Derivation of a viscous Boussinesq system for surface water waves

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October 30, 2014

Abstract

In this article, we derive a viscous Boussinesq system for surface water waves from Navier-Stokes equations for non-vanishing initial conditions. We use neither the irrotationality assumption, nor the Zakharov-Craig-Sulem formulation. During the derivation, we find the bottom shear stress and also the decay rate for shallow water. In order to justify our derivation, we derive the viscous Korteweg-de Vries equation from our viscous Boussinesq system and compare it to the ones found in the bibliography. We also extend the system to the 3-D geometry.

Subject Class: 76N20, 74J15, 76M45
Keywords: water waves, shallow water, Boussinesq system, viscosity, KdV equation

1 Introduction

1.1 Motivation

The propagation of water waves over a fluid is a long run issue in mathematics, fluid mechanics, hydrogeology, coastal engineering, ... In the case of an inviscid fluid, the topic stemmed many researches and even broadened with time. Various equations have been proposed to model this propagation of water waves. Since the full problem is very complex, the goal is to find reduced models on simplified domains with as little fields as possible, should they be valid only in an asymptotic regime.
This article is a step forward in the direction of a rigorous derivation of an asymptotic system for surface water waves in the so-called Boussinesq regime, taking into account the viscosity. While viscous effects can be neglected for most oceanic situations, they cannot be excluded for surface waves in relatively shallow channels.

[*]The author wants to thank Professor Jean-Claude Saut for initiating and following this research and Professor David Gérard-Varet for a fruitful discussion.

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In the inviscid potential case, the complete and rigorous justification of most asymptotic models for water waves has been thoroughly carried out and summarized in the book [15] and the bibliography therein. This book includes the proof of the consistency and stability of some models, the proof of the existence of solutions both of the water waves systems and of the asymptotic models on the relevant time scales and the proof of "optimal" error estimates between these two solutions. The curlfree assumption allows to use the Zakharov-Craig-Sulem formulation of the water waves system and facilitates the rigorous derivation of the models, through expansions of the Dirichlet to Neumann operator with respect to a suitable small parameter.
Things are more delicate when viscosity is taken into account and a complete justification of the asymptotic models is still lacking. The main difficulty, for not only a derivation but also for a rigorous proof, arises from the matching between the boundary layer solution coming from the bottom and the "Euler" solution in the upper part of the flow.
In this article, we derive an asymptotic system (Boussinesq system) for the viscous flow in a flat channel of water waves in the Boussinesq regime, that is to say in the long wave, small amplitude regime with an *ad hoc* balance between the two effects.

1.2 Literature

When deriving models of water waves in a channel and taking viscosity into account, numerous pitfalls must be avoided in order to be rigorous.
Since there are various dimensionless parameters, a linear study must be done so as to determine the most interesting regime between the parameters. One must also either assume linearized Navier-Stokes Equation (NSE), or justify that the nonlinear terms can be dropped. This is not so obvious because numerous authors extend the inviscid theory by assuming the velocity to be the sum of an inviscid velocity and a viscous one. Then they force only one condition (for instance the vanishing velocity on the bottom) to be satisfied by the total velocity, once the inviscid velocity is assumed unchanged by viscosity. This assumption deserves to be jutified.
At a certain level of the derivation, a heat-like equation arises. Most people solve it with a time Fourier transform while the only physical problem is a Cauchy one, so with an initial condition. The only possibility is to use either Laplace (in time) transform or a sine-transform (in the vertical dimension) with a complete treatment of the initial condition.
One must also derive the bottom shear stress because it is meaningful for the physicists who deal with sediment transport.
Last, the order up to which the expansion is done must be consistent throughout the article.
To the best of our knowledge, no article does all this. Yet various articles have been written on this topic. Let us review those that retained our attention and interest.

Boussinesq did take viscosity into account in 1895 [2]. Lamb [14] also derived the decay rate of the linear wave amplitude by a dissipation calculation (done in paragraph 348 of the sixth edition of [14]) and by a direct calculation based on the linearized NSE (paragraph 349 of the sixth edition of [14]). Both of them used linearized NSE on deep-water and computed the dispersion relation. We do not know who is the first. The imaginary part of the phase velocity gave the decay rate:

$$\frac{\partial A}{\partial t} = -2\nu k^2 A, \tag{1}$$

where A is the amplitude of the wave, ν the kinematic viscosity and k the wavenumber.
In [23], Ott and Sudan made a formal derivation (in nine lines) of a dissipative KdV equation (different from ours). They used the linear damping of shallow water waves already given by Landau-Lifschitz. This led them to an additional term to KdV, which looks like a half integral. They also found once again the damping in time of a solitary wave over a finite depth as $(1+T)^{-4}$ (already found by [12], and later by [11], [21], [10] (p. 374)).
J. Byatt-Smith studied the effect of a laminar viscosity (in the boundary layer where a laminar flow takes place) on the solution of an undular bore [3]. He found the (almost exact) Boussinesq system of evolution with a half derivative but with no treatment of the initial condition. He did an error when providing the solution to the heat equation: his convolution in time is over $(0, +\infty)$ instead of $\mathbb{R}$ (Fourier convolution) or $(0, t)$ (Laplace convolution).
In 1975, Kakutani and Matsuuchi [11] found a minor error in the computation of [23]. They started from the NSE and performed a clean boundary layer analysis. First, they made a linear analysis that gave the dispersion relation and, under some assumptions, the phase velocity as a function of both the wavenumber of the wave and the Reynold's number Re. They distinguished various regimes of Re as a function of the classical small parameter of the Boussinesq regime. Then, they derived the corresponding viscous KdV equation. We want to stress that, at the level of the heat equation, they used a time Fourier transform. As a consequence, they may not use any initial condition. So, the problem they solve is not the Cauchy's one.
In [20], one of the authors of the previous article [11] tried to validate the equation they had been led to. He showed that their "modified K-dV equation agrees with Zabushy-Galvin's experiment with respect to the damping of solitary waves, while it produces disagreement in their phases" (see the conclusions). One might object that the numerical treatment seems light because the space step was between one and 10 percent, the numerical relaxation was not very efficient, the unbounded domain was replaced by a periodic one though there is "non-locality of the viscous effect" (p. 685), there was no numerical validation of the full algorithm, and the regime was not the Boussinesq one (the dispersion's coefficient was about 0.002 and the viscous coefficient was 0.03). Moreover, the phase shift numerically measured was given with three digits while the space step was of the order of magnitude of some percents. The author, very fairly, added that "the phase shift obtained by the calculations is not confirmed by [the] experiments".
In 1987, Khabakhpashev [13] extended the derivation of the viscous KdV evolution equation to the derivation of a viscous Boussinesq system. He studied the dispersion relation and predicted a reverse flow in the bottom, in case of the propagation of a soliton wave. He used a Laplace transform (instead of Fourier as [11] did) with vanishing initial conditions since he assumed starting from rest. Although he stressed this assumption, he acknowledged that "the time required for the boundary layer to develop over the entire

thickness of the fluid [is] much greater than the characteristic time of the wave process". The equations were not made dimensionless, so the right regime was not discussed and a very inefficient numerical method was used (Taylor series expansion is replaced in the convolution term).
In the book [10] (part 5 pp. 356–391), Johnson found the same dispersion relation as [11], studied the attenuation of the solitary wave by a multiscale derivation, reached a heat equation, but solved it only with vanishing initial condition. He exhibited a convolution with a square root integrated on $(0, +\infty)$ (like Byatt-Smith [3]). Some numerical simulations (already partialy done by [3]) enabled him to recover the mecanism of undular bore slightly damped.

Later, Liu and Orfila wrote a seminal article [19] (LO hereafter) in which they studied water waves in an infinite channel (so without meniscus). They derived a Boussinesq system with an additionnal half integration (seen as a convolution), and an initial condition assumed to be vanishing, but implicitely added to the system when numerical simulation must be done.
More precisely, the authors took a linearized Navier-Stokes fluid, used the Helmholtz-Leray decomposition and defined the parameters (index LO denotes their parameters):

$$\begin{aligned}
&\alpha_{LO}^2 = \nu/\left(l\sqrt{gh_0}\right), \\
&\varepsilon_{LO} = A/h_0, \\
&\mu_{LO} = h_0/l,
\end{aligned}$$

where the following notations will be used throughout the present article: A is the characteristic amplitude of the wave, h_0 is the mean height of the channel, g is the gravitational acceleration, and l is the characteristic wavelength of the wave. They made expansions up to order α_{LO}, which square is a kind of a Reynold's number inverse. They used the classical Boussinesq approximation: $\varepsilon_{LO} \simeq \mu_{LO}^2$, but they also set the link between the viscosity and ε_{LO} by requiring $O(\alpha_{LO}) \simeq O(\varepsilon_{LO}^2) \simeq O(\mu_{LO}^4)$ without further justification. Although "the boundary layer thickness is of $O(\alpha_{LO})$", they stretched the coordinates by a larger factor $\alpha_{LO}/\mu_{LO} \simeq \mu_{LO}^3$ (see their (2.9)). More important, and maybe linked, they kept the $\alpha_{LO}\mu_{LO}$ terms (in their (2.8) or (2.21) for instance) and yet dropped $o(\alpha_{LO})$ terms ! This can explain why their final system (3.10-3.11) had a $\alpha_{LO}/\mu_{LO} = O(\varepsilon_{LO}^{3/2})$ term before the half integration, while we will justify an $O(\varepsilon_{LO})$ term for our system.
Let us stress that assuming $\alpha_{LO}^2 = \varepsilon_{LO}^4$ as did [19] amounts to $\mathrm{Re} = \varepsilon_{LO}^{-7/2}$ with our (further redefined) Reynold's number: Re$= \nu/(h_0\sqrt{gh_0})$, while we prove below that the regime at which gravity and viscosity are both relevant is $\mathrm{Re} = \varepsilon_{LO}^{-5/2}$. Our regime was also exhibited by [11], [3], [10]. So, [19] studied a regime different from ours.
Last, they claimed the shear stress at the bottom to be:

$$\tau_{bottom} = -\frac{1}{2\sqrt{\pi}} \int_0^t \frac{u(x,T)}{\sqrt{(t-T)^3}} \mathrm{d}T,$$

where $u(x,T)$ is the depth averaged horizontal velocity. Indeed this integral is infinite as they acknowledged in a later corrigendum where they claimed the right formula to be:

$$\tau_{bottom} = \frac{1}{\sqrt{\pi}}\frac{u(x,0)}{\sqrt{t}} + \frac{1}{\sqrt{\pi}}\int_0^t \frac{u_{,T}}{\sqrt{t-T}}\mathrm{d}T,$$

where $u_{,T}$ denotes the time derivative. However, they did not provide any justification. Moreover, their solution (2.15) to the heat equation, computed in [22] (pp. 153–159), assumed vanishing initial condition. So the treatment of the initial condition was not done. One of our goals in the present article is precisely to provide a better treatment of this initial condition. In this article [19], the authors also raised the question of the eligible boundary condition. Indeed, they considered to be well-known that for a laminar boundary layer, the phase shift between the bottom shear stress and the free stream velocities is $\pi/4$. So it prohibits any bottom condition of the Navier type $\tau_{xy} = -ku_{bottom}$ as is sometimes assumed (and not derived).

Although we presented some criticisms, we acknowledge the modeling, derivation and explanations of this article are insightful and, last but not least, very well written. Yet our criticisms apply to all subsequent articles of the same vein.

In [17], Liu *et al.* experimentaly validated LO's equations in the particular case of a solitary wave over a boundary layer. By Particle Image Velocimetry (PIV), they measured the horizontal velocity in the boundary layer over which the solitary wave run and confirmed the theory.

In [18], Liu *et al.* extended the derivation of the viscous Boussinesq system of [19] to the case of an unflat bottom. They compared the viscous damping and shoaling of a solitary wave, propagating in a wave tank from the experimental and numerical point of view. They provided a condition on the slope of the bottom and paid attention to the meniscus on the sidewalls of the rectangular cross section.

In [16], Liu and Chan used the same process to study the flow of an inviscid fluid over a mud bed modeled by a *very* viscous fluid. They also studied the damping rate of progressive linear waves and solitary waves. In [24], Park *et al.* validated this model with experiments. They also studied the influence of the ratio of the "mud bed thickness and the wave-induced boundary-layer thickness in the mud bed".

In 2008, Dias *et al.* [6] took the (linearized) NSE of a deep water flow with a free boundary and used the Leray-Helmholtz decomposition. Both Bernoulli's equation, through an irrotationnal pressure, and the kinematic boundary condition were modified. Then they made an *ad hoc* modeling for the nonlinear term. Starting from such a model, they provided the evolution equation for the enveloppe A of a Stokes wavetrain which, in case of an inviscid fluid, is the Non-Linear Schrödinger (NLS) equation. The provided equation happens to be a commonly used dissipative generalization of NLS.

Although it was published earlier (2007), [8] was a further development of [6] to a finite-depth flow. In this article, the authors still linearized NSE and generalized by including additional nonlinear terms.

In a later article [9], D. Dutykh linearized NSE and worked on dimensionned equations, considering the viscosity ν to be small (in absolute value). The author generalized by "including nonlinear terms" and reached a viscous Boussinesq system (his (11-12)). Making this system dimensionless triggered very odd terms and its zeroth order was no longer the wave equation. He further derived a KdV equation by making a change of variable in space (but not the associated change in time $\tau = \varepsilon t$). He also made a study of the dispersion relation by assuming an exponential function ansatz of the type $e^{i(kx-\omega t)}$, but then he froze the half derivative term. Indeed it is well-known that such an ansatz amounts to make a Fourier or Laplace transform. Here, the Fourier/Laplace transform of the half derivative of u is very simple: $\mid \xi \mid^{1/2} \hat{u}$ and could have been used instead of freezing this half derivative term.

In [4], Chen *et al.* investigated the well-posedness and decay rate of solutions to a viscous KdV equation which has a nonlocal term that is the same as Liu and Orfila's [19] and [9], but not the same as [11] nor the same as ours. The theoretical proofs were made with no dispersive term (u_{xxx}), but with a local dissipative term (u_{xx}). The tools were either theoretical by finding the kernel and studying its decay rate, or numerical. In the numerical study, they took the dispersive term into account. As expected, they noticed that the "local dissipative term produces a bigger decay rate when compared with the nonlocal dissipative term".
In [5], the authors proved the global existence of solution to the viscous KdV derived by [11] with the dispersive term and even, for sufficiently small initial conditions, without this dispersive term. In addition, they numerically investigated the decay rate for various norms.

In the present article, we first make a linear study of NSE in our domain (Section 2). We compute the dispersion relation and state various asymptotics that give different phase velocities, and so we give the decay rate in finite depth. In Section 3, we make the formal derivation of the viscous Boussinesq system by splitting the upper domain and the bottom one (the boundary layer). The explicit shear stress at the bottom is computed. In intermediate computations, there remains evaluations of the velocity at various heights which are expanded so as to replace these terms by the velocity at a generic height z without the assumption of irrotationality. This enables to state the viscous Boussinesq system. In Section 4, we state the 2-D system, and cross-check with [11] that we get a similar viscous KdV equation. We also discuss the various viscous KdV equations proposed in the bibliography.

2 Linear theory

In order to make a linear theory, we need first to obtain dimensionless equations. This is done in the next subsection. Then we investigate two asymptotics in the following subsections.

2.1 Dimensionless equations

Let us denote $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{w})$ the velocity of a fluid in a 2-D domain $\tilde{\Omega} = \{(\tilde{x}, \tilde{z}) \ / \ \tilde{x} \in \mathbb{R}, \ \tilde{z} \in (-h, \tilde{\eta}(\tilde{x}, \tilde{t}))\}$. So we assume the bottom is flat and the free surface is characterized by $\tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t})$ with $\tilde{\eta}(\tilde{x}, \tilde{t}) > -h$ (the bottom does not get dry). Let $\tilde{p}$ denote the pressure and $\tilde{\mathbf{D}}[\tilde{\mathbf{u}}]$ the symmetric part of the velocity gradient. The dimensionless domain is drawn in Figure 1. We also denote ρ the density of the fluid, ν the viscosity of the fluid, g the

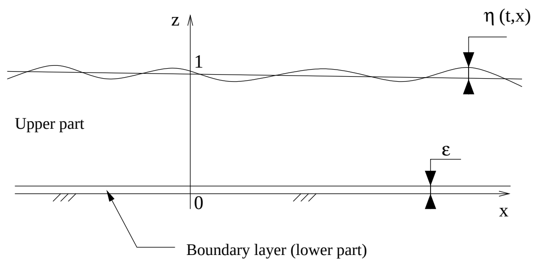

Figure 1: The dimensionless domain

gravitational acceleration, $\mathbf{k}$ the unit vertical vector, $\mathbf{n}$ the outward unit normal to the upper frontier of $\tilde{\Omega}$, $\tilde{p}_{atm}$ the atmospheric pressure. The original system reads:

$$\begin{cases} \rho \left(\dfrac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\mathbf{u}}.\tilde{\nabla}\tilde{\mathbf{u}} \right) - \nu \tilde{\Delta} \tilde{\mathbf{u}} + \tilde{\nabla} \tilde{p} = -\rho g \mathbf{k} & \text{in } \tilde{\Omega}, \\ \widetilde{\text{div}}\ \tilde{\mathbf{u}} = 0 & \text{in } \tilde{\Omega}, \\ \left(-\tilde{p}\mathbf{I} + 2\nu \tilde{\mathbf{D}}[\tilde{\mathbf{u}}] \right).\mathbf{n} = -\tilde{p}_{atm}\mathbf{n} & \text{on } \tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t}), \\ \tilde{\eta}_{\tilde{t}} + \tilde{u}\tilde{\eta}_{\tilde{x}} - \tilde{w} = 0 & \text{on } \tilde{z} = \tilde{\eta}(\tilde{x}, \tilde{t}), \\ \tilde{\mathbf{u}} = 0 & \text{on } \tilde{z} = -h, \end{cases} \tag{2}$$

where we write the second order tensors and the vectors with bold letters. Of course, we need to add an initial condition and conditions at infinity.
So as to get dimensionless fields and variables, we need to choose a characteristic horizontal length l which is the wavelength (roughly the inverse of the wave vector), a characteristic vertical length h which is the water's height, and the amplitude A of the propagating perturbation. Moreover, we denote U, W, P the characteristic horizontal velocity, vertical velocity and pressure respectively. We may then define:

$$\begin{array}{c} c_0 = \sqrt{gh},\ \alpha = \dfrac{A}{h},\ \beta = \dfrac{h^2}{l^2},\ U = \alpha c_0, \\ W = \dfrac{Ul}{h} = \dfrac{c_0 \alpha}{\sqrt{\beta}},\ P = \rho g A,\ \text{Re} = \dfrac{\rho c_0 h}{\nu}, \end{array} \tag{3}$$

where c_0 is the phase velocity. As a consequence, one may make the fields dimensionless and unscaled:

$$\tilde{u} = Uu,\ \tilde{w} = Ww,\ \tilde{p} = \tilde{p}_{atm} - \rho g \tilde{z} + Pp,\ \tilde{\eta} = A\eta, \tag{4}$$

and the variables:

$$\tilde{x} = lx,\ \tilde{z} = h(z-1),\ \tilde{t} = t\, l/c_0. \tag{5}$$

With these definitions, the new system with the new fields and variables writes in the new domain $\Omega_t = \{(x,z), x \in \mathbb{R},\ z \in (0, 1 + \alpha\eta(x,t))\}$ and the new outward unit normal still denoted $\mathbf{n}$:

$$\left\{\begin{array}{ll} u_t + \alpha u u_x + \frac{\alpha}{\beta} w u_z - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zz} + p_x = 0 & \text{in } \Omega_t, \\ w_t + \alpha u w_x + \frac{\alpha}{\beta} w w_z - \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{zz} + p_z = 0 & \text{in } \Omega_t, \\ \beta u_x + w_z = 0 & \text{in } \Omega_t, \\ (\eta - p)\mathbf{n} + \frac{1}{\mathrm{Re}} \begin{pmatrix} 2u_x\sqrt{\beta} & u_z + w_x \\ u_z + w_x & 2/\sqrt{\beta} w_z \end{pmatrix} .\mathbf{n} = 0 & \text{on } z = 1 + \alpha\eta, \\ \eta_t + \alpha u \eta_x - \frac{1}{\beta} w = 0 & \text{on } z = 1 + \alpha\eta, \\ \mathbf{u} = 0 & \text{on } z = 0. \end{array}\right. \tag{6}$$

Like Kakutani and Matsuuchi [11], we could have eliminated $\eta - p$ in one of the two equations of stress continuity at the free boundary. After simplification by $1/\mathrm{Re}$, this would have led us to the "simplified" system:

$$\left\{\begin{array}{l} \eta - p + (-\alpha\eta_x(u_z + w_x) - 2u_x\sqrt{\beta})/\mathrm{Re} = 0, \\ (1 - (\alpha\eta_x)^2)(u_z + w_x) = 4\alpha\sqrt{\beta}\eta_x u_x. \end{array}\right.$$

Notice that the number of dynamic conditions is linked to the Laplacian's presence. If, in a subdomain, the flow is inviscid (Euler or $\mathrm{Re} \to \infty$), then one must not keep the two above equations. Yet, once we have simplified the $1/\mathrm{Re}$ term above we might forget that the second equation must be taken off as if there remained a $1/\mathrm{Re}$ term before every term. So this "simplification" can be misleading.
Unlike us, the authors of [11] use the same characteristic length in the two space directions and so, for them, $h/l = 1$. Our vertical velocity (scaled by W) is not the same as in [11]. Our choice of scale for W raises some $\sqrt{\beta}$ terms that [11] avoids. It suffices to set $\beta = 1$ in our equations to get those of [11]. Although the authors make their system dimensionless, they did not really unscale the fields nor the variables. Our fields are unscaled and so are of the order of unity.
Our characteristic pressure is $P = \rho g A$ while [11] use $\rho g h$. This explains why [11] has an α more before the pressure p in their equations.

2.2 The dispersion relation

We are looking for small fields. So we linearize the system (6) and get:

$$\left\{\begin{array}{ll} u_t - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zz} + p_x = 0 & \text{in } \mathbb{R} \times [0,1], \\ w_t - \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{zz} + p_z = 0 & \text{in } \mathbb{R} \times [0,1], \\ \beta u_x + w_z = 0 & \text{in } \mathbb{R} \times [0,1], \\ \eta - p - \frac{2\sqrt{\beta}u_x}{\mathrm{Re}} = 0 & \text{on } z = 1, \\ u_z + w_x = 0 & \text{on } z = 1, \\ \eta_t - \frac{1}{\beta} w = 0 & \text{on } z = 1, \\ \mathbf{u} = 0 & \text{on } z = 0. \end{array}\right. \tag{7}$$

First, we eliminate the pressure from $(7)_1$ and $(7)_2$:

$$u_{zt} - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xxz} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zzz} - w_{xt} + \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xxx} + \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{xzz} = 0.$$

Then we eliminate u from the previous equation thanks to $(7)_3$ by differentiating with respect to x and after some simplifications:

$$(\partial_z^2 + \beta\partial_x^2)(\partial_z^2 + \beta\partial_x^2 - \mathrm{Re}\sqrt{\beta}\partial_t)w = 0. \tag{8}$$

If w is of the form $\mathcal{A}(z)\exp ik(x - ct)$ with a non-negative k and a (complex) phase velocity c, we can define a parameter with non-negative real part similar to the one used by [11]:

$$\mu^2 = \beta k^2 - \mathrm{Re}\sqrt{\beta}ikc. \tag{9}$$

Thanks to this notation, the solutions of (8) are such that

$$\mathcal{A}(z) = C_1 \cosh\sqrt{\beta}k(z-1) + C_2 \sinh\sqrt{\beta}k(z-1) \\ + C_3 \cosh\mu(z-1) + C_4 \sinh\mu(z-1). \tag{10}$$

Up to now we have eliminated u and p only in the volumic equations. We still have to use the boundary conditions of (7) to find the conditions on the remaining field w.
The first equation of $(7)_7$ is $u(0) = 0$. After a differentiation with respect to x and $(7)_3$, we get $w_z(0) = 0$.
The second equation of $(7)_7$ is $w(0) = 0$ and needs no treatment.
The equation $(7)_5$ can be differentiated with respect to x and, thanks to $(7)_3$ leads to $w_{zz} - \beta w_{xx} = 0$ at height $z = 1$.
The equation $(7)_6$ enables to compute/eliminate η.
The equation $(7)_4$ must be differentiated with respect to t for η to be replaced. Then we get

$$\frac{w}{\beta} - p_t + \frac{2}{\mathrm{Re}\sqrt{\beta}} w_{zt} = 0.$$

We may differentiate the previous equation with respect to x so as to have a p_x term which can be replaced thanks to $(7)_1$ to have new u terms. It suffices then to differentiate this equation and use the incompressibility $(7)_3$ to get the last condition. The full conditions on w are:

$$\begin{array}{l} w_z(0) = 0, \\ w(0) = 0, \\ w_{zz}(1) - \beta w_{xx}(1) = 0, \\ w_{xx}(1) - w_{ztt}(1) + \frac{3\sqrt{\beta}}{\mathrm{Re}} w_{xxzt}(1) + \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{zzzt}(1) = 0. \end{array} \tag{11}$$

The solutions (10) will satisfy a homogeneous linear system in the constants C_1, C_2, C_3, C_4. Its matrix is:

$$\begin{pmatrix} \sqrt{\beta}k\sinh(\sqrt{\beta}k) & -\sqrt{\beta}k\cosh(\sqrt{\beta}k) & \mu\sinh\mu & -\mu\cosh\mu \\ \cosh(\sqrt{\beta}k) & -\sinh(\sqrt{\beta}k) & \cosh\mu & -\sinh\mu \\ 2k^2\beta & 0 & \mu^2 + \beta k^2 & 0 \\ -k^2 & \sqrt{\beta}k^3c^2 + \frac{2i\beta k^4 c}{\mathrm{Re}} & -k^2 & \frac{2\mu\sqrt{\beta}ik^3c}{\mathrm{Re}} \end{pmatrix}. \tag{12}$$

It suffices to compute its determinant to get the dispersion relation:

$$\begin{aligned}4\beta k^2\mu(\beta k^2+\mu^2)+4\mu k^3\beta^{3/2}(\mu\sinh{(k\sqrt{\beta})}\sinh\mu-k\sqrt{\beta}\cosh{(k\sqrt{\beta})}\cosh\mu)\\ -(\beta k^2+\mu^2)^2(\mu\cosh{(k\sqrt{\beta})}\cosh\mu-k\sqrt{\beta}\sinh{(k\sqrt{\beta})}\sinh\mu)\\ -k\sqrt{\beta}\mathrm{Re}^2(\mu\sinh{(k\sqrt{\beta})}\cosh\mu-k\sqrt{\beta}\cosh{(k\sqrt{\beta})}\sinh\mu)=0.\quad(13)\end{aligned}$$

This relation is identical to the one of [11] except that our process of non-dimensionnalizing makes a difference between x and z. So instead of k (for [11]), we have $k\sqrt{\beta}$.

2.3 Asymptotic of the phase velocity (very large Re)

In this subsection, we prove the following Proposition:

Proposition 1. *Under the assumptions*

$$k\sqrt{\beta}Re\, c\to+\infty,\qquad(14)$$
$$k=\ O(1),\qquad(15)$$
$$\beta\to\ 0,\qquad(16)$$
$$Re\to+\infty,\qquad(17)$$
$$c=\ O(1)\ \ (\textit{and } c \textit{ bounded away from } 0),\qquad(18)$$

if there exists a complex phase velocity c solution of (13), then it is such that:

$$c=\sqrt{\frac{\tanh{(k\sqrt{\beta})}}{k\sqrt{\beta}}}-\frac{e^{i\pi/4}Re^{-1/2}(k\sqrt{\beta})^{1/4}}{2\tanh^{3/4}{(k\sqrt{\beta})}}+o(\beta^{-1/4}Re^{-1/2}).\qquad(19)$$

Moreover, the decay rate in our finite-depth geometry, which is viscous, is:

$$Im(kc)=\frac{-1}{2\sqrt{2}}\frac{k^{5/4}\beta^{1/8}}{\sqrt{Re}\tanh^{3/4}{(k\sqrt{\beta})}}+o(\beta^{-1/4}Re^{-1/2}).\qquad(20)$$

We denote $o(f)$ (respectively $O(f)$) a function which ratio with f tends to zero (respectively is bounded).
Our decay rate is not the same as Boussinesq's or Lamb's. The reason is that our geometry is not infinite. In the regime Re$=R\,\varepsilon^{-5/2}$ and $\beta=b\,\varepsilon$ with constant R,b it gets:

$$\mathrm{Im}(kc)=\frac{-\sqrt{k}}{2\sqrt{2}\sqrt{\mathrm{Re}\sqrt{\beta}}}+o(\beta^{-1/4}\mathrm{Re}^{-1/2})=\frac{-\sqrt{k}\varepsilon}{2\sqrt{2}\sqrt{R\sqrt{b}}}+o(\varepsilon).\qquad(21)$$

Our Proposition is stated in [11] but not fully proved. One must also notice that the viscosity modifies both the real and the imaginary part of the phase velocity at the same order.

Proof. The definition of μ ($\Re(\mu) \geq 0$) and assumptions (14, 15, 16, 18) enable to state that $\mu^2 \to \infty$ and the $k^2\beta$ term tends to zero. So we have:

$$\mu = e^{-i\pi/4}\sqrt{k\sqrt{\beta}\mathrm{Re}\, c} + O(\frac{\beta^{3/4}}{\mathrm{Re}}), \tag{22}$$

where the leading term tends to ∞ and its real part tends to $+\infty$, while the error term tends to zero. As a consequence, $\tanh\mu = 1 + O(e^{-\mu})$ and $1/\cosh\mu = O(e^{-\mu})$. Dividing (13) by $\cosh\mu$ and using a generic notation $P(\beta,\mu)$ for an unspecified polynomial in β, μ, we have:

$$\begin{aligned} O(P(\beta,\mu)e^{-\mu}) + 4\mu k^4\beta^2 \left(\mu\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} - \cosh(k\sqrt{\beta})\right) \\ -(k^2\beta+\mu^2)^2\left(\mu\cosh(k\sqrt{\beta}) - k\sqrt{\beta}\sinh(k\sqrt{\beta})\right) \\ -k^2\beta\mathrm{Re}^2\left(\mu\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} - \cosh(k\sqrt{\beta})\right) = 0. \end{aligned} \tag{23}$$

The leading term of the second monomial is $4k^4\beta^2\mu^2\sinh(k\sqrt{\beta})/(k\sqrt{\beta})$ while the leading term of the fourth (last) is $-k^2\beta\mathrm{Re}^2\mu\sinh(k\sqrt{\beta})/(k\sqrt{\beta})$. Seen the assumptions, their ratio is $4k^2\beta\mu\mathrm{Re}^{-2} = O(\beta^{5/4}\mathrm{Re}^{-3/2})$. Under the assumptions (16, 17), this ratio tends to zero. So, in a first step, we can neglect the second monomial with respect to the fourth. If we look for a non-vanishing solution, we need to have a compensation of the only two remaining leading terms. One may then rewrite (23) as:

$$-(\mu^4 + hot)(\mu\cosh(k\sqrt{\beta}) + hot) - k^2\beta\mathrm{Re}^2\left(\mu\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} + hot\right) + hot = 0,$$

where *hot* denotes higher order terms. This reads after easy computations:

$$c^2 = \frac{\tanh(k\sqrt{\beta})}{k\sqrt{\beta}} + hot. \tag{24}$$

Such a relation is well-known. It confirms the assumption (18). To pursue the expansion we come back to (23) and expand its various monomials starting with the second:

$$-4ik^5\beta^{5/2}\mathrm{Re}\, c\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} + O(\beta^3) + O(\beta^{9/4}\mathrm{Re}^{1/2}).$$

Indeed, even the leading term of this second monomial will be negligible in comparison with $O(\beta^{7/4}\mathrm{Re}^{3/2})$ that we will have further. The third monomial of (23) is more complex and we must keep:

$$+e^{-i\pi/4}\left(k\sqrt{\beta}\mathrm{Re}\, c\right)^{5/2}\cosh(k\sqrt{\beta}) - k^3\beta^{3/2}\mathrm{Re}^2c^2\sinh(k\sqrt{\beta}) + O(\beta^{7/4}\mathrm{Re}^{3/2}).$$

The fourth monomial of (23) is expanded:

$$-k^2\beta\mathrm{Re}^2\left(e^{-i\pi/4}\sqrt{k\sqrt{\beta}\mathrm{Re}\, c}\frac{\sinh(k\sqrt{\beta})}{k\sqrt{\beta}} - \cosh(k\sqrt{\beta})\right) + O(\beta^{7/4}\mathrm{Re}^{3/2}).$$

Using these expansions, the equation (23) can be rewritten:

$$e^{-i\pi/4}\left(k\sqrt{\beta}\mathrm{Re}\right)^{5/2}\sqrt{c}\cosh\left(k\sqrt{\beta}\right)\left[c^2-\frac{\tanh\left(k\sqrt{\beta}\right)}{k\sqrt{\beta}}+\frac{e^{i\pi/4}}{\sqrt{k\sqrt{\beta}\mathrm{Re}}\sqrt{c}}\right]$$
$$+O(P(\beta,\mu)e^{-\mu})-k^3\beta^{3/2}\mathrm{Re}^2c^2\sinh\left(k\sqrt{\beta}\right)+O(\beta^3)+O(\beta^{7/4}\mathrm{Re}^{3/2})=0.$$

We would like to justify that the term between square brackets vanishes. For that purpose, we must check that the various other monomials are negligible in comparison with the third of those written between the square brackets which expands into: $O((\sqrt{\beta}\mathrm{Re})^{5/2}[(\sqrt{\beta}\mathrm{Re})^{-1/2}])=O(\beta\mathrm{Re}^2)$ if we assume (18). Once it is checked (this is easy computation left to the reader), we can claim we proved that only the terms enclosed by square brackets remain:

$$c^2=\frac{\tanh\left(k\sqrt{\beta}\right)}{k\sqrt{\beta}}-\frac{e^{i\pi/4}}{\sqrt{k\sqrt{\beta}\mathrm{Re}\,c}}+o(\beta^{-1/4}\mathrm{Re}^{-1/2}), \tag{25}$$

and the proof is complete by computing the square root of (25) and replacing the first order of c into (25) which leads to (19).
The computation of the decay rate is straightforward.

□

We must stress that the complex phase velocity (19) contains two terms. The first is the classical gravitational term ($\sqrt{\tanh\left(k\sqrt{\beta}\right)/(k\sqrt{\beta})}$) which may be expanded when β tends to zero: $1-k^2\beta/6+O(\beta^2)$. The second is purely viscous and can be expanded: $-\sqrt{2}(1+i)(4\sqrt{k})^{-1}(\mathrm{Re}\sqrt{\beta})^{-1/2}+o(\mathrm{Re}\sqrt{\beta})^{-1/2})$. So the dependences of c both on the gravitational and on the viscous effects are of the same order of magnitude when β and $(\mathrm{Re}\sqrt{\beta})^{-1/2}$ are of the same order. In this regime of very large Re, studied hereafter, the dependence of Re on β is such that:

$$\mathrm{Re}\simeq\beta^{-5/2}. \tag{26}$$

2.4 Second asymptotics of the phase velocity (moderate Re)

The definition of μ^2 is $\mu^2=k^2\beta-ik\sqrt{\beta}\mathrm{Re}\,c$ and we still assume a long-wave asymptotics ($\beta\to0$). So one term or the other dominates in μ^2. The extremes are either $\mu^2\to\infty$ (see above) or $\mu^2\to0$.
In the present subsection, we investigate the latter case and exhibit a more precise expansion than the one justified in [11]. Indeed, we prove the following Proposition:

Proposition 2. *Under the assumptions*

$$k \textit{ is bounded from zero and infinity}, \tag{27}$$
$$\mu\to0 \textit{ and so } \sqrt{\beta}Re\,c\to0, \tag{28}$$
$$\beta\to0 \textit{ (long waves)}, \tag{29}$$
$$Re\to+\infty, \tag{30}$$

if there exists a complex phase velocitiy c solution of (13), then it is such that:

$$c = -\frac{ik\sqrt{\beta}Re}{3} - \frac{19ik^3\beta^{3/2}Re^3}{90} + o(\beta^{3/2}Re^3), \tag{31}$$

and necessarily (28) implies:

$$\sqrt{\beta}Re \to 0, \tag{32}$$

and so the phase velocity tends to zero.

Notice that if we assume $\sqrt{\beta}\mathrm{Re} \to 0$, the conclusion is the same and the proof much simpler.

Proof. Since $\mu \to 0$, we must expand all the functions in (13). In this expansion, we pay special attention to the fact that $\mathrm{Re} \to +\infty$ and it may not be considered as a constant parameter of an expansion in β (hidden in $O(\beta^2)$ as [11] did). After tedious expansions, there remains from (13):

$$\begin{aligned} &O(\beta\mathrm{Re}^2c^2\mu^5) + \beta\mathrm{Re}^2\mu^7(1/7! + o(\mu)) + O(\beta^{3/2}\mathrm{Re}\,c\mu^5) + O(\beta^2\mu^5) \\ &+O(\beta^3\mathrm{Re}^2c^2\mu) + O(\beta^{7/2}\mathrm{Re}\,c\mu) + O(\beta^4\mathrm{Re}^2\mu) \\ &+\mu\mathrm{Re}^2k^2\beta c\left[(c + \frac{ik\sqrt{\beta}\mathrm{Re}}{3}) - \frac{ik\sqrt{\beta}\mathrm{Re}\,c^2}{2} + \frac{4k^2\beta\mathrm{Re}^2c}{5} + 2k^2\beta c\right. \\ &\qquad\left. + \frac{8ik^3\beta^{3/2}\mathrm{Re}}{5} + \frac{ik^3\beta^{3/2}\mathrm{Re}\,\mu^2}{3\times 5!}\right] = 0. \end{aligned} \tag{33}$$

Thanks to the assumptions (27-28) we can use that $\sqrt{\beta}\mathrm{Re}\,c \to 0$. Then, if we denote T_i the ith term (among the eight) of this equation, and compare some of them, either to the first ($\mu\mathrm{Re}^2k^2\beta c^2$) or to the second ($-i\mu\mathrm{Re}^3k^3\beta^{3/2}c/3$) of the terms inside the square brackets, we have:

$$\begin{aligned} \frac{T_1}{\mu\mathrm{Re}^2k^2\beta c^2} &= O(\mu^4), & \frac{T_3}{-i\mu\mathrm{Re}^3k^3\beta^{3/2}c/3} &= O(\frac{\mu^4}{\mathrm{Re}^2}), \\ \frac{T_4}{T_2} &= O(\mathrm{Re}^{-2}), & \frac{T_5}{\mu\mathrm{Re}^2k^2\beta c^2} &= O(\beta^2), \\ \frac{T_6}{-i\mu\mathrm{Re}^3k^3\beta^{3/2}c/3} &= O(\frac{\beta^2}{\mathrm{Re}^2}), & \frac{T_7}{T_2} = O\left(\frac{\beta^3}{\mu^6}\right) &= O(1). \end{aligned}$$

As a consequence, the terms T_1, T_3, T_4, T_5, T_6 and T_7 can be taken off from (33). Then, if we simplify by $\beta\mathrm{Re}^2\mu$ and define D a constant, this equation writes:

$$\begin{aligned} \mu^6(D + O(\mu)) + \left[c^2 + \frac{ik(\sqrt{\beta}\mathrm{Re}\,c)}{3} - c^2\frac{ik(\sqrt{\beta}\mathrm{Re}\,c)}{2} + \frac{4k^2(\sqrt{\beta}\mathrm{Re}c)^2}{5} + 2k^2\beta c^2\right. \\ \left. + \frac{8ik^3\beta(\sqrt{\beta}\mathrm{Re}\,c)}{5} + \frac{ik^3\beta\mu^2(\sqrt{\beta}\mathrm{Re}\,c)}{3\times 5!}\right] = 0. \end{aligned} \tag{34}$$

Seen the assumptions (27-28), the highest order term in the square brackets is c^2 which must then vanish : $c \to 0$. Moreover, one may see that $\mu^6 = O(\beta^3) + O(\sqrt{\beta}\mathrm{Re}\,c)^3$ because of the complex definition of μ^2. As a consequence, the equation (34) simplifies first to

$$c = -\frac{ik\sqrt{\beta}\mathrm{Re}}{3} + o(\sqrt{\beta}\mathrm{Re}). \tag{35}$$

Since we proved that $c \to 0$, so does $\sqrt{\beta}\mathrm{Re}$ as stated in (32). Moreover, by the definition of μ^2 and because of (35), one may claim

$$\mu^2 = -ik\sqrt{\beta}\mathrm{Re}\, c(1 + O(1/\mathrm{Re}^2)) \sim -k^2\beta\mathrm{Re}^2/3.$$

In a second step, before pursuing the expansion of c, one may notice that $\mu^6 = O(\beta^3\mathrm{Re}^6)$ which may then be neglected in (34). So, there remains only the terms in the square brackets (simplified by c):

$$\begin{aligned} c + \frac{ik\sqrt{\beta}\mathrm{Re}}{3} + \frac{ik^3\beta^{3/2}\mathrm{Re}^3}{18} + o(\beta^{3/2}\mathrm{Re}^3) - \frac{4ik^3\beta^{3/2}\mathrm{Re}^3}{15} + o(\beta^{3/2}\mathrm{Re}^3) \\ - \frac{2ik^3\beta^{3/2}\mathrm{Re}}{3} + o(\beta^{3/2}\mathrm{Re}) + \frac{8ik^3\beta^{3/2}\mathrm{Re}}{5} + \frac{ik^3\beta^{3/2}\mathrm{Re}\,\mu^2}{3\times 5!} = O(\beta^3) + O(\beta^3\mathrm{Re}^6). \end{aligned} \tag{36}$$

Among all the terms, one may justify that only the first to fourth must be kept:

$$c + \frac{ik\sqrt{\beta}\,\mathrm{Re}}{3} + \frac{ik^3\beta^{3/2}\mathrm{Re}^3}{18} - \frac{4ik^3\beta^{3/2}\mathrm{Re}^3}{15} = o(\beta^{3/2}\,\mathrm{Re}^3),$$

which gives the announced result (31) and completes the proof. □

Our phase velocity is different from Kakutani and Matsuuchi's [11] because they assume constant Re (while it tends to infinity) and make expansions with the other parameter.

3 Formal derivation

We are going to consider the influence of viscosity on the solution of the Navier-Stokes equations in the domain Ω_t. On the basis of the linear theory of the previous section, we assume a large Re and

$$\mathrm{Re} \simeq \beta^{-5/2} \tag{37}$$

as announced in (26). This is the case when viscous and gravitationnal effects balance in their influence on the variation of the phase velocity. We further assume

$$\alpha \sim a\varepsilon, \qquad \beta \sim b\varepsilon, \tag{38}$$

where ε is an already defined common measure of smallness and a, b are two given positive numbers. So $\alpha/\beta \simeq 1$. Our main purpose here is to derive an asymptotic system of reduced size from the global Navier-Stokes equations in the whole *moving* domain. In the inviscid case, we would derive the classical Boussinesq system.

In order to prove our main result, we proceed in the same way as [11] and distinguish two subdomains: the upper part ($z > \varepsilon$) where viscosity can be neglected, and the lower part ($0 < z < \varepsilon$) which is a boundary layer at the bottom and where viscosity must be taken into account. All the other geometrical characteristics have already been depicted. Our first main Proposition is stated hereafter.

Proposition 3. *Let $\eta(x,t)$ be the free boundary's height. Let $u^{b,0}(x,\gamma)$ for $\gamma \in (0,+\infty)$ (resp. $u^{u,0}(x,z)$ for $z \in (0, 1+\alpha\eta(x,t))$) be the initial horizontal velocity in the boundary layer (resp. in the upper part of the domain). If $u^{b,0}(x,\gamma)$ is uniformly continuous in γ and $u^{b,0}(x,\gamma) - u^{u,0}(x, z=0)$ satisfies (58), then the solution of the Navier-Stokes equation with this given initial condition satisfies:*

$$
\begin{aligned}
&u_t + \eta_x + \alpha u u_x - \beta \eta_{xxx} \frac{(z^2-1)}{2} = O(\varepsilon^2),\\
&\eta_t + u_x(x,z,t) - \frac{\beta}{2}\eta_{xxt}(z^2 - \tfrac{1}{3}) + \alpha(u\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x * \frac{1}{\sqrt{t}}\\
&+\frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} \left(u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x,z=0)\right) \int_{\gamma'=0}^{\sqrt{\frac{R\sqrt{b}}{4t}}\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' = O(\varepsilon^2),
\end{aligned} \tag{39}
$$

where the convolution, denoted with $*$ is in time, the parameters α, β, Re have been defined and $z \in (0, 1+\alpha\eta(x,t))$.
If the initial velocity is a Euler flow, then $u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x, z=0) = 0$ (there is no viscous flow in the boundary layer) and the system writes:

$$
\left\{
\begin{aligned}
&u_t + \eta_x + \alpha u u_x - \beta \eta_{xxx} \frac{(z^2-1)}{2} &&= O(\varepsilon^2),\\
&\eta_t + u_x(x,z,t) - \frac{\beta}{2}\eta_{xxt}(z^2 - \tfrac{1}{3}) + \alpha(u\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x * \frac{1}{\sqrt{t}} &&= O(\varepsilon^2),
\end{aligned}
\right. \tag{40}
$$

where the convolution is still in time.

Before starting the proof, we must justify our non-obvious choice of method and a non-obvious term.

***Remark* 4.** *Of course, the domain in the boundary layer $\gamma \in [0,+\infty[$ is not physical. Indeed, u^b should be considered for γ between $\gamma = 0$ and $\gamma = 1$. We can extend its value up to γ large (with respect to 1), but small (with respect to $1/\varepsilon$ so as to ensure $z = \varepsilon\gamma < 1$). For instance, one may choose $\gamma = 1/\sqrt{\varepsilon}$ (equivalently $z = \sqrt{\varepsilon}$) or any value between $\gamma = 1$ and $\gamma = +\infty$ such that $z = \varepsilon\gamma \ll 1$.*
The same applies in the upper part. Indeed, $u^u(x,z,t)$ should be considered for $z \in (\varepsilon, 1)$ and $u^u(x, z=0, t)$ should be $u^u(x, z=\varepsilon, t)$.
One can then write the boundary condition at any height like $\gamma = 1/\sqrt{\varepsilon}$ and force that the final result does not rely on this choice.
As is classical in boundary layer analysis, these more justified notations would give the same result as our choice. So we will use the most straightforward and consider u^u for $z \in (0, 1+\alpha\eta(x,t))$ and u^b for $\gamma \in (0,+\infty)$.

***Remark* 5.** *The double integral term in (39) is new and surprising because of its dependence on the initial condition. One could wonder whether assuming vanishing initial conditions in the boundary ($u^{b,0} = u^{u,0}$ or equivalently that the initial flow is of Euler type), that would greatly simplify the computations, would be physical. A physical question is then to know whether an initial (inviscid) flow in the boundary layer (where Navier-Stokes applies) establishes (as a Navier-Stokes flow) fast or not.*

We claim that the characteristic time for the viscous effects to appear is roughly $T_{NSE} = \rho h_0^2/\nu$ or $T_{NSE} = \rho l^2/\nu$. Then, its ratio with the characteristic time of the inviscid gravity flow (l/c_0) is either Re $\sqrt{\beta} = \varepsilon^{-2}$ or Re $/\sqrt{\beta} = \varepsilon^{-3}$ respectively. In any case, it is large and the flow in the boundary layer does not establish fast enough. It does not enable to claim that a Euler initial condition is physically compatible with Navier-Stokes equations for moderate times. Khabakhpashev [13] already discussed it although he started from rest !

In the first subsection 3.1 we treat the upper part where convenient equations of (6) are kept. Then in subsection 3.2, after a rescaling, we solve in the boundary layer the convenient equations extracted from (6). The solutions are forced to match through a continuity condition at the boundary ($z = \varepsilon$) discussed in Subsection 3.3. At this stage, the system still has $u^u(1)$ and $\int_0^1 u^u$ terms. So Subsection 3.4 is devoted to making explicit and simple the dependence on z so as to get rid of these extra terms.

3.1 Resolution in the upper part

The upper part is characterized by $\varepsilon < z < 1 + \alpha\eta(x, t)$ and $x, t \in \mathbb{R}$. We start from the system for the fields in the upper part and write u, w, p instead of u^u, w^u, p^u for the sake of simplification. The height of the perturbation η is only defined in the upper part and so will always be denoted the same in the boundary layer. The system of PDE in the upper part is extracted from (6):

$$\begin{cases} u_t + \alpha u u_x + \frac{\alpha}{\beta} w u_z - \frac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} u_{zz} + p_x = 0, & \varepsilon < z < 1 + \alpha\eta, \\ w_t + \alpha u w_x + \frac{\alpha}{\beta} w w_z - \frac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \frac{1}{\mathrm{Re}\sqrt{\beta}} w_{zz} + p_z = 0, & \varepsilon < z < 1 + \alpha\eta, \\ \beta u_x + w_z = 0, & \varepsilon < z < 1 + \alpha\eta, \\ -\alpha\eta_x(\eta - p) + \frac{1}{\mathrm{Re}}(-2\alpha\sqrt{\beta} u_x \eta_x + (u_z + w_x)) = 0 & \text{on } z = 1 + \alpha\eta, \\ \eta - p + \frac{1}{\mathrm{Re}}(-\alpha\eta_x(u_z + w_x) - 2\sqrt{\beta} u_x) = 0 & \text{on } z = 1 + \alpha\eta, \\ \eta_t + \alpha u \eta_x - \frac{1}{\beta} w = 0 & \text{on } z = 1 + \alpha\eta. \end{cases} \tag{41}$$

Since we assume Re $\simeq \varepsilon^{-5/2}$, the terms $\sqrt{\beta}/\mathrm{Re}$ are of the order of ε^3 and the terms $1/(\mathrm{Re}\sqrt{\beta})$ of the order of ε^2. This simplifies $(41)_1$ and $(41)_2$ and justifies to take off the Laplacian. As a consequence, we must not keep the two dynamic conditions $(41)_4$ and $(41)_5$ since they are associated to a Laplacian. We decide to drop $(41)_4$.

Alternatively, one can stress that $(41)_5$ gives $\eta - p = O(\varepsilon^3)$ and so the lhs of $(41)_4$ is $O(\varepsilon^4) + O(\varepsilon^{5/2})$. Since we expand until the order two, one may claim the equation reduces to $0 = 0$. But one could also simplify by $1/\mathrm{Re}$ ($\simeq \varepsilon^{5/2}$) and be driven to a new equation. This equation would provide one more condition to the two equations for two fields. It is not surprising to see that the final solution would then be $u = 0$. The error is that we must drop one boundary condition unless we have one additionnal condition. The above argument to get rid of $(41)_4$ is sufficient.

On this topic, the literature uses the same equations, but the argument for dropping one boundary condition is rarely explicited. In [11], Kakutani and Matsuuchi claim "the

condition [$(41)_4$] is automatically satisfied" (p. 242 al. 3) which is either wrong (the equation disappears) or incomplete (what if they simplify by $\mathrm{Re} = \varepsilon^{5/2}$?).
In [8], Dutykh and Dias solve the same problem and write two equations (their (3) and (4)) among which they keep only one for the derivation without explaining this drop.

Let us come back to the resolution in the upper part. The equation $(41)_3$ gives w up to a constant that can be found in $(41)_6$:

$$w(x,z,t) = -\beta \int_{1+\alpha\eta}^{z} u_x(x,z',t)\,\mathrm{d}z' + \beta(\eta_t + \alpha u(1+\alpha\eta)\eta_x), \tag{42}$$

and we stress that this equation is exact. For the computations later, we need to expand this equation up to the third order:

$$w(x,z,t) = \beta(\eta_t + \int_0^1 u_x) - \beta \int_0^z u_x + \alpha\beta(u(1)\eta)_x + O(\varepsilon^3). \tag{43}$$

The second order of the previous equations suffices to determine p from $(41)_2$ up to a constant:

$$\begin{aligned} p(x,z,t) =& p(x,1+\alpha\eta,t) - \beta(\eta_{tt} + \int_0^1 u_{xt})(z-1) \\ &+ \beta \int_1^z \int_0^{z'} u_{xt}(x,z'',t)\,\mathrm{d}z''\,\mathrm{d}z' + O(\varepsilon^2). \end{aligned} \tag{44}$$

Thanks to $(41)_5$ the constant may be found ($p(1+\alpha\eta) = \eta + O(\varepsilon^3)$) and so:

$$\begin{aligned} p(x,z,t) =& \eta - \beta(\eta_{tt} + \int_0^1 u_{xt})(z-1) + \beta \int_1^z \int_0^{z'} u_{xt}(x,z'',t)\,\mathrm{d}z''\,\mathrm{d}z' + O(\varepsilon^2) \\ =& \eta - \beta\eta_{tt}(z-1) + \beta \int_1^z \int_1^{z'} u_{xt}(x,z'',t)\,\mathrm{d}z''\,\mathrm{d}z' + O(\varepsilon^2). \end{aligned} \tag{45}$$

Then the remaining field u satisfies $(41)_1$ at the first order:

$$u_t + \eta_x + \alpha u u_x + \alpha u_z(\eta_t + \int_z^1 u_x) - \beta\eta_{xtt}(z-1) - \beta\eta_{xxx}(z-1)^2/2 = O(\varepsilon^2), \tag{46}$$

where we have replaced the u_{xxt} by $-\eta_{xxx}$ as usual.
We still have to solve the equations in the lower part.

3.2 Resolution in the boundary layer

We need first to recall some classical properties of Laplace transforms.

3.2.1 Some useful properties

Before solving the equations in the lower part, we list here some classical properties of the Laplace transform. We start from the definition

$$\mathcal{L}(f)(p) = \hat{f}(p) = \int_{t\in\mathbb{R}^+} f(t)e^{-pt}\,\mathrm{d}t. \tag{47}$$

It is well-known that the Laplace transform of the derivative is given by

$$\mathcal{L}(f')(p) = -f(0) + p\mathcal{L}(f)(p). \tag{48}$$

If the two transforms $\mathcal{L}(f)(p)$ and $\mathcal{L}(g)(p)$ converge absolutely for $p = p_0$, and if both f and g are absolutely integrable and bounded in every finite interval that does not include the origin such as (p_1, p_2) where $0 < p_1 \leq p_2$, then the Laplace transform of the convolution exists for p such that $\Re(p) \geq \Re(p_0)$ ([7] Th. 10.1), even converges absolutely, and satisfies:

$$\mathcal{L}(f)(p)\mathcal{L}(g)(p) = \mathcal{L}(f * g)(p). \tag{49}$$

Below, we use the following definition of the convolution, linked to the Laplace transform:

$$f_1 * f_2(t) = \int_0^t f_1(u)f_2(t-u)\mathrm{d}u. \tag{50}$$

These formulas will be useful in the next subsection.

3.2.2 The fields in the boundary layer

The lower part of the domain ($0 < z < \varepsilon$) is a boundary layer. We start from the system for the bottom fields, written u, w, p instead of u^b, w^b, p^b for the sake of simplification and extracted from (6):

$$\begin{cases} u_t + \alpha u u_x + \dfrac{\alpha}{\beta} w u_z - \dfrac{\sqrt{\beta}}{\mathrm{Re}} u_{xx} - \dfrac{1}{\mathrm{Re}\,\sqrt{\beta}} u_{zz} + p_x = 0 & \text{for } 0 < z < \varepsilon, \\ w_t + \alpha u w_x + \dfrac{\alpha}{\beta} w w_z - \dfrac{\sqrt{\beta}}{\mathrm{Re}} w_{xx} - \dfrac{1}{\mathrm{Re}\,\sqrt{\beta}} w_{zz} + p_z = 0 & \text{for } 0 < z < \varepsilon, \\ \beta u_x + w_z = 0 & \text{for } 0 < z < \varepsilon, \\ u(z=0) = 0 \;\text{ and } w(z=0) = 0. \end{cases} \tag{51}$$

As is justified in subsection 2.3, the viscous and gravitational effects balance when $\mathrm{Re} \simeq \beta^{-5/2}$ (same as (26)). So we remind the reader of the assumptions $\mathrm{Re} = R\,\varepsilon^{-5/2}$, $\alpha = a\varepsilon$ and $\beta = b\varepsilon$ for constant R, a, b. We are naturally led to change the scale in z as in any boundary layer. Let us introduce a new vertical variable $\gamma = z/\varepsilon$. The new fields should be denoted in another way. Nevertheless, we will not change the notation for the sake of simplification. The new system writes:

$$\begin{cases} u_t + \alpha u u_x + \frac{a}{b\varepsilon} w u_\gamma - \frac{\sqrt{b}}{R}\varepsilon^3 u_{xx} - \frac{u_{\gamma\gamma}}{R\sqrt{b}} + p_x = 0, \\ w_t + \alpha u w_x + \frac{a}{b\varepsilon} w w_\gamma - \frac{\sqrt{b}}{R}\varepsilon^3 w_{xx} - \frac{w_{\gamma\gamma}}{R\sqrt{b}} + \frac{p_\gamma}{\varepsilon} = 0, \\ \varepsilon\beta u_x + w_\gamma = 0, \\ u(\gamma=0) = 0 \;\text{ and } w(\gamma=0) = 0. \end{cases} \tag{52}$$

One must notice that the Laplacian lets some remaining terms of zeroth degree in this system. So the viscosity is relevant in the boundary layer.
We can find the vertical velocity from $(52)_3$ and $(52)_4$:

$$w(x,\gamma,t) = -\varepsilon\beta \int_0^{\gamma} u_x(x,\gamma',t)\,\mathrm{d}\gamma'. \tag{53}$$

Carrying backward the previous equation in $(52)_2$, one has $p_\gamma = O(\varepsilon^3)$. So as to determine p, we need to use the continuity relation for the pressure $(p(x,\gamma=1,t) = p^u(x,z=\varepsilon,t))$ unless we cannot go on. Since we know the pressure in the upper part p^u from (45), we can write:

$$p(x,\gamma,t) = p(x,\gamma=1,t) + O(\varepsilon^3) = p^u(x,\varepsilon,t) + O(\varepsilon^3) = \eta(x,t) + O(\varepsilon). \tag{54}$$

Using this equation and (53) in $(52)_1$, we have at zeroth order:

$$u_t + \eta_x - \frac{u_{\gamma\gamma}}{R\sqrt{b}} = O(\varepsilon). \tag{55}$$

This equation must be completed with initial condition

$$u(x,\gamma,t=0) = u^{b,0}(x,\gamma), \tag{56}$$

and boundary condition:

$$\begin{cases} u(x,\gamma=0,t) = & 0, \\ u(x,\gamma\to+\infty,t) = & u^u(x,z=0,t) \text{ (continuity condition)}. \end{cases} \tag{57}$$

These are the equations to be solved.
Since we solve a Cauchy problem for a heat-like equation, we have an initial condition and so we must use the time-Laplace transform. In [11], the authors do not take an initial condition, and uses a time-Fourier transform. In all his articles, P.L. Liu, and coauthors (*e.g.* [19]), quote [22] (pp. 153–159) in which a sine-tranform (in γ) is used, but the initial condition is set to zero. In a separate calculation, not reproduced here, we used the same sine-transform in γ and paid attention to the initial condition. We were led to the very same result as the one stated hereafter.
We solve the system (55-57) in the following Lemma.

Lemma 6. *If the initial conditions $u^{b,0}(x,\gamma)$ and $u^{u,0}(x,z=0)$ are uniformly continuous in γ and satisfy*

$$\begin{aligned} &\int_0^{\infty} \mid u^{b,0}(x,\gamma) - u^{u,0}(x,z=0) \mid \mathrm{d}\gamma < \infty, \\ &\int_0^{\infty} \mid u_x^{b,0}(x,\gamma) - u_x^{u,0}(x,z=0) \mid \mathrm{d}\gamma < \infty, \end{aligned} \tag{58}$$

then the solution to (55-57) is

$$\begin{aligned} u(x,\gamma,t) = \; & u^u(x,z=0,t) + \frac{\sqrt{R\sqrt{b}}}{2}\int_0^{+\infty} f_0(x,\gamma')\frac{e^{-\frac{R\sqrt{b}(\gamma'-\gamma)^2}{4t}}}{\sqrt{\pi t}}\mathrm{d}\gamma' \\ & -u^u(x,0,.) * \mathcal{L}^{-1}(e^{-\sigma\gamma}) \\ & -\frac{\sqrt{R\sqrt{b}}}{2}\int_0^{+\infty} f_0(x,\gamma')\frac{e^{\frac{-R\sqrt{b}(\gamma'+\gamma)^2}{4t}}}{\sqrt{\pi t}}\mathrm{d}\gamma' + O(\varepsilon), \end{aligned} \tag{59}$$

where $f_0(x,\gamma) = u^{b,0}(x,\gamma) - u^{u,0}(x, z = 0)$*,* u^u *is the horizontal velocity in the upper part that satisfies (46) and* σ *is the only root with a positive real part of* $R\sqrt{b}\,p$*:*

$$\sigma = \sigma(p) = \sqrt{R\sqrt{b}p}. \tag{60}$$

where p *is the dual variable of time* t *and the convolution is in time.*

***Remark* 7.** *The solution of (55) may be known only up to any function of* x*. The boundary condition (57) enables to determine this function.*

***Remark* 8.** *The compatibility of the conditions (56) and (57) forces to have, when* γ *tends to* $+\infty$*:*

$$u^{b,0}(x,\gamma) \to u^{u,0}(x, z = 0),$$

and, when $\gamma \to 0$*:*

$$u^{b,0}(x, \gamma = 0) = 0.$$

Meanwhile we also prove the following Proposition

Proposition 9. *Under the same assumptions as in Lemma 6, the bottom shear stress is*

$$\begin{aligned}\tau^b = \left(\frac{\partial u^b}{\partial \gamma}\right)_{\gamma=0} = & \frac{\sqrt{R\sqrt{b}}u^u(x, z = 0, 0)}{\sqrt{\pi}}\text{p.v.}\frac{1}{\sqrt{t}} \\ & + \frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\int_0^t \frac{u_t^u(x, z = 0, t - s)}{\sqrt{s}}\mathrm{d}s,\end{aligned} \tag{61}$$

where p.v. *denotes the principal value as defined in the theory of distributions.*

First let us prove Proposition 9.

Proof. The initial condition f_0 may not make any difference (it can be seen through an explicit computation), so the correspondig term is taken off. Then a simple differentiation with respect to γ and the following formula (See [7] p. 320)

$$\mathcal{L}^{-1}\left(e^{-a\sqrt{p}}\right) = \frac{a}{2\sqrt{\pi}t^{3/2}}e^{-\frac{a^2}{4t}},$$

applied to (59) for any positive a leads to

$$\begin{aligned}\tau^b = & -\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\int_0^t u^u(x, z = 0, t - s)\frac{e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\sqrt{R\sqrt{b}}\gamma}{2\sqrt{\pi}s^{3/2}}\mathrm{d}s\right) + O(\varepsilon) \\ = & -\sqrt{R\sqrt{b}}\int_0^t \frac{u^u(x, z = 0, t - s)}{2\sqrt{\pi}s^{3/2}}e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\mathrm{d}s \\ & -\frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\int_0^t \frac{u^u(x, z = 0, t - s)}{s^{1/2}}\left(-\frac{R\sqrt{b}\gamma^2}{4s^2}e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\right)\mathrm{d}s + O(\varepsilon).\end{aligned}$$

The second term may be integrated by parts to get

$$-\frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\left(\frac{u^u(x,z=0,0)}{\sqrt{t}}e^{-\frac{R\sqrt{b}\gamma^2}{4t}}\right.$$
$$\left.-\int_0^t\left(-\frac{u^u_t(x,z=0,t-s)}{\sqrt{s}}-\frac{u^u(x,z=0,t-s)}{2s^{3/2}}\right)e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\mathrm{d}s\right),$$

which simplifies partially with the first term. At the end, there remains only

$$\frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\frac{u^u(x,z=0,0)}{\sqrt{t}}e^{-\frac{R\sqrt{b}\gamma^2}{4t}}+\frac{\sqrt{R\sqrt{b}}}{\sqrt{\pi}}\int_0^t\frac{u^u_t(x,z=0,t-s)}{\sqrt{s}}e^{-\frac{R\sqrt{b}\gamma^2}{4s}}\mathrm{d}s.$$

This justifies the formula as is classical in the theory of distributions. □

The scheme of the proof of Lemma 6 is to solve (55) up to two unknown functions, then to determine these functions so as to satisfy the initial and boundary conditions. This provides a necessary formula. We check in Appendix A that the solution satisfies the boundary and initial conditions. Let us prove Lemma 6.

Proof. Let us denote

$$f(x,\gamma,t)=u(x,\gamma,t)-u^u(x,z=0,t). \tag{62}$$

Since $f_t=u_t+\eta_x+O(\varepsilon)$ (thanks to (46)) and $f_\gamma=u_\gamma$, the equation (55) writes:

$$f_t-f_{\gamma\gamma}/(R\sqrt{b})=O(\varepsilon). \tag{63}$$

The initial condition is

$$f(x,\gamma,t=0)=u^{b,0}(x,\gamma)-u^{u,0}(x,z=0)=:f_0(x,\gamma), \tag{64}$$

and the boundary conditions read

$$\begin{aligned}&f(x,\gamma=0,t)=-u^u(x,0,t),\\&\lim_{\gamma\to+\infty}\lim_{\varepsilon\to0}u(x,\gamma,t)-u^u(x,z=\varepsilon,t)=\lim_{\gamma\to+\infty}f(x,\gamma,t)=0.\end{aligned} \tag{65}$$

The second condition is merely the continuity condition of the horizontal velocity at the border of the boundary layer. So we are driven to a heat equation in a half space with vanishing condition at infinity, and non-homogeneous initial and bottom conditions. Through a Laplace transform in time, denoted either $\mathcal{L}(f)$ or $\hat{f}$, (63) becomes

$$-f_0(x,\gamma)+p\hat{f}(p)-\frac{\hat{f}_{\gamma\gamma}}{R\sqrt{b}}=O(\varepsilon). \tag{66}$$

In order to solve this non-homogeneous ODE, we start with the homogeneous one and recall that we define σ as the only root with a positive real part of $R\sqrt{b}p$ in (60). Its solutions are

$$\hat{f}(x,\gamma,p)=C_1(x,p)e^{+\sigma\gamma}+C_2(x,p)e^{-\sigma\gamma}+O(\varepsilon).$$

By applying the method of parameters variation, we look for $C_1(x,\gamma,p)$, $C_2(x,\gamma,p)$ such that:

$$-C_{1,\gamma}\sigma e^{\sigma\gamma}+C_{2,\gamma}\sigma e^{-\sigma\gamma}=R\sqrt{b}f_0(x,\gamma)+O(\varepsilon),$$

and solving (66) amounts to solving the system of two equations with two unknown functions C_1 and C_2:

$$\left\{\begin{array}{rl} C_{1,\gamma}e^{\sigma\gamma}+C_{2,\gamma}e^{-\sigma\gamma}= & 0,\\ -C_{1,\gamma}e^{\sigma\gamma}+C_{2,\gamma}e^{-\sigma\gamma}= & \frac{R\sqrt{b}}{\sigma}f_0,\end{array}\right.$$

which solution is (thanks to assumption (58)):

$$\left\{\begin{array}{rl} C_1(x,\gamma,p)= & -\frac{R\sqrt{b}}{2\sigma}\int_{+\infty}^{\gamma} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma'+\tilde{C}_1(x,p),\\ C_2(x,\gamma,p)= & +\frac{R\sqrt{b}}{2\sigma}\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma'+\tilde{C}_2(x,p).\end{array}\right.$$

So, the full solution is

$$\begin{aligned}\hat{f}(x,\gamma,p)= & -\frac{R\sqrt{b}}{2\sigma}\int_{+\infty}^{\gamma} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{+\sigma\gamma}+\tilde{C}_1(x,p)e^{+\sigma\gamma}\\ & +\frac{R\sqrt{b}}{2\sigma}\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma}+\tilde{C}_2(x,p)e^{-\sigma\gamma}+O(\varepsilon).\end{aligned}$$

We look for $\tilde{C}_1$ first. Since f_0 is bounded, simple bounds prove that the first, third and fourth terms are bounded. So

$$\tilde{C}_1(x,p)=0.$$

The unknown function $\tilde{C}_2(x,p)$ is then given by the boundary condition $(65)_1$ at the bottom:

$$\tilde{C}_2(x,p)=-u^u(x,z=0,p)-\frac{R\sqrt{b}}{2\sigma}\int_0^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma'.$$

In a necessary way,

$$\begin{aligned}\hat{f}(x,\gamma,p)= & +\frac{R\sqrt{b}}{2\sigma}\int_{\gamma}^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{+\sigma\gamma}\\ & +\frac{R\sqrt{b}}{2\sigma}\int_0^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma}\\ & -\left(\hat{u}^u(x,z=0,p)+\frac{R\sqrt{b}}{2\sigma}\int_0^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma'\right)e^{-\sigma\gamma}+O(\varepsilon).\end{aligned} \tag{67}$$

From the definition of f, the existence of an inverse Laplace transform and formula (49), one knows that:

$$\begin{aligned}f(x,\gamma,t)= & \frac{R\sqrt{b}}{2}\int_{\gamma}^{+\infty} f_0(x,\gamma')\mathcal{L}^{-1}\left(\frac{e^{-\sigma(\gamma'-\gamma)}}{\sigma}\right)\mathrm{d}\gamma'\\ & +\frac{R\sqrt{b}}{2}\int_0^{\gamma} f_0(x,\gamma')\mathcal{L}^{-1}\left(\frac{e^{\sigma(\gamma'-\gamma)}}{\sigma}\right)\mathrm{d}\gamma'\\ & -u^u(x,z=0,.)*\mathcal{L}^{-1}\left(e^{-\sigma\gamma}\right)\\ & -\frac{R\sqrt{b}}{2}\int_0^{+\infty} f_0(x,\gamma')\mathcal{L}^{-1}\left(\frac{e^{-\sigma(\gamma'+\gamma)}}{\sigma}\right)\mathrm{d}\gamma'+O(\varepsilon).\end{aligned}$$

Owing to the formula (see [7])

$$\mathcal{L}^{-1}\left(\frac{e^{-\tilde{a}\sqrt{p}}}{\sqrt{p}}\right) = \frac{1}{\sqrt{\pi t}}e^{-\frac{\tilde{a}^2}{4t}},$$

if $\tilde{a} > 0$, one may justify the explicit form of u given in (59). Until the end of this article, we denote the function of time t:

$$A = A(t) = \sqrt{\frac{R\sqrt{b}}{4t}}. \tag{68}$$

We still have to check that the initial condition (64) and remaining of the boundary conditions $(65)_2$ are satisfied by u given by (59). This is completed in Appendix A.
So we completed the proof of the whole Lemma 6. □

From (53) and (59), we can then compute the vertical velocity

$$\begin{aligned} w^b(x,\gamma,t) = & -\varepsilon\beta \int_0^\gamma u_x^b(x,\gamma',t)\mathrm{d}\gamma' \\ = & -\varepsilon\beta u_x^u(x,0,t)\gamma + \varepsilon\beta u_x^u(x,0,.) * \mathcal{L}^{-1}\left(\frac{e^{-\sigma\gamma}-1}{-\sigma}\right) \\ & -\varepsilon\beta\frac{A(t)}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2(\gamma''-\gamma')^2}\,\mathrm{d}\gamma''\mathrm{d}\gamma' \\ & +\varepsilon\beta\frac{A(t)}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x,\gamma'')e^{-A^2(\gamma''+\gamma')^2}\,\mathrm{d}\gamma''\mathrm{d}\gamma' + O(\varepsilon^2\beta). \end{aligned} \tag{69}$$

We still have to satisfy the continuity conditions of all the fields u, w, p.

3.3 Continuity conditions

In the present subsection, we need to write explicitly the superscripts u and b for the upper part and bottom regions respectively. We write the computed fields at the same height ε that is the common frontier of both subdomains.
We already used the continuity of pressure that led us to (54). So the pressure is continuous.

Regarding the horizontal velocity, we must notice that the limit when $\gamma \to +\infty$ of $\lim_{\varepsilon\to 0}(u^b(x,\gamma,t) - u^u(x,\varepsilon\gamma,t)) = f(x,\gamma,t)$ has already been computed as vanishing (see Appendix A). So the boundary condition $(65)_2$ is already satisfied and the horizontal velocity is continuous.

Concerning the vertical velocity, we can use the velocity in the upper part w^u from (43) expanded in ε:

$$\begin{aligned} w^u(x,\varepsilon\gamma,t) = & \beta(\eta_t + \int_0^1 u_x^u) - \beta\int_0^{\varepsilon\gamma} u_x^u + \alpha\beta(u^u(1)\eta)_x + O(\varepsilon^3) \\ = & \beta(\eta_t + \int_0^1 u_x^u) - \beta\varepsilon\gamma u_x^u(z=0) + \alpha\beta(u^u(1)\eta)_x + O(\varepsilon^3). \end{aligned}$$

One may notice that as anywhere else, the $u^u(z=0)$ could be replaced by $u^u(z=\varepsilon)$ and $\int_0^1 u_x^u$ by $\int_\varepsilon^{1+\alpha\eta} u_x^u$ and so on. The formula would be the same and the final result would be the same.

The velocity in the bottom w^b is given in (69). The difference $w^u(x, \varepsilon\gamma, t) - w^b(x, \gamma, t)$ can be expanded in ε:

$$\begin{aligned}
w^u - w^b =& \beta(\eta_t + \int_0^1 u_x^u) - \beta\varepsilon\gamma u_x^u(z=0) + \alpha\beta(u^u(1)\eta)_x + O(\varepsilon^3) \\
&+ \varepsilon\beta u_x^u(x, z=0, t)\gamma - \varepsilon\beta u_x^u(x, 0, .) * \mathcal{L}^{-1}\left(\frac{e^{-\sigma\gamma} - 1}{-\sigma}\right) \\
&+ \varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')e^{-A^2(\gamma''-\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma' \\
&- \varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')e^{-A^2(\gamma''+\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma' + O(\varepsilon^3) \\
=& \beta(\eta_t + \int_0^1 u_x^u) + \alpha\beta(u^u(1)\eta)_x - \varepsilon\beta u_x^u(x, 0, .) * \mathcal{L}^{-1}\left(\frac{1}{\sigma}\right) \\
&+ \varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')e^{-A^2(\gamma''-\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma' \\
&- \varepsilon\beta\frac{A}{\sqrt{\pi}}\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')e^{-A^2(\gamma''+\gamma')^2}\mathrm{d}\gamma''\mathrm{d}\gamma' + O(\varepsilon^3),
\end{aligned} \tag{70}$$

up to functions that tend exponentially to zero when $\gamma \to +\infty$.

We still must simplify the two last double integrals. This is made in the following Lemma

Lemma 10. *If $A = A(t) = \sqrt{\frac{R\sqrt{b}}{4t}}$, γ is positive, $f_0(x, \gamma)$ is uniformly continuous in γ and satisfies (58), then*

$$\int_{\gamma'=0}^{\gamma}\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')\left(e^{-A^2(\gamma''-\gamma')^2} - e^{-A^2(\gamma''+\gamma')^2}\right)\mathrm{d}\gamma''\mathrm{d}\gamma'$$

tends to

$$\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')\int_{\gamma'''=-\gamma''}^{\gamma''} e^{-A^2\gamma'''^2}\mathrm{d}\gamma'''\mathrm{d}\gamma'', \tag{71}$$

when $\gamma \to +\infty$.

The proof relies on Fubini's theorem and changes of variables for the two integrals. The proof is only technical and left to the reader.
After simplification by β, the continuity of the vertical velocity (70) reads after making $\gamma \to +\infty$ thanks to Lemma 10:

$$\begin{aligned}
\eta_t + \int_0^1 u_x^u + \alpha(u^u(1)\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}u_x^u(x, 0, t) * \frac{1}{\sqrt{t}} \\
+ \frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty} f_{0,x}(x, \gamma'')\int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma'' = O(\varepsilon^2),
\end{aligned} \tag{72}$$

where the convolution is in time t and the formula $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{p}}\right) = 1/\sqrt{\pi t}$ [7] is used. If one had made the more rigorous expansion according to Remark 4, assuming u^u is defined only on $(\varepsilon, 1+\alpha\eta)$ and u^b is defined on $(0,1)$, one would have been led to

$$u_t + \int_\varepsilon^{1+\alpha\eta} u_x^u + \alpha u^u(1+\alpha\eta)\eta_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x^u(x, z=\varepsilon, t) * \frac{1}{\sqrt{t}}$$
$$+ \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{1/\sqrt{\varepsilon}} f_{0,x}(x,\gamma'') \int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' = O(\varepsilon^2). \quad (73)$$

3.4 The dependence on z of the fields

At this stage, we have reduced the equations but not as much as in the Euler case which leads to a Boussinesq system in 1+1 dimension. We have derived only a 2+1 dimension problem although we have eliminated w and p. The major difference with the Boussinesq derivation comes from the assumption of irrotationnality of Euler flows. This assumption would provide $u_z = O(\varepsilon)$. Such a condition would annihilate the dependence on z and greatly simplify the above computations.

Yet irrotationality and its corollary of a potential flow is incompatible with the number of conditions we set at the bottom, which are needed by the dissipativity of the Navier-Stokes equations. So we need to determine the dependence on z of u to have a more tractable system.

Starting from now, we drop the u superscripts for the fields in the upper part but keep the superscripts for the boundary layer. In summary, we assume Re $\simeq \varepsilon^{-5/2}$, and the assumptions of the first asymptotic stated in the subsection 2.3. Up to now, the reduced equations are collected from (46) and (72):

$$u_t + \eta_x + \alpha u u_x + \alpha u_z(\eta_t + \int_z^1 u_x) - \beta\eta_{xtt}(z-1) - \beta\eta_{xxx}(z-1)^2/2 = O(\varepsilon^2),\ \forall z \quad (74)$$

$$\eta_t + \int_0^1 u_x(z)\,\mathrm{d}z + \alpha(u(z=1)\eta)_x - \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} u_x(x, z=0, t) * \frac{1}{\sqrt{t}}$$
$$+ \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} \left(u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x, z=0)\right) \int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' = O(\varepsilon^2). \quad (75)$$

The equation (74) can be rewritten thanks to the order 0 of (75):

$$u_t + \eta_x + \alpha u u_x - \alpha u_z \int_0^z u_x - \beta\eta_{xtt}(z-1) - \beta\eta_{xxx}(z-1)^2/2 = O(\varepsilon^2), \forall z. \quad (76)$$

Notice that the η_{xxx} term comes from an integral of the shape $\int_1^z \int_1^{z'} u_{xxt}$. As an intermediate result one may see very easily that $\eta_{xx} = \eta_{tt} + O(\varepsilon)$ which is useful later.

We intend to prove the following Lemma:

Lemma 11. *A localized solution of (75), (76) is such that*

$$\int_0^1 u = u(x,z,t) - \beta\eta_{xt}\frac{z^2 - 1/3}{2} + O(\varepsilon^2), \tag{77}$$

$$u(x,0,t) = u(x,z,t) - \beta\eta_{xt}\frac{z^2}{2} + O(\varepsilon^2), \tag{78}$$

$$u(x,1,t) = u(x,z,t) + \beta\eta_{xt}\frac{1 - z^2}{2} + O(\varepsilon^2). \tag{79}$$

Proof. In a preliminary step, we prove

$$u_z(x,z,t) = \beta\,\eta_{xt}(x,t)\,z + O(\varepsilon^2). \tag{80}$$

To that end, we differentiate (76) with respect to z, so as to have:

$$u_{zt} + \alpha u\,u_{xz} - \alpha u_{zz}\int_0^z u_x - \beta\eta_{xtt} - \beta\eta_{xxx}(z-1) = O(\varepsilon^2),$$

and we can integrate this equation in time using that $\eta_{xx} = \eta_{tt} + O(\varepsilon)$:

$$u_z + \alpha\int_{t_0}^t (u\,u_{xz}) - \alpha\int_{t_0}^t (u_{zz}\int_0^z u_x) - \beta\eta_{xt} - \beta\eta_{xt}(z-1) = C_3(x,z) + O(\varepsilon^2), \tag{81}$$

where C_3 is a function of x, z but it does not depend on t. Since the solution is localized for any x, z, there exists a time t_0 at which u_z and η_{xt} vanish or are as small as wanted (in a local norm). So

$$C_3(x,z) = O(\varepsilon),$$

in a first attempt to determine C_3. But then the equation (81) implies $u_z = O(\varepsilon)$ and so the quadratic terms are all of second order in (81) since they contain at least one u_z. Hence

$$u_z(x,z,t) - \beta\eta_{xt}\,z = C_4(x,z) + O(\varepsilon^2).$$

Again since for all (x,z) there exists a time at which u and η vanish or are as small as wanted, then $C_4(x,z) = O(\varepsilon^2)$ and this completes the proof of (80). We can then go further by integrating between z' and z:

$$u(x,z,t) = u(x,z',t) + \beta\eta_{xt}\frac{z^2 - z'^2}{2} + O(\varepsilon^2),$$

and then, integrating in z' between $z' = 0$ and $z' = 1$, we can state (77). Setting $z' = 0$ (or $z' = \varepsilon$), we obtain (78) and setting $z' = 1$ we obtain (79).

□

So the system (75,76) can be rewritten thanks to (77-79), the formula $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{p}}\right) = 1/\sqrt{\pi t}$ [7], and the fact that, as in the Euler case the wave equation is the zeroth order ($\eta_{xx} =$

$\eta_{tt} + O(\varepsilon)$):

$$u_t + \eta_x + \alpha u u_x - \beta \eta_{xxx} \frac{(z^2 - 1)}{2} = O(\varepsilon^2), \tag{82}$$

$$\eta_t + u_x(x, z, t) - \frac{\beta}{2} \eta_{xxt} (z^2 - \frac{1}{3}) + \alpha (u\eta)_x - \frac{\varepsilon}{\sqrt{\pi R \sqrt{b}}} u_x * \frac{1}{\sqrt{t}}$$
$$+ \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} \left(u_x^{b,0}(x, \gamma'') - u_x^{u,0}(x, z = 0) \right) \int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' = O(\varepsilon^2), \tag{83}$$

where all the fields u are evaluated at (x, z, t) and the convolution is in time. This is the system stated in Proposition 3 and the proof is complete.

4 Generalization and checkings

In a first subsection, we state the 2-D Boussinesq system and check we may find the classical Boussinesq systems in the inviscid case. Then, in Subsection 4.2 we derive rigorously the viscous KdV equation and discuss its compatibility with the equation derived by Kakutani and Matsuuchi in [11], by Liu and Orfila in [19], and by Dutykh in [9].

4.1 The full 2-D Boussinesq systems family

One may start from the 3-D Navier-Stokes equations and derive in a way very similar to above a generalization of (82, 83):

$$\left\{ \begin{aligned} u_t + \eta_x + \alpha u u_x + \alpha v u_y - \beta(\eta_{xxx} + \eta_{xyy}) \frac{(z^2-1)}{2} &= O(\varepsilon^2), \\ v_t + \eta_y + \alpha u v_x + \alpha v v_y - \beta(\eta_{yxx} + \eta_{yyy}) \frac{(z^2-1)}{2} &= O(\varepsilon^2), \\ \eta_t + u_x + v_y - \frac{\beta}{2}(\eta_{xxt} + \eta_{yyt})(z^2 - \tfrac{1}{3}) & \\ + \alpha(u\eta)_x + \alpha(v\eta)_y + \tfrac{\varepsilon}{\sqrt{\pi R \sqrt{b}}} \eta_t * \left(\tfrac{1}{\sqrt{t}}\right) & \\ + \frac{2\varepsilon}{\sqrt{\pi}} \int_{\gamma''=0}^{+\infty} \left(u_x^{b,0}(x, \gamma'') - u_x^{u,0}(x, z=0) \right) \int_{\gamma'=0}^{A(t)\gamma''} e^{-\gamma'^2} \mathrm{d}\gamma' \mathrm{d}\gamma'' &= O(\varepsilon^2). \end{aligned} \right. \tag{84}$$

In case of a Euler initial condition, the last integral term vanishes, but this is not physical as is stressed in Remark 5.

It is well-known thanks to [1] that there is a family of Boussinesq systems, indexed by three free parameters. All these systems are equivalent in the sense that up ot order 1, they can be derived one from the other by using their own $O(\varepsilon^0)$ part and by replacing partially η_t, η_x and η_y by u_x, u_t, v_t. We are going to prove the same for our system. Namely, the order 0 of (84) enables to interpolate with $a_{int}, b_{int}, c_{int}$:

$$\left\{ \begin{array}{l} \eta_x = a_{int} \eta_x - (1 - a_{int}) u_t + O(\varepsilon), \\ \eta_y = b_{int} \eta_y - (1 - b_{int}) v_t + O(\varepsilon), \\ \eta_t = c_{int} \eta_t - (1 - c_{int})(u_x + v_y) + O(\varepsilon). \end{array} \right.$$

These formulas are reported in the full 2D system (84), where we drop the convolution term and the integral on the initial condition:

$$\left\{\begin{array}{ll} u_t + \eta_x + \alpha u u_x + \alpha v u_y - a_{int}\beta\Delta\eta_x\dfrac{(z^2-1)}{2} & \\ +(1-a_{int})\beta\Delta u_t\dfrac{(z^2-1)}{2} & = O(\varepsilon^2), \\ v_t + \eta_y + \alpha u v_x + \alpha v v_y - b_{int}\beta\Delta\eta_y\dfrac{(z^2-1)}{2} & \\ +(1-b_{int})\beta\Delta v_t\dfrac{(z^2-1)}{2} & = O(\varepsilon^2), \\ \eta_t + u_x + v_y - c_{int}\dfrac{\beta}{2}\Delta\eta_t(z^2-\frac{1}{3}) & \\ +(1-c_{int})\dfrac{\beta}{2}\Delta(u_x+v_y)(z^2-\frac{1}{3}) + \alpha(u\eta)_x + \alpha(v\eta)_y & = O(\varepsilon^2), \end{array}\right. \tag{85}$$

where we denote Δ the x, y Laplacian.
This is the general Boussinesq system as can be seen in [1] (p. 285 equation (1.6)). Indeed if we denote $a_{BCS}, b_{BCS}, c_{BCS}$ and d_{BCS} the interpolation parameters of this article, we can identify the 1D version of our interpolated (85) with

$$\begin{array}{ll} a_{BCS} = \frac{\beta}{2}(1-c_{int})(z^2-\frac{1}{3}), & b_{BCS} = \frac{\beta}{2}c_{int}(z^2-\frac{1}{3}), \\ c_{BCS} = -\beta a_{int}\frac{z^2-1}{2}, & d_{BCS} = -\beta(1-a_{int})\frac{z^2-1}{2}. \end{array}$$

The meaning of our height z is the same as the θ of [1] and the relation between $a_{BCS}, b_{BCS}, c_{BCS}$ and d_{BCS} (see (1.8) of this article) is satisfied.

4.2 About the KdV-like equation

Various authors have derived either a viscous Boussinesq system or a viscous KdV equation.
One may wonder what is the viscous KdV equation derived from our viscous Boussinesq system and compare it with what may be found in the literature. First, we state and prove the following Proposition.

Proposition 12. *If the initial flow is localized, the KdV change of variables applied to the system (82, 83) leads to*

$$2\tilde{\eta}_\tau + 3a\tilde{\eta}\tilde{\eta}_\xi + \frac{b}{3}\tilde{\eta}_{\xi\xi\xi} - \frac{1}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\tilde{\eta}_\xi(\xi+\xi',\tau)}{\sqrt{\xi'}}\mathrm{d}\xi' = O(\varepsilon), \tag{86}$$

for not too small times τ, where we set $\alpha = a\varepsilon$, Re$= R\,\varepsilon^{-5/2}$ and $\beta = b\varepsilon$.

In the formula (86), since it has been proved in [15] that KdV is a good approximation of Euler for times up to $1/\varepsilon^2$, and that the velocity is localized, it is a strong temptation to replace the integral term by

$$-\frac{1}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{+\infty}\frac{\tilde{\eta}_\xi(\xi+\xi',\tau)}{\sqrt{\xi'}}\mathrm{d}\xi'.$$

This is the term found in [11].

Proof. We start from the most general form of (82, 83) (same as (39)) and use the KdV change of variables

$$(\xi = x - t,\ \tau = \varepsilon t)\ \Leftrightarrow (x = \xi + \tau/\varepsilon,\ t = \tau/\varepsilon), \tag{87}$$

and change of fields

$$\Phi(x,z,t) = \tilde{\Phi}(x-t, z, \varepsilon t) \Rightarrow \Phi_t = -\tilde{\Phi}_\xi + \varepsilon\tilde{\Phi}_\tau(x-t,z,\varepsilon t), \tag{88}$$

where the generic field Φ is tilded when it depends on the (ξ, z, τ) variables.
There are only two difficult terms in the system (82, 83) (equivalent to (39)). The first is the convolution which we denote T_1:

$$\begin{aligned} T_1(x,z,t) &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{t'=0}^{t} \frac{u_x(x,z,t-t')}{\sqrt{t'}}\mathrm{d}t' \\ &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{t'=0}^{t} \frac{\tilde{u}_\xi(x-t+t', z, \varepsilon t - \varepsilon t')}{\sqrt{t'}}\mathrm{d}t', \end{aligned}$$

because of (87). But then it suffices to recognize the function of $(x-t, \varepsilon t) = (\xi, \tau)$ in the last equation to have the term after the KdV change of variables:

$$\begin{aligned} \tilde{T}_1(\xi,z,\tau) &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{t'=0}^{\tau/\varepsilon} \frac{\tilde{u}_\xi(\xi+t', z, \tau-\varepsilon t')}{\sqrt{t'}}\mathrm{d}t' \\ &= -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon} \frac{\tilde{u}_\xi(\xi+\xi', z, \tau)}{\sqrt{\xi'}}\mathrm{d}\xi' + O(\varepsilon^2). \end{aligned} \tag{89}$$

Since the t' variable is in place of a ξ, we changed the notation to ξ'. This term is odd because it has an integration variable (ξ') that has a physical meaning but bounds depending on time τ/ε. We discuss it below.
The second difficult term is the one that keeps the initial conditions and writes:

$$T_2(x,z,t) = +\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(x,\gamma'') - u_x^{u,0}(x,z=0)\right) \times \int_{\gamma'=0}^{\sqrt{\frac{R\sqrt{b}}{4t}}\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma''.$$

The change of variables (87) gives:

$$\begin{aligned} \tilde{T}_2(\xi,z,\tau) = &+\frac{2\varepsilon}{\sqrt{\pi}}\int_{\gamma''=0}^{+\infty}\left(u_x^{b,0}(\xi+\frac{\tau}{\varepsilon},\gamma'') - u_x^{u,0}(\xi+\frac{\tau}{\varepsilon}, z=0)\right) \\ &\times \int_{\gamma'=0}^{\sqrt{\frac{R\sqrt{b}\varepsilon}{4\tau}}\gamma''} e^{-\gamma'^2}\mathrm{d}\gamma'\mathrm{d}\gamma''. \end{aligned}$$

If the initial boundary layer is localized, for τ not too small, $u_x^{b,0}(\xi+\frac{\tau}{\varepsilon},\gamma'') - u_x^{u,0}(\xi+\frac{\tau}{\varepsilon}, z = 0)$ will be small in $L^1_{\gamma''}$ and so $\tilde{T}_2$ will be negligible in comparison with ε and so can be dropped. In addition, the inner integral's upper bound is very close to the lower bound. Then, we can claim that the Boussinesq system after the KdV change of variables and fields is

$$\left\{\begin{aligned} &-\tilde{u}_\xi + \varepsilon\tilde{u}_\tau + \tilde{\eta}_\xi + \alpha\tilde{u}\tilde{u}_\xi - \beta\tilde{\eta}_{\xi\xi\xi}\left(\frac{z^2-1}{2}\right) &&= O(\varepsilon^2), \\ &-\tilde{\eta}_\xi + \varepsilon\tilde{\eta}_\tau + \tilde{u}_\xi + \frac{\beta}{2}\tilde{\eta}_{\xi\xi\xi}\left(z^2 - \tfrac{1}{3}\right) + \alpha(\tilde{u}\tilde{\eta})_\xi && \\ &\qquad -\frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}}\int_{\xi'=0}^{\tau/\varepsilon}\frac{\tilde{u}_\xi(\xi+\xi',z,\tau)}{\sqrt{\xi'}}\mathrm{d}\xi' &&= O(\varepsilon^2). \end{aligned}\right. \tag{90}$$

We may notice that at the first order, and as in the derivation of the KdV equation,

$$\tilde{u}_\xi = \tilde{\eta}_\xi + O(\varepsilon) \Rightarrow \tilde{u} = \tilde{\eta} + O(\varepsilon),$$

thanks to a simple and classical integration (and a localized solution). But then the sum of the two equations of (90) gives:

$$\varepsilon \tilde{u}_\tau \,+\, \varepsilon \tilde{\eta}_\tau \,+\, \alpha \tilde{u}\tilde{u}_\xi \,+\, \alpha(\tilde{u}\tilde{\eta})_\xi \,+\, \frac{\beta}{3}\tilde{\eta}_{\xi\xi\xi} \,-\, \frac{\varepsilon}{\sqrt{\pi R\sqrt{b}}} \int_{\xi'=0}^{\tau/\varepsilon} \frac{\tilde{u}_\xi(\xi+\xi', z, \tau)}{\sqrt{\xi'}} \mathrm{d}\xi' \;=\; O(\varepsilon^2).$$

Using now the fact that $\tilde{u} = \tilde{\eta} + O(\varepsilon)$, dividing by ε, one states exactly the equation (86). The convolution that used to be on time is now on ξ' and the proof is complete. □

What can be found in the literature ?
As stated in the introduction, various authors already derived either a viscous Boussinesq system or a viscous KdV equation. Yet, none of them have the very same equation as us. We must clarify why there are such differences.
The first article is [23] in which Ott and Sudan obtained formally in nine lines:

$$+\alpha_3 \int_{\xi'=-\infty}^{+\infty} \frac{\tilde{u}_\xi(\xi', \tau)\,\mathrm{sgn}(\xi-\xi')}{\sqrt{\mid \xi - \xi' \mid}} \mathrm{d}\xi'.$$

but the authors used a Fourier transform [11] (p. 243) and they made an error pointed by [11]. Our formula differs from Ott and Sudan's by the sign and the bound !
Later, Kakutani and Matsuuchi [11] derived rather rigorously the KdV equation from Navier-Stokes and we set the same regime as them. Yet, they did not raise the problem of the initial condition. As a consequence, they used a time-Fourier transform to solve the heat-like equation. They proposed:

$$\frac{1}{4\sqrt{\pi R}} \int_{\xi'=-\infty}^{+\infty} \frac{\tilde{\eta}_\xi(\xi', \tau)(1-\mathrm{sgn}(\xi-\xi'))}{\sqrt{\mid \xi - \xi' \mid}} \mathrm{d}\xi'.$$

Their half order derivative term differs from ours only by the bound of the integral which is τ/ε for us and $+\infty$ for them.
Liu and Orfila in [19] (and subsequent articles) derived a Boussinesq system for a regime different from ours (Re$=R\,\varepsilon^{-7/2}$). They solved their heat equation with a sine-transform in the vertical coordinate by quoting [22] where a vanishing initial condition is assumed. Given their regime, their Boussinesq system is right. But when they derived a KdV equation (see [19] p. 89), they did not make explicit their change of variables in the term equivalent to our T_1. With the change of variable $\xi_{LO} = x - t, \tau_{LO} = (\alpha_{LO}/\mu_{LO})t$, they exhibit (see their (3.19) or (3.21)):

$$-\frac{1}{2\sqrt{\pi}} \int_0^t \frac{\eta_{\xi_{LO}}}{\sqrt{t-T}} \mathrm{d}T,$$

where there remains the former variable t inside the integral and in the bounds. Moreover, the dependence of $\eta_{\xi_{LO}}$ on the variables $(t, \tau_{LO}, ...\ ?)$ is not written. Is the T variable in

the integral a time variable ? One may wonder whether they did notice that the time convolution transforms into a *space* one.
Dutykh derived a Boussinesq system by a Leray-Helmholtz decomposition from a Linearized Navier-Stokes [9]. In order to derive the associated KdV (see Sec. 3.2), he assumed $u = \eta + \varepsilon P + \beta Q + ...$ and found P and Q. In this process, he used only the assumption that waves go right ($\eta_t + \eta_x = O(\varepsilon)$). So he *did not* use the change of time ($\tau = \varepsilon t$) and wrote a formula with unscaled time t (his (14)):

$$-\sqrt{\frac{\nu}{\pi}\frac{g}{h}}\int_0^t \frac{\eta_x}{\sqrt{t-\tau}}\mathrm{d}\tau.$$

Similar criticisms can be said on this formula in which the integral seems to be on time while it should be on the shifted space ξ.

5 Conclusion

In this article, we derive the viscous Boussinesq system for surface waves from Navier-Stokes equations with non-vanishing initial conditions (see Proposition 3). One of our by-product is the bottom shear stress as a function of the velocity (cf. Proposition 9) and the decay rate for shallow water (see Proposition 1). We also state the system in 3-D in (84), and derive the viscous KdV equation from our viscous Boussinesq system (cf. Proposition 12). The differences of our viscous KdV with other equations, already derived in the literature, are highlighted and explained.

A Boundary and initial conditions in Lemma 6

As is said in the proof of Lemma 6, we must check that u, given by the necessary equation (59), satisfies the initial condition (64) and the remaining of the boundary conditions $(65)_2$.
Concerning the initial condition (64). We try to find the limit when t tends to 0^+ and then $A = A(t)$ tends to $+\infty$. Since one assumes below $\gamma > 0$,

$$-u^u(x,0,.) * \mathcal{L}^{-1}(e^{-\sigma\gamma}) = -u^u(x,0,t) * \frac{\sqrt{R\sqrt{b}}}{2\sqrt{\pi}t^{3/2}}e^{-\frac{R\sqrt{b}}{4t}}$$

tends to zero exponentially (the convolution is the Laplace one and on time t). Then, one can come back to the formula of f (67) and make one change of variables in every integral:

$$\begin{aligned} f(x,\gamma,t) =& \frac{A}{\sqrt{\pi}}\int_{-\gamma}^{+\infty} f_0(x,\Gamma'+\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ &- \frac{A}{\sqrt{\pi}}\int_{\gamma}^{+\infty} f_0(x,\Gamma'-\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + O(\varepsilon), \end{aligned}$$

up to an exponentially tending to zero function when t tends to 0 thanks to $A(t)$. This can be rewritten

$$\begin{aligned} f(x,\gamma,t) =& \frac{A}{\sqrt{\pi}}\int_{\gamma}^{+\infty}\left(f_0(x,\Gamma'+\gamma)-f_0(x,\Gamma'-\gamma)\right)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ &+\frac{A}{\sqrt{\pi}}\int_{-\gamma}^{\gamma} f_0(x,\Gamma'+\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + O(\varepsilon), \end{aligned}$$

where we denote I_2 the second integral. The first integral may be bounded by

$$\begin{aligned} &\frac{2A}{\sqrt{\pi}}\sup_{\gamma>0}\mid f_0(x,\gamma)\mid \int_{\gamma}^{+\infty} e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ \leq& \frac{2}{\pi}\sup_{\gamma>0}\mid f_0(x,\gamma)\mid \int_{A\gamma}^{+\infty} e^{-\Gamma''^2}\mathrm{d}\Gamma'', \end{aligned}$$

which clearly tends to zero when t tends to zero ($A = A(t) \to +\infty$).
For the second integral denoted I_2, one may compute a similar integral where the integration variable of f_0 is frozen:

$$\begin{aligned} I_2' =& \frac{A}{\sqrt{\pi}}\int_{-\gamma}^{\gamma} f_0(x,\gamma)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ =& f_0(x,\gamma)\frac{1}{\sqrt{\pi}}\int_{-A\gamma}^{A\gamma} e^{-\Gamma''^2}\mathrm{d}\Gamma'', \end{aligned}$$

which clearly tends to $f_0(x,\gamma)$ if $\gamma > 0$ when $t \to 0^+$. So one may make the difference of the second integral I_2 with the previous integral (which tends to $f_0(x,\gamma)$) and find:

$$I_2 - I_2' = \frac{A}{\sqrt{\pi}}\int_{-\gamma}^{\gamma}\left(f_0(x,\Gamma'+\gamma)-f_0(x,\gamma)\right)e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' + o_{t\to 0^+}(1).$$

Here we must use the assumption of uniform continuity of the initial data:

$$\forall\epsilon > 0\ \exists\delta > 0\ /\ \mid \gamma' - \gamma \mid < \delta \Rightarrow \mid f_0(x,\gamma') - f_0(x,\gamma)\mid < \epsilon.$$

Then, for any $\epsilon > 0$, there exists a δ such that $I_2 - I_2'$ can be splitted into two parts and bounded by

$$\begin{aligned} &\frac{2A}{\sqrt{\pi}}\sup_{\gamma>0}\mid f_0(x,\gamma)\mid \int_{|\Gamma'|>\delta\bigcap|\Gamma'|<\gamma} e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' &&+\frac{A}{\sqrt{\pi}}\epsilon\int_{|\Gamma'|<\delta\bigcap|\Gamma'|<\gamma} e^{-A^2\Gamma'^2}\mathrm{d}\Gamma' \\ \leq\ &\frac{2A}{\sqrt{\pi}}\sup_{\gamma>0}\mid f_0(x,\gamma)\mid 2\gamma e^{-A^2\delta^2} &&+\ \epsilon. \end{aligned}$$

So we have proved that the f given by (67) or u given by (59) satisfies the initial condition ($A(t) \to +\infty$ when $t \to 0^+$).
Concerning the boundary condition $(65)_2$). Now we look for the limit when γ tends to $+\infty$. The formula (67) can be written:

$$\hat{f}(x,\gamma,p) = \frac{R\sqrt{b}}{2\sigma}\int_{\gamma}^{+\infty} f_0(x,\gamma')e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{+\sigma\gamma} + \frac{R\sqrt{b}}{2\sigma}\int_{0}^{\gamma} f_0(x,\gamma')e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma},$$

up to some exponentially tending to zero functions of γ. In this formula, the first integral is bounded by

$$\begin{aligned} & \frac{R\sqrt{b}}{2\sigma} \sup_{\gamma'\geq\gamma} \mid f_0(x,\gamma') \mid \int_{\gamma}^{+\infty} e^{-\sigma\gamma'}\mathrm{d}\gamma' e^{\sigma\gamma} \\ \leq \ & \frac{R\sqrt{b}}{2\sigma^2} \sup_{\gamma'\geq\gamma} \mid f_0(x,\gamma') \mid, \end{aligned}$$

which clearly tends to zero when γ tends to $+\infty$ because $f_0(x,\gamma)$ tends to zero when γ tends to $+\infty$.
For the second integral, one needs to cut it at a value Γ given by the definition of $f_0 \to 0$ when γ tends to $+\infty$ ($\forall\epsilon > 0\ \exists\Gamma > 0 \ / \mid \gamma \mid > \Gamma \Rightarrow \mid f_0 \mid < \epsilon$). We can bound it with:

$$\frac{R\sqrt{b}}{2\sigma}\int_0^{\Gamma} \mid f_0(x,\gamma') \mid e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma} + \frac{R\sqrt{b}}{2\sigma}\epsilon\int_{\Gamma}^{\gamma} e^{\sigma\gamma'}\mathrm{d}\gamma' e^{-\sigma\gamma}.$$

Since the first term tends to zero when γ tends to $+\infty$ (Γ fixed) and the second term is less than $R\sqrt{b}\epsilon/(2\sigma^2)$, the whole can be made smaller than any ϵ.
So the proof that $(65)_2$ is satisfied is complete.

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