepsilon(\eta_t+\varepsilon\eta_x u^u(z=1+\varepsilon\eta))-\varepsilon\textstyle\int_{1+\varepsilon\eta}^{z}u^u_x,
\end{aligned}
$$

then (u^u,w^u,p^u) satisfies $(2.3)_{1,3,5,6}$ up to $\mathrm{O}(\varepsilon^2)+\mathrm{O}(\varepsilon/\mathrm{Bo})$, and $(2.3)_4$ exactly in the upper part.

We must now check (2.3) in the boundary layer, where we define $z=\varepsilon\gamma$ and $f_0(x,\gamma)=u^{b,0}(x,\gamma)-u^{u,0}(x,z=0)$. Then one may define

$$
\begin{aligned}
u^b(x,\gamma,t)= &u^u(x,z=0,t)+\frac{\sqrt{R}}{2}\int_0^{+\infty}f_0(x,\gamma')\frac{e^{-\frac{R(\gamma'-\gamma)^2}{4t}}}{\sqrt{\pi t}}\mathrm{d}\gamma'\\
&-u^u(x,0,.)*\mathcal{L}^{-1}_{p\to t}(e^{-\sqrt{Rp}\gamma})-\frac{\sqrt{R}}{2}\int_0^{+\infty}f_0(x,\gamma')\frac{e^{\frac{-R(\gamma'+\gamma)^2}{4t}}}{\sqrt{\pi t}}\mathrm{d}\gamma'.
\end{aligned}
$$

It is proved in Lemma 6 of Le Meur (2015) that this function satisfies

$$
\begin{aligned}
&(u^b-u^u(z=0))_t-(u^b-u^u(z=0))_{\gamma\gamma}/R &= 0,\\
\Rightarrow\quad &u^b_t+\eta_x-u^b_{\gamma\gamma}/R-\eta_{xxx}/\mathrm{Bo} &= \mathrm{O}(\varepsilon)+\mathrm{O}(\varepsilon/\mathrm{Bo}).
\end{aligned}
$$

Because it is only the zeroth order of the conservation of momentum in the boundary layer $(2.3)_1$, and also because the limit between the boundary layer and the upper part is not made sufficiently precise through the lift "function" $u^u(x, z = 0, t)$, one may not prove more. This u^b also satisfies the initial conditions and the interface conditions:

$$\begin{cases} u^b(x,\gamma,t=0) = & u^{b,0}(x,\gamma), \\ u^b(x,\gamma=0,t) = & 0, \\ u^b(x,\gamma\to+\infty,t) = & u^u(x,z=0,t) \text{ (continuity condition).} \end{cases}$$

If we define:

$$\begin{aligned} p^b(x,\gamma,t) &= \eta(x,t) - \eta_{xx}/\mathrm{Bo} + \varepsilon\eta_{xx}/2 \\ w^b(x,\gamma,t) &= -\varepsilon^2 \int_0^\gamma u_x^b(x,\gamma',t)\,\mathrm{d}\gamma', \end{aligned}$$

then $(2.3)_1$ in the boundary layer reduces to:

$$\begin{aligned} &(u^b-u^u)_t - (u^b-u^u)_{\gamma\gamma}/R - \varepsilon u^u u_x^u + \varepsilon u^b u_x^b - \varepsilon u_\gamma^b \textstyle\int_0^\gamma u_x^b + \varepsilon\eta_{xxx}/2 + \mathrm{O}(\varepsilon^2) + \mathrm{O}(\varepsilon/\mathrm{Bo}), \\ &= -\varepsilon u^u u_x^u + \varepsilon u^b u_x^b - \varepsilon u_\gamma^b \textstyle\int_0^\gamma u_x^b + \varepsilon\eta_{xxx}/2 + \mathrm{O}(\varepsilon^2) + \mathrm{O}(\varepsilon/\mathrm{Bo}) = \mathrm{O}(\varepsilon), \end{aligned}$$

where u^u is evaluated at $z = 0$. We maynot recover more than $\mathrm{O}(\varepsilon)$ because of the heat equation solved only at this order. So (u^b, w^b, p^b) satisfies $(2.3)_1$ only up to $\mathrm{O}(\varepsilon)$. So as to go further, one should use correctors, solve a nonlinear heat equation, make a more precise study at the interface between the two subdomains and use functionnal spaces. Simple computations prove that (u^b, w^b, p^b) satisfies $(2.3)_{3,4,7}$ up to $\mathrm{O}(\varepsilon^2) + \mathrm{O}(\varepsilon/\mathrm{Bo})$.

Since the interface condition (for instance at $z = \sqrt{\varepsilon}$) between these two domains is satisfied, the proof is complete, up to order $\mathrm{O}(\varepsilon)$.

□

7 Conclusion

We modeled the water waves flow by the Navier-Stokes equations in a 3D geometry with a flat bottom and free surface with surface tension. We assumed the order of magnitude of the velocity in the transverse direction and deduced the scaling in this coordinate and discussed its influence on the results (Remark 3). The linear theory gave a new dispersion relation and a new phase velocity (Proposition 1). Surface tension was discussed and seemed often more relevant than viscosity, although these two parameters act very differently. We stated the associated Boussinesq system. This intermediate system enabled us to derive the KdV equation (Proposition 5) and the KP equation or system (Proposition 6). We justified why the zero-mass constraint does not raise any trouble for viscous KP, using Molinet et al. (2007). Using the previous computations, we derived the Boussinesq equation (Proposition 7). Reciprocally, we proved that, should they exist, the solutions to the non-isotropic Boussinesq system are consistent with the Navier-Stokes equations (Proposition 8). These systems and equations are derived for a viscous fluid and take the surface tension into account.

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