Derivation of an intermediate viscous Serre–Green–Naghdi equation

Denys Dutykh [a] Hervé Le Meur [b],
[a] Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA, 73000 Chambéry, France
[b] LAMFA, CNRS, UMR 7352, Université Picardie Jules Verne, 80039 Amiens, France

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Abstract

In this note we present the current status of the derivation of a viscous Serre–Green–Naghdi system. For this goal, the flow domain is separated into two regions. The upper region is governed by inviscid Euler equations, while the bottom region (the so-called boundary layer) is described by Navier-Stokes equations. We consider a particular regime linking the Reynolds number and the shallowness parameter. The computations presented in this note are performed in the fully nonlinear regime. The boundary layer flow reduces to a Prantdl-like equation. Further approximations seem to be needed to obtain a tractable model.

Résumé

Obtention des équations intermédiaires de Serre–Green–Naghdi visqueuses. Dans cette note nous présentons l'état actuel de l'obtention du système de type Serre–Green–Naghdi visqueux dans un canal. Nous séparons le domaine fluide en deux couches. La couche supérieure est décrite par les équations d'Euler tandis que la couche limite en-dessous, obéit aux équations de Navier-Stokes. Nous considérons un régime pleinement non linéaire où le nombre de Reynolds est lié au rapport de la longueur d'onde typique à la profondeur moyenne. La dynamique de l'écoulement dans la couche limite se ramène à une équation de type Prantdl. Des hypothèses supplémentaires sont nécessaires afin d'obtenir un modèle utilisable en pratique.

Version française abrégée

Nous tentons d'obtenir un modèle réduit aux équations de l'écoulement d'un fluide visqueux dans un canal peu profond. Nous ne supposons pas l'amplitude des vagues comme petite (régime non linéaire). Comme dans le cas du régime linéaire (cf. [5]), nous résolvons les équations dans la zone principale

Email addresses: Denys.Dutykh@univ-smb.fr (Denys Dutykh), Herve.Le.Meur@u-picardie.fr (Hervé Le Meur).

gouvernée par des équations d'Euler pour arriver à (13). Puis nous tentons la même résolution dans la couche limite, mais ne pouvons aller plus loin que (18). Cette dernière équation est de type Prandtl. Elle est connue pour un comportement très sensible à chacun de ses termes (cf. [4]) et donc laisse peu d'espoir pour être simplifiée afin d'obtenir un modèle 1D.

1. Introduction

The water wave theory has been essentially developed in the framework of the inviscid, and very often also irrotational, Euler equations. However, various viscous effects are inevitably present in laboratory experiments and even more in the real world. Thus, the conservative conventional models have to be supplemented with dissipative effects to improve the quality of their predictions. A straightforward energy balance asymptotic analysis shows that the main dissipation takes place at the bottom boundary layer [1, Section §2] (or at the lateral walls if they are also present [2]). In this way, the corresponding long wave Boussinesq-type systems have been derived taking into account the boundary layer effects [3]. In [5], the author derives the viscous Boussinesq model without the irrotationality assumption. Other articles already took the vorticity into account, even for fully nonlinear Boussinesq equations (here called Serre-Green-Naghdi or SGN) [6]. Fully nonlinear models are becoming very popular. In the present note we report the current status of the derivation of a viscous counterpart of the well-known SGN equations. The asymptotic regime relates the Reynolds number to the shallowness parameter.

2. Primary equations

Consider the flow of an incompressible liquid in a physical two-dimensional space over a flat bottom and with a free surface. We assume additionally that the fluid is homogeneous (i.e. the density $\rho = \text{const}$) and the gravity acceleration g is constant. For the sake of simplicity, in this study we neglect all other forces (such as the Coriolis force and friction). Hence, we deal with pure gravity waves. We introduce a Cartesian coordinate system $O\,\tilde{x}\,\tilde{y}$. The horizontal line $O\,\tilde{x}$ coincides with the still water level $\tilde{y} = 0$ and the axis $O\,\tilde{y}$ points vertically upwards. The fluid layer is bounded below by the horizontal solid bottom $\tilde{y} = -d$ and above by the free surface $\tilde{y} = \tilde{\eta}\,(\tilde{x},\,\tilde{t})$.

In order to make the equations dimensionless, we choose a characteristic horizontal length ℓ, vertical height of the free surface A and mean depth d. All this enables us to define a characteristic velocity $c_0 = \sqrt{gd}$. Then one may define dimensionless independent variables:

$$\tilde{x} = \ell x,\ \tilde{y} = dy,\ \tilde{t} = t\ell/c_0.$$

This enables us to define the dimensionless fields:

$$\tilde{u} = c_0 u,\ \tilde{v} = \frac{dc_0}{\ell} v,\ \tilde{p} = \tilde{p}_{\text{atm}} - \rho g d\, y + \rho g d\, p,\ \tilde{\eta}(\tilde{x},\tilde{y},\tilde{t}) = A\eta(x,y,t).$$

We also define some dimensionless numbers, characteristic of the flow:

$$\varepsilon = \frac{A}{d},\ \mu^2 = \frac{d^2}{\ell^2},\ \text{Re} = \frac{\rho c_0 d}{\nu}.$$

The system of Navier-Stokes equations can then be written in 2D and in dimensionless variables:

$$
\begin{cases}
u_t + uu_x + vu_y - 1/\mathrm{Re}\,(\mu u_{xx} + u_{yy}/\mu) + p_x &= 0 \\
\mu^2(v_t + uv_x + vv_y) - \mu^2/\mathrm{Re}\,(\mu v_{xx} + v_{yy}/\mu) + p_y &= 0 \\
u_x + v_y &= 0 \\
\left[-(p - \varepsilon\eta)\,\boldsymbol{I} + \dfrac{2}{\mathrm{Re}} \begin{pmatrix} \mu u_x & (u_y + \mu^2 v_x)/2 \\ (u_y + \mu^2 v_x)/2 & \mu v_y \end{pmatrix} \right]\Bigg|_{\varepsilon\eta} \boldsymbol{n} &= 0 \quad \text{on } y = \varepsilon\eta \\
\eta_t + u(y=\varepsilon\eta)\eta_x - v(y=\varepsilon\eta)/\varepsilon &= 0 \quad \text{on } y = \varepsilon\eta \\
u(y=-1) = v(y=-1) &= 0\ ,
\end{cases}
\tag{1}
$$

where we denote $u|_{\varepsilon\eta} = u(y=\varepsilon\eta) = u(x, y=\varepsilon\eta(x,t), t)$.

One could assume the fields to be small around the hydrostatic flow (which is lifted by the change of field from $\tilde{p}$ to p), so around $(u, v, p, \eta) \simeq 0$. Such an assumption is contradictory with our nonlinear assumption where ε is assumed not to be small. Yet, should we make this assumption, we would be led to a linear system identical (up to changes of variables) to System (7) of [5]. The study of the linear regime suggests to assume, not only in the Boussinesq regime:

$$
\mathrm{Re} \simeq \mu^{-5}\,. \tag{2}
$$

Below, we solve the problem in the bulk part where Euler's equations are justified to apply (Section 2.1), then try to solve the velocity in the boundary layer (Section 2.2). In this last section, we are led to Prandtl's equation that prohibits any further advance to the best of our knowledge.

[??? à nettoyer] What is the size of the boundary layer where the no-slip condition forces the fluid to have a large gradient of velocity ? In the same way as in [5], one may assume it is of size μ^2:

$$
y = -1 + \mu^2\,\gamma. \tag{3}
$$

One might be surprised that the gravity-viscosity layer be of size $O(\mu^2)$ (or a little larger) while one usually assumes the size of the viscous layer to be of size $O(\mathrm{Re}^{-1/2}) = O(\mu^{5/2})$. Indeed the classical term stems from the $1/\mathrm{Re}\,u_{yy}$ term which is replaced here by $1/(\mu\mathrm{Re})\,u_{yy}$. So $\mu\mathrm{Re}_{us} = \mathrm{Re}_{classical}$ and the size of the boundary layer is $\mathrm{Re}_{classical}^{-1/2} = (\mu\mathrm{Re}_{us})^{-1/2} = (\mu^{-4})^{-1/2} = \mu^2$!

2.1. *Resolution in the upper part (Euler)*

In the upper part, $y \gg -1 + \mu^2$ and μ^4 is small. So one may drop the Laplacian and keep from (1):

$$
\begin{cases}
u_t + uu_x + vu_y + p_x = O\left(u_{yy}/(\mu\mathrm{Re})\right) + O(\mu^6) \\
\mu^2(v_t + uv_x + vv_y) + p_y = O(\mu^6) \\
u_x + v_y = 0 \\
-p + \varepsilon\eta = O(u_y|_{\varepsilon\eta}/\mathrm{Re}) + O(\mu/\mathrm{Re}) \quad \text{on } y = \varepsilon\eta \\
(p - \varepsilon\eta)\varepsilon\eta_x + 2(-\mu u_x \varepsilon\eta_x + (u_y + \mu^2 v_x))/\mathrm{Re} = 0 \quad \text{on } y = \varepsilon\eta \\
\varepsilon\left(\eta_t + u|_{\varepsilon\eta}\eta_x\right) = v|_{\varepsilon\eta} \quad \text{on } y = \varepsilon\eta.
\end{cases}
\tag{4}
$$

First, one may notice that the viscosity terms are no more present inside the domain. It is argued in [5] that one may (and even must) then drop the fifth equation from this system, due to the fact that the fluid is indeed no more viscous in this part of the domain.

It is classical to use $(4)_3$ to get

$$
v = v|_{\varepsilon\eta} - \int_{\varepsilon\eta}^{y} u_x\,\mathrm{d}y', \tag{5}
$$

where $v|_{y=\varepsilon\eta}$ is given by $(4)_6$. One may use this vertical velocity in $(4)_2$ to compute p_y. Thanks to $(4)_4$, one has:

$$\begin{aligned} p =&\varepsilon\eta - \mu^2 \Bigg[(y-\varepsilon\eta)\left((v|_{\varepsilon\eta})_t + u_x|_{\varepsilon\eta}\varepsilon\eta_t\right) + \left(\int_{\varepsilon\eta}^y u\right)\left((v|_{\varepsilon\eta})_x + u_x|_{\varepsilon\eta}\varepsilon\eta_x - \left(\int_{\varepsilon\eta}^y u_x\right) v|_{\varepsilon\eta}\right) \\ &- \int_{\varepsilon\eta}^y \int_{\varepsilon\eta}^{y'} u_{xt} - \int_{\varepsilon\eta}^y \left(u\int_{\varepsilon\eta}^{y'} u_{xx}\right) + \int_{\varepsilon\eta}^y \left(u_x \int_{\varepsilon\eta}^{y'} u_x\right)\Bigg] + O\left(\frac{u_y|_{\varepsilon\eta}}{\mathrm{Re}}\right) + O\left(\frac{\mu}{\mathrm{Re}}\right) + O(\mu^6). \end{aligned} \quad (6)$$

So we have both v (thanks to (5)) and p (thanks to (6)) and may rewrite $(4)_1$ with the only fields u and η:

$$\begin{aligned} u_t + uu_x + u_y\left(v|_{\varepsilon\eta} - \int_{\varepsilon\eta}^y u_x\right) + \varepsilon\eta_x - \mu^2 \Bigg[&(y-\varepsilon\eta)\left((v|_{\varepsilon\eta})_t + u_x|_{\varepsilon\eta}\varepsilon\eta_t\right) \\ &+ \left(\int_{\varepsilon\eta}^y u\right)\left((v|_{\varepsilon\eta})_x + u_x|_{\varepsilon\eta}\varepsilon\eta_x\right) - \left(\int_{\varepsilon\eta}^y u_x\right) v|_{\varepsilon\eta} - \int_{\varepsilon\eta}^y\int_{\varepsilon\eta}^{y'} u_{xt} \\ &- \int_{\varepsilon\eta}^y \left(u\int_{\varepsilon\eta}^{y'} u_{xx}\right) + \int_{\varepsilon\eta}^y \left(u_x\int_{\varepsilon\eta}^{y'} u_x\right)\Bigg]_x = O\left(\frac{(u_y|_{\varepsilon\eta})_x}{\mathrm{Re}}\right) + O\left(\frac{\mu}{\mathrm{Re}}\right) + O\left(\frac{u_{yy}}{\mu\mathrm{Re}}\right). \end{aligned} \quad (7)$$

In order to take off the dependence on y of this equation, we integrate between $y = -1 + \mu^2\gamma_\infty$ and $y = \varepsilon\eta(x,t)$ and we define:

$$\mathcal{H}_{\mu,\gamma_\infty} = 1 + \varepsilon\eta - \mu^2\gamma_\infty, \text{ and } \bar{u}(x,t) = \frac{1}{\mathcal{H}_{\mu,\gamma_\infty}} \int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta} u(x,y)\,\mathrm{d}y. \quad (8)$$

We also need a lemma that will enable to commute the integration and the x differentiation under an assumption:

Lemma 2.1 *Let F a $\mathcal{C}^1$ function defined in $\Omega = \{(x,y)/x \in \mathbb{R}, -1+\mu^2\gamma_\infty < y < \varepsilon\eta(x)\}$, such that if $\forall x,\ F(x, y = \varepsilon\eta) = 0$, then*

$$\int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta} \frac{\partial F}{\partial x}(x,y)\mathrm{d}y = \frac{\partial}{\partial x}\int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta} F(x,y)\mathrm{d}y. \quad (9)$$

The proof is very simple and left to the interested reader.

Thanks to Lemma 2.1, one may commute the x differentiation of the square bracket in Equation (7) with the integral since the terms in the square brackets vanish at $y = \varepsilon\eta$. An integration by parts of the $\int u_y\left(v|_{\varepsilon\eta} - \int_{\varepsilon\eta}^y u_x\right)\mathrm{d}y$ term, and the treatment of $\int (u^2)_x$ leads to (below, we write $\mathcal{H} = \mathcal{H}_{\mu,\gamma_\infty}$):

$$\begin{aligned} \mathcal{H}\bar{u}_t + &\left(\int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta} u^2\right)_x + \mathcal{H}\varepsilon\eta_x + (\bar{u} - u|_{-1+\mu^2\gamma_\infty})\left(\varepsilon\eta_t + (\mathcal{H}\bar{u})_x\right) - \bar{u}\,(\mathcal{H}\bar{u})_x \\ &- \mu^2\Bigg[-\frac{\mathcal{H}^2}{2}\left((\partial_t + \bar{u}\partial_x)(v|_{\varepsilon\eta}) + \varepsilon\eta_t(u_x|_{\varepsilon\eta} - \bar{u}_x) + \varepsilon\eta_x(\bar{u}u_x|_{\varepsilon\eta} - \bar{u}_x u|_{\varepsilon\eta})\right) \\ &+ \int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta}\int_{\varepsilon\eta}^y (u-\bar{u})\mathrm{d}y'\mathrm{d}y \times \left((v|_{\varepsilon\eta})_x + u_x|_{\varepsilon\eta}\varepsilon\eta_x\right) - \int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta}\int_{\varepsilon\eta}^y (u-\bar{u})_x\mathrm{d}y'\,\mathrm{d}y\, v|_{\varepsilon\eta} \\ &- \int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta}\int_{\varepsilon\eta}^y\left[\int_{\varepsilon\eta}^{y'} u_{xt} + u\int_{\varepsilon\eta}^{y'} u_{xx} - u_x\int_{\varepsilon\eta}^{y'} u_x\right]\mathrm{d}y'\mathrm{d}y\Bigg]_x = O\left(\frac{(u_y|_{\varepsilon\eta})_x}{\mathrm{Re}}\right) + O\left(\frac{u_{yy}}{\mu\mathrm{Re}}\right). \end{aligned} \quad (10)$$

We need now the following (double) assumption:

$$u(x,y,t) = \bar{u}(x,t) + \mu^2\tilde{u}(x,y,t), \quad (11)$$

with

$$\int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta} \tilde{u} = 0 \text{ and } \bar{u}(x,t) = \frac{1}{\mathcal{H}_{\mu,\gamma_\infty}} \int_{-1+\mu^2\gamma_\infty}^{\varepsilon\eta} u(x,y)\,\mathrm{d}y. \tag{12}$$

The mean $\bar{u}$ is the same as before. Notice that the expansion of a function around its mean value $\bar{u}$ is not an assumption. The real assumption is that the discrepancy with the mean is small ($O(\mu^2)$). An other way to formulate this assumption is to look at an expansion in μ^2, in which one assumes that the zeroth order term does not depend on y and that the next order term is zero-mean value. This gives two different ways to see its consequences. Last but not least, this assumption is proved to be true in Lemma 11 (Eq. (77)) of [5] in case of a Boussinesq flow (where ε is small) without the assumption of irrotationality in the Euler part of the flow. We remind the reader that we still assume we solve the Euler equations and not yet the Navier-Stokes ones. So we are coherent.

Upon this assumption, (10) simplifies to:

$$\begin{aligned}&\mathcal{H}\bar{u}_t + \mathcal{H}\bar{u}\bar{u}_x + \mathcal{H}\mathcal{H}_x + (\bar{u} - u|_{-1+\mu^2\gamma_\infty})\,(\mathcal{H}_t + (\mathcal{H}\bar{u})_x)\\ &- \mu^2\left[-\frac{\mathcal{H}^2}{2}(\partial_t + \bar{u}\partial_x)\,(v|_{\varepsilon\eta}) - \frac{\mathcal{H}^3}{6}(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - \bar{u}\bar{u}_x)\right]_x = O\left(\frac{(u_y|_{\varepsilon\eta})_x}{\mathrm{Re}}\right) + O\left(\frac{u_{yy}}{\mu\mathrm{Re}}\right) + O(\mu^4).\end{aligned} \tag{13}$$

Remark 1 The attention may be drawn to the fact that

$$\mathcal{H}_t + (\mathcal{H}\bar{u})_x = \varepsilon\eta_t + \mathcal{H}_x\bar{u} + \mathcal{H}\bar{u}_x = v|_{\varepsilon\eta} + \mathcal{H}\bar{u}_x + O(\mu^2) = v|_{-1+\mu^2\gamma_\infty} + O(\mu^2).$$

In the Euler case, $v|_{-1+\mu^2\gamma_\infty} = 0$ *since the flow does not cross the boundary. So we would not need to compute* $u|_{-1+\mu^2\gamma_\infty}$.

2.2. *Resolution in the boundary layer*

We write the system that applies in the layer, extracted from (1):

$$\begin{cases} u_t + uu_x + vu_y - \mu u_{xx}/\mathrm{Re} - u_{yy}/(\mu\mathrm{Re}) + p_x = 0,\\ \mu^2\,(v_t + u\,v_x + v\,v_y) - \mu^3 v_{xx}/\mathrm{Re} - \mu v_{yy}/\mathrm{Re} + p_y = 0,\\ u_x + v_y = 0,\\ u(y=-1) = v(y=-1) = 0.\end{cases} \tag{14}$$

This system may be rewritten with the change of variables justified in (3) $y = -1 + \mu^2\,\gamma$, where $\gamma > 0$ and may be up to a large (but not too large) γ_∞. We also use the assumption (2) on Re such that $\mathrm{Re} = R\,\mu^{-5}$ where R is a constant. We should have tilded the fields but would have dropped the tilde soon after. So we omit them. When precision is needed, we denote $u^{BL} = u(x,\gamma,t)$ the horizontal velocity in the boundary layer. The system writes:

$$\begin{cases} u_t + u\,u_x + v\,u_\gamma/\mu^2 - \mu^6/R\,u_{xx} - u_{\gamma\gamma}/R + p_x = 0,\\ \mu^2\left(v_t + u\,v_x + v\,v_\gamma/\mu^2\right) - (\mu^8/R)v_{xx} - (\mu^2/R)v_{\gamma\gamma} + p_\gamma/\mu^2 = 0,\\ u_x + v_\gamma/\mu^2 = 0,\\ u(x,\gamma=0,t) = v(x,\gamma=0,t) = 0.\end{cases} \tag{15}$$

As is classical, we first compute v (owing to (15_3) and (15_4)):

$$v(x,\gamma,t) = 0 - \mu^2 \int_{\gamma=0}^{\gamma} u_x\,\mathrm{d}\gamma'. \tag{16}$$

Then we can compute the differentiated pressure from $(15)_2$ that proves $p_\gamma = O(\mu^4)$. As a consequence,

$$p^{BL}(x,\gamma,t) = p^{BL}(\gamma \to \gamma_\infty) + O(\mu^4),$$

where $p^{BL}(\gamma \to \gamma_\infty)$ is determined thanks to a matching condition with the bottom of the upper part (Euler part). From (6), and owing to the already stated assumption (11, 12) the pressure in the boundary layer is, up to $O(\mu^4)$:

$$p^{BL}(x,\gamma,t) = \varepsilon\eta(x,t) + \mu^2 \left[\mathcal{H}(\partial_t + \bar{u}\partial_x)(v|_{\varepsilon\eta}) + \mathcal{H}^2/2(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - \bar{u}_x\bar{u}_x)\right] + O(\mu^4). \quad (17)$$

Last, we may gather v^{BL} (from (16)), p^{BL} (from (17)) and rewrite $(15)_1$:

$$u_t^{BL} + u^{BL}\, u_x^{BL} - u_\gamma^{BL} \int_0^\gamma u_x^{BL}(\gamma')\mathrm{d}\gamma' - \frac{u_{\gamma\gamma}^{BL}}{R} + \varepsilon\eta_x + \mu^2 \left[\mathcal{H}(\partial_t + \bar{u}\partial_x)(v|_{\varepsilon\eta}) + \mathcal{H}^2/2(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - \bar{u}_x\bar{u}_x)\right]_x = O(\mu^4). \quad (18)$$

At that stage of the derivation, we recognize a Prandtl's equation. We do not know how to derive a simpler model. It is well-known that Prandtl's equation still resists to the best of physicists and mathematicians. It was proved to be ill-posed in [4] and partially well-posed later. Moreover, so as to couple this equation with (13), one should assume a link between u^{BL} and $u|_{-1+\mu^2\gamma_\infty}$ like identity by continuity.

3. Conclusions

We stopped our derivation, in the fully nonlinear regime, at equation (13), (18), which is only an intermediate state because we still have functions of x, γ, t. We would like to stress it out that Equations (13), (18) are still Galilean invariant despite the presence of the boundary layer. The proposed model enjoys this property because we did not introduce any drastic simplifications yet at this level. To make further progress, the Prantdl-type equation should be further simplified but it seems highly speculative. One strategy could consist in assuming a particular profile of the velocity u^{BL} in the coordinate γ similar to the one computed in [5] in the Boussinesq regime, but it is incoherent. Further research is needed to reach an effective 1D model.

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