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Complexité des flots géodésiques intégrables sur le tore

sous la direction de Jean-Pierre Marco *et* Elisha Falbel

Soutenue le Vendredi 13 Juillet 2012 devant le jury composé de :

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M. Alexey Bolsinov	Loughborough University
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Après les rapports de :

M. Patrick Bernard	Université Paris-Dauphine
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À la mémoire de Vati qui aurait tant aimé que
je lui explique la géométrie symplectique,
et à celle d'Andrée Delamarre, la première.

À mes parents, pour leur confiance au-delà
de toute réussite ou de tout échec.

Et à Jean-Pierre.

'Tell me one thing', said Harry. 'Is this real? Or has this been happening inside my head?'

Dumbledore beamed at him, and his voice sounded loud and strong in Harry's ears even though the bright mist was descending again, obscuring his figure.

'Of course it is happening inside your head, Harry, but why on earth should that mean that it is not real?'

Joan Rowling, *Harry Potter and the Deathly Hallows.*

Une page se tourne...

Une page se tourne en effet, emportant avec elle cinq longues années de ma vie, pendant lesquelles toute mon énergie ou presque était consacrée à ce but unique : écrire une thèse de mathématiques. Pendant les dernières années, j'ai souvent rêvé à la vie « d'après la thèse » . Voilà maintenant quelques semaines qu'elle est écrite et que je pense à l'avenir, et pourtant, aujourd'hui, c'est avec beaucoup d'émotion et de plaisir que je me retourne pour embrasser du regard les années écoulées. Ceux qui me connaissent le mieux savent combien ces années ont pu être difficiles et douloureuses par moments... et si j'avais été plus seule, si je n'avais pas bénéficié d'un entourage aussi patient et compréhensif, si je n'avais pas fait de si belles rencontres, je ne serais pas en train d'écrire ces lignes.

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J'ai eu la chance d'effectuer ma thèse dans ce que j'imagine être l'équipe de recherche la plus sympathique de France et de Navarre, à savoir l'équipe ~~Patrick Roger~~ Analyse Algébrique. Le rituel de la pause café d'après repas m'a offert des moments de détente particulièrement agréables (et gourmands). Merci à Grégory (et ses tartes aux agrumes), Nicolas (et son humour piquant), Erwan (et sa cachaça imbuvable), Emmanuel, Frédéric, Maxime, Patrice et Vincent, qui comptent parmi ceux dont je suis le plus proche mathématiquement (et je regrette de ne pas avoir davantage discuté de maths avec vous) ainsi qu'à toute l'équipe.

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Le quotidien d'une thèse se partage avec les camarades thésards et longue est la liste de ceux qui ont parcouru une partie ou la totalité du chemin avec moi. Mes premières pensées sont bien sûr pour ma petite sœur de thèse Lara qui m'a supportée avec beaucoup de patience pendant toutes ces années, sans me tenir rigueur pour mes sautes d'humeur et essayant toujours avec gentillesse d'apaiser mes nombreuses crises d'angoisse.

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C'était un matin de septembre. Dans un vieil amphi de l'ancienne annexe de la rue Cuvier, j'assistais pour la première fois à un cours de Jean-Pierre Marco. Il m'est impossible de décrire l'émerveillement qui me gagna : j'ai vécu ces trois heures en état d'apesanteur. L'émerveillement que j'ai éprouvé ce matin-là ne m'a pas quittée et il n'est pas exagéré de dire que cette matinée a marqué un tournant dans ma vie. Je mesure la chance inouïe que j'ai eue de travailler avec Jean-Pierre mais je suis incapable de le remercier ici : toute la richesse et la beauté de la langue française ne sauraient exprimer la profondeur de ma gratitude.

Résumé

Nous cherchons les métriques sur le tore $\mathbb{T}^2$ qui minimisent la *complexité*. L'entropie topologique pouvant s'annuler pour des systèmes géodésique de complexités *a priori* non équivalentes, nous cherchons les minimums de l'*entropie polynomiale* $\mathrm{h_{pol}}$ parmi des systèmes géodésiques à entropie nulle : les métriques plates, et les métriques pour lesquelles le flot géodésique admet une intégrale première non dégénérée au sens de Bott.

Dans un premier temps, nous calculons l'entropie polynomiale des systèmes hamiltoniens intégrables au sens de Bott avec une condition de « cohérence dynamique » supplémentaire. Un tel système vit sur un niveau d'énergie compact de dimension 3 d'une variété symplectique de dimension 4. Nous montrons que l'entropie polynomiale ne peut prendre que les valeurs 0, 1 ou 2.

Ensuite, nous montrons que l'entropie polynomiale d'un système géodésique sur une variété riemannienne compacte M est minorée par le degré de croissance polynomiale du groupe fondamental de M moins 1. De là, nous déduisons que les métriques plates sur les tores $\mathbb{T}^n$ minimisent l'entropie polynomiale.

Enfin, nous montrons que, parmi les systèmes géodésiques sur $\mathbb{T}^2$ qui sont Bott-intégrables et dynamiquement cohérents, ceux qui découlent des métriques plates sont des minimums stricts locaux de $\mathrm{h_{pol}}$.

Abstract

We look for metrics on the torus $\mathbb{T}^2$ that minimize the complexity. Since the topological entropy may vanish for systems with *a priori* non equivalent complexity, we look for the minima of the *polynomial entropy* $\mathrm{h_{pol}}$ among geodesic system whose topological entropy vanishes: flat metrics, metrics whose cogeodesic flow admits a Bott nondegenerate first integral.

We begin with computing the polynomial entropy for Hamiltonian systems that are integrable in the Bott sense, with an additionnal condition of "dynamical coherence". Such a system lie in a 3-dimensional compact energy level, contained in a 4-dimensional symplectic manifold. We prove that the polynomial entropy must take the only three values 0, 1 or 2.

Then, we show that the polynomial entropy of a geodesic system associated with a compact Riemannian M manifold is bounded below by the degree of growth of the fundamental group of M minus 1. From this, one deduces that flat metrics on the tori $\mathbb{T}^n$ do minimize the polynomial entropy.

Finally, we show that, among geodesic systems on the torus $\mathbb{T}^2$ that are dynamically coherent, the one associated with flat metrics are strict local minimizers of $\mathrm{h_{pol}}$.

Contents

Introduction

Une variété différentielle M étant donnée, quelles sont les métriques les plus « simples » dont on puisse munir M ?

Cette question assez vague mérite des précisions, la première étant celle du qualificatif « simples ». Ici, nous associerons la simplicité/complexité d'une métrique riemannienne à celle de son flot géodésique ; c'est-à-dire que nous nous intéresserons à une caractérisation *dynamique* de la géométrie.

La complexité d'un système dynamique peut être vue comme comme la difficulté à pouvoir décrire l'ensemble de ses trajectoires. L'*entropie topologique* est un outil permettant de *quantifier* cette complexité. Si (X, d) un espace métrique compact et f une application continue de X dans lui-même, on peut définir pour tout $n \in \mathbb{N}$ la distance d_n^f de la manière suivante :

$$d_n^f(x, y) = \max_{0 \leq k \leq n-1} d(f^k(x), f^k(y)).$$

Ces nouvelles distances sont topologiquement équivalentes à d, elles détectent l'écartement des points sous l'action de f. On note $G_n(\varepsilon)$ le nombre minimal de d_n^f-boules de rayon ε nécessaires pour recouvrir X. On peut voir $G_n(\varepsilon)$ comme le nombre minimal de trajectoires nécessaires pour connaître à ε près toutes les trajectoires du système jusqu'au temps n. On aura donc envie de dire que le système est d'autant plus complexe que ce nombre croît avec n. L'entropie topologique $\mathrm{h_{top}}$ est le supremum sur l'ensemble des $\varepsilon > 0$ des taux de croissance exponentielle de ces nombres.

La topologie d'une variété différentielle M a une influence sur la « complexité » de ses métriques potentielles : Dinaburg ([Din71]) a montré que si le groupe fondamental de M est à croissance exponentielle, l'entropie topologique d'un flot géodésique associé à une métrique quelconque sur M est positive.

La complexité d'un flot géodésique peut aussi être évaluée par un autre invariant de nature plus géométrique : l'entropie volumique $\mathrm{h_{vol}}$ qui correspond à l'asymptotique du taux de croissance exponentielle du volume des boules dans le revêtement universel.

A. Manning a prouvé ([Man79]) que $\mathrm{h_{vol}} \leq \mathrm{h_{top}}$ avec égalité si la variété est à courbure sectionnelle négative ou nulle.

G. Besson, G. Courtois et S. Gallot ([BCG96]) ont montré que sur une variété différentielle compacte connexe M pouvant porter une métrique localement symétrique de courbure négative, les métriques localement symétriques de courbure négative sont les minimums stricts de l'entropie topologique. Ce résultat avait déjà été prouvé pour les surfaces de genre supérieur ou égal à 2 par Katok ([Kat88]). Rappelons en effet que Milnor a montré que si une variété différentielle porte une métrique à courbure négative, son groupe fondamental est à croissance exponentielle.

Ainsi, pour les surfaces de genre supérieur ou égal à 2, on connaît les « métriques les plus simples ». Nous nous intéressons dans cette thèse au cas radicalement opposé des surfaces de genre 1.

Une première remarque est que l'entropie topologique peut s'annuler. C'est le cas par exemple pour une métrique plate g_0 sur $\mathbb{T}^2$. En effet, une telle métrique réalise un cas d'égalité dans l'inégalité de Manning, et la croissance du volume des boules étant quadratique, l'entropie volumique est nulle. Les métriques plates ne sont cependant pas les seules à annuler l'entropie topologique ; G. Paternain a en effet montré ([Pat91]) qu'un flot géodésique sur une surface (en restriction au fibré unitaire tangent) qui admet une intégrale première dont les points critiques forment des sous-variétés strictes a une entropie topologique nulle.

Ceci nous amène à étudier des « mesures polynomiales » de la complexité - les *entropies polynomiales forte* $\mathrm{h_{pol}}$ *et faible* $\mathrm{h^*_{pol}}$ - introduites par J-P. Marco ([Mar09]) et à chercher les métriques qui minimisent ces quantités parmi celles dont le flot associé possède une intégrale première satisfaisant les hypothèses du théorème de Paternain.

Les orbites d'un flot géodésique rendant extrémale l'action lagrangienne donnée par la métrique, elles apparaissent naturellement comme solutions d'un flot hamiltonien défini sur le fibré cotangent de $\mathbb{T}^2$ muni de sa structure symplectique standard. C'est le point de vue que nous adoptons dans notre étude.

La présence d'une intégrale première fait d'un flot géodésique sur le tore un système *intégrable au sens de Liouville.* Nous nous sommes donc intéressés dans un premier temps à la complexité de ces systèmes. De tels systèmes semblent en effet dynamiquement assez simples et d'ailleurs, pour beaucoup (mais pas pour tous, voir à ce sujet les travaux de L. Butler [But99], A.Bolsinov et I.Taïmanov [BT00]), l'entropie topologique est nulle.

Toutefois, il est probablement évident pour un dynamicien qu'un oscillateur harmonique est moins complexe qu'un oscillateur anharmonique (c'est-à-dire un système hamiltonien associé à la fonction $\mathbb{T} \times \mathbb{R} \to \mathbb{R} : (\theta, r) \mapsto \frac{1}{2}r^2$), ce dernier étant à son tour moins compliqué qu'un pendule simple (système hamiltonien associé à la fonction $\mathbb{T} \times \mathbb{R} \to \mathbb{R} : (\theta, r) \mapsto \frac{1}{2}r^2 + \cos\theta$), ou, de manière un peu équivalente, que le flot géodésique d'un tore plat est plus simple que celui d'un tore de révolution.

Les entropies polynomiales $\mathrm{h_{pol}}$ et $\mathrm{h^*_{pol}}$ se révèlent être des outils particulièrement pertinents pour l'étude de ces systèmes. J-P. Marco a en effet montré que ces deux quantités coïncident pour les systèmes sous forme action-angle (comme les oscillateurs harmonique et anharmonique ou le tore plat) et détectent le nombre « effectif » de degrés de liberté du système. Une conséquence immédiate de ce phénomène est que les entropies polynomiales du flot géodésique d'un tore plat (en restriction au fibré unitaire) sont égales à 1.

J-P. Marco ([Mar09]) a aussi montré que l'entropie polynomiale forte $\mathrm{h_{pol}}$ d'un système hamiltonien défini sur une surface par une fonction de Morse détectait la présence des singularités hyperboliques. Son résultat est le suivant.

Théorème. (Marco) *Soit $\mathscr{S}$ une surface symplectique compacte éventuellement à bord. Soit $H : \mathscr{S} \to \mathbb{R}$ une fonction lisse de Morse, constante sur les composantes connexes du bord ∂S. Soit ϕ_H le flot hamiltonien associé à H. Alors*

$$\mathrm{h_{pol}}(\phi_H) \in \{0, 1, 2\}.$$

De plus $\mathrm{h_{pol}}(\phi_H) = 2$ *si et seulement si H a un point critique d'indice* 1.

Nous nous sommes inspirés de ce résultat pour calculer les entropies polynomiales de systèmes Hamiltoniens ϕ_H intégrables au sens de Liouville sur une variété symplectique de dimension 4 qui admettent une intégrale première f *non dégénérée au sens de Bott* sur un niveau d'énergie fixé $\mathscr{E}$ et qui vérifient une condition supplémentaire de *cohérence dynamique*. Nous disons qu'un tel système $(\mathscr{E}, \phi_H, f)$ est *dynamiquement cohérent*. Notre résultat est similaire à celui de J.P Marco.

Théorème A. *Soit $(\mathscr{E}, \phi_H, f)$ un système dynamiquement cohérent. Alors*

$$\mathrm{h}_{\mathrm{pol}}(\phi_H) \in \{0, 1, 2\}.$$

De plus $\mathrm{h}_{\mathrm{pol}}(\phi_H) = 2$ si et seulement si ϕ_H possède une orbite hyperbolique.

L'étude des systèmes dynamiquement cohérents n'est pas gratuite au regard de la recherche des métriques les plus simples sur le tore. Nous montrons en effet que le flot géodésique d'un tore de révolution *générique* est dynamiquement cohérent et possède au moins une orbite hyperbolique.

De manière analogue à Manning, nous obtenons une minoration de l'entropie polynomiale d'un flot géodésique au moyen du *degré de croissance* du volume des boules dans le revêtement universel. Si (M, g) est une variété riemannienne, nous considérons le nombre

$$\tau(M) = \inf \left\{ s \geq 0 \,|\, \limsup_{r \to \infty} \frac{1}{r^s} \operatorname{Vol} B(x, r) = 0 \right\} = \limsup_{r \to \infty} \frac{\log \operatorname{Vol} B(x, r)}{\log r} \leq \infty.$$

où $B(x, r)$ est une boule dans le revêtement universel. Ce nombre est indépendant de x et de g, c'est un invariant topologique de M. Notre second résultat est le suivant, en notant ϕ_g le flot géodésique associé à (M, g) en restriction au fibré unitaire :

Théorème B. $\tau(M) \leq \mathrm{h}_{\mathrm{pol}}(\phi_g) + 1$

Il en résulte que les métriques plates sur le tore sont des minimums de l'entropie polynomiale. Une question bien naturelle est de savoir si ce sont les seules. À cette interrogation, nous obtenons la réponse partielle suivante. Nous notons $\mathscr{DC}$ l'ensemble des métriques sur $\mathbb{T}^2$ dont le flot géodésique est dynamiquement cohérent.

Theorem C. *Si g_0 est une métrique plate sur $\mathbb{T}^2$, il existe un voisinage $\mathscr{U}$ de g_0 en topologie C^5 tel que si $g \in \mathscr{U} \cap \mathscr{DC}$ alors*

- *ou bien g est plate,*
- *ou bien g possède une orbite hyperbolique.*

En conséquence, appliquant le théorème A, nous voyons que les seules métriques de flot géodésique dynamiquement cohérent contenues dans $\mathscr{U}$ et d'entropie polynomiale 1 sont les métriques plates.

Résumé des chapitres

Nous présentons ici la structure de la thèse et un résumé des différents chapitres. Les deux premiers chapitres sont des chapitres introductifs dans lesquels nous rappelons les différents objets et outils étudiés ainsi que la plupart des résultats antérieurs que nous utilisons. Les chapitres 3, 4 et 5 concernent respectivement les théorèmes A, B et C.

Chapitre 1. Ce chapitre est consacré aux notions de flots géodésiques d'une part et de système hamiltonien intégrable d'autre part. Nous décrivons les systèmes non dégénérés au sens de Bott et introduisons la notion de systèmes dynamiquement cohérents.

Chapitre 2. Dans ce chapitre, nous rappelons la définition et les principales propriétés de l'entropie topologique et des entropies polynomiales faible $\mathrm{h}^*_{\mathrm{pol}}$ et forte $\mathrm{h}_{\mathrm{pol}}$. Nous voyons en particulier que $h^*_{pol} \leq h_{pol}$ et qu'elles coïncident pour les systèmes sous forme action-angle. Nous soulignons que l'entropie faible h^*_{pol} possède une propriété de σ-additivité analogue à celle de l'entropie topologique, mais que l'entropie polynomiale forte h_{pol} ne vérifie pas cette propriété. Ces résultats sont issus des travaux de J.-P Marco ([Mar09]).

Chapitre 3. Ce chapitre correspond à [LM]. Nous calculons les entropies polynomiales forte et faible pour les systèmes dynamiquement cohérents. Un tel système vit sur une variété compacte orientable $\mathscr{E}$ de dimension 3 ($\mathscr{E}$ est le niveau d'énergie d'un hamiltonien sur une variété symplectique de dimension 4). Une telle variété admet une partition en « domaines action-angles » (domaines de la forme $\mathbb{T}^2 \times I$, où I est un intervalle), orbites périodiques elliptiques, niveaux-en-huit (sous-variétés stratifiées composée d'une orbite hyperbolique et de ses variétés invariantes), tores et bouteilles de Klein. Dans une première partie, nous étudions la dynamique au voisinage des singularités. Une deuxième partie est consacrée au calcul de l'entropie polynomiale faible h^*_{pol}. Nous utilisons l'étude précédente pour exhiber une partition de $\mathscr{E}$ en sous-domaines invariants par le flot sur lesquels h^*_{pol} est aisée à calculer. Nous obtenons le résultat suivant :

Théorème A*. *Soit* $(\mathscr{E}, \phi_H, f)$ *un système dynamiquement cohérent.*
Alors $h^*_{pol}(\phi_H) \in \{0, 1\}$.

La dernière partie est consacrée à la preuve du Théorème A. Comme pour le théorème A*, nous exhibons une partition de $\mathscr{E}$ en sous-domaines invariants sur lesquels nous sommes en mesure de calculer h_{pol}. Nous voyons que la principale difficulté réside dans le voisinage des niveaux-en-huit (et dans l'absence de propriété de σ-additivité de h_{pol}). Nous considérons en fait une classe un peu plus générale de sous-variétés stratifiées que les niveaux-en-huit : celle des *polycycles simples* (union connexe de plusieurs orbites hyperboliques et de leurs variétés invariantes à laquelle nous imposons une propriété géométrique et une propriété dynamique). Notre stratégie est la suivante : nous nous limitons à des voisinages partiels des polycycles simples (que nous appelons *domaines de désingularisation*) et nous conjuguons le flot sur ces domaines à celui d'un système modèle *ad hoc* pour lequel on sait calculer h_{pol}. Le point crucial de la démonstration est la construction de la conjugaison entre les deux flots.

Chapitre 4. Ce chapitre correspond à [L-1]. La première partie est consacrée au Théorème B. Nous commençons par montrer que le nombre $\tau(M)$ est un invariant topologique de M, c'est le « degré de croissance » du groupe fondamental de M. Nous donnons ensuite la preuve du Théorème B, qui est très analogue à celle de Manning pour l'entropie topologique. Nous voyons immédiatement que les métriques plates sur les tores réalisent des cas d'égalité.

Dans une deuxième partie nous montrons que les tores de révolution *génériques* offrent des cas d'inégalité stricte. Nous montrons en effet qu'ils sont dynamiquement cohérents et possèdent une orbite hyperbolique ; leur entropie polynomiale est donc égale à 2.

Burago et Ivanov ont montré que, si $B(x, r)$ désigne le volume dans le revêtement universel d'une boule sur un tore $\mathbb{T}^n$ muni d'une métrique riemannienne g, la limite $\lim_{r \to +\infty} \frac{\mathrm{Vol}\, B(x,r)}{r^n}$ existe, est indépendante de x et est égale au volume $\mathscr{V}_g$ de la boule unité de la *norme stable* associée à g. Le caractère intégrable du flot géodésique sur le tore de révolution permet de déterminer explicitement le volume $\mathscr{V}_g$. La norme stable coïncide en effet avec la (racine de) la *fonction* β *de Mather*, que l'on peut ici calculer. Nous rappelons brièvement les définitions de la norme stable et de la fonction β de Mather et nous montrons que le volume $\mathscr{V}_g$ est le volume du compact délimité par la courbe des fréquences du hamiltonien sous forme action-angle dans le domaine action-angle formé des tores de

Liouvilles qui sont des graphes au-dessus de la section nulle.

Chapitre 5. Ce dernier chapitre, qui correspond à [L-2], est consacré au Théorème C. La preuve du Théorème C est en partie basée sur une remarque fondamentale liée au théorème de Hopf selon lequel toute métrique sur $\mathbb{T}^2$ qui n'a pas de points conjugués est plate : utilisant une propriété de minimisation des orbites d'un flot hamiltonien sur $T^*\mathbb{T}^n$ contenues dans des graphes lagrangiens de fonctions différentiables $\mathbb{T}^n \to \mathbb{R}^n$, nous voyons qu'une métrique sur $\mathbb{T}^n$ est plate si et seulement si son fibré unitaire tangent est feuilleté en tores (invariants par le flot géodésique) qui sont des graphes différentiables au-dessus de la base.

Nous montrons ensuite que si une métrique dont le flot géodésique est dynamiquement cohérent est suffisamment proche d'une métrique plate, alors ou bien le fibré unitaire admet un feuilletage en tores lagrangiens qui sont des graphes différentiables au-dessus de la base, ou bien il existe un niveau-en-huit dans le feuilletage induit par l'intégrale de Bott. Ceci repose sur une propriété de graphes pour les tores invariants d'un système hamiltonien obtenu par une petite pertubation d'un système en action-angle défini par une forme quadratique. Plus précisément :

Lemme. *Soit* $H : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R} : (\theta, r) \mapsto h(r)$ *où* h *est une forme quadratique définie positive. Soit* $f : \mathbb{T}^2 \times \mathbb{R}^2$ *une fonction de classe* C^5 *telle que* $||f||_{C^5} = 1$. *Pour* $\varepsilon > 0$, *on pose* $H_\varepsilon : H + \varepsilon f$, *et on note* ϕ_ε *le flot hamiltonien associé à* H_ε. *Il existe* ε_0 *tel que pour tout* $\varepsilon \leq \varepsilon_0$,

1. *il existe des tores* ϕ_ε*-invariants dans* $H_\varepsilon^{-1}(\{1\})$ *qui sont les graphes de fonctions* C^1 $\mathbb{T}^2 \to \mathbb{R}^2$
2. *si* $\mathcal{T} \subset H_\varepsilon^{-1}(\{1\})$ *est un tore* ϕ_ε*-invariant homotope à* $\mathbb{T}^2 \times \{0\}$, *alors* $\mathcal{T}$ *est le graphe d'une fonction continue* $\mathbb{T}^2 \to \mathbb{R}^2$
3. *il n'y a pas de bouteille de Klein* ϕ_ε*-invariant dans* $H_\varepsilon^{-1}(\{1\})$.

Le point 1 est exactement le théorème KAM et le point 3 est une conséquence facile du théorème KAM et de la particularité, propre à la dimension 3, de pouvoir « bloquer » la dynamique par les tores de KAM. L'intérêt du lemme est contenu dans le point 2. Celui-ci se montre à l'aide du théorème KAM et d'un théorème de Birkhoff sur les distorsions monotones de l'anneau qui préservent l'aire. Nous rappelons donc brièvement ces deux résultats au début du chapitre.

Pour aller plus loin...

Les résultats ci-dessus soulèvent plus de questions qu'ils n'en résolvent et nombreuses sont les perspectives de recherche dans leur prolongement. Celles-ci s'orientent suivant deux directions.

Métriques d'entropie minimale.

Le résultat du chapitre 4 sur la minimisation de l'entropie polynomiale par les métriques plates s'étend en fait à la classe *a priori* plus générale des métriques dont le Hamiltonien géodésique est sous forme action-angle. Le fibré unitaire tangent $S^*\mathbb{T}^2$ est alors feuilleté en tores invariants par le flot géodésique et homotopes à la section nulle. Si la métrique n'est pas plate, l'un au moins de ces tores a un pli : il n'est pas un graphe au-dessus de la section nulle. Cette observation soulève deux questions très naturelles :

- de telles métriques existent-elles ?
- si elles existent, dans quelle mesure la donnée du feuilletage en tores invariants permet-elle d'en caractériser la géométrie ?

Concernant le résultat de minimisation locale par les métriques plates, l'étape suivante est de lever la condition de cohérence dynamique avec laquelle nous avons travaillé. Celle-ci impose que les cercles critiques de l'intégrale de Bott soient des orbites périodiques non dégénérées. Il nous faut donc comprendre la dynamique au voisinage d'orbites critiques dégénérées. Ceci nous amène à la seconde orientation de recherche : l'étude de la complexité des systèmes intégrables.

Entropie polynomiale des systèmes intégrables.

La dimension 2 et la dimension 4 offrent déjà, à elles seules, un vaste champ de recherche.

Systèmes sur les surfaces. Un premier pas pour calculer l'entropie polynomiale au voisinage d'orbites critiques dégénérées d'un système de Bott est celui du calcul de l'entropie au voisinage de points fixe dégénérés pour un système hamiltonien défini sur une surface par une fonction qui n'est plus de Morse.

Systèmes avec une intégrale de Bott. Nous avons calculé les entropies polynomiales d'un tel système en restriction à un niveau d'énergie fixé $\mathscr{E}$. Qu'advient-il des entropies quand on s'autorise à faire varier l'énergie ? Une première étape pour répondre à cette question consisterait à calculer les entropies polynomiales sur des voisinages compacts de $\mathscr{E}$, saturés pour H et tel que l'intégrale première est de Bott sur chacun des niveaux de H. Déjà dans ce cas, la géométrie induite par le feuilletage de Liouville peut devenir assez compliquée si on ne se limite pas aux niveaux-en-huit ou aux polycycles simples mais que l'on étudie le cas général de polycycles qui peuvent bifurquer.

Chapitre 1

Integrable Geodesic flows

1.1 Geodesic flows.

Let (M, g) be a Riemannian manifold endowed with its Riemannian connection ∇. The *length* of a differentiable curve $\gamma : [a, b] \to M$ is defined as

$$\ell(\gamma) = \int_a^b ||\dot{\gamma}(t)|| dt,$$

and its *action* or *energy* as

$$\mathcal{A}_g(\gamma) = \frac{1}{2} \int_a^b ||\dot{\gamma}(t)||^2 dt.$$

By Hölder's inequality, for every differentiable curve $\gamma : [a, b] \to M$

$$\ell(\gamma)^2 \leq 2(b - a)\mathcal{A}_g(\gamma), \tag{1.1}$$

with equality if and only if $||\dot{\gamma}||$ is constant.

The *Riemannian distance* (associated with g) on M is defined as

$$d_g(m, m') = \inf \ell(\gamma),$$

where the infimum is taken over all piecewise differentiable curves γ joining m to m'. With the distance d_g, the Riemannian manifold M is a metric space whose topology is the same as the given manifold topology of M.

The key feature of the length of a curve is that it is independent of its parametrization, that is, if $\psi : [a, b] \to [\alpha, \beta]$ is a diffeomorphism, $\ell(\gamma) = \ell(\gamma \circ \psi)$. One easily sees that any differentiable curve admits a reparametrization of the form $\gamma : [0, \ell] \to M$ with $||\dot{\gamma}|| \equiv 1$. One says that γ is parametrized by arc length.

Let γ be a differentiable curve joining m to m', parametrized by arc length. Assume that $\ell(\gamma) = d_g(m, m') = \ell$: one says that γ is *minimizing between m and m'.* Then by (1.1), for every curve $c : [0, \ell] \to M$ joining m to m', one has

$$\mathcal{A}_g(\gamma) \leq \mathcal{A}_g(c),$$

and γ minimizes $\mathcal{A}_g$.

With any C^2 function $L : TM \to \mathbb{R}$, one can associate the *Lagrangian action* of L defined on the set of C^1 paths on M by

$$\mathcal{A}_L : \gamma \to \int L(\gamma, \dot{\gamma}).$$

Then $\mathcal{A}_g(\gamma) = \mathcal{A}_{L_g}(\gamma) = \int L_g(\gamma, \dot{\gamma})$ where $L_g : TM \to \mathbb{R} : (m, v) \mapsto \frac{1}{2} g_x(v, v)$. We say that L_g is the *geodesic Lagrangian.*

A *variation* of a curve $\gamma : [a, b] \to M$ is a differentiable map

$$\Gamma : [a, b] \times]-\varepsilon, \varepsilon[\to M, \quad \varepsilon > 0$$

such that $\Gamma(t, 0) = \gamma(t)$ for all $t \in [a, b]$. The variation is called *proper* if the endpoints are fixed, that is, $\Gamma(a, s) = \gamma(a)$ and $\Gamma(b, s) = \gamma(b)$ for all $s \in]-\varepsilon, \varepsilon[$. Denote by γ_s the curves $\gamma_s := \Gamma(\,.\,, s)$. The vector field $V(t) = \frac{\partial \Gamma}{\partial s}(0, t)$ along γ is the *variation field* of Γ. It is called proper if $V(a) = V(b) = 0$. If Γ is proper, V is proper. The converse is not necessarily true but if V is proper one can always associate a proper variation with V. A curve γ is a *critical point* of a Lagrangian action $\mathcal{A}_L$ if for all proper variations Γ of γ, one has

$$\frac{d}{ds} \mathcal{A}_L(\gamma_s) = 0.$$

The critical points of $\mathcal{A}_g$ are the *geodesic segments* of (M, g). As a consequence, the curves of shortest length are geodesic segments.

Lemma 1.1.1. *Let $L : TM \to \mathbb{R}$ be a C^2 function. The critical points of $\mathcal{A}_L$ are the solutions of the following Euler-Lagrange equations*

$$\frac{d}{dt} \frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t)) = \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t)).$$

In coordinate charts, the Euler-Lagrange equations of $\mathcal{A}_g$ read

$$\ddot{\gamma}^k(t) + \dot{\gamma}^i(t) \dot{\gamma}^j(t) \Gamma^k_{ij}(\gamma(t)) = 0, \tag{1.2}$$

where the Γ^k_{ij} stand for the Christoffel symbols.

Thus the geodesic segments may also be considered as curves whose *acceleration* vanishes identically. Indeed, if D_t denotes the covariant derivative along a curve γ, one can see that (still in a coordinate chart) the equations (1.2) are equivalent to $D_t \dot{\gamma} \equiv 0$, that is, $\dot{\gamma}$ is parallel along γ. An immediate consequence of the previous definition of geodesics is that they are constant speed curves. Indeed $\frac{d}{dt} ||\dot{\gamma}(t)||^2 = 2 \langle D_t \dot{\gamma}, \dot{\gamma} \rangle = 0$, since $\dot{\gamma}$ is parallel along γ.

1.1.1 The geodesic flow as a Hamiltonian flow.

In each coordinate chart $U \subset M$, the Euler-Lagrange equations lead to a (local) vector field X^{L_g} on TU :

$$X^{L_g} : \begin{cases} \dot{x_k} = v_k \\ \dot{v_k} = -\Gamma^k_{ij} v_i v_j. \end{cases}$$

It turns out that X^{L_g} is a global and intrinsically defined vector field on TM. Indeed, the Euler-Lagrange equations lead to an ordinary differential equation of first order on the cotangent bundle T^*M in the following way. Consider the *Legendre mapping*

$$\begin{array}{cccc} \mathcal{L} : & TM & \longrightarrow & T^*M \\ & (m, v) & \mapsto & (m, \frac{\partial L}{\partial v}(m, v)) = (m, p), \end{array}$$

where $p = \frac{\partial L}{\partial v}(m, v)$ is the 1-form on $T_m M$:

$$p : w \mapsto g_m(w, v).$$

The Legendre mapping is a fiber bundle diffeomorphism. Let us denote by $(m,p) \mapsto (m, G(m,p))$ its inverse and consider the *Hamiltonian function* H_g on T^*M defined by

$$H_g(m,p) = \langle p, G(m,p)\rangle - L\left(m, G(m,p)\right).$$

Observe that if $p = \frac{\partial L}{\partial v}(m,v)$, $H_g(m,p) = L_g(m,v)$.

More generally, with a C^2 function $L : TM \to \mathbb{R}$, one can associate a C^2 function $H : T^*M \to \mathbb{R}$, called the *Fenchel-Legendre transform of* L and defined in the following way :

$$\begin{array}{rccl} H: & T^*M & \to & \mathbb{R} \\ & (m,p) & \mapsto & \sup_{v\in T_mM}\{\langle p, m\rangle_m - L(m,v)\}. \end{array}$$

One proves (see [Fat]) for example) that, if L is convex in the fibers, the equality

$$\langle p, m\rangle_m = H(m,p) + L(m,v)$$

is reached if and only if $p = \frac{\partial L}{\partial v}(m,v)$.

The Hamiltonian function H_g defines a norm on the cotangent bundle by setting, for $p \in T^*_mM$, $||p|| = \sqrt{2H_g(m,p)}$. The Legendre mapping $\mathcal{L}$ then becomes a fiber isometry. The *Hamiltonian vector field* associated with H_g is defined as

$$X^{H_g} := \begin{cases} \dot{m} = \dfrac{\partial H_g}{\partial p} \\ \dot{p} = -\dfrac{\partial H_g}{\partial m}. \end{cases}$$

We will see in the next section that such a vector field is intrinsically defined in the whole cotangent space T^*M (definition 1.2.1). With a differentiable curve $\gamma : [a,b] \to M$, we associate the smooth curve $\gamma^* : t \to \mathcal{L}(\gamma(t), \dot{\gamma}(t))$ on T^*M. One easily checks that γ is solution of the Euler-Lagrange equations if and only if γ^* is an integral curve for the vector field X^{H_g}. Therefore the vector field X^{L_g} may be defined as

$$X^{L_g} = \mathcal{L}^* X^{H_g}.$$

The vector fields X^{H_g} and X^{L_g} are respectively called the *cogeodesic vector field* and the *geodesic vector field.* In the following, we will omit the subscript g and just write L and H for L_g and H_g.

The local flows $(\phi^t_H)_t$ and $(\phi^t_L)_t$ associated with X^H and X^L are the *cogeodesic flow* and the *geodesic flow.* Recall that the local flow of a vector field X on TM is the local one-parameter group $(\phi_t)_t$ defined by

$$\frac{d}{dt}\phi^t(\nu)_{|t_0} = X(\phi^{t_0}(\nu)),$$

for all $\nu \in TM$ and all $t_0 \in I_\nu$ where I_ν is the interval of $\mathbb{R}$ containing 0 on which the maximal solution of the ordinary differential equation $\dot{y} = X(y)$ satisfying $y(0) = \nu$ is defined. We denote by $\pi : TM \to M$ and $\pi^* : T^*M \to M$ the canonical projections. If $t \mapsto \phi^t(\nu)$ is a solution of the dynamical system associated with X, its projection $\{(\pi\circ\phi^t(\nu) \,|\, t \in I_\nu\}$ which lies in M is a *trajectory* of the dynamical system. The subset $\{\phi^t(\nu) \,|\, t \in I_\nu\}$ is the *orbit* of ν under the flow action. The *geodesics* of M are the trajectories of the geodesic flow or equivalently of the cogeodesic flow. In order to study them, one may indifferently work with the geodesic or the cogeodesic flow. It turns out that the cogeodesic flow is a particular case of *Hamiltonian system* and it is that point of view we will adopt. In the next section we will make a short incursion into symplectic geometry, which is the natural setting for the study of Hamiltonian systems, and see how we can learn about their *integrabilty*.

Before let us state some more features of the geodesic and the cogeodesic flows. For all m in M there exists an open neighborhood V_m of 0 in T_mM such that for all $v \in V_m$, $I_{(m,v)}$ contains the interval $[0, 1]$. The *exponential map at* m is defined as

$$\begin{array}{rccl} \exp_m : & V_m & \to & M \\ & v & \mapsto & \pi \circ \phi_L^1((m, v)). \end{array}$$

Moreover, there exists an open neighborhood W_m of 0 contained in V_m and an open neighborhood U_m of m in M such that $\exp_m : W_m \to U_m$ is a diffeomorphism. One can also define the exponential map on T_m^*M by $\exp_m^*(p) = \pi \circ \phi_H^1((m, p))$. It satisfies the same properties.

Remark 1.1.1. Since the geodesics are constant speed curves, each orbit of the geodesic flow is contained into a circle bundle

$$S_eM := \{(m, v) \,|\, m \in M,\, v \in T_mM,\, ||v|| = e\}.$$

In the same way, since the Legendre mapping is a fiber isometry, each orbit of the cogeodesic flow is contained into a circle bundle

$$S_e^*M := \{(m, p) \,|\, m \in M,\, p \in T_m^*M,\, ||p|| = e\}.$$

As we will show in the next section one can directly see (that is without using the Riemannian connexion) that the orbits of the cogeodesic flow are contained into the circle bundles S_e^*M, and therefore that the geodesics are constant speed curves.

If M is compact, these circle bundles are also compact and the solutions (of both flows) are defined for all $t \in \mathbb{R}$. One says that the flows are *complete*. For all m in M the exponential maps are then defined over the whole tangent spaces T_mM and T_m^*M A manifold whose geodesic flow (or equivalently cogeodesic flow) is complete is called *geodesically complete*. In the next chapters, we will only work with compact manifolds but for the sake of completness we state the following result due to Hopf and Rinow.

Theorem 1. ***Hopf-Rinow's Theorem.*** *A Riemannian manifold is geodesically complete if and only of it is complete as a metric space.*

In the following, we will consider only Riemannian manifolds which are complete. Thus a geodesic will always be a curve defined on $\mathbb{R}$.

1.1.2 Minimization and conjugate points.

Let us now make more precise the rough idea that the geodesic segments are "curves of shortest length". One says that a curve $\gamma : [a, b] \to M$ joins m to m' if $\gamma(a) = m$ and $\gamma(b) = m'$.

A geodesic γ is said to be *minimizing* if for each closed interval I, the geodesic segment $\gamma_{|_I}$ is minimizing between its endpoints. Let us mention the following observation of Morse in dimension 2, generalized by Mather in any dimension.

Lemma 1.1.2. *Two distinct minimizing geodesics meet at most once.*

It is not true that every geodesic is minimizing, as one sees by looking at the geodesics on the sphere : these are the great circles so beyond their half period, none is minimizing. Nevertheless they satisfy a property of local minimization.

A curve $\gamma : I \to M$ is said to be *locally minimizing* if any $t \in I$ admits a neighborhood $J \subset I$ such that for all $(t_0, t_1) \in J^2$, $\gamma_{|J}$ is minimizing between $\gamma(t_0)$ and $\gamma(t_1)$.

Theorem 2. *Every geodesics is locally minimizing.*

Let us now state the following result for complete manifolds which is a corollary of the above theorem and of the Hopf-Rinow theorem.

Corollary 1.1.1. *Assume that M is complete. Then any two points m and m' can be joined by a geodesic of shortest length.*

Now let us see an obstruction for a geodesic to be minimizing between two given points. Let $\gamma : [a, b] \to M$ be a segment of geodesic joining m to m'. A variation $\Gamma : [a, b]\times]-\varepsilon, \varepsilon[\to M$ of γ is called a *geodesic variation* if the curves γ_s are geodesic segments for all s. The variation field associated with a geodesic variation is a *Jacobi field*.

One says that m' is *conjugate* to m along γ if γ admits a geodesic variation whose Jacobi field is proper. Notice that the geodesic variation may be not proper.

Proposition 1.1.1. *Let $\gamma : [a, b] \to M$ be a segment of geodesic.*

1. *If there is no point conjugate to $\gamma(a)$ along γ, there exists $\varepsilon > 0$ with the property that for any piecewise differentiable curve $c : [a, b] \to M$ with the same endpoints and such that $\forall t \in [a, b]$, $d(c(t), \gamma(t)) \leq \varepsilon$, we have*

$$\ell(\gamma) \leq \ell(c),$$

with equality if and only if c is a reparametrization of γ.

2. *Conversely, if there exists $\tau \in]a, b[$ such that $\gamma(\tau)$ is conjugate to $\gamma(a)$ along γ, then there exists a proper variation $\Gamma : [a, b]\times] - \varepsilon, \varepsilon[\to M$ of γ with*

$$\ell(\gamma_s) \leq \ell(\gamma), \quad \forall\, 0 < s < \varepsilon.$$

The following proposition is a classical consequence of the possiblity of “rounding the corners”.

Proposition 1.1.2. *Let $\gamma : [a, b] \to M$ be a segment of geodesic. If there exists $\tau \in]a, b[$ such that $\gamma(\tau)$ is conjugate to $\gamma(a)$ along γ, then γ cannot be minimizing between $\gamma(a)$ and $\gamma(b)$.*

These two results imply that in absence of conjugate points a segment of geodesic minimizes the length among sufficiently close curves. Nevertheless, it does not necessarily realize the shortest length between its endpoints as we see by considering geodesics on a flat cylinder that wind around more than once.

The most important property of conjugate points is that they are precisely the images of the singularities of $\exp_m$.

Proposition 1.1.3. *Let $m \in M$, $v \in T_m M$ and $m' = \exp_m(v)$. Then $\exp_m$ is a diffeomorphism in a neighborhood of v if and only if m' is not conjugate to m along the segment of geodesic $\gamma : t \mapsto \pi \circ \phi^t(v)$, $t \in [0, 1]$.*

To conclude let us state two results that relate the non existence of conjugate points with the curvature of M.

Theorem 3. ***Cartan-Hadamard's Theorem***. *Assume that (M,g) is a complete manifold with sectional curvature nonpositive. Then, there are no conjugate points on M.*

The following statement was conjectured by Hopf, who proved it in dimension 2 in [Hop48]. The proof in arbitrary dimension is due to Burago and Ivanov ([BI94]).

Theorem 4. (Hopf, Burago-Ivanov). *Assume that g is a Riemannian metric on the torus $\mathbb{T}^n$ wich does not have conjugate points. Then g is flat.*

1.2 Bott integrals and dynamical coherence

Let us recall that a *symplectic vector space* is a pair (E,Ω) where E is a finite dimensional real vector space and Ω a nondegenerate skew-symetric bilinear form. The vector space E is necessarily even dimensional. The symplectic orthogonal of a subspace F of E is defined as $F^\perp := \{x \in E \,|\, \Omega(x,y) = 0, \forall\, y \in F\}$, and F is *Lagrangian* (resp. *symplectic*) if $F = F^\perp$ (resp. $F \cap F^\perp = \{0\}$). Since Ω is nondegenerate, a Lagrangian subspace has dimension n.

Example 1.2.1. $\mathbb{R}^{2n}$ with the symplectic form $dx \wedge dy := \sum_{i=1}^n dx_i \wedge dy_i$.

A *symplectic manifold* is a differentiable manifold M endowed with a closed nondegenerate 2-form Ω. In particular each tangent space (T_mM, Ω_m) is a symplectic vector space. This implies that M is even-dimensional and orientable : Ω^n is a volume form on M, called the symplectic volume form. A *symplectomorphism* of M is a diffeomorphism $\psi : M \to M$ that preserves the symplectic form (and therefore the volume form), that is $\psi^*\Omega = \Omega$ (and $\psi^*\Omega^n = \Omega^n$).

Example 1.2.2. The cotangent bundle T^*N of a differentiable manifold N endowed with the 2-form $\Omega = d\lambda$ where λ is the Liouville 1-form defined as follows. Let $\eta \in T^*N$ and denote by π the canonical projection from T^*N to N. Then

$$\lambda(\eta) := \eta \circ d_\eta\pi : T_\eta T^*N \to \mathbb{R}.$$

The Liouville form η is actually uniquely caracterized by the property that $\sigma^*\eta = \sigma$ for every 1-form σ on N. In local coordinates $(x_1, ..., x_n)$ on N, with associated coordinates $(y_1, ..., y_n)$ on the fibers T_x^*N, the Liouville form reads $\lambda = ydx := \sum_{i=1}^n y_i dx_i$.

A properly embedded submanifold N of M is said to be Lagrangian (resp. symplectic) if for all $q \in N$ the subspace T_qN of the symplectic vector space (T_qM, Ω_q) is Lagrangian (resp. symplectic).

Consider a Riemannian metric g on the torus $\mathbb{T}^n$, and denote by X^H the cogeodesic vector field on $T^*\mathbb{T}^n$. The theory of Hamilton-Jacobi equations enables one to prove the following result (see [Fat]).

Theorem 5. *Let γ be the projection of a solution of X^H which is contained in a C^1 Lagrangian graph $\mathcal{G} := \{(x, \eta + d_x u) \,|\, x \in \mathbb{T}^n\}$. Then, for any $a < b$ in $\mathbb{R}$, the curve $\gamma_{|[a,b]}$: $[a,b] \to \mathbb{R}$ minimizes the action $\mathcal{A}_g$ among absolutely continuous curves. In particular, γ does not have conjugate points.*

The following theorem shows that in symplectic geometry there are no local invariants other than the dimension.

Theorem 6. ***Darboux's Theorem.*** *Let (M, Ω) be a symplectic manifold. Each point q of M admits a neighborhood U with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that in these coordinates $\Omega = \sum_{i=1}^n dx_i \wedge dy_i$.*

As a consequence, each point q of M possesses a neighborhood U in M, such that Ω is exact in U. The next theorem, due to Weinstein, shows that this property holds in the neighborhood of a Lagrangian submanifold.

Theorem 7. ***The Lagrangian neighborhood Theorem.*** *Let (M, Ω) be a symplectic manifold and L be a Lagrangian submanifold of M. There exists a neighborhood $\mathcal{N}(Z)$ of the zero section Z of T^*L, a neighborhood $U \subset M$ of L and a diffeomorphism $\phi : \mathcal{N}(Z) \to U$ such that*

$$\phi^*\omega = -d\lambda, \quad \phi_{|L} = \mathrm{Id},$$

where λ is the canonical Liouville form on T^*L.

Let H be a smooth function on M.

Definition 1.2.1. The *Hamiltonian vector field* X^H is defined by $dH = \iota_{X^H}\Omega$.

Remark 1.2.1. In Darboux coordinates on a neighborhood of each point of M, the vector field X^H reads

$$X^H(x,y) := \begin{cases} \dot{x}_i = \dfrac{\partial H}{\partial y_i} \\ \dot{y}_i = -\dfrac{\partial H}{\partial x_i}. \end{cases}$$

One also says that X^H is the symplectic gradient of the function H.

Denote by $(\phi_H^t)_t$ the (local) flow associated with X^H. Since in all this thesis we will only work with complete vector fields, we will assume that X^H is complete. Then ϕ^t is a diffeomorphism whose inverse is ϕ^{-t}. The one-parameter group of diffeomorphisms $\phi_H = (\phi_H^t)_{t\in\mathbb{R}}$ is the *Hamiltonian flow* associated with X^H.

Proposition 1.2.1. *For all t, ϕ_H^t is a symplectomorphism.*

Remark 1.2.2. The Hamiltonian vector field is tangent to the levels of H or, in an equivalent way, the Hamiltonian H is constant along the orbits of the Hamiltonian flow. We will see that the functions which are constant along the orbits play an important role in the understanding of Hamiltonian systems. We will often call *energy levels* the levels $H^{-1}(\{e\})$ for a value e of H.

In the particular case of the cogeodesic flow, the energy levels of the Hamiltonian are the circle bundles S_e^*M. The orbits of the cogeodesic vector field are then contained in these circle bundle. Hence the orbits of the geodesic flow are contained into the circle bundle S_eM : we recognize the property for the geodesics of being constant speed curves.

Definition 1.2.2. A *first integral* of the vector field X^H is a differentiable function $f : M \to \mathbb{R}$ such that for all x, $t \mapsto f \circ \phi_t(x)$ is constant.

An equivalent definition of a first integral is that $df(X^H) = 0$. One may define the Poisson bracket of two differentiable functions on M as $\{f, g\} = df(X^g)$. One sees that the Poisson bracket satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

Thus the space $\mathrm{Diff}(M,\mathbb{R})$ endowed with the Poisson bracket becomes a Lie algebra. Moreover, the Lie bracket of the Hamiltonian vector fields of two differentiable functions f,g on M satisfies

$$[X^f, X^g] = X^{\{f,g\}},$$

that is, the set of Hamiltonian vector fields forms a subalgebra of the Lie algebra of vector fields on M. Note that the Hamiltonian vector fields X^f and X^H then commute.

1.2.1 Hamiltonian integrability

We are now in a position to define the *integrability* of a Hamiltonian system. Let us start with a simple example. We denote by $\mathbb{A}^n$ the n-dimensional annulus, that is, $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n = T^*\mathbb{T}^n$ with coordinates (α, I). It is endowed with its canonical symplectic form $\Omega_{\mathrm{can}} = d\alpha \wedge dI$.

Example 1.2.3. Hamiltonian system in action-angle form :
Let $H : \mathbb{A}^n \to \mathbb{R}$, such that $H(\alpha, I) = h(I)$ where $h : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. The Hamiltonian vector field X^H reads :

$$\begin{cases} \dot{\alpha} = \omega(I) \\ \dot{I} = 0, \end{cases}$$

where ω refers to the map $\mathbb{R}^n \to \mathbb{R}^n : I \mapsto \nabla h(I)$ (with the identification of $\mathbb{R}^n$ with its dual). The Lagrangian tori $\mathbb{T}^n \times \{I\}$ are invariant under the flow. On each of these tori, the Hamiltonian system ϕ_H induces a *quasi-periodic motion*, that is, a linear flow with frequency $\omega(I)$. We say that the system $(\phi_H, \mathbb{T}^n \times \{I\})$ is a *Kronecker system*. The Hamiltonian system ϕ_H on $\mathbb{A}^n$ is said to be in *action-angle form*. In the beginning of chapter 5, we give a more precise description of these systems.

The Arnol'd-Liouville theorem asserts that if a Hamiltonian system admits "enough" first integrals, it will locally behaves as a Hamiltonian system in action-angle form. A proof of this theorem is given in appendix A.

Theorem 8. ***Arnol'd-Liouville's Theorem.*** *Let (M, Ω) be a symplectic manifold. Assume that there exists $F = (f_1, ..., f_n) : M \to \mathbb{R}^n$ such that*

- $\{f_i, f_j\} = 0,$
- $\mathrm{rank}\, dF = n.$

Let c be a regular value of F and assume that $F^{-1}(c)$ has a compact connected component C.

Then, there exists a neighborhood U of C and an open domain B of $\mathbb{R}^n$ such that, if $\mathbb{T}^n \times B$ is endowed with the canonical symplectic form Ω_{can}, there exists a symplectic diffeomorphism

$$\begin{array}{cccc} \Psi : & U & \to & \mathbb{T}^n \times B \\ & x & \mapsto & (\alpha, I), \end{array}$$

where the variables I depend only on the values of F.

Corollary 1.2.1. *Let H be a Hamiltonian on M which admits n first integrals $f_1, ..., f_n$ such that the map $F = (f_1, ..., f_n)$ satisfies the Arnol'd-Liouville theorem hypotheses. Then, with the above notation, the Hamiltonian function $H \circ \Psi^{-1}$ on the symplectic manifold $(\mathbb{T}^n \times B, \Omega_{\mathrm{can}})$ is in action-angle form.*

In other words, C is symplectomorphic to a Lagrangian torus and the foliation on U given by the levels of F is a foliation by ϕ_H-invariant Lagrangian tori called the *Liouville tori* of F. The restriction of the flow on each of these tori is conjugate to a Kronecker flow. Roughly speaking, it suffices to know n quantities which are preserved by the system to be able to give a description of its behaviour.

Definition 1.2.3. F is a *moment map*, or an *integral map* of the vector field X^H. The domains U defined in theorem 8 are called *action-angle domains* and the variables (α, I) are *action-angle coordinates* on U.

When $\operatorname{rank} dF = n$ on a open dense domain of M, and when the energy levels are compact, the vector field X^H is said to be *integrable in the Liouville sense.* The qualitative behaviour of the orbits which are contained in the Liouville tori is well understood. Now, some orbits are also contained in the critical loci of F, and for these ones, one does not have any information. It turns out that some *non-degeneracy* conditions on the first integrals f_i narrow the kind of phenomenons which occur and hence provide a description of the qualitative behaviour of the whole set of orbits.

1.2.2 Bott non degeneracy and dynamical coherence.

Consider a symplectic 4-dimensional manifold (M, Ω) and a differentiable Hamiltonian function $H : M \to \mathbb{R}$, with its associated vector field X^H and its associated Hamiltonian flow ϕ_H. We fix a (connected component of a) compact regular energy level $\mathscr{E}$ of H. It is an orientable compact connected submanifold of dimension 3.

Definition 1.2.4. A first integral $F : M \to \mathbb{R}$ of the vector field X^H is said to be *nondegenerate in the Bott sense* on $\mathscr{E}$ if the critical points of $f := F_{|\mathscr{E}}$ form nondegenerate smooth submanifolds of $\mathscr{E}$, that is, the Hessian $\partial^2 f$ of f is nondegenerate on normal subspaces to the submanifolds.

In the following we consider the restrictions of the vector field and the flow to $\mathscr{E}$, they are still denoted by X^H and ϕ_H. Since $\mathscr{E}$ is compact, ϕ_H is complete. The singularities of f are well known. The following proposition is proved in [Mar93] and [Fom88].

Proposition 1.2.2. *The critical submanifolds may only be circles, tori or Klein bottles.*

Remark that the critical circles for f are periodic orbits of the flow ϕ_H. Their *index* is the number of negative eigenvalues of the restriction of $\partial^2 f$ to a supplementary plane to $\mathbb{R}X^H$.

Consider a critical circle $\mathscr{C}$ for f such that $f(\mathscr{C}) = c$. Let us summarize the two possibilities that occur (see [Mar09] for more details).

• If $\mathscr{C}$ has index 0 or 2, there exists a neighborhood U of $\mathscr{C}$ such that $f^{-1}\{c\} \cap U = \mathscr{C}$ and such that the levels $f^{-1}(\{c'\})$ for c' close to c are tori whose intersection with a normal plane Σ to $\mathscr{C}$ are "circles with common center $\Sigma \cap \mathscr{C}$".

• If $\mathscr{C}$ has index 1, there exists a neighborhood U of $\mathscr{C}$ such that $f^{-1}\{c\} \cap U$ is a stratified submanifold homeomorphic to a "fiber bundle" with basis a circle and with fiber a "cross". The whole connected component $\mathscr{P}$ of $f^{-1}(\{c\})$ containing $\mathscr{C}$ is a finite union of critical circles and cylinders $\mathbb{T} \times \mathbb{R}$ whose boundary is made of either one or two critical circles. All the critical circles contained in $\mathscr{P}$ are homotopic and have index 1. Such a stratified submanifold is called a *polycycle.* In [Fom88], Fomenko assumes that a polycycle contains only one critical circle, this assumption may be compared to the assumption on a Morse function to reach a critical value at only once critical point. In this case, we say that $\mathscr{P}$ is an "eight-level", and we write ∞-level. An ∞-level can be orientable or nonorientable.

Definition 1.2.5. The triple $(\mathscr{E}, \phi_H, f)$ is called a *nondegenerate Bott system* if all the polycycles are ∞-levels.

Before introducing the notion of dynamical coherence, le us recall some *dynamical* characterizations of periodic orbits. Consider a vector field X on a 3-dimensionnal manifold N, and denote by $\phi := (\phi^t)$ its flow. Let γ be a periodic orbit of ϕ with period T and let $q \in \gamma$. Then $\gamma := \{\phi^t(q) \,|\, t \in \mathbb{R}\}$ and for any $t \in \mathbb{R}$, one has

$$D\phi^{-t}(\phi_t(q)) \circ D\phi^T(\phi^t(q)) \circ D\phi^t(q) = D\phi^T(q)$$

Therefore the eigenvalues of $D\phi^T(q)$ do not depend on q. Now X is an eigenvector for $D\phi^T$ associated with the eigenvalue 1. We denote by λ_1, λ_2 the other two eigenvalues. The closed orbit γ is said to be *nondegenerate* if $\lambda_1, \lambda_2 \neq 1$.

Assume now that N is an isoenergy level of a Hamiltonian function on a symplectic 4-dimensional manifold, and that (ϕ^t) is its Hamiltonian flow. Due to the conservation of the volume, one has $\lambda_1\lambda_2 = 1$. There are two types of nondegenerate closed orbit :

- *elliptic* if λ_1 and λ_2 are complex conjugate numbers lying on the unit circle $\mathbb{U}$.
- *hyperbolic* if $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $|\lambda_i| \neq 1$.

Assume that γ is a hyperbolic periodic orbit and that $|\lambda_1| < 1$ and $|\lambda_2| > 1$. For $q \in \gamma$, we denote by $E_1(q)$ and $E_2(q)$ the eigenspaces associated with the eigenvalues λ_1 and λ_2. Given a Riemannian metric on N, and denoting by d the associated distance, we define the *stable and unstable sets* of γ respectively by

$$W^s := \{x \in N \,|\, d(\phi^t(x), \gamma) \underset{t\to+\infty}{\to} 0\}, \quad W^u := \{x \in N \,|\, d(\phi^{-t}(x), \gamma) \underset{t\to+\infty}{\to} 0\}.$$

One proves that W^s and W^u are differentiable submanifolds immersed in N, containing γ, and that meet transversely along γ : for $q \in \gamma$, $T_qW^s = E_1(q) \times \mathbb{R}X$ and $T_qW^u = E_2(q) \times \mathbb{R}X$.

Let us come back now to the Bott system $(\mathscr{E}, \phi_H, f)$. We first remark that a periodic orbit that is nondegenerate must be a critical circle for f. Indeed, consider a T-periodic orbit γ of X^H that is not critical. Then, both X^H and X^F are eigenvectors for $D\phi_H^T$ associated with the eigenvalues 1. They are independent by assumptions on γ, so by conservation of the volume, the third eigenvalue must be equal to 1.

More precisely, we see that an elliptic orbit is a critical circle with index 0 or 2 and that a hyperbolic orbit is a critical circle with index 1. Indeed if γ is a hyperbolic periodic orbit, then γ has invariant manifolds W^s and W^u. Obviously, $f(W^s) = f(W^u) = f(\gamma)$. This is possible only if γ is contained in a ∞-level, that is, γ has index 1. Conversely, consider a critical circle $\mathscr{C}$ with index 1 contained in a ∞-level $\mathscr{P}$. Fix $q \in \mathscr{C}$, and let Σ be a transverse section to $\mathscr{C}$ at q. By assumption, $\mathscr{P} \cap \Sigma$ is a "cross". We denote by F_1 and F_2 the tangent directions to $\mathscr{P} \cap \Sigma$ at q. Then if T is the period of $\mathscr{C}$, F_1 and F_2 are invariant under $D\phi_H^T$, that is, $D\phi_H^T$ has two real eigenvalues, and $\mathscr{C}$ cannot be an elliptic orbit.

Conversely, a critical circle is not always a nondegenerate periodic orbit. This leads to the following definition.

Definition 1.2.6. A nondegenerate Bott system $(\mathscr{E}, \phi_H, f)$ is said to be *dynamically coherent* if the critical circles $\mathscr{C}$ are either elliptic periodic orbits or hyperbolic periodic orbits.

To conclude this chapter, we state the following result proved separately and with different methods by Paternain ([Pat91]) and Marco ([Mar93]). Here χ is the Euler-Poincaré characteristic.

Theorem 9. (Marco, Paternain) *Let S be a compact connected surface. Assume that S supports a geodesic flow that is nondegenerate in the Bott sense. Then $\chi(S) \geq 2$.*

Chapitre 2

Complexity of dynamical systems

2.1 Topological entropy

In this section, we will briefly recall the definition and some facts about the topological entropy. For a more complete introduction to the subject, see [KH95].

Let (X, d) be a compact metric space and f a continuous map $X \to X$. We construct new metrics d_n^f on X, which depend on the iterations of f, by setting

$$d_n^f(x, y) = \max_{0 \le k \le n-1} \{d(f^k(x), f^k(y))\}.$$

These metrics are the *dynamical metrics* associated with f. They are equivalent to d. Obviously, if f is an isometry or is contracting, d_f^n coincides with d. In fact, the metrics d_n^f detect if f tends to increase the distances between two points. Let $G_n(\varepsilon)$ be the minimal number of balls of radius ε (with respect to the metric d_n^f) which are necessary to cover X, that is, the minimal number of initial conditions which are necessary to follow the n first iterates of any point of X within a precision ε. If f is expanding, this number will increase with n.

Let us look at a simple example.

Example 2.1.1. A diffeomorphism of the plane with hyperbolic fixed point. Let $f : (x, y) \mapsto (\lambda x, \mu y)$ with $0 < \lambda < 1$ and $1 < \mu$. Let $a = (x_a, y_a)$ and $b = (x_b, y_b)$ be two points of $\mathbb{R}^2$. One has $d(a, b) = \text{Max}\,\{|x_a - x_b|, |y_a - y_b|\}$. When n is large enough,

$$d_n^f(a, b) = \mu^n |y_a - y_b| = c\mu^n d(a, b).$$

Thus, if ε is fixed, the area of a ball with radius ε is exponentially decreasing with n and for any compact $K \in \mathbb{R}^2$, the minimal number of balls necessary to cover K is exponentially increasing.

One may expect this kind of phenomenon to occur in compact spaces. This leads to define the topological entropy as the exponential growth rate of the number $G_n(\varepsilon)$.

2.1.1 Definition and basic properties.

For $x \in X$ and $n \in \mathbb{N}$, denote by $B_n^f(x, \varepsilon)$ the ball centered at x and of radius ε for the metric d_n^f. We denote respectively by $G(\varepsilon)$ and $G_n^f(\varepsilon)$ the minimal number of balls of radius ε necessary to cover X for the metric d and the metrics d_n^f.

Definition 2.1.1. The *topological entropy* of f is defined as

$$\mathrm{h_{top}}(f) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n}\log(G_n^f(\varepsilon)).$$

Instead of the balls of radius ε, we could consider the sets with diameter smaller than or equal to ε for the metric d_f^n. We denote by $D_n(\varepsilon)$ the smallest number of sets X_i such that

$$\mathrm{diam}_{d_f^n} X_i \leq \varepsilon, \quad \text{and} \quad \bigcup_i X_i \supset X.$$

We also define the sets *ε-separated* for the metrics d_n^f (we will write (n,ε)-separated). A set E is said to be ε-separated for a metric d if for all (x,y) in E^2, $d(x,y) \geq \varepsilon$. Denote by $S_n^f(\varepsilon)$ the maximal cardinal of a (n,ε)-separeted set contained in X. Then, one has

$$D_n^f(2\varepsilon) \leq G_n^f(\varepsilon) \leq D_n^f(\varepsilon) \quad \text{and} \quad S_n^f(2\varepsilon) \leq G_n^f(\varepsilon) \leq S_n^f(\varepsilon).$$

Indeed, the diameter of a ball of radius ε is smaller than or equal to 2ε, so each covering of X with such balls induces a covering by sets of diameter 2ε, and conversely, a set with diameter ε is contained into a ball of radius ε. On the other hand, if S is (n,ε)-separated, the union of the open balls centered at the points of S and of radius ε covers X, and conversely, no open ball of radius ε can contain two points whose distance is 2ε.

Therefore:

$$\mathrm{h_{top}}(f) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n}\log(S_n^f(\varepsilon)) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n}\log(D_n^f(\varepsilon)).$$

We emphasize the following important property of the topological entropy.

Property 2.1.1. *The topological entropy is a C^0-conjugacy invariant and does not depend on the choice of topologically equivalent metrics on X.*

Remark 2.1.1. As a consequence, if ϕ_L and ϕ_H respectively refer to the geodesic flow and the cogeodesic flow of a compact manifold M in restriction to the unit tangent bundle SM and to the unit cotangent bundle S^*M, $\mathrm{h_{top}}(\phi_L) = \mathrm{h_{top}}(\phi_H)$. Indeed, the Legendre transform induces a conjugacy between ϕ_L and ϕ_H.

For $A \subset X$, we denote by $G_n^f(A,\varepsilon)$ the minimal number of ε-balls for the distance d_n^f in a finite covering of A, and we set $\mathrm{h_{top}}(f,A) := \sup_{\varepsilon>0} \limsup_{n\mapsto\infty} \frac{1}{n}\log(G_n^f(A,\varepsilon))$. If A is invariant by f, we denote by $f_{|A}$ the restriction of f to A. We now give without proof some basic properties of the topological entropy.

Property 2.1.2.

1. *If $A \subset X$ is invariant under f, $\mathrm{h_{top}}(f_{|\overline{A}}) = \mathrm{h_{top}}(f_{|A}) = \mathrm{h_{top}}(f,A)$ and $\mathrm{h_{top}}(f_{|_A}) \leq \mathrm{h_{top}}(f)$.*

2. *If (Y,d') is another compact metric space and if $g : Y \to X$ is a continuous factor of f i.e. there exists a continuous surjective map $h : X \to Y$ such that $h \circ f = g \circ h$, then $\mathrm{h_{top}}(g) \leq \mathrm{h_{top}}(f)$.*

3. *If g is a continuous transformation of a compact metric space Y, then $\mathrm{h_{top}}(f \times g) = \mathrm{h_{top}}(f) + \mathrm{h_{top}}(g)$, where $X \times Y$ is endowed with the product metric.*

4. *For $m \in \mathbb{N}$, $\mathrm{h_{top}}(f^m) = m\mathrm{h_{top}}(f)$ and if f is invertible, $\mathrm{h_{top}}(f^{-m}) = m\mathrm{h_{top}}(f)$.*

5. *Let $A = \cup_{i=1}^n A_i$ where A_i is invariant under f, $\mathrm{h_{top}}(f_{|_A}) = \max_i(\mathrm{h_{top}}(f_{|_{A_i}})$.*

The last property also holds for a *countable* union of invariant closed subsets, but it is not as obvious as for a finite union. We refer to [Pes97] for the proof.

Proposition 2.1.1. ***The σ-union property for*** $\mathrm{h_{top}}$**.** *If $X = \cup_{i\in\mathbb{N}} F_i$ where F_i is closed and invariant under f,*

$$\mathrm{h_{top}}(f) = \sup_{i\in\mathbb{N}}(\mathrm{h_{top}}(f_{|F_i}).$$

Until now, we have defined the topological entropy for discrete dynamical systems. Let $\phi = (\phi^t)_{t\in\mathbb{R}}$ be a flow on a compact metric space X. For all $t \in \mathbb{R}$, we can construct as before the dynamical metrics associated with ϕ by setting $d_t^\phi(x, y) = \sup_{0\leq s<t}\{d(\phi^s(x), \phi^s(y))\}$. Then the quantities $G_t^\phi(\varepsilon)$, $D_t^\phi(\varepsilon)$ and $S_t^\phi(\varepsilon)$ for t in $\mathbb{R}$ are defined as in the discrete case.

Definition 2.1.2. The *topological entropy* of ϕ is defined by

$$\mathrm{h_{top}}(\phi) = \sup_{\varepsilon>0} \limsup_{t\to\infty} \frac{1}{t}\log(G_t^\phi(\varepsilon)).$$

Proposition 2.1.2. $\mathrm{h_{top}}(\phi) = \mathrm{h_{top}}(\phi^1)$.

2.1.2 The Variational Principle.

Now, assume that X is a probability space with measure μ and that f is a continuous map $X \to X$ preserving μ. We can study the action of f on a partition of X by measurable subsets. Such a partition is called measurable. Let ξ be a measurable partition. The *information function* I_ξ associated with ξ is the function $I_\xi : x \mapsto -\log(\mu(C_\xi(x)))$ where $C_\xi(x)$ is the element of ξ that contains x.

Definition 2.1.3. The *entropy of a partition* ξ is defined by : $H_\mu(\xi) = \int_X I_\xi d\mu$.

Since f is continuous, it is measurable and we can look at the new partition whose elements are the intersections of the elements of ξ and those of $f^{-1}(\xi)$. If η and ξ are two partitions, the joint partition of η and ξ is defined by:

$$\eta \bigvee \xi = \{C \cap D \,|\, C \in \eta, D \in \xi, \mu(C \cap D) > 0\}.$$

Definition 2.1.4. Let ξ be a measurable partition. The n^{th} *joint partition with respect to ξ and f* is defined by $\xi_{-n}^f := \xi \bigvee f^{-1}(\xi) \bigvee ... \bigvee f^{-n}(\xi)$.

This partition is *finer* than the previous one and the information it gives is more precise as f “move the points a lot” , the intersections between the elements of ξ and those of $f^{-1}(\xi)$ being then smaller. This leads to the following definitions.

Definition 2.1.5.

1. The *metric entropy of f with respect to ξ* is given by

$$h_\mu(f, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_{-n}^f).$$

2. The *metric entropy of f with respect to μ* is $h_\mu(f) := \sup\{h_\mu(f, \xi), \xi\}$ where the supremum is taken over the set of measurable partitions ξ such that $H_\mu(\xi) < \infty$.

Theorem 10. ***The Variational Principle.*** *If f is a homeomorphism of a compact metric space (X, d) then $\mathrm{h_{top}}(f) = \sup\{h_\mu \,|\, \mu \in \mathcal{M}(f)\}$, where $\mathcal{M}(f)$ is the set of Borel probability measures which are invariant under f.*

Remark 2.1.2. If f is a continuous map of a compact metric space X, the set $\mathcal{M}(f)$ is nonempty, compact and convex for the weak-$\star$ topology and its extremal points are ergodic measures. Therefore, Choquet's theorem gives a decomposition of each f-invariant measure by integral on ergodic measures in the following way: there exists a partition (modulo zero-measure sets) of X in f-invariant sets $(X_\alpha)_{\alpha\in A}$ where A is a Lebesgue space and a family $(\mu_\alpha)_{\alpha\in A}$ of ergodic measures with support in X_α such that for all L^1 function F on, X, one has

$$\int_X F d\mu = \int_A \int_{M_\alpha} F d\mu_\alpha d\alpha.$$

One deduces an analogous decomposition for metric entropies:

$$h_\mu(f) = \int_A h_{\mu_\alpha}(f) d\alpha.$$

Also, if we denote $\mathcal{M}_e(f)$ the set of f-invariant ergodic measures, one has:

$$\mathrm{h}_{\mathrm{top}}(f) = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}_e(f)\}.$$

2.1.3 Vanishing Entropy for integrable Hamiltonian systems.

Let (M,Ω) be a symplectic $2n$-dimensional manifold and H a C^2 function $M \to \mathbb{R}$. We denote by $\phi_H = (\phi_H^t)_{t\in\mathbb{R}}$ the Hamiltonian flow associated with H.

Proposition 2.1.3. *Assume that $M = \mathbb{A}^n$ and that H is in action-angle form. If M^* is any compact subset of M which is invariant under ϕ_H, then $\mathrm{h}_{\mathrm{top}}(\phi_H, M^*) = 0$.*

Proof. Since H is in action-angle form, $\mathbb{A}^n$ is foliated by invariant tori $\mathbb{T}^n \times I$ and an invariant compact subset M^* of $\mathbb{A}^n$ has the form $\mathbb{T}^n \times K$ where K is a compact subset of $\mathbb{R}^n$. Let ν be a ϕ_H-invariant ergodic measure on M^*. Consider a partition of K by subsets $K_1^{(1)}, \cdots K_{k_n}^{(1)}$ with diameter smaller than or equal to $\frac{1}{n}$. Then, any $\mathbb{T}^n \times K_j^{(1)}$ is ϕ_H-invariant. Since ν is ergodic, there exists a unique $j_1 \in \{1\ldots, k_n\}$ such that $\nu\left(\mathbb{T}\times K_{j_1}^{(1)}\right) = 1$ and $\nu\left(\mathbb{T}\times K_j^{(1)}\right) = 0$ whenever $j \neq j_1$. We consider now a partition of $\mathbb{T}^n \times K_{j_1}^{(1)}$ by subsets $K_1^{(2)}, \cdots K_{k'_n}^{(2)}$ with diameter smaller than or equal to $\frac{1}{n^2}$. As before, there exists a unique $j_2 \in \{1, \ldots, k'_n\}$ such that $\nu\left(\mathbb{T}^n\times K_{j_2}^{(2)}\right) = 1$ and $\nu\left(\mathbb{T}^n\times K_j^{(2)}\right) = 0$ whenever $j \neq j_2$. We construct in this way a decreasing sequence of ϕ_H-invariant subsets $\mathbb{T}^n \times K_{j_m}^{(m)}$ such that $\nu\left(\mathbb{T}\times K_{j_m}^{(m)}\right) = 1$ and $\operatorname{diam} K_{j_m}^{(m)} = n^{-m}$. The intersection $\bigcap_{m\in\mathbb{N}} \overline{K_{j_m}^{(m)}}$ is a single point $r_0 \in \mathbb{R}^n$. As a consequence, $\operatorname{Supp}\nu \subset \mathbb{T}^n \times \{r_0\}$. Now the restriction of ϕ_H to $\mathbb{T}^n\times\{r_0\}$ is an isometry, so $\mathrm{h}_{\mathrm{top}}(\phi_H, \mathbb{T}^n\times\{r_0\}) = 0$ and the variational principle shows that $h_\nu(\phi_H)$ also vanishes. Hence, the entropy of ϕ_H with respect to any invariant measure is zero, and using the variational principle once again, we conclude that $\mathrm{h}_{\mathrm{top}}(\phi_H, M^*) = 0$ □

A natural question is then to look at what happens when the Hamiltonian system is integrable in the Liouville sense. Let M^* be a compact subset of M which is invariant under ϕ_H. Such a compact is locally foliated by Liouville tori on which ϕ_H induces an isometry and an argument analogous to the previous one let us see that the topological entropy is read on the singular loci of the moment map. G. Paternain gave in [Pat94] some nondegeneracy conditions on the singular loci of the moment map under which the topological entropy vanishes. We won't give further details, but we state the following theorem for Hamiltonian systems with two degrees of freedom due to Kozlov ([Koz83]) and Paternain ([Pat91]).

Theorem 11. (Kozlov, Paternain) *Let M be a symplectic 4-dimensional manifold and let X^H be a Hamiltonian vector field on M with associated flow ϕ_H. Assume that X^H is integrable in the Liouville sense with a first integral F. Let $\mathscr{E}$ be a compact isoenergy level of H. Assume that $f = F_{|\mathscr{E}}$ satisfies either of the following conditions*

1. *f is real analytic,*

2. *f is C^1 and the connected components of the critical sets of f form (strict) submanifolds.*

Then $\mathrm{h_{top}}(\phi_H, \mathscr{E}) = 0$.

Remark 2.1.3. Nondegenerate Bott systems are included in condition 2.

2.2 Polynomial entropies

2.2.1 Definitions.

As before f is a continuous map $X \to X$, where (X, d) is a compact metric space (X, d).

Definition 2.2.1. The *strong polynomial entropy* $\mathrm{h_{pol}}$ is defined by

$$\mathrm{h_{pol}}(f) = \sup_{\varepsilon} \inf\{\sigma > 0 |\limsup \frac{1}{n^\sigma} G_n^f(\varepsilon) = 0\} = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\mathrm{Log}\, G_n^f(\varepsilon)}{\mathrm{Log}\, n}.$$

As the topological entropy, $\mathrm{h_{pol}}$ may be defined with numbers D_n^f and S_n^f:

$$\mathrm{h_{pol}}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\mathrm{Log}\, S_n^f(\varepsilon)}{\mathrm{Log}\, n} = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\mathrm{Log}\, D_n^f(\varepsilon)}{\mathrm{Log}\, n}.$$

In order to introduce the weak polynomial entropy, let us state some notations. For $\varepsilon > 0$ consider the set

$$\mathscr{B}_\varepsilon^f := \{B_n^f(x, \varepsilon) \,|\, (x, n) \in X \times \mathbb{N}\},$$

of all open balls of radius ε for all the distances d_n^f. We denote by $\mathscr{C}^f(\varepsilon)$ the set of the coverings of X by balls of $\mathscr{B}_\varepsilon^f$, and by $\mathscr{C}_{\leq N}^f(\varepsilon)$ the subset of $\mathscr{C}^f(\varepsilon)$ formed by the coverings $(B_{n_i}^f(x_i, \varepsilon))_{i\in I}$ such that $n_i \leq N$. Given an element $C = (B_{n_i}^f(x_i, \varepsilon))_{i\in I}$ in $\mathscr{C}^f(\varepsilon)$ and a nonnegative real parameter s, we set

$$M(C, s) = \sum_{i\in I} \frac{1}{n_i^s} \in [0, \infty].$$

Note that since a ball may admit several representations of the form $B_{n_i}^f(x_i, \varepsilon)$, the number $M(C, s)$ depends on the family C and not only of its image. Let $N \in \mathbb{N}^*$. The compactness of X allows us to define

$$\Delta^f(\varepsilon, s, N) = \inf\{M(C, s) \,|\, C \in \mathscr{C}_{\geq N}^f(\varepsilon)\} \in [0, \infty].$$

Obviously $\Delta^f(\varepsilon, s, N) \leq \Delta^f(\varepsilon, s, N')$ when $N' \leq N$, so one can define

$$\Delta^f(\varepsilon, s) = \lim_{N\to\infty} \Delta^f(\varepsilon, s, N) = \sup_{N\in\mathbb{N}^*} \Delta^f(\varepsilon, s, N).$$

The definition of the weak polynomial entropy is based on the following lemma.

Lemma 2.2.1. *There exists a unique critical value* $s_c^f(\varepsilon)$ *such that*

$$\Delta^f(\varepsilon, s) = 0 \ \text{ if } \ s > s_c^f(\varepsilon) \quad \text{and} \quad \Delta^f(\varepsilon, s) = \infty \ \text{ if } \ s < s_c^f(\varepsilon).$$

Since $s_c^f(\varepsilon) \leq s_c^f(\varepsilon')$ when $\varepsilon' \leq \varepsilon$, one states the following definition.

Definition 2.2.2. The *weak polynomial entropy* $\mathrm{h}^*_{\mathrm{pol}}$ of f is the limit of the critical value $s_c(\varepsilon)$ when ε goes to 0:

$$\mathrm{h}^*_{\mathrm{pol}}(f) := \lim_{\varepsilon\to 0} s_c^f(\varepsilon) = \sup_{\varepsilon>0} s_c^f(\varepsilon) \in [0, \infty].$$

The relation between the polynomial entropy and the weak polynomial entropy can be made more precise. Denote by $\mathscr{C}^f_{=N}(\varepsilon)$ the subset of $\mathscr{C}^f(\varepsilon)$ of coverings of the form $(B^f_{n_i}(x_i, \varepsilon))_{i\in I}$ with $n_i = N$. We set

$$\Gamma^f(\varepsilon, s, N) = \inf\{M(C, s) \,|\, C \in \mathscr{C}^f_{=N}(\varepsilon)\} \in [0, \infty],$$

and

$$\Gamma^f(\varepsilon, s) = \limsup_{N\to\infty} \Gamma^f(\varepsilon, s, N).$$

Then, one has

$$\Gamma^f(\varepsilon, s) = \limsup_{N\to\infty} \frac{1}{N} G_N^f(\varepsilon).$$

As before, one checks that there exists a critical value $\overline{s}_c^f(\varepsilon)$ such that

$$\Gamma^f(\varepsilon, s) = 0 \ \text{ if } \ s > \overline{s}_c^f(\varepsilon) \quad \text{and} \quad \Gamma^f(\varepsilon, s) = \infty \ \text{ if } \ s < \overline{s}_c^f(\varepsilon).$$

Therefore

$$\mathrm{h}_{\mathrm{pol}}(f) = \lim_{\varepsilon\to 0} \overline{s}_c^f(\varepsilon) = \sup_{\varepsilon>0} \overline{s}_c^f(\varepsilon).$$

Since $\mathscr{C}^f_{=N}(\varepsilon) \subset \mathscr{C}^f_{\leq N}(\varepsilon)$, $\Delta^f(\varepsilon, s, N) \leq \Gamma^f(\varepsilon, s, N)$ which yields $\Delta^f(\varepsilon, s) \leq \Gamma^f(\varepsilon, s)$ and to

$$\mathrm{h}^*_{\mathrm{pol}}(f) \leq \mathrm{h}_{\mathrm{pol}}(f).$$

As the topological entropy, $\mathrm{h}_{\mathrm{pol}}$ and $\mathrm{h}^*_{\mathrm{pol}}$ do not depend on the choice of topologically invariant metrics on X and are C^0-conjugacy invariant. Therefore, if ϕ_L and ϕ_H are the respective geodesic flow and cogeodesic flow of a compact Riemannian manifold (restricted to the unit tangent bundles), $\mathrm{h}_{\mathrm{pol}}(\phi_L) = \mathrm{h}_{\mathrm{pol}}(\phi_H)$ and $\mathrm{h}^*_{\mathrm{pol}}(\phi_L) = \mathrm{h}^*_{\mathrm{pol}}(\phi_H)$. Before giving some properties of the polynomial entropies, let us emphasize the important following fact.

Proposition 2.2.1. *When* $\mathrm{h}_{\mathrm{top}}(f) > 0$*, the strong polynomial entropy and the weak polynomial entropy are both infinite.*

Let us now state briefly the definition of the polynomial entropies for flows. For each $t \geq 1$, we denote by $\mathscr{C}^\phi_{\geq t}(\varepsilon)$ the set of coverings of X of the form $C = (B_{\tau_i}(x_i, \varepsilon))_{i\in I}$ with $\tau_i \geq t$ and, for such a covering C, we set $M(C, s) = \sum_{i\in I} \frac{1}{\tau_i^s}$ for $s \geq 0$. Finally we introduce the quantity

$$\delta^\phi(\varepsilon, s, t) = \inf\{M(C, s) \,|\, C \in \mathscr{C}^\phi_{\geq t}(\varepsilon)\}$$

which is monotone nondecreasing with t, and we set $\Delta^\phi(\varepsilon, s) = \lim_{t\to\infty} \delta^\phi(\varepsilon, s, t)$. As in the discrete case, one sees that there exists a unique $s_c^\phi(\varepsilon)$ such that $\Delta^\phi(\varepsilon, s) = 0$ if

$s > s_c^\phi(\varepsilon)$ and $\Delta^\phi(\varepsilon, s) = +\infty$ if $s < s_c^\phi(\varepsilon)$. Hence, the weak polynomial entropy for the continuous system ϕ is defined as

$$\mathrm{h}^*_{\mathrm{pol}}(\phi) = \lim_{\varepsilon\to 0} s_c^\phi(\varepsilon) = \sup_{\varepsilon>0} s_c^\phi(\varepsilon).$$

One checks that

$$\mathrm{h}^*_{\mathrm{pol}}(\phi) = \mathrm{h}^*_{\mathrm{pol}}(\phi_1).$$

We now denote by $G_t^\phi(\varepsilon)$ the minimal number of d_t^ϕ-balls of radius ε in a covering of X, and we set

$$\mathrm{h}_{\mathrm{pol}}(\phi) = \sup_{\varepsilon>0} \limsup_{t\to\infty} \frac{\mathrm{Log}\, G_t^\phi(\varepsilon)}{\mathrm{Log}\, t} = \inf\{\sigma \geq 0 \mid \lim_{t\to\infty} \frac{1}{t^\sigma} G_t^\phi(\varepsilon) = 0\}.$$

As before, one has

$$\mathrm{h}_{\mathrm{pol}}(\phi) = \mathrm{h}_{\mathrm{pol}}(\phi_1).$$

2.2.2 Properties.

As before, we give without proofs some basic properties of the polynomial entropies. All the results in this section are proved in [Mar09].

Property 2.2.1. *Here, the symbol* $\bar{\mathrm{h}}$ *will stand indifferently for* $\mathrm{h}_{\mathrm{pol}}$ *or* $\mathrm{h}^*_{\mathrm{pol}}$.

1. *If A is a subset of X invariant under f, $\bar{\mathrm{h}}(f_{|A}) \leq \bar{\mathrm{h}}(f)$.*

2. *If (Y, d') is another compact metric space and if $g : Y \to X$ is a continuous factor of f then $\bar{\mathrm{h}}(g) \leq \bar{\mathrm{h}}(f)$.*

3. *If $g : Y \to Y$ is a continuous map on a compact metric space Y, and if $X \times Y$ is endowed with the product metric, then $\mathrm{h}_{\mathrm{pol}}(f \times g) = \mathrm{h}_{\mathrm{pol}}(f) + \mathrm{h}_{\mathrm{pol}}(g)$.*

4. *For $m \in \mathbb{N}$, $\bar{\mathrm{h}}(f^m) = \bar{\mathrm{h}}(f)$ and if f is invertible, $\bar{\mathrm{h}}(f^{-m}) = \bar{\mathrm{h}}(f)$.*

5. *If $A = \cup_{i=1}^n A_i$ where A_i is invariant under f, $\bar{\mathrm{h}}(f_{|A}) = \max_i(\bar{\mathrm{h}}(f_{|A_i})$.*

J-P. Marco proved in [Mar09] that there is no analogous to the property of σ-union for the polynomial entropy $\mathrm{h}_{\mathrm{pol}}$, but that this property holds true for the weak polynomial entropy.

Proposition 2.2.2. ***The σ-union property for*** $\mathrm{h}^*_{\mathrm{pol}}$**.** *If $X = \cup_{i\in\mathbb{N}} F_i$ where F_i is closed and invariant under f,*

$$\mathrm{h}^*_{\mathrm{pol}}(f) = \sup_{i\in\mathbb{N}}(\mathrm{h}^*_{\mathrm{pol}}(f_{|F_i})).$$

2.2.3 Hamiltonian systems in action-angle form.

It turns out that polynomial entropies are particularly relevant for the study of Hamiltonian systems. The first remarkable fact is that for Hamiltonian systems in action-angle form the weak polynomial entropy and the polynomial entropy do coincide. Indeed, J-P Marco proved that they actually detect the "effective" number of degrees of freedom.

Theorem 12. (Marco) *Consider a C^2 Hamiltonian function H on $\mathbb{T}^n \times B$, where B is a closed ball of $\mathbb{R}^n$, which depends only on the action variable I. Denote by ϕ_H the Hamiltonian flow associated with H. Then, if h denotes the fonction on B such that $H(\alpha, I) = h(I)$, one has*

$$\mathrm{h}_{\mathrm{pol}}(\phi_H) = \mathrm{h}^*_{\mathrm{pol}}(\phi_H) = \max_{I \in B} \operatorname{rank} \omega(I).$$

where $\omega : I \mapsto dh(I)$

Notice that $\mathrm{h}_{\mathrm{pol}}$ and $\mathrm{h}^*_{\mathrm{pol}}$ are smaller than or equal to the half dimension of the ambient manifold. We conclude this part by the following slight generalisation of Marco's Theorem.

Proposition 2.2.3. *Consider a C^2 Hamiltonian function H on $T^*\mathbb{T}^n$ which depends only on the action variable I. Denote by h the fonction on $\mathbb{R}^n$ such that $H(\alpha, I) = h(I)$. Let $\mathcal{S}$ be a compact submanifold of $\mathbb{R}^n$, possibly with boundary. Then the compact $\mathbb{T}^n \times \mathcal{S}$ is invariant under the flow ϕ and one has*

$$\mathrm{h}_{\mathrm{pol}}(\phi_{|\mathbb{T}^n \times \mathcal{S}}) = \mathrm{h}^*_{\mathrm{pol}}(\phi_{|\mathbb{T}^n \times \mathcal{S}}) = \max_{I \in \mathcal{S}} \operatorname{rank} \omega(I),$$

where $\omega : I \mapsto d(h_{|\mathcal{S}})(I)$.

Remark 2.2.1. In the particular cases where h is strictly convex and where $\mathcal{S}$ is a compact energy level $\mathcal{S} = h^{-1}(\{e\})$, one has

$$\mathrm{h}_{\mathrm{pol}}(\phi_{|\mathbb{T}^n \times \mathcal{S}}) = \mathrm{h}^*_{\mathrm{pol}}(\phi_{|\mathbb{T}^n \times \mathcal{S}}) = n - 1.$$

Proof. Recall that given a compact metric space (X, d), the ball dimension $D(X)$ is by definition

$$D(X) := \limsup_{\varepsilon \to 0} \frac{\operatorname{Log} c(\varepsilon)}{|\operatorname{Log} \varepsilon|}$$

where $c(\varepsilon)$ is the minimal cardinal of a covering of X by ε-balls. We will use the fact that the ball dimension of a compact manifold is equal to its usual dimension and that the ball dimension of the image of a compact submanifold by C^1 map of rank ℓ is $\leq \ell$.

We endow $\mathbb{R}^n$ with the product metric defined by the Max norm $\|\ \|$ and the submanifold $\mathcal{S}$ with the induced metric. We endow the torus $\mathbb{T}^n$ with the quotient metric. Since the pairs of points (α, α') of $\mathbb{T}^n \times \mathbb{T}^n$ we will have to consider are close enough to one another, we still denote by $\|\alpha - \alpha'\|$ their distance. Finally we endow the product $\mathbb{T}^n \times \mathcal{S}$ with the product metric of the above ones.

Assume that $\operatorname{rank} \omega = \ell$ and denote by Ω the image $\omega(\mathcal{S})$.

We will first prove that $\mathrm{h}_{\mathrm{pol}}(\varphi) \leq \ell$. Let $\varepsilon > 0$. Observe that, for $N \geq 1$, if two points (α, I) and (α', I') of $\mathbb{T}^n \times \mathcal{S}$ satisfy

$$\|\alpha - \alpha'\| < \frac{\varepsilon}{2}, \qquad \|\omega(I) - \omega(I')\| \leq \frac{\varepsilon}{2N}, \qquad \|I - I'\| < \varepsilon \tag{2.1}$$

then $d_N^{\phi}((\alpha, I), (\alpha', I')) < \varepsilon$. Let us introduce the following minimal coverings :

- a minimal covering $C_{\mathbb{T}^n}$ of $\mathbb{T}^n$ with balls of radius $\varepsilon/2$, so its cardinal i^* depends only on ε,
- a minimal covering $(\widehat{B}_j)_{1 \leq j \leq j^*}$ of $\mathcal{S}$ by balls of radius $\varepsilon/2$, so again j^* depends only on ε,
- for $N \geq 1$, a minimal covering $(\widetilde{B}_k)_{1 \leq k \leq k^*}$ of the image Ω with balls of radius $\varepsilon/(2N)$.

The last two coverings form a covering $C_{\mathcal{S}} = (\widehat{B}_j \cap \omega^{-1}\tilde{B}_k))_{j,k}$ of $\mathcal{S}$ such that any two points I, I' in the same set $\widehat{B}_j \cap \omega^{-1}(\tilde{B}_k)$ satisfy the last two conditions of (2.1). Hence we get a covering of $\mathbb{T}^n \times \mathcal{S}$ whose elements are contained in balls of d_N^ϕ-radius ε by considering the products of the elements of $C_{\mathbb{T}^n}$ and $C_{\mathcal{S}}$.

Note that, since the ball dimension of Ω is smaller than or equal to ℓ, given any $\ell' > \ell$, for N large enough, $k^* \le (2N/\varepsilon)^{\ell'}$. Thus:

$$G_N(\phi, \varepsilon) \le i^*\, j^*\, k^* \le c(\varepsilon) N^{\ell'}$$

and $\overline{s}_c^\phi(\varepsilon) \le \ell'$. Since $\ell' > \ell$ is arbitrary $\mathrm{h}_{\mathrm{pol}}(\phi) \le \ell$.

Now it suffices to prove that $\mathrm{h}^*_{\mathrm{pol}}(\phi) \ge \ell$. We begin with describing the (N, ε)-balls of the system more precisely. Let (α, I) in $\mathbb{T}^n \times \mathcal{S}$ be given, and fix $\varepsilon > 0$. In all what follows the balls of the form $B(I, \varepsilon)$ and $B((\alpha, I), \varepsilon)$ will respectively refer to the balls of $\mathcal{S}$ for the induced metric of $\mathbb{R}^n$ and the balls of $\mathbb{T}^n \times \mathcal{S}$ for the product metric defined above. Then a point (α', I') in $\mathbb{T}^n \times \mathcal{S}$ belongs to the ball $B_N((\alpha, I), \varepsilon)$ if and only if

$$\|I' - I\| < \varepsilon, \qquad \|k(\omega(I') - \omega(I)) + (\alpha' - \alpha)\| < \varepsilon, \quad \forall k \in \{0, \dots, N-1\}.$$

In component form, the second condition reads:

$$|\alpha_i' - \alpha_i| < \varepsilon, \quad \omega_i(I') \in \left]\frac{(\alpha_i - \alpha_i') - \varepsilon}{N-1} + \omega_i(I), \frac{(\alpha_i - \alpha_i') + \varepsilon}{N-1} + \omega_i(I)\right[, \qquad 1 \le i \le n.$$

Thus $B_N((\theta, r), \varepsilon)$ has the following fibered structure over the ball $B(\alpha, \varepsilon)$

$$B_N((\alpha, I), \varepsilon) = \bigcup_{\alpha' \in B(\alpha, \varepsilon)} \{\alpha'\} \times F_{\alpha'},$$

where the fiber over the point α' is the curved polytope

$$F_{\alpha'} = \omega^{-1}\left(\prod_{1 \le i \le n} \left]\frac{(\alpha_i - \alpha_i') - \varepsilon}{N-1} + \omega_i(r), \frac{(\alpha_i - \alpha_i') + \varepsilon}{N-1} + \omega_i(r)\right[\right) \bigcap B(r, \varepsilon).$$

Now, fix a covering $C = (B_{n_i}((\alpha_i, I_i), \varepsilon))_{i \in I}$ of $\mathscr{C}_{\ge N}(\mathbb{T}^n \times \mathcal{S})$, and denote by F_0^i the fiber of $\alpha = 0$ in the ball $B_{n_i}((\alpha_i, I_i), \varepsilon)$ (which may be empty). Then the set $\{0\} \times \mathcal{S}$ is contained in the union of the fibers F_0^i. We denote by ν the Lebesgue volume of this set.

Since $\operatorname{rank} \omega = \ell$, there exists a constant $c > 0$ such that the Lebesgue volume of the fiber F_0^i satisfies

$$\mathrm{Vol}\,(F_0^i) \le c\Big(\frac{2\varepsilon}{n_i - 1}\Big)^\ell.$$

The sum of the volumes of the fibers must be larger than ν, so

$$\sum_{i \in I} c\Big(\frac{2\varepsilon}{n_i - 1}\Big)^\ell \ge \nu.$$

Then, if $s < \ell$

$$M(C, s) = \sum_{i \in I} \frac{1}{n_i^s} = \frac{1}{c(2\varepsilon)^\ell} \sum_{i \in I} c\Big(\frac{2\varepsilon}{n_i - 1}\Big)^\ell \frac{(n_i - 1)^\ell}{n_i^s} \ge \frac{\nu}{c(2\varepsilon)^\ell} \frac{1}{2^\ell} N^{\ell - s}$$

and therefore

$$\Delta(\mathbb{T}^n \times \mathcal{S}, \varepsilon, s) = \lim_{N \to \infty} \delta(\mathbb{T}^n \times \mathcal{S}, \varepsilon, s, N) = +\infty.$$

Thus $s_c^\phi(\mathbb{T}^n \times \mathcal{S}, \varepsilon) \ge \ell$, and hence $\mathrm{h}^*_{\mathrm{pol}}(\phi) \ge \ell$. $\square$

Chapter 3

Polynomial entropies for dynamically coherent systems

In this chapter, we compute the polynomial entropies for a dynamically coherent system. Consider a 4-dimensional symplectic manifold (M, Ω) and a smooth Hamiltonian function $H : M \to \mathbb{R}$, with its associated vector field X^H and its associated Hamiltonian flow ϕ_H. Fix a (connected component of a) compact nondegenerate energy level $\mathscr{E} := H^{-1}(\{e_0\})$ of H and assume there exists a first integral $F : M \to \mathbb{R}$ of the vector field X^H such that, if $f := F_{|\mathscr{E}}$, the system $(\mathscr{E}, \phi_H, f)$ is dynamically coherent.

We begin with the description of the dynamics in the neighborhood of the singularities of f in section 3.1. In section 3.2, we prove that $\mathrm{h}^*_{\mathrm{pol}}(\phi) \in \{0, 1\}$ and in section 3.3, we prove that $\mathrm{h}_{\mathrm{pol}}(\phi) \in \{0, 1, 2\}$.

3.1 The dynamics in the neighborhood of the singularities

We will first see that the dynamics in the neighborhood of a critical torus or of an elliptic orbit is easily deduced from the dynamics in the regular set of f. Moreover, up to a 2-sheeted covering, the dynamics in the neighborhood of a Klein bottle is the same as the one near a critical torus.

The complexity mostly occurs in the neighborhood of the ∞-levels. As before, one sees that up to a 2-sheeted covering, the dynamics in the neighborhood of a nonorientable ∞-level is the same as the one in the orientable case. As for this last one, we will actually study the more general case of the dynamics near a *simple polycycle* (see the definition in paragraph 3.1.3).

3.1.1 Critical tori and Klein bottles

Proposition 3.1.1. *Let $\mathcal{T} \subset \mathscr{E}$ be a critical torus of f. There exists a neighborhood U of $\mathcal{T}$ in M that is an action-angle domain.*

Proof. We first show hat there exists a neighborhood U of $\mathcal{T}$ such that the foliation induced by F is made by homotopic tori. Then we see that the construction of the action-angle variables can be done as in the usual case by taking a family of bases of the homology of each torus that depends smoothly on the tori (see Annex A, or [Dui80], or [Aud01]).

• We begin by studying the foliation induced by f on $\mathscr{E}$. We endow M with a Riemannian metric g and we denote by $||\cdot||$ the norm associated with g. We can assume without loss of generality that $f(\mathcal{T}) = 0$ and that f has index 0 on $\mathcal{T}$ (that is, the Hessian of f in restriction to a transverse line to $\mathcal{T}$ is positive definite). Therefore, by the Morse-Bott theorem (see ([Ban04]) for a recent proof), for any $q \in \mathcal{T}$, there exist a neighborhood U_q of q in $\mathscr{E}$, a neighborhood $O_q \times I_q$ of $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$ and diffeomorphism $\Phi_q : U_q \to O_q \times I_q, p \mapsto (\varphi, \xi)$ such that $f \circ \Phi_q^{-1}(\varphi, \xi) = \xi^2$.

Now, for $q \in \mathcal{T}$, one has the decomposition $T_q\mathscr{E} = T_q\mathcal{T} \oplus \mathbb{R}N$, where N is the unit normal vector to $\mathcal{T}$. Consider the vector bundle $\widetilde{F}$ over $\mathcal{T}$ with fiber $F_q := N(q)$. Since $\mathcal{T}$ is orientable, $\widetilde{F}$ is trivial. Using the tubular neighborhood theorem, there exist a neighborhood I of 0 in $\mathbb{R}$ and a diffeomorphism $\Psi : U \to \mathbb{T}^2 \times I, q \mapsto (\theta, x)$, that satisfies $\Psi(\mathcal{T}) := \{(\theta, 0) \,|\, \theta \in \mathbb{T}^2\}$.

Fix $q \in \mathcal{T}$. We can assume that $U_q \subset U$. The map

$$P := \Psi \circ \Phi_q^{-1} : O_q \times I_q \to \mathbb{T}^2 \times I, (\varphi, \xi) \mapsto (\theta(\varphi, \xi), x(\varphi, \xi))$$

is a diffeomorphism on its image. Let $q' = (\varphi, 0) \in U_q \cap \mathcal{T}$. The curve $\xi \mapsto (\theta(\varphi, \xi), x(\varphi, \xi))$ is transverse to $\mathcal{T}$ at the point q'. So $\frac{\partial x}{\partial \xi}(\varphi, 0) \neq 0$ for all $(\varphi, 0) \in U_q \cap \mathcal{T}$. Assume that $\frac{\partial x}{\partial \xi} > 0$ on $U_q \cap \mathcal{T}$. Then for $c > 0$ small enough, $f^{-1}(\{c\}) \cap U_q$ has two connected components C^+ and C^- with $C^+ \subset \{x > 0\}$ and $C^- \subset \{x < 0\}$. Both are graphs over $U_q \cap \mathcal{T}$. Now $f^{-1}(\{c\}) \cap \{x > 0\} = f^{-1}(\{c\}) \cap \{x \geq 0\}$ is closed. But its complementary $f^{-1}(\{c\}) \cap \{x \leq 0\}$ is also closed so $f^{-1}(\{c\})$ is not connected. Therefore $f^{-1}(\{c\}) \cap U$ has two connected components which are both graphs over $\mathcal{T}$. So U is foliated by homotopic tori that invariant under the Hamiltonian flow. Fix $c_0 > 0$ and let us introduce the following domains:

$$D_0^+ := \bigcup_{0 \leq c \leq c_0} \{f^{-1}(c) \cap U \cap \{x \geq 0\}\}, \quad D_0^- := \bigcup_{0 \leq c \leq c_0} \{f^{-1}(c) \cap U \cap \{x \leq 0\}\}.$$

The foliation induced by f on each on these domain is trivial.

• We will now see that this property holds true for an energy level $\mathscr{E}'$ close to $\mathscr{E}$. Since $\mathscr{E}$ is a regular level of H there exists a neighborood U of $\mathscr{E}$ such that the Riemannian gradient ∇H does not vanish on U. We can assume without loss of generality that $H(\mathscr{E}) = 0$.

Consider the vector field $X := \frac{\nabla H}{||\nabla H||^2}$ on U and denote by (ϕ_t) its associated flow. Since X is C^1, for t small enough, ϕ_t is a diffeomorphism. Moreover, since $X.H \equiv 1$, $\phi_t(\mathscr{E}) \subset H^{-1}(t)$.

Consider now an open neighborhood V of $\mathcal{T}$ in $\mathscr{E}$. We define a one-parameter family of vector fields $(Y_t)_t$ in V in the following way

$$Y_t := \phi_t^* \nabla(f).$$

Then Y_t depends in C^1 way of t, and we observe that $\mathcal{T}$ is a normally hyperbolic manifold for $Y_0 = \nabla(f)$. Therefore one can apply the Hirsch-Pugh-Shub theorem of persistence of normally hyperbolic manifold [HPS77]: for t close enough to 0, Y_t admits a normally hyperbolic torus $\widetilde{\mathcal{T}}_t$ which is C^1 close to $\mathcal{T}$. We set $\mathcal{T}_t = \phi_t(\widetilde{\mathcal{T}}_t)$. Since ϕ_t is a diffeomorphism, $\mathcal{T}_t$ is a critical torus of F contained in $H^{-1}(\{t\})$ and the Hessian $\partial^2(F_{|\mathcal{T}_t})$ of the restriction of F to $\mathcal{T}_t$ has the same type than the Hessian of the restriction of f to $\mathcal{T}$. The first argument holds true and one gets two domains D_t^+ and D_t^- as before.

• Consider the two domains $\widehat{D}^+ := \bigcup_t D_t^-$ and $\widehat{D}^+ := \bigcup_t D_t^-$. One can construct action variables in each of these two domains using the Arnol'd method "by quadrature" (see Annex A.2). One immediately checks that the action variables can be glued continuously along the union $\bigcup_t \mathcal{T}_t$ (with the convention $\mathcal{T}_0 = \mathcal{T}$). Therefore, the angle variables may be constructed by considering any Lagrangian section of the moment map (H, F) as in Annex 1 step 5. □

Remark 3.1.1. If $\mathcal{K}$ is a critical Klein bottle, one proves that there exists a neighborhood U of $\mathcal{K}$ in M that admits a natural two-sheeted covering $\widetilde{U}$, such that the symplectic form Ω, and the functions H and F can be lifted to $\widetilde{U}$ ([Zun96]). Therefore, the study of the dynamics near a Klein bottle boils down to the study near a critical torus. Indeed, if we denote by $\widetilde{\phi}_H^t$ the lifted flow and by $\pi : \widetilde{U} \to U$ the canonical projection, $\phi_H^t \circ \pi = \pi \circ \widetilde{\phi}_H^t$. We will see that it enough to determine the polynomial entropies in the neighborhood of $\mathcal{K}$.

3.1.2 Elliptic orbits

Consider a critical circle $\mathcal{C}$ of f which is an elliptic periodic orbit. The following proposition stated in [BBM10] (and references therein) shows there exist generalized action-angle coordinates in the neighborhood of $\mathcal{C}$.

Proposition 3.1.2. *In a neighborhood U of $\mathcal{C}$, there exist canonical coordinates (φ, I, p, q) with $\{\varphi, I\} = \{q, p\} = 1$ such that*

- *H and F only depend on I and $p^2 + q^2$,*
- *$(\varphi + 2\pi, I, p, q) = (\varphi, I, p, q)$,*
- *$\mathcal{C}$ is defined by $I = 0$ and $(p, q) = (0, 0)$.*

Moreover if $J := \frac{1}{2}(p^2 + q^2)$ one has

$$\frac{\partial H}{\partial I} \neq 0, \quad \det \begin{pmatrix} \dfrac{\partial H}{\partial I} & \dfrac{\partial H}{\partial J} \\ \dfrac{\partial F}{\partial I} & \dfrac{\partial F}{\partial J} \end{pmatrix} \neq 0.$$

Corollary 3.1.1. *There exist a neighborhood O of $(0, 0) \in \mathbb{R}^2$ and a projection*

$$\pi : \mathbb{T}^2 \times O \to U, (\varphi, \psi, I, J) \mapsto (\varphi, I, \sqrt{2J}\cos\psi, \sqrt{2J}\sin\psi)$$

such that the following diagram commutes

$$\begin{tikzcd} \mathbb{T}^2 \times O \arrow[r, "\phi^t"] \arrow[d, "\pi"'] & \mathbb{T}^2 \times O \arrow[d, "\pi"] \\ U \arrow[r, "\phi_H^t"'] & U. \end{tikzcd}$$

where (ϕ_t) is the flow associated with the vector field

$$\dot{I} = \dot{J} = 0, \quad \dot{\varphi} = \frac{\partial H}{\partial I}(I, J), \quad \dot{\psi} = \frac{\partial H}{\partial J}(I, J). \tag{3.1}$$

3.1.3 ∞-levels and simple polycyles

We gather in this section some “classical” statements on the ∞-levels which are explicitely stated in or could certainly be extracted from the studies of Bolsinov-Fomenko and Zung, but, in this last case, for which we were unable to find precise references. We will only give short proofs, since they are essentially based on standard methods of symplectic geometry.

Consider an ∞-level $\mathscr{P}$ and denote by $\mathcal{C}$ the crictical circle contained in $\mathscr{P}$. Let us set out the following easy lemma, which will prove useful in the following.

Lemma 3.1.1. *Let C be a connected component of $\mathscr{P} \setminus \mathcal{C}$. Then C is homeomorphic to the cylinder $\mathbb{T} \times \mathbb{R}$ and*

– there exists $(u_e, v_e) \in \mathbb{R}^2$ such that the restriction to C of $X^{u_eH+v_eF}$ is periodic (moreover, (u_e, v_e) is unique up to a multiplicative factor);

– there exists $(\widehat{u}_e, \widehat{v}_e) \in \mathbb{R}^2$ such that, given $a \in \mathcal{C}$, there exists a unique orbit of the restriction to C of $X^{\widehat{u}_eH+\widehat{v}_eF}$ whose α and ω limit sets are equal to $\{a\}$ (as above, this pair is unique up to rescaling by a nonzero factor).

Proof. Observe that on the complement $\mathscr{P} \setminus \mathcal{C}$ the Hamiltonian vector fields X^H and X^F are independent. Therefore, by the Liouville theorem, given a connected component C of $\mathscr{P} \setminus \mathcal{C}$, one gets an action $\Phi : \mathbb{R}^2 \times C \to C$ by setting

$$((t,s),a) \mapsto \Phi^{tH} \circ \Phi^{sF}(a).$$

Clearly, C cannot be compact and for topological reasons (a doubling argument) it cannot be diffeomorphic to $\mathbb{R}^2$. The rest of the proof is immediate (see [LM87]). $\square$

Let us now denote by $\mathscr{D}(\delta)$ the quotient of the manifold

$$P(\delta) = (\mathbb{R}\times\,]-\delta,\delta[) \times (]-\delta,\delta[^2),$$

endowed with the product symplectic structure, by the action $\mathbb{Z} \times P(\delta) \to P(\delta)$ defined by

$$(m,(x,I,p,q)) \mapsto (x+m,I,p,q),$$

so that $\mathscr{D}(\delta) = (\mathbb{T}\times\,]-\delta,\delta[) \times (]-\delta,\delta[^2)$, endowed with its usual structure. We then denote by $\widehat{\mathscr{D}}(\delta)$ the "twisted" version $\mathscr{D}(\delta)$, that is, the quotient of $P(\delta)$ by the action $\mathbb{Z} \times P(\delta) \to P(\delta)$ defined by

$$(m,(x,I,p,q)) \mapsto (x+m,I,(-1)^m p,(-1)^m q).$$

The following proposition is essentially stated in [BBM10] (see also the references therein).

Proposition 3.1.3. *There exists a neighborhood U of the critical orbit $\mathcal{C}$ in M and a symplectic diffeomorphism Φ from U to $\mathscr{D}(\delta)$ or $\widehat{\mathscr{D}}(\delta)$ (that is Φ passes to the quotient under the previous action), such that, denoting by (φ, I, q, p) the components of Φ and considering them as local coordinates on U (so that $\{\varphi, I\} = \{q,p\} = 1$), the following properties hold true:*

- *H and F only depend on I and $J := qp$,*
- *$\mathcal{C}$ is defined by $I = 0$ and $(p,q) = (0,0)$.*

Moreover, one has the independence relations

$$\frac{\partial H}{\partial I} \neq 0, \quad \frac{\partial H}{\partial J} \neq 0, \quad \det \begin{pmatrix} \dfrac{\partial H}{\partial I} & \dfrac{\partial H}{\partial J} \\ \dfrac{\partial F}{\partial I} & \dfrac{\partial F}{\partial J} \end{pmatrix} \neq 0.$$

and if T is the period of the periodic orbit $\mathcal{C}$, the eigenvalues of $D\phi^T$ are $1, e^{2\pi\alpha}$ in the case when $\mathscr{P}$ is orientable and $e^{-2\pi\alpha}$ and $1, -e^{2\pi\alpha}$ and $-e^{-2\pi\alpha}$ in the case when $\mathscr{P}$ is nonorientable, where $\alpha = (\frac{\partial H}{\partial I})^{-1}\frac{\partial H}{\partial J}{}_{|I=J=0}$.

Remark 3.1.2. In both orientable and nonorientable cases the orbit $\mathcal{C}$ is hyperbolic if and only if $\frac{\partial H}{\partial J}_{|I=J=0} = 0$.

Observe also that, relatively to the previous coordinates, $\mathscr{P} \cap U$ admits the (local) equation

$$\varphi \in \mathbb{T},\ I = 0,\ J = 0.$$

In the case where the image of Φ is $\mathscr{D}(\delta)$, the complement $(\mathscr{P} \cap U) \setminus \mathcal{C}$ therefore admits 4 connected components diffeomorphic to cylinders $\mathbb{T}\times\ ]0,1[$ and $\mathscr{P}$ is said to be an *orientable* ∞*–level*. In the case where the image of Φ is $\widehat{\mathscr{D}}(\delta)$ the complement $(\mathscr{P} \cap U) \setminus \mathcal{C}$ admits 2 connected components diffeomorphic to Möbius strips, and $\mathscr{P}$ is said to be a nonorientable ∞–level. In both cases, we say that U is a *normalizing neighborhood* for $\mathcal{C}$.

The previous local proposition admits a local counterpart which enables one to discriminate between the orientable and non orientable ∞-levels.

Corollary 3.1.2. *Let $\mathscr{P}$ be an ∞–level and let U be a normalizing neighborhood for the critical orbit $\mathcal{C}$. Then if $\mathscr{P}$ is orientable the complement $\mathscr{P} \setminus U$ admits two connected components, while if $\mathscr{P}$ is nonorientable it admits only one connected component.*

Proof. Observe that a component C of $\mathscr{P} \setminus \mathcal{C}$ has two "ends" (in the topological sense), which are homeomorphic to cylinders. Each connected component of $(U \cap \mathscr{P}) \setminus \mathcal{C}$ is obviously contained in a component of $\mathscr{P} \setminus \mathcal{C}$, and is indeed an end for such a component, from which the proposition easily follows. This can be made more precise by using lemma 3.1.1, together with the normal form of proposition 3.1.3 in U. $\square$

Let us now pass to the definition of simple polycycles. Recall that, in the most general case, a polycycle $\mathscr{P}$ is the connected union of critical circles of index 1 and of cylinders $\mathbb{T} \times \mathbb{R}$ whose boundary is made of one or two critical circles.

Consider an orientable polycycle $\mathscr{P}$ and let $\{\mathcal{C}_1, \ldots, \mathcal{C}_q\}$ be its critical circles. For any $\mathcal{C}_k$, we denote by U_k the neighborhood of $\mathcal{C}_k$ given by the previous proposition and by $(\varphi_k, I_k, q_k, p_k)$ the corresponding coordinates. We set $J_k = p_k q_k$. The following corollary is an easy consequence of proposition 3.1.3.

Corollary 3.1.3. *Fix $e \in H(U_k)$ and set $\mathscr{U}_{k,e} := U_k \cap H^{-1}(\{e\})$.*

1. *the coordinates (φ_k, q_k, p_k) form a local chart of $\mathscr{U}_{k,e}$,*
2. *the compact set $\mathcal{C}_{k,e} := \{(\varphi_k, 0, 0) \,|\, \varphi_k \in \mathbb{T}\}$ is a periodic orbit.*

Proof. We denote by D the open domain in $\mathbb{R}^3$ given by the coordinates (I_k, q_k, p_k) and by $\widehat{D}$ its image in $\mathbb{R}^2$ by the map $(I_k, q_k, p_k) = (I_k, q_k p_k)$. Let $\pi_J : (I_k, q_k p_k) \mapsto (q_k p_k)$.

(1) By the implicit function theorem, there exists a function $\mathscr{I}_k : \pi_J(\widehat{D}) \times H(U_k) \to \mathbb{R}$ such that

$$H(I_k, J_k) = e \Longleftrightarrow I_k = \mathscr{I}_k(J_k, e),$$

so $(\varphi_k, I_k, q_k, p_k) \in \mathscr{U}_{k,e}$ if and only if $I_k = \mathscr{I}_k(p_k q_k, e)$.

(2) The vector field X^H restricted to $\mathscr{U}_{k,e}$ reads:

$$\begin{aligned}
\dot{\varphi}_k &= \frac{\partial H}{\partial I_k}(I_k, J_k) && = \frac{\partial H}{\partial I_k}(\mathscr{I}(q_k p_k), q_k p_k) \\
\dot{q}_k &= -\frac{\partial H}{\partial J_k}(I_k, J_k)\frac{\partial J_k}{\partial q_k}(J_k) && = -p_k \frac{\partial H}{\partial J_k}(\mathscr{I}(q_k p_k), q_k p_k) \\
\dot{q}_k &= \frac{\partial H}{\partial J_k}(I_k, J_k)\frac{\partial J_k}{\partial p_k}(J_k) && = q_k \frac{\partial H}{\partial J_k}(\mathscr{I}(q_k p_k), q_k p_k),
\end{aligned}$$

Obviously the compact set $\{(\varphi_k, 0, 0) \,|\, \varphi_k \in \mathbb{T}\}$ is a periodic orbit. $\square$

We say that a polycycle $\mathscr{P}$ is *continuable* if there exists $\delta_0 > 0$ such that for all $e \in]e_0 - \delta_0, e_0 + \delta_0[\subset \bigcap_{1\leq k\leq q} H(U_k)$, the hyperbolic orbits $\mathcal{C}_{k,e}$, $1 \leq k \leq q$ lie in the same polycycle $\mathscr{P}_e$, which is diffeomorphic to $\mathscr{P}$. The one-parameter family $\mathscr{P}_e$ is a (differentiable) deformation of $\mathscr{P}$ and we set

$$\widehat{\mathscr{P}} = \bigcup_{e\in J(\delta_0)} \mathscr{P}_e \subset H^{-1}(]e_0 - \delta_0, e_0 + \delta_0[).$$

Proposition 3.1.4. *An orientable ∞-level is continuable.*

Proof. Let $\mathscr{P}$ be an orientable ∞-level in $\mathscr{E}$ with hyperbolic orbit $\mathcal{C}$. For $e \in]e_0 - \delta_0, e_0 + \delta_0[$, we set $\mathscr{E}_e = H^{-1}(e)$ and $f_e = F_{|\mathscr{E}_e}$. We keep the notation of the proof of proposition 3.1.3.

• The submanifold of U of equation $p = q = 0$ is entirely foliated by periodic orbits $\mathcal{C}(I_0)$ of X^H, of equation

$$\varphi \in \mathbb{T},\ I = I_0,\ p = 0,\ q = 0,$$

for $I_0 \in]-\delta, \delta[$. One can moreover assume $\delta > 0$ small enough so that $I \mapsto H(I, 0) := e(I)$ is a diffeomorphism from $]-\delta, \delta[$ onto its image E, with inverse $e \mapsto I(e)$. We set $c(e) := F(I(e), 0)$ for $e \in E$. We first want to prove briefly that the connected component $\mathscr{P}_e$ of $f_e^{-1}(c(e))$ which contains $\mathcal{C}_e := \mathcal{C}(I(e))$ is an orientable ∞–level. For $0 < \delta' \leq \delta$, we set $U(\delta') = \Phi^{-1}(\mathscr{D}(\delta'))$.

• The complement $\mathscr{P} \setminus \mathcal{C}$ has two connected components, and by the normal form of proposition 3.1.3 this is also the case for $\mathscr{P} \setminus U(\delta/2)$. By the implicit function theorem, $\mathscr{P}_e \setminus U(\delta/2)$ also admits two connected components, which are smoothly varying with the energy e.

• Observe that

$$\mathscr{P}_e \cap U = \{\varphi \in \mathbb{T},\ I = I(e),\ J = 0\},$$

and that $\mathscr{P}_e \setminus U(\delta/2)$ is the subset of $\mathscr{P}_e \cap U$ with equation

$$|p| \geq \delta/2, \qquad |q| \geq \delta/2.$$

• An immediate gluing argument shows that $\mathscr{P}_e \setminus \mathcal{C}_e$ admits two connected components, and that $\mathscr{P}_e$ is an ∞–level for f_e (being compact and connected). So $\mathscr{P}_e$ is an orientable ∞–level, which depends smoothly on e in the sense of stratified manifolds. □

We are now in position to state the following definition.

Definition 3.1.1. We call *simple polycycle* a continuable polycycle that satisfies the following properties:

1. there exist an open subset O in $\mathbb{R}^2$, a neighborhood U of $\mathscr{P}$, saturated for F, and a diffeomorphism
$$\mathscr{B} : \mathbb{T} \times O \times]e_0 - \delta_0, e_0 + \delta_0[\to U$$
such that
   (a) the submanifold $V = \mathscr{B}^{-1}(\{0\} \times O \times]e_0 - \delta_0, e_0 + \delta_0[)$ is transverse to F,
   (b) $\mathscr{B}(\mathbb{T} \times O \times \{e\}) \subset H^{-1}(e), \quad \forall e \in]e_0 - \delta_0, e_0 + \delta_0[$,
   (c) $F(\mathscr{B}(\varphi, x, e)) = F(\mathscr{B}(\varphi', x, e)), \quad \forall (\varphi, \varphi', x, e) \in \mathbb{T}^2 \times O \times]e_0 - \delta_0, e_0 + \delta_0[$,

2. there exist two functions $\mathscr{I}$ and $\mathscr{J}$ in U, such that one can find coordinates $(\varphi_k, I_k, q_k, p_k)$ in a neighborhood U_k of each $\mathcal{C}_k$ such that $\mathscr{I}$ and $\mathscr{J}$ coincide with I_k and J_k.

Proposition 3.1.5. *An orientable ∞-level is a simple polycycle.*

Proof. Let $\mathscr{P}$ be an orientable ∞-level in $\mathscr{E}$ and denote by $\mathcal{C}$ its critical circles.

• Let $\mathscr{U}$ be the connected component of $(H,F)^{-1}(O)$ which intersects the submanifold U of equation $p = q = 0$ and, for any regular value $(e,\rho) \in O$ of (H,F), let $\mathcal{T}_{e,\rho}$ be the Liouville torus of equation $H = e$, $F = \rho$ which intersects U. Hence $\mathscr{U}$ is the union of the levels $\mathscr{P}_e$ for $e \in E$ and of the family $(\mathcal{T}_{e,\rho})$. The functions I and J immediately continue to $\mathscr{U}$ in a canonical way: if $z \in \mathscr{U}$, then $(H(z), F(z)) \in O$ and there exists a unique pair $(I,J) \in]-\delta,\delta[\times]-\delta^2,\delta^2[$ such that $(H(I,J), F(I,J)) = (e,\rho)$. One defines the continuations $\mathscr{I}$ and $\mathscr{J}$ as the functions

$$z \mapsto \mathscr{I}(z) = I, \qquad z \mapsto \mathscr{J}(z) = J.$$

Observe that the vector field $X^{\mathscr{I}}$ is 1–periodic, since it is indeed 1–periodic over the intersection of each Liouville torus $\mathcal{T}_{e,\rho}$ or each regular component of $\mathscr{P}_e$ with the domain U, by the normal form of proposition 3.1.3 (the periodicity everywhere is therefore a consequence of the Liouville theorem).

• Let $C^{\bullet}$ and $C_{\bullet}$ be the connected components of $\mathscr{P} \setminus \mathcal{C}$. Fix the point a of coordinate $\varphi = 0$ on $\mathcal{C}$, in the previous system. Using lemma 3.1.1, one can fix two curves $\sigma^{\bullet}$ and $\sigma_{\bullet}$ (orbits of $X^{\widehat{u}_0 H + \widehat{v}_0 F}$) on $C^{\bullet}$ and $C_{\bullet}$, whose α and ω limit sets are equal to $\{a\}$. Using the Liouville theorem, the normal form of proposition 3.1.3 and the continuation arguments for $\mathscr{I}$, one checks that these curves are transverse to the vector fields $X^{\mathscr{I}}$ and X^H on each connected component, and that they intersect each orbit of $X^{\mathscr{I}}$ only once. By usual transversality arguments, one can find 3–dimensional surfaces $\Sigma^{\bullet}$ and $\Sigma_{\bullet}$, containing $\sigma^{\bullet}$ and $\sigma_{\bullet}$, which are transverse to $X^{\mathscr{I}}$ and X^H. One can "glue and smooth" these hypersurfaces to the subset of equation $\varphi = 0$ in U. As a result, one obtains a 3–dimensional surface Σ, transverse to $X^{\mathscr{I}}$ and X^H, intersecting each orbit of $X^{\mathscr{I}}$ only once, such that $\Sigma \cap U = \{\varphi = 0\}$. One can moreover assume without loss of generality that Σ is saturated for H and F.

• The intersection $\Sigma \cap H^{-1}(e)$ is symplectic, being transverse to X^H. By usual deformation arguments one shows that, reducing Σ if necessary, there exists a diffeomorphism χ from $E \times \mathscr{O}$ onto Σ, where E is a suitable interval of $\mathbb{R}$ and $\mathscr{O}$ is an open subset of $\mathbb{R}^2$ (a "fattened eight"), such that $\chi(e, \,.\,)$ is symplectic for $e \in E$. Now, given $z \in \mathscr{U}$, taking the 1–periodicity of $X^{\mathscr{I}}$ into account, there exists a unique $\tau(z) \in \mathbb{T}$ such that $\Phi^{\tau(z)\mathscr{I}}(z) \in \Sigma$. One immediately checks that the map

$$z \mapsto \Big(\tau(z), H(z), \chi\big(H(z), \Phi^{\tau(z)\mathscr{I}}(z)\big)\Big)$$

is a symplectic diffeomorphism from $\mathscr{U}$ onto $\mathscr{M} = \mathbb{T} \times E \times \mathscr{O}$. From this and the previous remarks one easily deduces that an orientable ∞–level is a simple polycycle. $\square$

We emphasize the following property of simple polycycles.

Property 3.1.1. *Let $T_{k,e}$ be the period of the critical circles $\mathcal{C}_{k,e}$. Then $T_{k,e}$ only depends on e, that is, all the critical circles that lie in the same polycycle $\mathscr{P}_e$ have the same period.*

Proof. Fix k and consider the symplectic cylinder

$$\mathscr{C}_k := \bigcup_{e\in H(U_k)} \mathcal{C}_{k,e}.$$

We first observe that the restriction of H to $\mathscr{C}_k$ depends only on I_k. We set $H_{|\mathscr{C}_k}(\varphi_k, I_k) = h_k(I_k)$ and the vector field $X^{H_{|\mathscr{C}_k}}$ reads

$$\dot\varphi_k = h_k'(I_k), \quad \dot I_k = 0.$$

Therefore, since h_k is a diffeomorphism, $T_{k,e} := h_k'(\mathscr{I}_k(0,e))^{-1}$. Let $\chi_k := h_k^{-1}$. Then $T_{k,e} := \chi_k'(e)$. Now if $k' \neq k$, $\chi_{k'}(e) = I_{k'}(0,e) = \mathscr{I}(0,e) = I_k(0,e) = \chi_k(e)$, that is, χ_k does not depend on k. We denote it by χ. Then, for any $1 \leq k \leq p$, $T_{k,e} = \chi'(e)$. □

To conclude this section, it only remains to investigate the case of nonorientable ∞–levels. As we have stated at the begining of the chapter, we will show, that up to a two-sheeted cover, the dynamics in the neighborhood of a nonorientable ∞–level is the same as in the orientable case.

We say that a triple $(\mathscr{U}, H, F)$ where $\mathscr{U}$ is a neighborhood of an orientable ∞-level for H and F, chosen as in proposition 3.1.4, is an *orientable model.*

Proposition 3.1.6. *Let $\mathscr{P}$ be a non orientable ∞–level for the Hamiltonian H and the first integral F. Then there exist a neighborhood $\mathscr{U}$ of $\mathscr{P}$ in M, an orientable model $(\widetilde{\mathscr{U}}, \widetilde{H}, \widetilde{F})$ and a two sheeted symplectic covering $\eta : \widetilde{\mathscr{U}} \to \mathscr{U}$ such that $\eta_*\widetilde{H} = H$*

Proof. As above, and for the same reasons, one can continue the functions I and J of the normal form of proposition 3.1.3 to functions $\mathscr{I}$ and $\mathscr{J}$ defined over a suitable neighborhood $\mathscr{U}$ of $\mathscr{P}$. The main difference is that now the vector field $X^{\mathscr{I}}$ is everywhere 2–periodic, with minimal period 1 only on the critical orbits $\mathcal{C}_e$.

• As above, one can also find a transverse hypersurface Σ with the same properties as in the orientable case, except that Σ intersects each orbit of $X^{\mathscr{I}}$, which is not a critical orbit $\mathcal{C}_e$, exactly twice (observe that the *local* topological structure of a non orientable ∞–level is the same as that of an orientable one). One can assume without loss of generality that Σ is globally invariant under the time-one map $\Phi^{\mathscr{I}}$ and fixed by $\Phi^{2\mathscr{I}}$ (this amounts to using symmetric surfaces $\Sigma_\bullet$ and $\Sigma^\bullet$ from the very beginning of the construction). We denote by σ the involution of Σ defined by $\Phi^{\mathscr{I}}$.

• One gets a symplectic diffeomorphism from $\mathscr{U}$ onto a symplectic manifold $\mathscr{M}$, of the previous form

$$z \mapsto \Big(\tau(z), H(z), \chi\big(H(z), \Phi^{\tau(z)\mathscr{I}}(z)\big)\Big),$$

but now the range $\mathscr{M}$ is the quotient of the manifold $\mathbb{R} \times E \times \mathscr{O}$ by the $\mathbb{Z}$–action:

$$(m, (\tau, e, x)) \mapsto (\tau + m, e, \sigma^m(x)).$$

Clearly, the map $\mu : \mathbb{R} \times E \times \mathscr{O} \to \mathscr{M}$ defined by

$$\mu(\tau, e, x) \mapsto (2\tau, e, x)$$

passes to the quotient and yields a map $\mu : \mathbb{T} \times E \times O \to \mathscr{M}$ which is a double covering.

• Using the construction of the last section the reverse way, one gets an open symplectic manifold $\widetilde{\mathscr{U}}$, a Hamiltonian function $\widetilde{H}$ and a first integral $\widetilde{F}$ on $\mathscr{U}$ which admits an orientable ∞–level on $\widetilde{\mathscr{E}} := \widetilde{H}^{-1}(\{e_0\})$, such that the dynamics induced by H on $\mathscr{U}$ is a factor of that defined by $\widetilde{H}$ on $\widetilde{\mathscr{U}}$. More precisely, there exists a symplectic two-sheeted covering $\eta : \widetilde{\mathscr{U}} \to \mathscr{U}$ which satisfies the compatibility relation $\eta_* \widetilde{H} = H$. This proves our claim. $\square$

From now, and until the end of this thesis, we will assume that all the critical circles in an ∞-level or in a simple polycycle are hyperbolic orbits.

3.1.4 Maximal action-angle domains.

We denote by $\mathcal{R}(f)$ the set of regular values of f and by $\mathrm{Crit}(f)$ the set of its critical values. If $c \in \mathrm{Crit}(f)$, we denote by $\mathscr{R}_c$ the union of the connected components of $f^{-1}(\{c\})$ that does not contain any critical point. We define the *regular set* of f as

$$\mathscr{R} := f^{-1}(\{\mathcal{R}\}) \cup \left(\bigcup_{c \in \mathrm{Crit}(f)} \mathscr{R}_c \right).$$

We denote by $\mathscr{T}_c$ the set of all critical tori of f and we introduce the domain $\widehat{\mathscr{R}} := \mathscr{R} \cup \mathscr{T}_c$.

A connected component $\mathcal{A}$ of $\widehat{\mathscr{R}}$ satisfies the following properties:

- there exist $a, b \in \mathrm{Crit}(f)$ with $a < b$ and $\mathcal{A} = f^{-1}(]a, b[)$,
- for all $x \in]a, b[$, $f^{-1}(x) \cap \mathcal{A}$ is diffeomorphic to $\mathbb{T}^2$,
- there is a critical point of f in each connected component of $\partial \mathcal{A}$.

Therefore $\mathcal{A}$ is diffeomorphic to $\mathbb{T}^2 \times]0, 1[$. There exists a neighborhood $\widehat{\mathcal{A}}$ of $\mathcal{A}$ in M and a simply connected neighborhood B of $]a, b[\times \{e_0\}$ in $\mathbb{R}^2$ such that, for any $b \in B$, $(H, F)^{-1}(b) \cap \widehat{\mathcal{A}}$ is diffeomorphic to $\mathbb{T}^2$. Therefore, the construction "by quadrature" of action-angle variables due to Arnol'd (see Annex A) can be performed in $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}$ is a action-angle domain.

We say that $\mathcal{A}$ is a *maximal action-angle domain of* $(\mathscr{E}, \phi_H, f)$. The connected component of $\partial \mathcal{A}$ can be either an elliptic orbit, a Klein bottle or containded in a ∞-level.

3.2 The weak polynomial entropy $\mathrm{h}^*_{\mathrm{pol}}$

This section is devoted to the proof of the following theorem.

Theorem A*. *Let $(\mathscr{E}, \phi_H, f)$ be a dynamically coherent system. Then*

$$\mathrm{h}^*_{\mathrm{pol}}(\phi_H) \in \{0, 1\}.$$

3.2.1 Maps with contracting fibered structure

Definition 3.2.1. Let (E, d_E), (X, d_X) be compact metric spaces and consider two continuous maps $\varphi : E \to E$ and $\psi : X \to X$. We say that (E, φ) has a contracting fibered structure over (X, ψ) when the following conditions hold true.

(i) E is metrically fibered over X : there exists a surjective continous map $\pi : E \to X$, a metric space (F, d_F) and a finite open covering $(U_i)_{1 \le i \le m}$ of X such that for each i there exists an isometry $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ (this latter space being equipped with the product metric). We write $\phi_i(z) = (\pi_i(z), \varpi_i(z)) \in U_i \times F$.

(ii) (X, ψ) is a factor of (E, φ) relative to π: $\psi \circ \pi = \pi \circ \varphi$.

(iii) If z, z' are two points of E such that there exists i and j in $\{1, \dots, n\}$ such that $z, z' \in \pi_i^{-1}(U_i)$ and $\phi(z), \phi(z') \in \pi_j^{-1}(U_j)$, then

$$d_F(\varpi_j(\varphi(z)), \varpi_j(\varphi(z'))) \leq d_F(\varpi_i(z), \varpi_i(z')).$$

A simple example of a map with contracting fibered structure is the one of a diffeomorphism φ of a manifold M that admits a compact invariant manifold N which is normally hyperbolic: then its stable manifold $W^+(N)$ admits an invariant foliation by the stable manifolds of the points of N, and there exists a projection π from a neighborhood E of N in $W^+(N)$ to N which associate with each point x the unique point $a \in N$ such that $x \in W^+(a)$. It is not difficult to see that one can choose a Riemanniann metric on M and the neighborhood E is such a way that E is invariant under φ and (E, φ) admits a contracting fibered structure over (N, φ_N).

Proposition 3.2.1. *Let (E, d_E), (X, d_X) be metric spaces, and $\varphi : E \to E$, $\psi : X \to X$ be continuous maps, such that (E, φ) admits a contracting fibered structure over (X, ψ). Then*

$$\mathrm{h_{pol}}(\varphi) = \mathrm{h_{pol}}(\psi).$$

Proof. We already know that $\mathrm{h_{pol}}(\varphi) \geq \mathrm{h_{pol}}(\psi)$ by the factor property. To prove the converse inequality, consider a finite open covering $(U_i)_{i \in I}$ of X adapted to the fibered structure and let $\varepsilon_0 > 0$ be the Lebesgue number of this covering (so each set of diameter less than ε_0 for d_X is contained in one of the U_i).

Let now $N \geq 1$ be fixed, choose $\varepsilon < \varepsilon_0/2$ and consider a ball $B^X \subset X$ of d_N^ψ–radius less than ε. In particular, B^X has diameter less than ε_0 (for δ), so B^X is contained in an element U_{i_0} of the covering. Consider then a ball B^F of radius ε, in the fiber (F, d_F). As $B^X \subset U_{i_0}$, one can define the set

$$P = \phi_{i_0}^{-1}(B^X \times B^F).$$

We want to prove that P has diameter less than 2ε for the distance d_N^φ.

For z, z' in P, let $x = \pi_{i_0}(z)$ and $x' = \pi_{i_0}(x')$, so x and x' lie in B^X. Note that for $0 \leq k \leq N$, $\psi^k(B^X)$ has diameter less then ε_0 and so is contained in some open set U_{i_k} of the covering. So, for $0 \leq k \leq N-1$, the fibered structure yields the equality:

$$d_E(\varphi^k(z), \varphi^k(z')) = \mathrm{Max}\, \Big(d_X(\psi^k(x), \psi^k(x')), d_F(\varpi_{i_k}(\varphi^k(z)), \varpi_{i_k}(\varphi^k(z')))\Big).$$

Now by induction, using the inclusion $\psi^k(B^X) \subset U_{i_k}$:

$$d_F\Big(\varpi_{i_k}(\varphi^k(z)), \varpi_{i_k}(\varphi^k(z'))\Big) \leq d_F\Big(\varpi_{i_0}(z), \varpi_{i_0}(z')\Big) < 2\varepsilon$$

and on the other hand $d_X(\psi^k(x), \psi^k(x')) < 2\varepsilon$ since $x, x' \in B^X$, so

$$d_E(\varphi^k(z), \varphi^k(z')) < \varepsilon.$$

This proves that P has d_N^φ–diameter less than 2ε. We denote by $P(B^X, B^F)$ this set.

We now fix a minimal covering $B_1^X, \cdots, B_n^X)$ of X by balls of radius ε for d_N^ψ, and a finite covering $B_1^F \cdots, B_m^F)$ of the fiber F by balls of radius ε for d_F. To each pair (B_i^X, B_j^F), we associate the subset $P_{ij} = P(B_i^X, B_j^F)$ of E. It is easy to see that $(P_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ is a covering of E by subsets of diameter less than 2ε for d_N^φ. Then

$$G_N^\varphi(2\varepsilon) \leq m \cdot G_N^\psi(\varepsilon),$$

which yields $\mathrm{h_{pol}}(\varphi) \leq \mathrm{h_{pol}}(\psi)$. □

3.2.2 Proof of theorem A*.

We are now in a position to prove Theorem A*.

Proof of theorem A.* We assume without loss of generality that $H(\mathscr{E}) = 0$. For two values $a < b$ of f, we denote by $cc(f^{-1}(]a, b[))$ (resp $cc(f^{-1}(\{a\}))$) a connected component of $f^{-1}(]a, b[)$ (resp $f^{-1}(\{a\})$). Such a domain is obvioulsy invariant by ϕ_H. The strategy of the proof consists in choosing a suitable finite covering of $\mathscr{E}$, by such domains D and to compute $h^*_{pol}(\phi_H, D) := h^*_{pol}((\phi_H)_{|D})$ for each of them. We choose the domains such that only the following four different cases occur:

1. $cc(f^{-1}(]a, b[))$ is a maximal action angle domain.
2. there exists $c \in]a, b[$ such that $f^{-1}(c) \cap cc(f^{-1}(]a, b[))$ is a Klein bottle, and $]a, b[\setminus\{c\} \subset \mathcal{R}(f)$.
3. $cc(f^{-1}(\{a\}))$ is an elliptic orbit.
4. $cc(f^{-1}(\{a\}))$ is a ∞-level.

• **Cases (1) and (2):** Using remark 3.1.1 and property 2.2.1 2, we see that (2) boils down to (1). Let $\widehat{\mathcal{A}} \subset M$ be an action angle domain such that $cc(f^{-1}([a, b])) \subset \mathcal{A}$. There exists an open domain $U \subset \mathbb{R}^2$ and a symplectic diffeomorphism $\Psi : \mathcal{A} \to \mathbb{T}^2 \times U, m \mapsto (\theta, r)$ such that $r = R \circ (H, f)$ where R is a diffeomorphism between two open domains of $\mathbb{R}^2$. Moreover ϕ_H is conjugate to the Hamiltonian flow $\psi := (\psi_t)$ on $\mathbb{T}^2 \times U$ associated with $H \circ \Psi^{-1}$. One has $\Psi(cc(f^{-1}(]a, b[))) = \mathbb{T}^2 \times R(0,]a, b[)$. Consider the sequences $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$ defined by

$$a_n = a + \frac{1}{n}(b - a), \quad b_n = b - \frac{1}{n}(b - a)$$

For $n \geq 2$, we set $K_n := \Psi(cc(f^{-1}([a_n, b_n]))) = \mathbb{T}^2 \times R(0, [a_n, b_n])$. So

$$cc(f^{-1}(]a, b[)) = \bigcup_{n\geq 2} \Psi^{-1}(K_n).$$

By propositions 2.2.2 and 2.2.3, one gets

$$h^*_{pol}(\phi_H, cc(f^{-1}(]a, b[))) = \sup_{n\in\mathbb{N}} h^*_{pol}(\psi, \mathbb{T}^2 \times K_n) \in \{0, 1\}.$$

• **Case (3):** Let $\mathcal{C} := cc(f^{-1}(\{a\}))$ be the elliptic orbit. The time-one map of the flow ϕ_H restricted to $\mathcal{C}$ is conjugate to a rotation, so $h^*_{pol}(\phi_H, \mathcal{C}) = h_{pol}(\phi_H, \mathcal{C}) = 0$.

• **Case (4):** Let $\mathscr{P} =:= cc(f^{-1}(\{a\}))$ be the ∞-level. Let $\mathcal{C}$ be the hyperbolic orbit contained in $\mathscr{P}$ and denote by W^s its stable manifold. Then, as before, $h_{pol}(\phi_H, \mathcal{C}) = 0$. Now $\mathscr{P} \setminus \mathcal{C}$ has two connected components W_1 and W_2. For $i = 1, 2$, there exists a domain $D_i \subset W_i \cup \mathcal{C}$ such that

$$\mathcal{C} \subset D_i \subset W^s \quad \text{and} \quad W_i = \bigcup_{n\in\mathbb{N}} (\phi^1_H)^{-n}(D_i).$$

We can assume that D_i is small enough so that $(D_i, (\phi_H)_{|W_i\cup\mathcal{C}})$ admits a contracting fibered structure over $(\mathcal{C}, (\phi_H)_{|\mathcal{C}})$. Therefore $h_{pol}(\phi_H, D_i) = h_{pol}(\phi_H, \mathcal{C}) = 0$, which yields $h^*_{pol}(\phi_H, D_i) = 0$. Applying proposition 2.2.2, one gets

$$h^*_{pol}(\phi_H, \bigcup_{n\in\mathbb{N}} (\phi^1_H)^{-n}(D_i)) = 0,$$

and $h^*_{pol}(\phi_H, \mathscr{P}) = 0$. □

3.3 The strong polynomial entropy $\mathrm{h}_{\mathrm{pol}}$.

This section is devoted to the following main result of this chapter.

Theorem A. *Let $(\mathscr{E}, \phi_H, f)$ be a dynamically coherent system. Then*

$$\mathrm{h}_{\mathrm{pol}}(\phi_H) \in \{0,1,2\}.$$

Moreover, $\mathrm{h}_{\mathrm{pol}}(\phi_H) = 2$ if and only if ϕ_H possesses a hyperbolic orbit.

3.3.1 Sketch of proof

For two values $a < b$ of f, we denote by $cc(f^{-1}([a,b]))$ an arbitrary connected component of $f^{-1}([a,b])$. Such a domain is obviously invariant by ϕ_H. The strategy of the proof consists in choosing a suitable finite covering of $\mathscr{E}$ by such domains and to compute $\mathrm{h}_{\mathrm{pol}}(\phi_H, cc(f^{-1}([a,b])))$ for each of them. We choose the domains such that the following four different cases only occur:

1. $cc(f^{-1}([a,b]))$ is contained in a action-angle domain.
2. there exists $c \in]a,b[$ such that $f^{-1}(c) \cap cc(f^{-1}(]a,b[))$ is a Klein bottle, and $[a,b] \setminus \{c\} \subset \mathcal{R}(f)$
3. $f^{-1}(a) \cap cc(f^{-1}([a,b]))$ is an elliptic orbit and $]a,b] \subset \mathcal{R}(f)$.
4. $f^{-1}(a) \cap \overline{cc(f^{-1}(]a,b]))}$ is contained in an ∞-level and $]a,b] \subset \mathcal{R}(f)$.

• **Case (1) and (2)**. As before, we just have to study the case (1). By proposition 2.2.3, one can immediately conclude that:

$$\mathrm{h}_{\mathrm{pol}}(\phi_H, cc(f^{-1}([a,b])) \in \{0,1\}.$$

• **Case (3)**. We will use corollary 3.1.1 and proposition 2.2.3. We denote by $\mathcal{C}$ the elliptic orbit $\mathcal{C} = f^{-1}(\{a\}) \cap cc(f^{-1}([a,b]))$. We assume that $cc(f^{-1}([a,b]))$ is contained in a neighborhood U of $\mathcal{C}$ with coordinates (φ, I, q, p) as in proposition 3.1.2.

Let $O := I(U) \times J(U)$ and consider the flow ϕ on $\mathbb{T}^2 \times O$ associated with the vector field (3.1), defined in corollary 3.1.1. Then, by corollary 3.1.1,

$$\mathrm{h}_{\mathrm{pol}}(\phi_H, cc(f^{-1}([a,b]))) \leq \mathrm{h}_{\mathrm{pol}}(\phi^1, \pi^{-1}(cc(f^{-1}([a,b])))).$$

Now by proposition 2.2.3, $\mathrm{h}_{\mathrm{pol}}(\phi^1, \pi^{-1}(cc(f^{-1}([a,b])))) \in \{0,1\}$.

1) Assume that $\mathrm{h}_{\mathrm{pol}}(\phi^1, \pi^{-1}(cc(f^{-1}([a,b])))) = 0$, then $\mathrm{h}_{\mathrm{pol}}(\phi_H, cc(f^{-1}([a,b]))) = 0$.

2) Assume that $\mathrm{h}_{\mathrm{pol}}(\phi^1, \pi^{-1}(cc(f^{-1}([a,b])))) = 1$ and show that $\mathrm{h}_{\mathrm{pol}}(\phi_H, f^{-1}([a,b])) = 1$. It suffices to show that $\mathrm{h}_{\mathrm{pol}}(\phi_H, f^{-1}([a,b])) \geq 1$. We consider (I,J) as functions on the values e, ρ of H and F and conversely, we consider H as functions on the variables I, J. We set

$$\mathcal{S} : \{(I(0,\rho), J(0,\rho)) \,|\, \rho \in [a,b]\}, \quad \text{and} \quad \omega : \mathcal{S} \to \mathbb{R}^n : (I,J) \mapsto d_{(I,J)}(H_{|\mathcal{S}}).$$

Since $\mathrm{h}_{\mathrm{pol}}(\phi^1, \pi^{-1}(cc(f^{-1}([a,b])))) = 1$, by proposition 2.2.3, there exists a value $c \in [a,b]$ such that $\operatorname{rank} \omega(I(0,c), J(0,c)) = 1$. By lower semi-continuity of the rank , there exists a neighborhood $V \subset [a,b]$ of c such that for each $c' \in V$, $\operatorname{rank}\omega(I(0,c'),J(0,c')) = 1$. So we can assume that $c \in]a,b]$. Fix $\varepsilon > 0$ such that $[c-\varepsilon, c+\varepsilon] \subset [a,b]$. Then

$$\mathrm{h}_{\mathrm{pol}}(\phi_H, f^{-1}([a,b])) \geq \mathrm{h}_{\mathrm{pol}}(\phi_H, f^{-1}([c'-\varepsilon, c'+\varepsilon])) = 1,$$

the last equality coming from proposition 2.2.3.

• **Case (4)**. We first observe that, as in the case of Klein bottles, by 2.2.1 2., it suffices to study the case where $\mathscr{P}$ is orientable. We will indeed study the more general case where $\mathscr{P}$ is a simple polycycle.

Given a simple polycycle $\mathscr{P}$ contained in $f^{-1}(0)$, there are regular values $\pm a$ of f and a neighborhood $\mathscr{V} \subset f^{-1}(]-a,a[)$ of $\mathscr{P}$ in $\mathscr{E}$ such that each connected component of $f^{-1}(]0,\pm a[) \cap \mathscr{V}$ is contained in a maximal action-angle domain. Given such a domain, $\mathscr{D}$, then $\overline{\mathscr{D}} \cap \mathscr{P}$ is a stratified submanifold of $\mathscr{E}$, which is the "ordered" union of a finite number of hyperbolic orbits, $\mathcal{C}_1, \ldots, \mathcal{C}_n$, and cylinders $W_{k,k+1}$ for $1 \leq k \leq n$, such that $W_{k,k+1}$ is one common connected component of $W^-(\mathcal{C}_k)$ and $W^+(\mathcal{C}_{k+1})$ (with the convention $n+1 = 1$, see [Mar09] for some details in the ordering in the planar case).

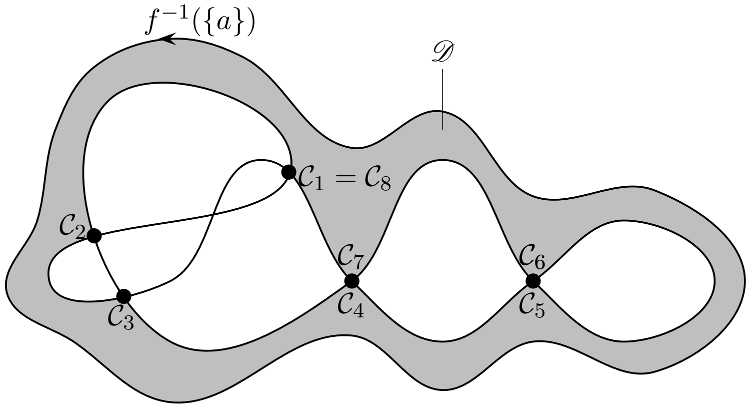


Figure 3.1

Since $\mathscr{P}$ is simple, by definition, one can choose $a > 0$ small enough so that for each domain $\mathscr{D}$ as above, setting $\mathscr{D}_a = \mathscr{D} \cap f^{-1}(]0, \pm a])$ (according to the initial sign), there exists a homeomorphism

$$\chi : \overline{\mathscr{D}_a} \to \mathbb{T}^2 \times [0,1]$$

which is smooth outside the union of the hyperbolic orbits contained in $\overline{\mathscr{D}_a} \cap \mathscr{P}$ and such that $\chi^{-1}(\mathbb{T}^2 \times \{c\})$ is a Liouville torus for each $c \in]0,1]$. Such a domain $\overline{\mathscr{D}_a}$ will be called a *partial neighborhood* for $\mathscr{P}$ and such a homeomorphism χ will be called a *compatible* homeomorphism.

We remark that if a is small enough, there exists a finite set of such partial neighborhoods $\overline{\mathscr{D}_a}$ whose union cover $\mathscr{P}$.

We will choose a suitable finite covering of a neighborhood of $\mathscr{P}$ by partial neighborhoods and we will conjugate the flow ϕ_H restricted to any of them of $\mathscr{P}$ to the flow ϕ of a "model" system on $\mathscr{A} := \mathbb{T}^2 \times [0,1]$ for which we will be able to estimate the polynomial entropy.

In section 3.3.2, we define the model system ϕ and construct a conjugacy between ϕ and the restriction of ϕ_H to a partial neighborhood. In section 3.3.3, we show that $\mathrm{h}_{\mathrm{pol}}(\phi) = 2$.

3.3.2 Construction of the conjugacy to a p-model system.

Consider the compact annulus $\widehat{\mathscr{A}} := \mathbb{T} \times [0,1]$ with coordinates (θ, r).

If $\psi = (\psi^t)_{t\in\mathbb{R}}$ is a flow on $\widehat{\mathscr{A}}$ whose orbits are the circles $\mathbb{T} \times \{r\}$, we define the *separation function* for two points $a = (\theta, r)$ and $a' = (\theta', r)$ on the same orbits as follows. Consider a lift $\widetilde{\psi} := (\widetilde{\psi}^t)$ of ψ to $\mathbb{R} \times [0,1]$ and two lifts $\tilde{a}, \tilde{a}'$ of a, a' located in the same fundamental domain of the covering. We set $\widetilde{\psi}^t(\tilde{a}) = (x(t), r)$ and $\widetilde{\psi}^t(\tilde{a}') = (x'(t), r)$. Then the separation function of a and a' is the function $E_{a,a'} : \mathbb{R} \to \mathbb{R}$ defined by

$$E_{a,a'}(t) = |x'(t) - x(t)|.$$

Obviously, $E_{a,a'}$ is independent of the lifts. It is a smooth, nonnegative and periodic function.

Notation 3.3.1. *If $\psi = (\psi^t)_{t\in\mathbb{R}}$ is a flow on a set X, we will often write $\psi(t, x)$ instead of $\psi^t(x)$.*

We define a *fundamental domain* for the flow ψ on $\widehat{\mathscr{A}}$ as a subset $\mathscr{K}$ of $\widehat{\mathscr{A}}$ of the form $\psi([0,1], \Delta) = \cup_{t\in[0,1]}\psi^t(\Delta)$, where Δ is a vertical segment of equation $\theta = \theta_0$.

Fix $p \in \mathbb{N}^*$. For $1 \le k \le p$, we set $z_k := (\frac{k}{p}, 0)$ and $\mathscr{O}_k := \{|\frac{k}{p} - \theta| < \frac{1}{8p}\}$.

Definition 3.3.1. We call *planar p–model* on $\widehat{\mathscr{A}}$ any continuous flow $\psi := (\psi_t)_{t\in\mathbb{R}}$ that satisfies the following conditions:

- (C1) If $r > 0$, the orbit of any point (θ, r) is the circle $\mathbb{T} \times \{r\}$ and there exists $\ell > 0$ such that, for any lift $\widetilde{\psi} := (\widetilde{\psi}^t)$ of ψ in $\mathbb{R}\times]0,1]$, and any $(x, r) \in \mathbb{R}\times]0,1]$, one has

$$0 < \frac{x(t) - x(t')}{(t - t')} \le \ell, \quad t' \neq t \in \mathbb{R},$$

  where $(x(t), r) = \widetilde{\psi}^t(x, r)$.

- (C2) There exists a neighborhood $\mathscr{O}_k$ of z_k such that the restriction of ψ to $\mathscr{O}_k$ is a flow associated with a vector field of the form

$$V_k(\theta, r) = \lambda_k(r)\sqrt{\left(\theta - \frac{k}{p}\right)^2 + \mu_k(r)}\frac{\partial}{\partial\theta} \tag{3.2}$$

  where λ_k and μ_k are positive C^1 functions on $[0,1]$ with $\lambda_k(0) > 0$, $\mu_k(0) = 0$ and $\mu_k'(0) > 0$. Moreover if we set $\psi^t(\theta, 0) = (\theta(t), 0)$, then $(\theta(t) - \theta(t'))(t - t') > 0$, for any $\theta \in \mathbb{T} \setminus \{\frac{k}{p}, 1 \le k \le p\}$.

- (C3) **Torsion condition**: If $\widetilde{\Psi} := (\widetilde{\psi}^t)_{t\in\mathbb{R}}$ is any lift of Ψ in $\mathbb{R} \times [0,1]$, one has, for any $x \in \mathbb{R}$ and $0 \le r_1 < r_2 \le 1$:

$$x_1(t) < x_2(t)$$

  where $(x_i(t), r_i) = \widetilde{\psi}^t(x, r_i)$.

- (C4) **Tameness condition**: There exists a fundamental domain $\mathscr{K}$ for ψ such that, given two points a and a' in $\mathscr{K}$ on the same orbit, there exists t_0 such that $E_{a,a'}(t_0)$ is maximum and the points $\psi^{t_0}(a)$ and $\psi^{t_0}(a')$ are located inside the domain $\mathscr{K}$.

Let us comment this definition. The conditions (C1) and (C2) say that each orbit with positive r is periodic and that the points z_k are the only fixed points of the flow. The torsion condition says that the vertical is twisted to the right by the maps $\widetilde{\psi}^t$ for $t > 0$. The tameness condition is essentially a technical condition that facilitate the computation of $\mathrm{h}_{\mathrm{pol}}$. Due to the torsion condition, the period $T(r)$ of the periodic orbit $\mathbb{T} \times \{r\}$, $r > 0$, is a decreasing function of r, so that the minimal period T^* is achieved when $r = 1$.

Definition 3.3.2. Let α be a C^1 positive function and let ψ be a planar p-model flow on $\mathbb{T} \times [0, 1]$. The *p-model system on $\mathscr{A} := \mathbb{T}^2 \times [0, 1]$ associated with α and ψ* is the continuous flow $\alpha \otimes \psi : ((\alpha \otimes \psi)^t)_{t \in \mathbb{R}}$ defined by

$$(\alpha \otimes \psi)^t(\varphi, \theta, r) = (\varphi + t\alpha(r)\,[\mathbb{Z}], \psi^t(\theta, r)).$$

We call *minimal period* of the p-model system the minimal period T^* of the associated planar p-model flow.

The following proposition, stated here a little bit improperly, will be made precise and proved in section 3.3.2.

Proposition 3.3.1. *With the previous assumptions, given a p-model system on $\mathscr{A}$ with large enough minimal period, then* $\mathrm{h}_{\mathrm{pol}}(\alpha \otimes \psi) = 2$.

We say that a partial neighborhood $\overline{\mathscr{D}_a}$ of a simple polycycle is a *desingularization domain* if there exists a compatible homeomorphism $\chi : \overline{\mathscr{D}_a} \to \mathbb{T}^2 \times [0, 1]$ which conjugates ϕ_H to the flow $\alpha \otimes \psi$ of a p-model on $\mathscr{A}$, that is,

$$\forall (t, z) \in \mathbb{R} \times \overline{\mathscr{D}_a}, \quad \chi \circ \phi_H(t, z) = \alpha \otimes \psi(t, \chi(z)).$$

We can now state our main result.

Theorem A-bis *Given a simple polycycle $\mathscr{P}$ of the dynamically coherent system $(\mathscr{E}, \phi_H, f)$, there exists $a > 0$ small enough so that any partial neighborhood $\overline{\mathscr{D}_a}$ is a desingularization domain.*

Definition 3.3.3. Given a partial neighborhood $\overline{\mathscr{D}_a}$ for $\mathscr{P}$ and a positive function $\tau : [0, a] \to \mathbb{R}$, we call *proper section associated with* τ a 2–dimensional C^1 submanifold S of $\overline{\mathscr{D}_a}$ such that for each $z \in \overline{\mathscr{D}_a} \setminus S$, there exists a unique pair $(t_z^-, t_z^+) \in \mathbb{R}^{*-} \times \mathbb{R}^{*+}$ with $t_z^+ - t_z^- = \tau(f(z))$ and $\Phi_H(]t_z^-, t_z^+[, z) \cap \Sigma = \emptyset$, and such that the Poincaré map defined for each $z \in S$ by

$$\wp(z) = \Phi_H(\tau(f(z)), z)$$

is a homeomorphism of S. By C^1 submanifold we mean here a C^0 submanifold whose intersection with $\mathscr{D}_a$ is C^1.

Given a planar p–model $(\mathbb{T} \times [0, 1], \psi)$ and a continuous positive function $\alpha : [0, 1] \to \mathbb{R}$, the associated time-$\alpha$ map is the map $\psi^\alpha : \mathbb{T} \times [0, 1] \to \mathbb{T} \times [0, 1]$ such that

$$\psi^\alpha(\theta, r) = \psi(\alpha(r), (\theta, r)).$$

The proof of Theorem A-bis will rely on the following two lemmas.

Lemma I. *Let T be the common period of the hyperbolic orbits contained in $\mathscr{P}$. Let $\overline{\mathscr{D}_a}$ be a partial neighborhood of $\mathscr{P}$. Then, there exists a proper section in $\mathscr{D}_a$ associated with a C^1 function $\tau : [0, a] \to \mathbb{R}_+^*$ such that* $\lim_{\rho \to 0} \tau(\rho) = T$.

Lemma II. *Let $\overline{\mathscr{D}_a}$ be a partial neighborhood of $\mathscr{P}$ and let S be a proper section associated with τ. Set $\alpha : [0,1] \to \mathbb{R} : r \mapsto \tau(ar)$. Then the return map $\wp$ of S is C^0-conjugated to the time-α map of a planar p-model $(\mathbb{T} \times [0,1], \psi)$.*

a. Normal coordinates in the neighborhood of hyperbolic orbits

Fix a hyperbolic orbit $\mathcal{C}_k$ in $\overline{\mathscr{D}_a}$. Set $\mathscr{U}_k : \mathscr{U}_{k,e_0} = U_k \cap \mathscr{E}$ as defined in corollary 3.1.3 with coordinates (φ_k, q_k, p_k). The vector field X^H reads

$$\begin{aligned} \dot\varphi_k &= \frac{\partial H}{\partial I_k}(I_k, J_k) & &= \frac{\partial H}{\partial I_k}(\mathscr{I}(q_k p_k), q_k p_k) \\ \dot q_k &= -\frac{\partial H}{\partial J_k}(I_k, J_k)\frac{\partial J_k}{\partial q_k}(J_k) & &= -p_k \frac{\partial H}{\partial J_k}(\mathscr{I}(q_k p_k), q_k p_k) \\ \dot q_k &= \frac{\partial H}{\partial J}(I_k, J_k)\frac{\partial J_k}{\partial p_k}(J_k) & &= q_k \frac{\partial H}{\partial J_k}(\mathscr{I}(q_k p_k), q_k p_k) \end{aligned}$$

Let $\bar f : J_k \mapsto F(\mathscr{I}_k(J_k, e_0), J_k)$

$$\partial^2 f = \begin{vmatrix} q^2 \bar f'(q_k p_k) & q_k p_k \bar f''(q_k p_k) \\ p^2 \bar f'(q_k p_k) & p_k q_k \bar f''(q_k p_k) \end{vmatrix}.$$

Since the surface $\mathscr{U}_k \cap \{(0, q_k, p_k) \,|\, (q_k, p_k) \in \mathscr{D}\}$ is transverse to $\mathcal{C}_k$ in $\mathscr{E}$, the determinant above does not vanish and $\bar f'(0) \neq 0$.

Denoting by $\bar f^{-1}$ the inverse of $\bar f : J_k \mapsto f(I_k(J_k, e_0), J_k)$, we set

$$\omega_k : J_k \mapsto \frac{\partial H}{\partial I}(\mathscr{I}(\bar f^{-1}(J_k)), \bar f^{-1}(J_k)), \quad \lambda_k : J_k \mapsto \frac{\partial H}{\partial J}(\mathscr{I}(\bar f^{-1}(J_k)), \bar f^{-1}(J_k)).$$

One easily checks that these functions are C^1.

By remark 3.1.2, $\lambda_k \neq 0$. Permutating p_k and $-p_k$ and q_k and $-q_k$ if necessary, we can assume that $\overline{\mathscr{D}_a} \cap \mathscr{U}_k$ is defined by $p_k \geq 0, q_k \geq 0$, for any $1 \leq k \leq p$ and that $\lambda_k > 0$. Therefore, $\overline{\mathscr{D}_a} \cap \mathscr{U}_k$ can be parametrized by $(\varphi_k, u = q_k - p_k, \rho = J_k(p_k q_k))$.

Since $q_k + p_k = \sqrt{(q_k - p_k)^2 + 4 q_k p_k} = \sqrt{u^2 + 4J_k^{-1}(\rho)}$ and setting $\mu_k(\rho) = 4J_k^{-1}(\rho)$, the vector field reads

$$\dot\varphi_k = \omega_k(\rho), \quad \dot u := \lambda_k(\rho)\sqrt{u^2 + \mu_k(\rho)}, \quad \dot\rho = 0. \tag{$\star$}$$

In the following, we denote by $\mathscr{U}_k$ the domain contained in $\mathscr{U}_k \cap \overline{\mathscr{D}_a}$ defined by $|u| \leq \overline{u}$, for $\overline{u} > 0$ small enough independent of k.

b. Construction of the proper section: proof of Lemma I.

This section is devoted to the proof of lemma I. Since a partial neighborhood is the closure of an action-angle domain, we will have to study the proper sections for action-angle systems. Since action-angle systems admit a foliation by invariant Kronecker tori, we begin by studying the proper sections (suitably defined) for Kronecker flows on $\mathbb{T}^2$.

b-1. *Proper sections for minimal Kronecker flows.* Consider a constant vector field $X = (x_1, x_2)$ with $x_2 \neq 0$ on $\mathbb{T}^2$. We set $\Phi : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{T}^2 : (t, \theta) \mapsto \phi^t(\theta)$. We denote by π the canonical projection $\mathbb{R}^2 \to \mathbb{T}^2$.

Definition 3.3.4. A closed curve $S \subset \mathbb{T}^2$ transverse to X is a *proper section* if

- for all $\theta \in \mathbb{T}^2 \setminus S$, there exists $(t_\theta^-, t_\theta^+) \in \mathbb{R}_-^* \times \mathbb{R}_+^*$ such that $\phi^{t_\theta^+}(\theta) \in S$, $\phi^{t_\theta^-}(\theta) \in S$ and $\Phi(]t_\theta^-, t_\theta^+[, \theta) \cap S = \emptyset$,
- the number $\tau := t_\theta^+ - t_\theta^-$ is independent of θ,

We say that τ is the *transition time* associated with S.

We denote by $\mathcal{D}_r$ the set of vector lines in $\mathbb{R}^2$ with rational slope. For $(q,p) \in \mathbb{Z} \times \mathbb{N}$, we denote by $D_{q,p}$ the vector line with direction vector (q,p). Let $\mathscr{S}$ be the subset of $\mathbb{Z} \times \mathbb{N}$ defined by $(q,p) \in \mathscr{S}$ if $|q| \wedge p = 1$. Obviously, the map $\mathcal{D}_r \to \mathscr{S} : D_{q,p} \mapsto (q,p)$ is bijective.

Proposition 3.3.2. *For any $(q,p) \in \mathscr{S}$ such that $X \notin D_{q,p}$, the projection $\pi(D_{q,p})$ is a proper section.*

Proof. Fix $(q,p) \in \mathscr{S}$ and let us study the dynamics in the lift $\mathbb{R}^2$ of $\mathbb{T}^2$. One has:

$$\begin{aligned}\pi^{-1}(\pi(D_{q,p})) = & \bigcup_{(m,n)\in\mathbb{Z}^2} ((m,n) + D_{q,p}) && = \bigcup_{(m,n)\in\mathbb{Z}^2} \left\{ \left(x, \frac{p}{q}(x-m) + n \right) \mid x \in \mathbb{R} \right\} \\ = & \bigcup_{(m,n)\in\mathbb{Z}^2} \left((0, n - \frac{p}{q}m) + D \right) && = \bigcup_{n\in\mathbb{Z}} \left((0, \frac{n}{q}) + D_{q,p} \right),\end{aligned}$$

the last equality coming from $\mathbb{Z} + \frac{p}{q}\mathbb{Z} = \frac{1}{q}\mathbb{Z}$ since $(q,p) \in \mathscr{S}$. For $n \in \mathbb{Z}$, we denote by D_n the affine line $(0, \frac{n}{q}) + D_{q,p}$. Let $z \in \pi^{-1}(\pi(D_{q,p}))$. Let $m \in \mathbb{Z}$ such that $z \in D_m$. The time τ needed to come back to $\pi^{-1}(\pi(D_{q,p}))$ following the orbit $z + \mathbb{R}X$ is the time needed to cut the line D_{m+1} or the line D_{m-1}. This time is independent of the choice of z on D_m and of the choice of $m \in \mathbb{Z}$. A simple computation yields $\tau = |x_2 q - p x_1|^{-1}$. From this, one immediately deduces that for any $z \in \mathbb{R}^2$, $t_z^+ - t_z^- = \tau$. $\square$

b-2. *Proper sections for action-angle systems on $\mathbb{T}^2 \times \mathbb{R}^2$.* Let O be an open domain in $\mathbb{R}^2$ and let $h : O \to \mathbb{R}$ be a C^2 function. Consider the Hamiltonian system X^H on $\mathbb{T}^2 \times O$ defined by $H(\theta, r) = h(r)$. We denote by ϕ_H its associated flow. Fix a regular value e of h and set $\mathcal{H}_e := h^{-1}(\{e\})$. For $r \in O$, we set $\mathcal{T}_r := \mathbb{T}^2 \times \{r\}$. The torus $\mathcal{T}_r$ is ϕ_H-invariant and X^H is constant on $\mathcal{T}_r$, so it can be canonically identified with an element of $\mathbb{R}^2$. We denote by $\pi_r : \mathbb{R}^2 \to \mathcal{T}_r$ the canonical projection.

Definition 3.3.5. Let $D \in \mathcal{D}_r$ and assume that $X^H(r) \notin D$, $\forall r \in O$.

1) Let $\Theta : O \to \mathbb{T}^2$ be a smooth map and set $\mathscr{L} := \{(\Theta(r), r) \mid r \in O\} \subset \mathbb{T}^2 \times O$. The *proper section associated with $\mathscr{L}$ and D* is the submanifold:

$$\widehat{S} := \bigcup_{r\in O} (\Theta(r) + \pi_r(D)).$$

2) Let $\Theta_e : \mathcal{H}_e \to \mathbb{T}^2$ be a smooth map and set $\mathscr{L}_e := \{(\Theta_e(r), r) \mid r \in \mathcal{H}_e\} \subset \mathbb{T}^2 \times \mathcal{H}_e$. The *proper section associated with $\mathscr{L}_e$ and D* is the submanifold:

$$S := \bigcup_{r\in\mathcal{H}_e} (\Theta_e(r) + \pi_r(D)).$$

Remark 3.3.1. Fix r in $\mathcal{H}_e$. For any $\theta \in \mathcal{T}_r$, the circle $\Theta_e(r) + \pi_r(D)$ is a proper section for the Kronecker flow induced by ϕ_H on $\mathcal{T}_r$. We denote by $\tau(r)$ its associated transition time. Obviously, the function $\tau : r \mapsto \tau(r)$ is smooth. We say that τ is the *transition function* associated with S.

b-3. *Proper sections in a partial neighborhood of a simple polycycle.* Now we go back to our Bott system and our simple polycyle $\mathscr{P}$, with its neighborhood U endowed with globally defined functions $\mathscr{I}$ and $\mathscr{J}$. Observe that

$$\mathscr{I}(z) = \int_{C(z)} \lambda, \tag{3.3}$$

where $C(z)$ is any circle $\mathcal{C}_{k,e}(\rho)$ such that $z \in \mathcal{T}_{e,\rho}$. This function is well defined. Obviously, $\mathscr{I}$ only depends on the values e and ρ of H and F. By construction its vector field $X^{\mathscr{I}}$ is 1-periodic and the critical circles $\mathscr{C}_{k,e}$ are orbits of its flow $(\phi^t_{\mathscr{I}})$.

Consider the partial neighborhood $\mathscr{V} \subset \widetilde{\mathscr{U}}$ of $\mathscr{P}$ in M that contains $\overline{\mathscr{D}}_a$, that is, $\mathscr{V} \cap U_k = \widetilde{\mathscr{U}} \cap \{p_k q_k \geq 0\}$ (for a suitable compatibe choice of the variables p_k, q_k). Assume that we got another function A defined on $\mathscr{V}$ such that:

- A only depends on the values e and ρ of H and F,
- A is independent of $\mathscr{I}$,
- A generates a 1-periodic flow (ϕ^t_A).

Then, the pair $(\mathscr{I}, A)$ is a pair of *action variables* as defined in [Dui80] and we can construct a symplectic diffeomorphism $\Psi : \mathring{\mathscr{V}} \to \mathbb{T}^2 \times B : z \mapsto (\theta_i, \theta_a, \mathscr{I}, A)$, where $\mathring{\mathscr{V}} = \mathscr{V} \cap \{p_k q_k > 0\}$ and B is an open domain in $\mathbb{R}^2$ (see Appendix A for the construction in a general case). As a consequence, if $\widetilde{H} := H \circ \Psi^{-1}$ and if $(\tilde{\phi}^t)$ is the Hamiltonian flow associated with $\widetilde{H}$ in $\mathbb{T}^2 \times B$, then $\Psi \circ \phi^t_H = \tilde{\phi}^t \circ \Psi$, for all $t \in \mathbb{R}$.

Proposition 3.3.3. *Fix $u^* > 0$ and set $\zeta_1 := \{0\} \times \{-u^*\} \times [0, a] \subset \mathscr{U}_1$. Assume that X^H is transverse to $\phi_A([0, T], \zeta_1)$. Then $\phi_A([0, T], \zeta_1)$ is a proper section for ϕ_H in $\mathscr{D}_a$ associated with a C^1 function $\tau :]0, a] \to \mathbb{R}^*_+$.*

Proof. We begin with showing that $\Psi(\phi_A([0, 1], \zeta_1))$ is a proper section for $\tilde{\phi}$ in $\Psi(\mathscr{D}_a)$. The restrictions on $\mathscr{D}_a$ of $\mathscr{I}$ and A only depend on the values ρ of F and we write $\mathscr{I}(\rho)$, $A(\rho)$. Since ζ_1 is only parametrized by ρ, $\Psi(\zeta_1)$ as the following graph form:

$$\Psi(D) = (\theta_i(\mathscr{I}(\rho), A(\rho)), \theta_a(\mathscr{I}(\rho), A(\rho)), \mathscr{I}(\rho), A(\rho)).$$

Set $B_{e_0} := \{(\mathscr{I}(\rho), A(\rho))) \mid \rho \in]0, a]\}$. Consider A as a function on $\mathbb{T}^2 \times B_{e_0}$ and let $\tilde{\phi}_A$ be the Hamiltonian flow associated with A in $\mathbb{T}^2 \times B_{e_0}$. Then

$$\tilde{\phi}_A([0, T], (\theta_i(\mathscr{I}, A), \theta_a(\mathscr{I}, A), \mathscr{I}, A)) = \bigcup_{\mathscr{I}, A} \{(\theta_i(\mathscr{I}, A), \theta_a(\mathscr{I}, A)) + \pi_{\mathscr{I}, A}(D_{0,1})\} \times \{\mathscr{I}, A\}.$$

Let us denote by $\tilde{\tau}$ the transition function associated with $\Psi(\phi_A([0, 1], \zeta_1)$. Obviously, since Ψ conjugates the flows ϕ_H and $\tilde{\phi}$, $\phi_A([0, 1], \zeta_1)$ is a proper section for ϕ_H with well defined transition function $\tau(f(z)) = \tilde{\tau}(\mathscr{I}(f(z)), A(f(z)))$, thanks to the transversality assumption . $\square$

We will now construct such an action variable A. Then we show that the proper section got in the previous proposition has a well defined continuation to the polycycle $\mathscr{P}$, with a C^1 transition function τ.

b-3.1. *Construction of A.* The construction of A is based on the following lemma.

Lemma 3.3.1. *There exists a 3-dimensional submanifold Π in $\mathscr{V}$ which is transverse to $\widehat{\mathscr{P}} := \bigcup_{e \in [e_0 - \delta_0, e_0 + \delta_0]} \mathscr{P}_e$ and such that for any $1 \leq k \leq p$, $\Pi \cap U_k := \{\varphi_k = 0\}$.*

Proof. Consider the submanifold $\widehat{\Pi} := \mathscr{B}(\{0\} \times O \times]e_0 - \delta_0, e_0 + \delta_0[)$ (see the definition of a simple polycycle). Let $k \in \{1 \ldots, p\}$. Since $\widehat{\Pi}$ is transverse to F, $\widehat{\Pi}$ is transverse to $\mathcal{C}_k$. We set $(\varphi_k^0, I_k(e_0, 0), 0, 0) := \widehat{\Pi} \cap \mathcal{C}_k$. Consider the symplectic diffeomorphism $(\varphi_k, I_k, q_k, p_k) \mapsto (\varphi_k - \varphi_k^0, I_k, q_k, p_k)$ in U_k. We still denote by φ_k the first variable. In these new coordinates, $\Pi \cap \mathcal{C}_k := (0, I_k(e_0, 0), 0, 0)$. By transversality, there exists a neighborhood $V_k \subset U_k$ in which $\widehat{\Pi}$ has the following graph form:

$$\widehat{\Pi} \cap V_k := \{(\varphi_k(I_k, p_k, q_k), I_k, p_k, q_k)\}.$$

Consider the symplectic diffeomorphism

$$(\varphi_k, I_k, q_k, p_k) \mapsto (\varphi_k, u_k := \frac{1}{\sqrt{2}}(q_k - p_k), v_k := \frac{1}{\sqrt{2}}(q_k + p_k)). \tag{3.4}$$

Fix $\overline{u} > 0$ such that $\{(\varphi_k, I_k, u_k, v_k) \in U_k \,|\, |u_k| \leq \overline{u}\} \subset V_k$. Let η_k be a bump function on U_k with support in the domain $\{u \leq \overline{u}\} \subset V_k$ and consider the submanifold Π defined by

$$\Pi \cap U_k := \{(1 - \eta_k)\varphi_k(I_k, p_k, q_k), I_k, p_k, q_k)\}, \quad 1 \leq k \leq p,$$

and which coincides with $\widehat{\Pi}$ outside the U_k. One easily checks that Π satisfies the hypotheses of the lemma. $\square$

For $(\mathscr{J}, e) \in \mathscr{J}(\mathscr{V}) \times H(\mathscr{V}) \setminus \{(\mathscr{J}(\rho(e), e) \,|\, e \in]e_0 - \delta_0, e_0 + \delta_0[\}$, we set

$$\gamma_{\mathscr{J},e} := \mathcal{T}_{e,\rho} \cap \Pi,$$

where $\mathcal{T}_{e,\rho}$ is the Liouville torus with $\mathscr{J}(\rho, e) = \mathscr{J}$. For $z \in \mathscr{V}$, we set $\gamma(z) := \gamma_{\mathscr{J}(z),H(z)}$.

The function

$$\begin{aligned} A : \mathscr{V} &\longrightarrow \mathbb{R} \\ z &\longmapsto \int_{\gamma(z)} \lambda \end{aligned} \tag{3.5}$$

is well defined and C^2. Obviously, A only depends on e and ρ and one immediately checks that A is independent of $\mathscr{J}$. By construction its vector field X^A is 1-periodic. We denote by S the proper section given by proposition 3.3.3 and by τ its associated transition function.

b-3.1. *Continuation of S to $\mathscr{P}$.* For $0 < u < \overline{u}$, and $1 \leq k \leq p$, we define the surfaces $\Gamma_k^-(u) := \{u_k = u\} \subset U_k \cap \mathscr{D}_a$ and $\Gamma_k^+(-u) := \{u_k = -u\} \subset U_k \cap \mathscr{D}_a$. The continuation of S in $\mathscr{P}$ necessitates five steps.

• **Step 1:** *For $1 \leq k \leq p$, there exists $u_k^* > 0$ such that the Poincaré maps associated with ϕ^A between the surfaces $\Gamma_k^+(-u_k^*)$ and $\Gamma_k^-(u_k^*)$ are well defined and read*

$$P_k(\varphi_k, u_k^*, \rho) = (\varphi_k + \vartheta_k(\rho), u_k^*, \rho) \quad 1 \leq k \leq p,$$

where $\vartheta_k : [0, a] \to \mathbb{R}$ is continuous, C^1 on $]0, a]$ and satisfies $\lim_{\rho \to 0} \vartheta_k(\rho) = 0$.

Proof. For $(e, \rho) \in H(\mathscr{V}) \times F(\mathscr{V})$, $k \in \{1, \ldots, p\}$, and $u \in]0, \overline{u}]$ we set

$$\gamma_{\rho,e}(k, u) := \gamma_{\mathscr{J}(\rho,e),e}(k, u) := (\gamma_{\mathscr{J}(\rho,e),e} \cap \{[-u, u]\}) \subset (\gamma_{\mathscr{J}(\rho,e),e} \cap (U_k \cap \mathscr{E}_e)),$$

so that $\gamma_{\rho,e}(k, u)$ is the part of $\gamma_{\rho,e}$ limited by the sections $\Gamma_k^{\pm}$ inside U_k. Since $\rho \mapsto \mathscr{J}(\rho, e)$ is a diffeomorphism for fixed e, we will work with $\mathscr{J}$ in the rest of this proof and we will write $\gamma_{\mathscr{J},e}$ instead of $\gamma_{\mathscr{J}(\rho,e),e}$. In the coordinates $(\varphi_k, I_k, u_k, v_k)$ introduced in (3.4) the 1-form λ_k reads $\lambda_k := I_k d\varphi_k + v_k du_k$. Therefore,

$$\int_{\gamma_{\mathscr{J},e}(k,u)} \lambda_k = \int_{-u}^{u} \sqrt{s^2 + 2\mathscr{J}}\, ds.$$

In particular, when $\mathscr{J} = 0$:

$$\int_{\gamma_{0,e}(k,u)} \lambda_k = \int_{-u}^{u} |s| ds = u^2.$$

For $\tilde{u}, u^*$ in $]0, \overline{u}]$, we set

$$\mathcal{R}_{\tilde{u},u^*}(\mathscr{J}, e) := \int_{\gamma_{\mathscr{J},e}} \lambda - \int_{\gamma_{\mathscr{J},e}(1,\tilde{u})} \lambda_1 - \sum_{k=2}^{p} \int_{\gamma_{\mathscr{J},e}(k,u^*)} \lambda_k.$$

One immediately checks that $\mathcal{R}_{\tilde{u},u^*}$ is C^2. Then, writing A as function of the variables $\mathscr{J}$ and e:

$$A(\mathscr{J}, e) := \int_{\gamma_{\mathscr{J},e}(1,\tilde{u})} \lambda_1 + \sum_{k=2}^{p} \int_{\gamma_{\mathscr{J},e}(k,u^*)} \lambda_k + \mathcal{R}_{\tilde{u},u^*}(\mathscr{J}, e).$$

Set, for $e \in]e_0 - \delta_0, e_0 + \delta_0[$, $u_1(e) := u^* + \frac{(e-e_0)}{2u^*} \dfrac{\partial A}{\partial e}(0, e_0)$. Then, one immediately checks that

$$\frac{\partial \mathcal{R}_{u_1(e),u^*}}{\partial e}(0, e_0) = \frac{\partial A}{\partial e}(0, e_0) - 2u^* \frac{1}{2u^*} \frac{\partial A}{\partial e}(0, e_0) = 0. \tag{3.6}$$

Observe that $u_1(e_0) = u^*$. We set $u_k^* = u^*$, for $2 \le k \le p$. In the following, we will omit the lower indices $u_1(e), u^*$ and write $\mathcal{R}(\mathscr{J}, e)$ instead of $\mathcal{R}_{u_1(e),u^*}(\mathscr{J}, e)$.

In the coordinates, $(\varphi_k, I_k, q_k, p_k)$ the vector field X^A, restricted to U_k, reads

$$\dot{\varphi}_k = \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J}, e) \frac{\partial H}{\partial I_k}(I_k, \mathscr{J}), \qquad \dot{I}_k = 0$$

$$\begin{aligned} \dot{q}_k = \frac{\partial}{\partial p_k} \int_{-u_1(e)}^{u_1(e)} \sqrt{s^2 + 2\mathscr{J}}\, ds + \sum_{\ell=2}^{p} \frac{\partial}{\partial p_k} \int_{-u^*}^{u^*} \sqrt{s^2 + 2\mathscr{J}}\, ds \\ + \frac{\partial \mathcal{R}}{\partial \mathscr{J}}(\mathscr{J}, e) \frac{\partial \mathscr{J}}{\partial p_k}(p_k, q_k) + \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J}, e) \frac{\partial H}{\partial \mathscr{J}}(I_k, \mathscr{J}) \frac{\partial \mathscr{J}}{\partial p_k}(p_k, q_k) \end{aligned}$$

$$\begin{aligned} = \Big(\int_{-u_1(e)}^{u_1(e)} \frac{ds}{\sqrt{s^2 + 2\mathscr{J}}} + \sum_{\ell=2}^{p} \int_{-u^*}^{u^*} \frac{ds}{\sqrt{s^2 + 2\mathscr{J}}} \\ + \frac{\partial \mathcal{R}}{\partial \mathscr{J}}(\mathscr{J}, e) + \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J}, e) \frac{\partial H}{\partial \mathscr{J}}(I_k, \mathscr{J}) \Big) q_k, \end{aligned}$$

$$\begin{aligned} \dot{p}_k = -\frac{\partial}{\partial q_k} \int_{-u_1(e)}^{u_1(e)} \sqrt{s^2 + 2\mathscr{J}}\, ds + \sum_{\ell=2}^{p} \frac{\partial}{\partial q_k} \int_{-u^*}^{u^*} \sqrt{s^2 + 2\mathscr{J}}\, ds \\ + \frac{\partial \mathcal{R}}{\partial \mathscr{J}}(\mathscr{J}, e) \frac{\partial \mathscr{J}}{\partial q_k}(p_k, q_k) + \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J}, e) \frac{\partial H}{\partial \mathscr{J}}(I_k, \mathscr{J}) \frac{\partial \mathscr{J}}{\partial q_k}(p_k, q_k) \end{aligned}$$

$$= -\Big(\int_{-u_1(e)}^{u_1(e)} \frac{ds}{\sqrt{s^2+2\mathscr{J}}} + \sum_{\ell=2}^{p}\int_{-u^*}^{u^*}\frac{ds}{\sqrt{s^2+2\mathscr{J}}} \\ + \frac{\partial \mathcal{R}}{\partial \mathscr{J}}(\mathscr{J},e) + \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J},e)\frac{\partial H}{\partial \mathscr{J}}(I_k,\mathscr{J})\Big)p_k.$$

From now on, we will limit ourselves to the level $H = e_0$. Set

$$\kappa(\mathscr{J}) := \Big(p\int_{-u^*}^{u^*}\frac{ds}{\sqrt{s^2+2\mathscr{J}}} + \frac{\partial \mathcal{R}}{\partial \mathscr{J}}(\mathscr{J},e_0) + \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J},e_0)\frac{\partial H}{\partial \mathscr{J}}(I_k,\mathscr{J})\Big)$$

and $R_k(\mathscr{J}) := \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J},e_0)\frac{\partial H}{\partial I_k}(I_k,\mathscr{J})$. Then, in the coordinates (φ_k,q_k,p_k), the restriction of X^A to $\mathscr{D}_a$ reads:

$$\dot\varphi_k = R_k(\mathscr{J}),\quad \dot q_k = \kappa(\mathscr{J})q_k,\quad \dot p_k = -\kappa(\mathscr{J})p_k.$$

Consider the renormalized vector field X on $\mathscr{D}_a$ defined by

$$X = \frac{1}{\kappa(\mathscr{J})}X^A.$$

Its flow ϕ has the same orbits as the flow ϕ_A associated with X^A, so the Poincaré maps between $\Gamma_k^+(u_k^*)$ and $\Gamma_k^-(u_k^*)$ associated with ϕ and ϕ_A do coincide. Observe that, if we introduce the transition time

$$T(\mathscr{J}) := \int_{-u^*}^{u^*}\frac{ds}{\sqrt{s^2+2\mathscr{J}}},\qquad 1\le k\le p,$$

then the map $P_k : z\mapsto \phi(T(\mathscr{J}),z)$ for $z\in\Gamma_k^+(u_k^*)$ is the Poincaré map associated with X between $\Gamma_k^+(u_k^*)$ and $\Gamma_k^-(u_k^*)$.

Observe also that $\mathscr{J}$ and A are in involution. Indeed, the orbits of ϕ_A are contained in the level sets $H=e$, $\mathscr{J}=J$. Therefore, $\mathscr{J}$ is also constant along the orbits of X. Hence, one gets

$$P_k(\varphi_k,u^*,\rho) = \left(\varphi_k + \int_0^{T(\mathscr{J}(\rho))}\frac{R_k(\mathscr{J}(\rho))}{\kappa(\mathscr{J}(\rho))}dt, u^*,\rho\right) = \left(\varphi_k + \frac{T(\mathscr{J}(\rho))R_k(\mathscr{J}(\rho))}{\kappa(\mathscr{J}(\rho))}, u^*,\rho\right).$$

Now

$$\kappa(\mathscr{J}) := \sum_{\ell=1}^{p} T(\mathscr{J}) + B(\mathscr{J}),$$

where $B : \mathscr{J}\mapsto \frac{\partial \mathcal{R}}{\partial \mathscr{J}}(\mathscr{J},e_0) + \frac{\partial \mathcal{R}}{\partial e}(\mathscr{J},e_0)\frac{\partial H}{\partial \mathscr{J}}(I_k,\mathscr{J})$ is a bounded function. Since $\lim_{\mathscr{J}\to 0}T(\mathscr{J}) = +\infty$ and $R_k(0)=0$, one gets $\lim_{\mathscr{J}\to 0}\dfrac{T(\mathscr{J})R_k(\mathscr{J})}{\kappa(\mathscr{J})} = 0$, which concludes the proof by setting $\vartheta_k : \rho\mapsto \dfrac{T(\mathscr{J}(\rho,e_0))R_k(\mathscr{J}(\rho,e_0))}{\kappa(\mathscr{J}(\rho,e_0))}$. $\square$

In the following, we denote by $\Gamma_k^\pm$ the surfaces $\Gamma_k^\pm(u_k^*)$ and by P_k the Poincaré map associated with ϕ_A between Γ_k^+ and Γ_k^-. We denote by D_k the subdomain of $\mathscr{D}_a\cap U_k$ bounded by Γ_k^+ and Γ_k^-, that is, the domain $D_k := \{|u|\le u_k^*\}$. Observe that (φ_k,u,ρ) is a system of coordinates in D_ℓ.

For $1 \le k \le p$, we call admissible arc on $\Gamma_k^{\pm}$ a curve

$$\zeta := \{(\varphi_k(\rho)), \mp u^*, \rho)) \mid \rho \in [0,a]\},$$

which is C^1 on $]0,a]$ and C^0 in $[0,a]$.

Step 2: *Let ζ be an admissible arc on Γ_k^+. Then $P_k(\zeta)$ is an admissible arc on Γ_k^-. Moreover, $\phi_A([0,1],\zeta) \cap D_k$ has the following graph form:*

$$\phi_A([0,1],\zeta) \cap D_\ell := \{\varphi_k(u,\rho), u, \rho) \mid (u,\rho) \in [-u^*, u^*] \times [0,a]\}.$$

Proof. Let ζ be an admissible arc on Γ_k^+. Then, $P_k(\zeta) := (\varphi_k(\rho) + \vartheta_k(\rho), u^*, \rho)$. Now, by step 1, the function $[0,a] \to \mathbb{R} : \rho \mapsto \vartheta_k(\rho)$ is C^0 on $[0,a]$ and C^1 on $]0,a]$, so $P_k(\zeta)$ is an admissible arc.

To see the second part, we first observe that, in D_k, $\dot{u} > 0$. Then for any $u_0 \in [-u^*, u^*]$, the Poincaré map (associated with ϕ_A) between Γ_k^+ and the surface $\Gamma(u_0) := \{u = u_0\}$ is well defined. As before, this Poincaré map coincides with the Poincaré map associated with the flow ϕ and we denote by T_{k,u_0} the associated time of the last one. Then

$$T_{k,u_0} := \int_{-u^*}^{u_0} \frac{ds}{\sqrt{s^2 + 2\mathscr{J}}}.$$

which yields

$$\phi_A([0,1],\zeta) \cap D_k := \left\{\varphi_k(\rho) + \frac{T_{k,u}(\mathscr{J}(\rho,e_0))R(\mathscr{J}(\rho,e_0))}{\kappa(\mathscr{J}(\rho,e_0))}, u, \rho) \mid (u,\rho) \in [-u^*, u^*] \times [0,a]\right\}.$$

As before, one immediately check that $(u,\rho) \mapsto \varphi_k(\rho)) + \dfrac{T_{k,u}(\mathscr{J}(\rho,e_0))R(\mathscr{J}(\rho,e_0))}{\kappa(\mathscr{J}(\rho,e_0))}$ is continuous on $[0,a]$ and C^1 on $]0,a]$. □

Step 3: *The flow ϕ_A induces a Poincaré map $P_{k,k+1}$ between Γ_k^- and Γ_{k+1}^+. Moreover, if ζ is an admissible arc on Γ_k^-, $P_{k,k+1}(\zeta)$ is an admissible arc on Γ_{k+1}^+.*

Proof. First, we show that the flow ϕ_H defines a Poincaré map between Γ_k^+ and Γ_k^- with associated transition-time $\sigma_{k,k+1}$ that only depends on ρ.

We set $C_k^+ = \Gamma_k^+ \cap W_k^+$ and $C_k^- = \Gamma_k^- \cap W_k^+$. Note that the ω-limit set of C_k^- with respect to ϕ_H is the hyperbolic orbit $\mathcal{C}_{k+1}$. Now, since C_k^- and $\mathcal{C}_{k+1}$ are not in the same connected component of $W_k^- \setminus C_{k+1}^+$, for all $z \in C_k$ there exists $\sigma(z) > 0$ such that $\phi_H(\sigma(z),z) \in C_{k+1}^+$. Since $\dot{u} = \lambda_k(\rho)\sqrt{u^2 + \mu_k(\rho)} > 0$ in $\mathscr{D}_a \cap \mathscr{U}_k$, the intersection time $\sigma(z)$ is unique. In the same way, for any $z \in C_{k+1}^+$, there exists a unique $\sigma(z) < 0$ such that $\Phi_H(\sigma(z),z) \in C_k^-$. Therefore the Poincaré map between C_k^- and C_{k+1}^+ is well defined. By compactness of C_k^-, there exist $0 < t_1 < t_2$ such that the associated transition time σ takes its values in $[t_1, t_2]$.

One easily deduces from the previous study that, for $a > 0$ small enough, the Poincaré map between $\Gamma_k^- \cap \{\rho \in [0,a]\}$ and $\Gamma_{k+1}^+ \cap \{\rho \in [0,a]\}$ is well defined. We denote by $\sigma_{k,k+1}$ its transition time. The function $\sigma_{k,k+1}$ is smooth.

Now, observe that the sections Γ_k^- and Γ_{k+1}^+ are invariant under the flow $(\phi_{\mathscr{I}}^t)$ associated with the first integral $\mathscr{I}$. Indeed, the flow $X^{\mathscr{I}}$ reads (in the variables (φ_k, p_k, q_k)): $\dot{\varphi}_k = 1$, $\dot{p}_k = 0 = \dot{q}_k$, which yields

$$\dot{\varphi}_k = 1, \quad \dot{u} = 0, \quad \dot{\rho} = 0,$$

Moreover, since ϕ_H and $\phi_{\mathscr{I}}$ commute, for any $t > 0$ and any $\varphi_k \in \mathbb{T}$:

$$\begin{aligned}\phi_H(\sigma_{k,k+1}(\varphi_k, u^*, \rho), (\varphi_k + t, u^*, \rho)) \\ = \phi_H(\sigma_{k,k+1}(\varphi_k, u^*, \rho), \phi_{\mathscr{I}}(t, (\varphi_k, u^*, \rho))) \\ = \phi_{\mathscr{I}}(t, \phi_H(\sigma_{k,k+1}(\varphi_k, u^*, \rho), (\varphi_k, u^*, \rho)))\end{aligned} \quad (3.7)$$

which belongs to Γ^+_{k+1}. Therefore $\sigma(\varphi_k + t, u^*, \rho) = \sigma(\varphi_k, u^*, \rho)$. We set $\sigma_{k,k+1}(\rho) := \sigma_{k,k+1}(\varphi_k, u^*, \rho)$, for any $\varphi_k \in \mathbb{T}$.

We denote by $\mathring{D}_{k,k+1}$ the connected component of $\mathscr{D}_a \setminus (\Gamma^-_k \cup \Gamma^+_{k+1})$ that does not contain $\mathcal{C}_k$ and we set $D_{k,k+1} := \mathring{D}_{k,k+1} \cup \Gamma^-_k \cup \Gamma^+_{k+1}$.

Set $\Gamma_k := \{u = 0\} \subset D_k$. The transition time σ_k (associated with ϕ_H) between $\Gamma_k \setminus \{\rho = 0\}$ and $\Gamma^+_k \setminus \{\rho = 0\}$ is well defined and is independent of φ_k. Indeed,

$$\sigma_k(\rho) := \int_{-u^*}^{0} \frac{ds}{\lambda_k(\rho)\sqrt{s^2 + \mu_k(\rho)}}.$$

Since $\lim_{\rho\to 0} \sigma_k(\rho) = +\infty$, there exists $a > 0$ small enough so that for any $1 \le k \le p$, $\phi_H([0,1], \Gamma^+_k) \subset \{u \in]-u^*, 0]\}$. Therefore the map

$$\begin{array}{lcl}\Gamma^-_k \times [0,2] & \to & D_{k,k+1} \cup \phi_H(]0,1], \Gamma^+_k) \\ ((\varphi_k, u^*, \rho), x) & \mapsto & \phi_H(x\sigma_{k,k+1}(\rho), (\varphi_k, u^*, \rho))\end{array}$$

is a diffeomorphism. In the following, we consider (φ_k, x, ρ) as a system of coordinates in $D_{k,k+1} \cup \phi_H(]0,1], \Gamma^+_k)$.

As before, we will work with the vector field X and its flow ϕ instead of X^A and ϕ_A, and we will show that ϕ induces a Poincaré map between Γ^-_k and Γ^+_{k+1}. Since ρ is invariant under ϕ, the vector field X reads:

$$X(\varphi_k, x, \rho) := (X_\varphi(\varphi_k, x, \rho), X_x(\varphi_k, x, \rho), 0),$$

where X_φ and X_x are C^1 functions.

Now, let $u \in [u^*, \overline{u}]$. Then,

$$x(\varphi_k, u, \rho) = \frac{1}{\sigma_{k,k+1}(\rho)} \int_{u^*}^{u} \frac{ds}{\sqrt{s^2 + 2\mathscr{I}(\rho)}},$$

that is, x is a strictly increasing function of u in $D_{k,k+1} \cap \mathscr{U}_k$. Hence, since $\dot{u}_k = p_k + q_k > 0$ on Γ^-_k,

$$X_x(\varphi_k, 0, \rho) > 0, \quad \forall\, (\varphi_k, \rho) \in \mathbb{T} \times [0, a]. \quad (3.8)$$

Moreover, since x is independent of φ_k in the domain $\{u \in [u^*, \overline{u}]\}$, for any $\varphi_k \in \mathbb{T}$, $X_x(\varphi_k, 0, \rho) = X_x(0, 0, \rho)$.

With $x_0 \in [0,1]$, we associate the diffeomorphism

$$\begin{array}{llcl}\eta_{x_0}: & D_{k,k+1} & \to & D_{k,k+1} \cup \phi_H([0,1], \Gamma^+_k) \\ & (\varphi_k, x, \rho) & \mapsto & (\varphi_k, x + x_0, \rho).\end{array}$$

Observe that for any $\varphi_k \in \mathbb{T}$, $D_{(\varphi_k,0,\rho)}\eta_{x_0}(X(\varphi_k, 0, \rho) = X(\varphi_k, x_0, \rho))$. So, for any $\varphi_k \in \mathbb{T}$, $X(\varphi_k, x_0, \rho) = X(0, x_0, \rho)$. This property holds for any $x \in [0,1]$. As a consequence, using (3.8), one sees that there exists $\beta > 0$ such that, for all $z \in D_{k,k+1}$, $X_x(z) > \beta$. Therefore, if we set

$$t_{k,k+1}(\rho) = \int_0^1 \frac{ds}{X_x(0, s, \rho)},$$

the map

$$\begin{array}{rlcl} P_{k,k+1}: & \Gamma_k^- & \to & \Gamma_{k+1}^+ \\ & (\varphi_k, 0, \rho) & \mapsto & \phi_A(t_{k,k+1}(\rho), \varphi_k, 0, \rho)). \end{array}$$

is a Poincaré map between Γ_k^- and Γ_{k+1}^+. We denote by $\varphi_A : \mathbb{T} \times [0,a] \to \mathbb{T}$ the smooth function defined by $\phi_A(t_{k,k+1}(\rho)(\varphi_k, 0, \rho) = (\varphi_A(\varphi_k, \rho), 1, \rho)$ and by $\varphi_H : \mathbb{T} \times [0,a] \to \mathbb{T}$ the smooth function defined by $\phi_H(\sigma_{k,k+1}(\rho)(\varphi_k, u^*, \rho) = (\varphi_H(\varphi_k, \rho), -u^*, \rho)$. Hence,

$$P_{k,k+1}(\varphi_k, u^*, \rho) = (\varphi_H(\varphi_A(\varphi_k, \rho), \rho), -u^*, \rho).$$

As a consequence, if $\zeta := \{(\varphi_k(\rho), u^*, \rho) \,|\, \rho \in [0,a]\}$ is an admissible arc on Γ_k^-, its image $P_{k,k+1}(\varphi_k(\rho), u^*, \rho) = (\varphi_H(\varphi_A(\varphi_k(\rho), \rho), \rho), -u^*, \rho)$ is an admissible on Γ_{k+1}^+. $\square$

Consider now the surface $S := \phi_A([0,1], \zeta_1)$ as defined in proposition 3.3.3. Observe that, according to the previous graph form (Step 2), the vector field X^H is transverse to S inside the domains D_k. By construction of S, this immediately implies that X^H is everywhere transverse to S. Indeed, the action-angle form proves that it is enough that X^H be transverse to S at only one point in each Liouville torus of the regular foliation. So S is a proper section for X^H.

Step 4: *The surface S can be continuated to $\mathscr{P}$ as a continuous surface.*

Proof. Using alternatively the steps 1 and 3, one sees that $\zeta_k^\pm := \phi_A([0,1], \zeta_1) \cap \Gamma_k^\pm$ is an admissible arc for any $1 \le k \le p$.

Using step 2, one sees that $\phi_A([0,1], \zeta_1) \cap D_k$ is a continuous submanifold with the graph form $\phi_A([0,1], \zeta_1) \cap D_k := \{\varphi_k(u,\rho), u, \rho) \,|\, (u,\rho)\}$.

It remains to check that $\phi_A([0,1], \zeta_1) \cap D_{k,k+1}$ is a continuous submanifold. For $\rho \in [0,a]$, we write $\zeta_k^-(\rho) := (\varphi_k(\rho), u^*, \rho)$. Observe that

$$\phi_A([0,1], \zeta_1) \cap D_{k,k+1} = \bigcup_{\rho \in [0,a]} \bigcup_{x \in [0,1]} \phi_X(x t_{k,k+1}(\rho), \zeta_k^-(\rho))$$

The map $[0,a] \times [0,1] \to S : \phi_X(x t_{k,k+1}(\rho), \zeta_k^-(\rho))$ is a homemorphism onto $\phi_A([0,1], \zeta_1) \cap D_{k,k+1}$, which is C^1 on $[0,1] \times]0,a]$. This proves that S admits a C^0 continuation on $\rho = 0$. $\square$

We still denote by S the continuation of S in $\overline{\mathscr{D}}_a$.

Step 5: *The surface S is a proper section for ϕ_H associated with a C^1 function $\tau : [0,a] \to \mathbb{R}_+^*$.*

Proof. We will first show that the transition function $\tau :]0,a] \to \mathbb{R}$ defined in proposition 3.3.3 has a well defined C^1 continuation on $[0,a]$. Then we will prove that this function is a transition function for S with respect to ϕ_H.

We consider the lift $\mathbb{R} \times [-u^*, u^*] \times [0,a]$ of $\mathbb{T} \times [-u^*, u^*] \times [0,a] \subset \mathscr{U}_k$. We still denote by ϕ_H, ϕ_A and ϕ the lifted Hamiltonian flows on $\mathbb{R} \times [-u^*, u^*] \times [0,a]$. Let us denote by T the common period of the orbits $\mathcal{C}_k$. We set $X^H(z) := (X_\varphi^H(z), X_u^H(z), 0)$.

By continuity of X^H, we can assume that u^* and a are small enough so that for any $z \in \mathbb{R} \times [-u^*, u^*] \times [0,a]$, $X_\varphi^H(z) \in [\frac{2}{T}, \frac{1}{2T}]$. Hence, there exists $\alpha > 0$ such that

$$\phi_H(\frac{T}{2}, \zeta_1) \subset \{\varphi_1 \le 1 - \alpha\}, \quad \phi_H(2T, \zeta_1) \subset \{\varphi_1 \ge 1 + \alpha\}.$$

On the other hand, we can also assume that a is small enough so that there exists $u_0 \in]0, u^*]$ such that

$$\phi_H([0, 2T], \zeta_1) \subset \{u \in [-u^*, -u_0]\}.$$

Let $\tilde{\zeta}_1 := \zeta_1 + (1, 0, 0) \subset \mathbb{R} \times [-u^*, u^*] \times [0, a]$ and set $\tilde{\zeta}_1(\rho) := (1, -u^*, \rho)$. Finally set $t(\rho) := \displaystyle\int_{-u^*}^{-u_0} \frac{ds}{\sqrt{u^2 + 2\mathscr{J}(\rho)}}$. By construction, since in the coordinates (φ_1, u, ρ) the vector field X reads

$$\dot{\varphi}_1 = \frac{R(\mathscr{J}(\rho))}{\kappa(\mathscr{J}(\rho))}, \quad \dot{u} = \sqrt{u^2 + 2\mathscr{J}(\rho)}, \quad \dot{\rho} = 0,$$

for any $\rho \in [0, a]$, $\phi_A(t(\rho), (1, -u^*, \rho)) \in \{u = u_0\}$. Let $\widetilde{S}_1$ be the smooth surface

$$\widetilde{S}_1 := \bigcup_{\rho \in [0,a]} \phi_X([0, t(\rho)], \tilde{\zeta}_1(\rho)).$$

By step 2, $\widetilde{S}_1$ has the following graph form:

$$\widetilde{S}_1 := \{(\varphi_1(u, \rho), u, \rho) \,|\, (u, \rho) \in [-u^*, -u_0] \times [0, a]\}$$

Since the function $\rho \to t(\rho)$ is decreasing, therefore

$$\widetilde{S}_1 := \phi_X([0, t(0)], \tilde{\zeta}_1) \cap \{u \in [-u^*, -u_0]\}.$$

Moreover, since $\lim_{\rho\to 0} \frac{R(\mathscr{J}(\rho))}{\kappa(\mathscr{J}(\rho))} = 0$, there exists $a > 0$ small enough so that for any $\rho \in [0, a]$, $t(0)\dfrac{R(\mathscr{J}(\rho))}{\kappa(\mathscr{J}(\rho))} < \alpha$. As a consequence,

$$\widetilde{S}_1 \subset B := [1 - \alpha, 1 + \alpha] \times [-u^*, -u_0] \times [0, a],$$

and $B \setminus \widetilde{S}_1$ has two connected components. Hence for any $\rho \in [0, a]$, $\phi_H(\mathbb{R}, (0, us, \rho))$ cut $\widetilde{S}_1$ once and only once. Let $\tau : [0, a] \to \mathbb{R}^+$ be such that $\phi_H(\tau(\rho), (0, -us, \rho) \in \widetilde{S}_1$. Then τ is the restriction to $\{-u^*\} \times [0, a]$ of the transition time associated to the Poincaré map (with respect to ϕ_H) between the smooth surfaces $\{\varphi = 0\}$ and $\widetilde{S}_1$. This map is smooth since the both surfaces are smooth and transverse to ϕ_H. Hence, τ is smooth on $[0, a]$. Finally, if $\pi : \mathbb{R} \to \mathbb{T}$ is the canonical projection, $\pi(\widetilde{S}_1) \subset S$. Obviously, the function τ defined above coincide in $S \cap \{\rho > 0\}$ with the transition function τ defined in proposition 3.3.3. It remains to check, that for any $z \in S \cap \mathscr{P}$, the transition time $\tau(z)$ of z is $\tau(0)$. Fix $z \in S \cap \mathscr{P}$. Let $\gamma : [0, a] \to S$ be a continuous map such that $\gamma(0) = z$ and for any $r \in]0, a]$, $\gamma(r) \subset S \cap \{\rho = r\}$. Then for any $\rho \in]0, a]$, $\phi_H(\tau(\rho), \gamma(\rho)) \in S$. Since S is closed, by continuity of ϕ_H, τ and γ, $\gamma(z) \in S$. $\square$

c. Construction of the conjugacy between $\wp$ and ψ^α: proof of Lemma II.

This section is devoted to the proof of lemma II. Let S be a proper section with time τ as in lemma I. In each $\mathscr{U}_k \cap \overline{\mathscr{D}_a}$, S has the following graph form

$$S := \{(\varphi_k(u, \rho)\, w, \rho) \,|\, \varphi \in \mathbb{T}, (u, \rho) \in [-\overline{u}, \overline{u}] \times [0, a]\}.$$

The proof consists in the construction of a suitable planar p-model ψ. Our strategy is the following.

• We first construct a *fundamental domain* for the map $\wp$ in each subdomain $\mathscr{U}_k \cap S$ of S. A fundamental domain means a domain Δ_k bounded by a "vertical" curve D_k with equation $u = u_0$ in S and its image $\wp^{-1}(D_k)$ (and the natural horizontal boundaries).

• We show that there exists an integer m_k such that the domains Δ_k, $\wp(\Delta_k)$, ..., $\wp^{m_k}(\Delta_k)$ cover the connected component R_k of S between $S \cap \mathscr{U}_k$ and $S \cap \mathscr{U}_{k+1}$ and such that $\wp^{m_k}(\Delta_k) \subset S \cap \mathscr{U}_{k+1}$.

This being done, the construction of the planar p-model necessitates three steps.

• For $1 \leq k \leq p$, we construct a vector field X_k with associated flow $\psi_{(k)}$ in a suitable neighborhood $\mathscr{O}_k$ of the point $(\frac{k}{p}, 0) \in \mathbb{T} \times [0,1]$, such that there exists a homeomorphism χ_k between Δ_k and a fundamental domain of X_k.

• For $1 \leq k \leq p$, we construct a vector field of $\widehat{X}_k$ with associated flow $\widehat{\psi}_{(k)}$ in a suitable subdomain $\mathscr{R}_k$ of $\mathbb{T} \times [0,1]$ such that we can glue together the flows $\psi_{(k)}$ and $\widehat{\psi}_{(k)}$ (for $1 \leq k \leq p$) to get a flow ψ on $\mathbb{T} \times [0,1]$ with a time map ψ^{α} conjugated to $\wp$.

• We check that the flow ψ is a planar p-model.

c-1. *Construction of the fundamental domains* Δ_k. The construction of the fundamental domains Δ_k is based on the construction in each domain $\mathscr{U}_k$ of a pair of sections that are transverse both to the flow and to S.

Lemma 3.3.2. *There exists $u^* \in [0, \overline{u}]$ such that, for any $1 \leq k \leq p$, the sections $\Sigma_k^+ := \{u = -u^*\}$ and $\Sigma_k^- := \{u = u^*\}$ satisfy the following conditions:*

• ***(C1):*** *The Poincaré return map between $\Sigma_k^+ \setminus \{\rho = 0\}$ and $\Sigma_k^- \setminus \{\rho = 0\}$ is well defined and its associated time τ_k does not depend on φ and is a decreasing function of ρ.*

• ***(C2):*** *The Poincaré return map between Σ_k^- and Σ_{k+1}^+ is well defined and its associated time $\sigma_{k,k+1}$ does not depend on φ and is a decreasing function of ρ.*

Proof. Let us prove (C1). Fix $u_0 \leq \overline{u}$ and consider the surfaces $\Sigma_k^{\pm}$ defined as above for $1 \leq k \leq p$. By $(\star)$, one immediately checks that they are transverse to the flow and that the transition time between $\Sigma_k^+ \setminus \{\rho = 0\}$ and $\Sigma_k^- \setminus \{\rho = 0\}$ is given by

$$\tau_k(\rho) = \frac{2}{\lambda(\rho)} \operatorname{Argsh} \frac{u_0}{\sqrt{\mu(\rho)}}.$$

Moreover, by direct computation, one sees that $\lim_{\rho \to 0} \tau_k'(\rho) = -\infty$. As a consequence, if a is small enough, $\tau_k'(\rho) < 0$ for any $\rho \in [0, a]$. In the following, we assume that a is small enough so that, for any $1 \leq k \leq p$, τ_k is decreasing on $[0, a]$ and (C1) is realized.

In Step 3 of the continuation of S, we proved the existence of a Poincaré map between Σ_k^- and Σ_{k+1}^+ with associated transition time $\sigma_{k,k+1}$ that only depends on ρ. It remains to check the decreasing condition on the time $\sigma_{k,k+1}$. Note that the function $\rho \mapsto \sigma_{k,k+1}'(\rho)$ is uniformely bounded on $[0, a]$. Fix $0 < u^* < u_0$ and consider, for $1 \leq k \leq p$, the sections $\Sigma_k^+(u^*) := \{u = -u^*\}$ and $\Sigma_k^-(u^*) := \{u = u^*\}$. Let $\tilde{\sigma}_{k,k+1}$ be the transition time between $\Sigma_k^-(u^*)$ and $\Sigma_{k+1}^+(u^*)$. Then

$$\tilde{\sigma}_{k,k+1}(\rho) := \frac{2}{\lambda(\rho)} \left(\operatorname{Argsh} \frac{u_0}{\sqrt{\mu(\rho)}} - \operatorname{Argsh} \frac{u^*}{\sqrt{\mu(\rho)}} \right) + \sigma_{k,k+1}(\rho).$$

Let $g : \rho \mapsto \frac{2}{\lambda(\rho)} \left(\operatorname{Argsh} \frac{u_0}{\sqrt{\mu(\rho)}} - \operatorname{Argsh} \frac{u^*}{\sqrt{\mu(\rho)}} \right)$. By elementary computation one sees that:

$$g'(\rho) \sim_{\rho \to 0} \frac{\lambda(\rho)\mu'(\rho)}{4} \left(\frac{1}{u_0^2} - \frac{1}{(u^*)^2} \right).$$

Therefore, for u^* small enough $\rho \to \tilde{\sigma}_{k,k+1}(\rho)$ is decreasing and (C2) is also realized. Obviously, one can choose u^* small enough so that (C1) and (C2) are realized for any $1 \leq k \leq p$. □

Remark 3.3.2. We can assume that u^* is small enough so that, for any $k \in \{1, \cdots, p\}$ and any $\rho \in [0, a]$, $\sigma_{k,k+1}(\rho) > 2$.

For $1 \leq k \leq p$, we set $\delta_k := S \cap \Sigma_k^- := \{(\varphi_k(u^*, \rho) \,|\, \rho \in [0, a]\}$. Since for all k, $\lim_{\rho\to 0} \tau_k(\rho) = +\infty$, there exists $a > 0$ small enough such that for any $\rho \in [0, a]$ and any k, $\tau(\rho) \leq \tau_k(\rho)$. Therefore, $\wp^{-1}(\delta_k) \subset \mathscr{U}_k \cap S$.

The fundamental domain Δ_k is defined as the subdomain of $S \cap \mathscr{U}_k$ bounded by $\wp^{-1}(\delta_k)$ and δ_k. Fix $(\varphi_k(u^*, \rho), u^*, \rho) \in \delta_k$. Then

$$\wp^{-1}(\varphi_k(u^*, \rho), u^*, \rho) = (\varphi_k(u_k(\rho), \rho), u_k(\rho), \rho)$$

where $u_k(\rho)$ is defined by

$$\tau(\rho) := \int_{u_k(\rho)}^{u^*} \frac{du}{\lambda_k(\rho)\sqrt{u^2 + \mu_k(\rho)}}.$$

Therefore,

$$\Delta_k = \{(\varphi_k(u, \rho), u, \rho) \,|\, u \in [u_k(\rho), u^*], \rho \in [0, a]\}.$$

We denote by R_k the connected component of $S \setminus (\mathscr{U}_k \cup \mathscr{U}_{k+1})$ that contains $\wp(\Delta_k)$, that is, the connected component of $S \setminus (\mathscr{U}_k \cap \mathscr{U}_{k+1})$ with nonempty intersection with W_k^-.

Lemma 3.3.3. *For $1 \leq k \leq p$, there exists $m_k \in \mathbb{N}^*$ such that*

- $\wp^{m_k}(\Delta_k) \subset (S \cap \mathscr{U}_{k+1})$,
- $R_k \subset \bigcup_{j=1}^{m_k} \wp^j(\Delta_k)$.

Proof. For $a > 0$ we set $T_a := \{\tau(\rho) \,|\, \rho \in [0, a]\}$. There exists $0 < t_0 \leq t_1$ such that $T_a := [t_0(a), t_1(a)]$. Notice that, if $a' \leq a$, then $t_0(a) \leq t_0(a') \leq t_1(a') \leq t_1(a)$. By compactness of S_k and continuity of $\sigma_{k,k+1}$, one can define $\sigma_k := \max_{z\in\Sigma_k^+} \sigma_{k,k+1}(z) > 0$. There exists $m_k \in \mathbb{N}^*$ such that $(m_k - 1)t_0(a) \geq \sigma_k$. Since $\lim_{\rho\to 0} \tau_{k+1}(\rho) = +\infty$, one can assume that a is small enough so that $m_k t_1(a) < \tau_{k+1}(\rho) - \sigma_{k,k+1}(\rho)$ for any $\rho \in [0, a]$. That is,

$$\Phi_H([(m_k - 1)t_0(a), m_k t_1(a)], \Sigma_k^-) \subset \mathscr{U}_k.$$

In particular, for all $z \in \delta_k$, $\wp^{m_k}(z) \subset \Phi(m_k[t_0(a), t_1(a)], \Sigma_k^-) \subset \mathscr{U}_k$, that is, $\wp^{m_k}(\delta_k) \subset \mathscr{U}_k \cap S$. In the same way, for all $z \in \wp^{-1}(\delta_k)$, $\wp^{m_k}(z) \subset \Phi((m_k - 1)[t_0(a), t_1(a)], \Sigma_k^-) \subset \mathscr{U}_k$, that is, $\wp^{m_k-1}(D_k) \subset \mathscr{U}_k \cap S$.

We set $\overset{\circ}{\Delta}_k := \Delta_k \setminus (\delta_k \cup \wp^{-1}(\delta_k))$. Since $\wp^{m_k}$ is a diffeomorphism, $\wp^{m_k}(\overset{\circ}{\Delta}_k)$ is one of the two connected components of $S \setminus (\wp(\delta_k) \cup \wp^{m_k-1}(\delta_k))$. Since the second one has nonempty intersection with all the hyperbolic orbits Γ_j, $\wp^{m_k}(\overset{\circ}{\Delta}_k)$ must be the first one which is contained in $\mathscr{U}_{k+1}$ and the first point is proved.

To prove the second point, we first remark that R_k is contained in the connected component of $S \setminus (\wp(D_k) \cup \wp^{m_k-1}(D)$. With the same argument as in the beginning of the proof, we can assume that a is small enough so that $\phi_H([-m_k, 0], \Delta_k) \subset \mathscr{U}_k$. So for $2 \leq j \leq m_k$,

$$\wp^{-j}(\delta_k) = \{(\varphi_k(w_j(\rho), \rho), w_j(\rho), \rho)\}$$

where $u_j(\rho)$ is defined by $\displaystyle\int_{u_j(\rho)}^{w^*}\frac{du}{u^2+\mu(\rho)} = j\tau(\rho)$. Therefore, $\displaystyle\bigcup_{j=1}^{m_k}\wp^{-j}(\Delta_k)$ is a connected 2-dimensional submanifold of S with boundaries $\wp^{-m_k}(\delta_k)$ and $\wp^{-1}(\delta_k)$. As a consequence, $\displaystyle\bigcup_{j=1}^{m_k}\wp^{j}(\Delta_k) = \wp^{m_k}\left(\bigcup_{j=1}^{m_k}\wp^{-j}(\Delta_k)\right)$ is a connected 2-dimensional submanifold of S with boundaries $\wp(\delta_k)$ and $\wp^{m_k-1}(\delta_k)$. Obviously, it is the one which contains R_k. □

Consider the compact annulus $\widehat{\mathscr{A}} := \mathbb{T}\times[0,1]$ with coordinates (θ,r). For $1\le k\le p$, we set $z_k := (\frac{k}{p},0)$, $\mathscr{O}_k := [\frac{k}{p}-u^*,\frac{k}{p}+u^*]\times[0,1]$ and $\mathscr{R}_k := [\frac{k}{p}+u^*,\frac{k+1}{p}-u^*]\times[0,1]$. Finally we set

$$\alpha : [0,1]\to\mathbb{R}_+^* : r\mapsto\tau(ar).$$

c-2. *Construction of X_k and χ_k.* Let X_k be the vector field defined on $\mathscr{O}_k$ by

$$X_k(\theta,r) := \lambda_k(ar)\sqrt{\left(\theta-\frac{k}{p}\right)^2+\mu_k(ar)}\,\frac{\partial}{\partial\theta}.$$

We denote by $(\psi^t_{(k)})$ its local flow in $\mathscr{O}_k$ and by $\psi^\alpha_{(k)}$ its associated time-α map, that is, $\psi^\alpha_{(k)}(\theta,r)=\psi_{(k)}(\alpha(r),(\theta,r))$.

For $1\le k\le p$, we set

$$\begin{array}{llll}\chi_k: & \mathscr{U}_k\cap\Sigma & \to & \mathscr{O}_k\\ & (\varphi_k(u,\rho),u,\rho) & \mapsto & (u+\frac{k}{p},\frac{\rho}{a}).\end{array}$$

The proof of the following lemma is immediate.

Lemma 3.3.4. *Let $k\in\{1,\cdots,p\}$.*

1. $\psi^\alpha_{(k)}(\chi_k(\wp^{-1}(\delta_k)))=\chi_k(\delta_k)$,
2. $\psi^\alpha_{(k+1)}(\chi_{k+1}(\wp^{m_k-1}(\delta_k)))=\chi_{k+1}(\wp^{m_k}(\delta_k))$,
3. *for all $z\in\mathscr{U}_k\setminus\Delta_k$, $\psi^\alpha_{(k)}(z)=\chi_k(P(z))$.*

c-3. *Construction of $\widehat{X}_k$ and of the flow ψ.* For any $k\in\{1,\cdots,p\}$, we consider a function $\xi_k:\mathscr{R}_k\to\mathbb{R}$ such that

$$\int_{\frac{k}{p}+u^*}^{\frac{k+1}{p}-u^*}\frac{d\theta}{\xi_k(\theta,r)}=\sigma_{k,k+1}(ar)\qquad (**)$$

Let $\widehat{X}_k$ be the vector field on $\mathscr{R}_k$ defined by

$$\widehat{X}_k(\theta,r)=\xi_k(\theta,r)\frac{\partial}{\partial\theta},$$

and denote by $(\widehat{\psi}_{(k)})$ its local flow. By construction, for any $r\in[0,1]$

$$\widehat{\psi}_{(k)}\left(\sigma_{k,k+1}(ar),\left(\frac{k}{p}+u^*,r\right)\right)=\left(\frac{k+1}{p}-u^*,r\right),$$

We define a flow ψ on $\mathbb{T}\times[0,1]$ by gluing together the flows $\psi_{(k)}$ and $\widehat{\psi}_{(k)}$. We begin by constructing, for $1\le k\le p$, a local flow ψ_k on $\mathscr{O}_k\bigcup\mathscr{R}_k$ in the following way. For $(\theta,r)\in\mathscr{O}_k\setminus\{z_k\}$ there exists a unique $t(\theta,r)>0$ such that $\psi_{(k)}(t(\theta,r))\in\chi_k(\delta_k)$. We set

- $\psi_k^t(\theta,r) = \psi_{(k)}(t,(\theta,r))$ if $t \le t(\theta,r)$,
- $\psi_k^t(\theta,r) = \widehat{\psi}_{(k)}(t - t(\theta,r), (\frac{k}{p}+u^*, r))$ if $t(\theta,r) \le t \le t(\theta,r) + \sigma_{k,k+1}(ar)$.

Lemma 3.3.5. *ψ_k is a continuous flow on $\mathscr{O}_k \bigcup \mathscr{R}_k$.*

Proof. The continuity of ψ_k is obvious by construction. One just has to check that ψ_k is a flow, that is, $\psi_k(s+t,(\theta,r)) = \psi_k^s \circ \psi_k^t(\theta,r) = \psi_k^t \circ \psi_k^s(\theta,r)$. The only possible difficulty occurs when $t(\theta,r) < t+s < t(\theta,r) + \sigma_{k,k+1}(ar)$.

Assume that $s \le t$ and that $t(\theta,r) < t+s < t(\theta,r) + \sigma_{k,k+1}(ar)$. We set $\theta^* := \frac{k}{p} + u^*$. We first remark that $\psi_k(s+t,(\theta,r)) = \widehat{\psi}_k(t+s-t(\theta,r),(\theta^*,r))$. They are three possibilities.

• $t < s < t(\theta,r)$. Then $t(\psi_{(k)}(s,(\theta,r))) = t(\theta,r) - s$ and $t > t(\psi_{(k)}(s,(\theta,r)))$. Hence

$$\begin{aligned}
\psi_k^t \circ \psi_k^s(\theta,r) = \psi_k(t, \psi_k^s(\theta,r)) &= \widehat{\psi}_{(k)}(t - t(\psi_{(k)}(s,(\theta,r))), (\theta^*,r)) \\
&= \widehat{\psi}_{(k)}(t - (t(\theta,r) - s), (\theta^*,r)) \\
&= \widehat{\psi}_{(k)}(t + s - t(\theta,r), (\theta^*,r)) \\
&= \psi_k(s+t,(\theta,r)).
\end{aligned}$$

In the same way, $\psi_k^s \circ \psi_k^t(\theta,r) = \psi_k(s+t,(\theta,r))$.

• $t \le t(\theta,r) \le s$. Then $\psi_k(s,(\theta,r)) = \widehat{\psi}_{(k)}(s - t(\theta,r),(\theta,r))$, which yields

$$\psi_k^t \circ \psi_k^s(\theta,r) = \widehat{\psi}_{(k)}(t, \widehat{\psi}_{(k)}(s - t(\theta,r),(\theta^*,r))) = \widehat{\psi}_{(k)}(t+s-t(\theta,r),(\theta^*,r)) = \psi_k(s+t,(\theta,r)).$$

On the other hand,

$$\psi_k^s \circ \psi_k^t(\theta,r) = \widehat{\psi}_{(k)}(s - t(\psi(s,(\theta,r))),(\theta^*,r)) = \widehat{\psi}_{(k)}(t+s-t(\theta,r),(\theta^*,r)) = \psi_k(s+t,(\theta,r)).$$

• $t(\theta,r) \le t \le s$. Then, as before $\psi_k^t \circ \psi_k^s(\theta,r) = \psi_k(s+t,(\theta,r))$. Conversely,

$$\psi_k^s \circ \psi_k^t(\theta,r) = \widehat{\psi}_{(k)}(s, \widehat{\psi}_{(k)}(t - t(\theta,r),(\theta^*,r))) = \psi_k(s+t,(\theta,r)).$$

This concludes the proof. □

c-4. *Construction of ψ and conjugacy between ψ^α and ϕ_H.* Now, we construct a global flow ψ on $\mathscr{A}$ by gluing together the previous flows defined on $\mathscr{O}_k \bigcup \mathscr{R}_k$ with the usual convention $p+1=1$. One checks as in the previous lemma that this defines a flow on $\mathbb{T} \times [0,1]$. We denote by ψ^α its time-α map.

Lemma 3.3.6. *For all $z \in \Delta_k$*

$$(\psi^\alpha)^{m_k}(\chi_k(z)) = \chi_{k+1}(\wp^{m_k}(z)).$$

Proof. Fix $z := (\varphi_k(u,\rho),u,\rho) \in \Delta_k$. Let $u' \in [u_{k+1}(\rho),u^*]$ be such that $\wp^{m_k}(z) = (\varphi_{k+1}(u',\rho),u',\rho)$. We set

$$t_1(z) := \int_u^{u^*} \frac{ds}{\lambda_k(\rho)\sqrt{s^2+\mu_k(\rho)}}, \quad t_2(z) := \int_{-u^*}^{u'} \frac{ds}{\lambda_k(\rho)\sqrt{s^2+\mu_k(\rho)}}$$

Then

$$\begin{aligned}\chi_{k+1}(\wp^{m_k}(z)) &= \left(\frac{k+1}{p}+u', \frac{\rho}{a}\right) & &= \psi\left(t_1(z), \left(\frac{k+1}{p}-u^*, \frac{\rho}{a}\right)\right)\\ &= \psi\left(\sigma_{k,k+1}(\rho)+t_1(z), \frac{k}{p}+u^*\right) & &= \psi\left(\sigma_{k,k+1}(\rho)+t_1(z)+t_2(z), \frac{k}{p}+u\right)\\ &= \psi(\tau(\rho), \chi_k(z)).\end{aligned}$$

This concludes the proof. □

We define a map $\chi : S \to \mathbb{T} \times [0,1]$ by setting

- for $1 \leq k \leq p$ and $z \in \mathscr{O}_k$, $\chi(z) = \chi_k(z)$,
- for $1 \leq k \leq p$ and $z \in \wp^j(\Delta_k)$ with $1 \leq j \leq m_k$, $\chi(z) = (\psi^\alpha)^j(\chi_k(\wp^{-j}(z)))$.

Lemma 3.3.7. *The map χ is well defined and is a homeomorphism.*

Proof. To see that χ is well defined, one has to check that, for $1 \leq k \leq p$, both definitions of χ coincide when $z \in \wp^{m_k}(z)$. Fix $z \in \wp^{m_k}(z)$ and let $z_0 = \wp^{-m_k}(z) \in \Delta_k$. Then

$$(\psi^\alpha)^{m_k}(\chi_k(\wp^{-m_k}(\wp^{m_k}(z_0))) = (\psi^\alpha)^{m_k}(\chi_k(z_0)) = \chi_{k+1}(\wp^{m_k}(z_0)) = \chi_{k+1}(z),$$

the third equality coming from lemma 3.3.6.
Obviously, χ is continuous on $S \setminus \left(\bigcup_{1=k}^{p}\bigcup_{1=j}^{m_k} \wp^j(\delta_k)\right)$. Fix k and $j \in \{1, \cdots, m_k\}$ One just has to check that for $z_0 \in \wp^j(\delta_k)$,

$$\lim_{\substack{z\to z_0\\ z\in\wp^j(\Delta_k)}} \chi(z) = \lim_{\substack{z\to z_0\\ z\in\wp^{j+1}(\Delta_k)}} \chi(z)$$

Let $z_1 = \wp^{-j}(z_0)$. Then

$$\lim_{\substack{z\to z_0\\ z\in\wp^j(\Delta_k)}} \chi(z) = (\psi^\alpha)^j(\chi_k(\wp^{-j}(z_0))) = (\psi^\alpha)^j(\chi_k(z_1)).$$

On the other hand,

$$\lim_{\substack{z\to z_0\\ z\in\wp^{j+1}(\Delta_k)}} \chi(z) = (\psi^\alpha)^{j+1}(\chi_k(\wp^{-j-1}(z_0))) = (\psi^\alpha)^j(\psi^\alpha(\chi_k(\wp^{-1}(z_1)))) = (\psi^\alpha)^j(\chi_k(z_1)),$$

the last equality coming from lemma 3.3.4. By construction, χ is a homeomorphism. □

Lemma 3.3.8. *For all $z \in S$, $\chi \circ \wp(z) = \psi^\alpha \circ \chi(z)$.*

Proof. By lemma 3.3.4, it remains to check the conjugacy for $z \in \bigcup_{1=k}^{p}\bigcup_{j=0}^{m_k} \wp^j(\Delta_k)$. Fix $k \in \{1, \ldots, p\}$, $j \in \{0, \cdots m_k\}$ and $z \in \wp^j(\Delta_k)$. Then:

$$\begin{aligned}\chi \circ \wp(z) = (\psi^\alpha)^{j+1}(\chi_k(\wp^{-j-1}(\wp(z))) &= (\psi^\alpha)^{j+1}(\chi_k(\wp^{-j}(z))\\ &= \psi^\alpha((\psi^\alpha)^j(\chi_k(\wp^{-j}(z))) = \psi^\alpha(\chi(z))\end{aligned} \tag{3.9}$$

which concludes the proof. □

It remains to prove that we can construct ψ such that it satisfies conditions (C1), (C2), (C3) and (C4) of definition 3.3.1. Conditions (C1) and (C2) are obviously realized. To get conditions (C3) and (C4), we have to be more precise about the choice of the functions ξ_k defined on the domains $\mathscr{R}_k$.

Proof of lemma II. For $u \in [0, \frac{1}{p}]$, we set $\delta_u := \{\frac{1}{p} + u\} \times [0,1] \subset \mathscr{R}_1$. Using remark 3.3.2, we see there exists $u_0 \geq u^*$ such that $\psi([0,1], \delta_{u_0}) \subset \overset{\circ}{\mathscr{R}}_1$. Let $u_1 \in]u_0, \frac{1}{p}[$ be such that $\psi([0,1], \delta_{u_0}) \subset [\frac{1}{p} + u_0, \frac{1}{p} + u_1] \times [0,1]$. Set $\beta = \frac{1}{p} - 2u^*$.

As for ξ_1, we choose a continuous and piecewise C^1 function on $\mathscr{R}_1 = [\frac{1}{p} + u_0, \frac{1}{p} + u_1] \times [0,1]$, constant and equal to

$$M := \max(\beta/2, \max_{\substack{1 \leq k \leq p \\ r \in [0,1]}} \lambda_k(ar)\sqrt{(u^*)^2 + \mu_k(ar)}) \tag{3.10}$$

over $[\frac{1}{p} + u_0, \frac{1}{p} + u_1] \times [0,1]$, and we choose the values of ξ_1 for $(\theta, r) \notin [\frac{1}{p} + u_0, \frac{1}{p} + u_1] \times [0,1]$ in order to satisfy the relation $(**)$ for each fixed r, which is possible since $\sigma_{1,2}(r)$ is bounded below by 2. For $k \geq 2$, let us choose

$$\xi_k(\theta, r) = \frac{\beta}{\sigma_{k,k+1}(r)}.$$

We moreover require that

$$\xi_1(\theta, r) \leq \xi_1(\theta, r')$$

if $r \leq r'$ and $\theta \in [\frac{1}{p} + u^*, \frac{2}{p} - u^*]$. Such a choice is obviously possible since the function $\sigma_{1,2}$ is decreasing. Let us check that conditions (C3) and (C4) are realized.

• We begin with the tameness condition (C4). Set $\mathscr{K} = \psi([0,1], \delta_{u_0})$ and fix two points $z = (\theta, r)$ and $z' = (\theta', r)$ in $\mathscr{K}$, on the same orbit, with lifts (x, r) and (x', r) in the universal covering $\mathbb{R} \times [0,1]$. We write as usual $\tilde{\psi}$ for the lifted flow. There exists a unique $t_0 \in]0,1]$ such that $(x', r) = \tilde{\psi}^{t_0}(x, r)$. Now, setting $(x(t), r)$ and $(x'(t), r)$ for $\tilde{\psi}^t(x, r)$ and $\tilde{\psi}^t(x', r)$, the separation function is defined by $E_{z,z'}(t) = x'(t) - x(t)$.

By construction, for any $(\theta, r) \in \mathscr{K}$, one has

- $M \geq \max X_k(\theta', r))$ for any $(\theta', r) \in \mathscr{O}_k$ and any $k \in \{1, \dots, p\}$,
- $M \geq \max \widehat{X}_k(\theta', r))$ for any $(\theta', r) \in \mathscr{R}_k$ and any $k \in \{1, \dots, p\}$.

Therefore,

$$E_{z,z'}(t) \leq \int_t^{t+t_0} M ds = t_0 M, \tag{3.11}$$

Obviously, the maximum of $E_{z,z'}$ is achieved when $t = 0$ and the tameness condition is proved for the fundamental domain $\mathscr{K}$.

• The torsion condition (C3) is easy. It suffices to check that (C3) is satisfied in each domain $\mathscr{O}_k$ and $\mathscr{R}_k$, that is, one has to verify that for $r' \geq r$ in $[0,1]$,

1. for all θ such that $(\theta, r) \in \mathscr{O}_k$, $X_k(\theta, r') > X_k(\theta, r)$,
2. for all θ such that $(\theta, r) \in \mathscr{R}_k$, $\widehat{X}_k(\theta, r) > \widehat{X}_k(\theta', r)$.

The first point is an immediate consequence of the fact that μ_k is an increasing function. The second point is an immediate consequence of the fact that $\sigma_{k,k+1}$ is decreasing. □

d. Proof of Theorem A-bis.

Consider the p-model $\alpha\otimes\psi$ on $\mathscr{A}$. Recall that for any $z\in\mathscr{D}_a\setminus S$, there exists a unique pair $(t_z^-,t_z^+)\in\mathbb{R}^-\times\mathbb{R}^+$, such that $\phi_H(t_z^-,z)\in S$, $\phi_H(t_z^+,z)\in S$ and $\phi_H(]t_z^-,t_z^+[,z)\cap S=\emptyset$. Moreover if $z\in f^{-1}(\rho)$, $t_z^+-t_z^-=\tau(\rho)$.

Consider the map $\widetilde{\chi}:\overline{\mathscr{D}_a}\to\mathscr{A}$ defined by

- $\widetilde{\chi}(z)=(0,\chi(z))$ if $z\in S$
- $\widetilde{\chi}(z)=\alpha\otimes\psi(-t_z^-,\widetilde{\chi}(\phi_H(t_z^-,z)))$

Proof of Theorem A-bis. We will prove that $\widetilde{\chi}$ is a compatible homeomorphism that conjugates ϕ_H and $\alpha\otimes\psi$. We begin by checking the continuity of $\widetilde{\chi}$. Obviously, $\widetilde{\chi}$ is continuous on $\mathscr{D}_a\setminus S$. Let us check the continuity in S. Let $\varepsilon=\frac{1}{3}\operatorname*{Min}_{\rho\in[0,a]}\tau(\rho)$. Let $V^+:=\{z\in\mathscr{D}_a\,|\,t_z^+\in[0,\varepsilon]\}$ and $V^+:=\{z\in\mathscr{D}_a\,|\,t_z^-\in[-\varepsilon,0]\}$. Then $V^+\cup V^-$ is a neighborhood of S and $V^-\cap V^+=S$. Fix $z_0\in S\cap f^{-1}(\rho)$. One has to check that

$$\lim_{\substack{z\to z_0\\ z\in V^+}}\widetilde{\chi}(z)=\widetilde{\chi}(z_0)=\lim_{\substack{z\to z_0\\ z\in V^-}}\widetilde{\chi}(z).$$

Now, if $z\to z_0$ and $z\in V^+$, then $t_z^+\to 0$ and $t_z^-\to-\tau(\rho)$. Therefore

$$\begin{aligned}\lim_{\substack{z\to z_0\\ z\in V^+}}\widetilde{\chi}(z)&=\lim_{\substack{z\to z_0\\ z\in V^+}}\alpha\otimes\psi(-t_z^-,\widetilde{\chi}(\phi_H(t_z^-,z)))\\
&=\alpha\otimes\psi\left(-\tau(\rho),(0,\chi(\phi_H(-\tau(\rho),z_0)))\right)\\
&=\alpha\otimes\psi(-\tau(\rho),(0,\chi\circ\wp^{-1}(z_0)))\\
&=(0,\psi^\alpha\circ\chi\circ\wp^{-1}(z_0))\\
&=(0,\chi\circ\wp\circ\wp^{-1}(z_0))\\
&=(0,\chi(z_0)).\end{aligned}$$

On the other hand, if $z\to z_0$ and $z\in V^-$, then $t_z^-\to 0$. Therefore

$$\lim_{\substack{z\to z_0\\ z\in V^-}}\widetilde{\chi}(z)=\alpha\otimes\psi(0,\widetilde{\chi}(\phi_H(0,z_0))=\alpha\otimes\psi(0,(0,\chi(z_0)))=(0,\chi(z_0)).$$

By construction, $\widetilde{\chi}$ is a homeomorphism. It remains to check that it conjugates $\alpha\otimes\psi$ and ϕ_H. Fix $z\in\mathscr{D}_a\cap f^{-1}(\{\rho\})$ and $t\in\mathbb{R}$. Let $m\in\mathbb{Z}$ and $s\in[0,\tau(\rho)[$ such that $t=m\tau(\rho)+s$. Let $z_0:=\phi_H(t_z^-,z)\in S$. Then

$$\begin{aligned}\widetilde{\chi}\circ\phi_H(t,z)&=\widetilde{\chi}\circ\phi_H(m\tau(\rho)+s,z)\\
&=\widetilde{\chi}\circ\phi_H(m\tau(\rho)+s-t_z^-,z_0)\\
&=\widetilde{\chi}\circ\phi_H(s-t_z^-,\wp^m(z_0))\\
&=\alpha\otimes\psi(s-t_z^-,(0,\chi\circ\wp^m(z_0)))\\
&=\alpha\otimes\psi(s-t_z^-,(0,(\psi^\alpha)^m\circ\chi(z_0)))\\
&=\alpha\otimes\psi(s-t_z^-+m\tau(\rho),(0,\chi(z_0)))\\
&=\alpha\otimes\psi(s+m\tau(\rho),(\alpha\otimes\psi(-t_z^-,(0,\chi(z_0))))\\
&=\alpha\otimes\psi(s+m\tau(\rho),(\alpha\otimes\psi(-t_z^-,\widetilde{\chi}(\phi_H(t_z^-,z))))\quad=\alpha\otimes\psi(t,\widetilde{\chi}(z)),\end{aligned}$$

which concludes the proof. ☐

3.3.3 The polynomial entropy of a p-model system

This section is devoted to the proof of proposition 3.3.1 in the precise setting we have now at our disposal: namely, the p-model system at hand will be that we constructed in the previous sections. The main remark is that we will have the possibility to choose the minimal period of the p-model by simply reducing the parameter a of our partial neighborhood. We first emphasize the following remarks.

Remark 3.3.3. 1) For $r \in]0,1]$, we denote by $T(r)$ the period of the orbit $\mathbb{T} \times \{r\}$. By condition (C1)

$$T(r) \sim_{r=0} -\sum_{\ell=1}^{p-1} \frac{\lambda_\ell(r)}{\ln(\mu_\ell(r))} \sim_{r=0} -\sum_{\ell=1}^{p-1} \frac{\lambda_\ell(r)}{\ln r},$$

the last equivalent coming from $\mu_k(0)' \neq 0$.
2) Let β be such that $[\frac{k}{p}-\beta, \frac{k}{p}+\beta] \subset \mathscr{O}_k$, for any $1 \leq k \leq p$, and set

$$\tau_k(r,\beta) := \int_{\frac{k}{p}+\beta}^{\frac{k}{p}-\beta} \frac{d\theta}{\lambda_k(r)\sqrt{(\frac{k}{p}-\theta)^2+\mu_k(r)}}.$$

One has:

$$\frac{T(r)}{\tau_k(r,\beta)} \sim_{r=0} \sum_{\ell=1}^{p-1} \frac{\lambda_\ell(0)}{\lambda_k(0)}.$$

3) Due to the torsion condition, $r \mapsto T(r)$ is a strictly decreasing function from $]0,1]$ to $[q^*,+\infty[$, where q^* is the period of the motion ψ on $\mathbb{T} \times \{1\}$.

Notation 3.3.2. *For the sake of simplicity, in the whole proof of proposition 3.3.1, we write ϕ^t instead of $(\alpha \otimes \psi)^t$.*

Proof of proposition 3.3.1. We denote by a an element $(\theta,r) \in \widehat{\mathscr{A}}$. For $\varphi \in \mathbb{T}$ and $k \in \mathbb{N}$, we set $\varphi_r(k) := \varphi + k\alpha(r)$. Therefore

$$\phi^k(\varphi,a) = (\varphi_r(k), \psi^k(a)).$$

We denote by δ the natural quotient distance on $\mathbb{T}$ and by $\hat{d}$ the product metric on $\mathbb{T} \times [0,1]$. We denote by $d := \delta \times \hat{d}$, the product metric on $\mathscr{A}$. For $k \in \mathbb{N}$, we denote by d_k the dynamical distances in $\mathbb{T}^2 \times [0,1]$ associated with the motion ϕ and by $\hat{d}_k$ the distances in $\mathbb{T} \times [0,1]$ associated with ψ.

Remark that for φ in $\mathbb{T}$ and $a=(\theta,r), a'=(\theta',r')$ in $A \subset \mathbb{T} \times [0,1]$, one has:

$$\begin{aligned} d_N^\phi((\varphi,a),(\varphi',a')) &= \max_{0\leq k\leq N} d(\phi^k(\varphi,a),\phi^k(\varphi',a')) \\ &= \max_{0\leq k\leq N} \max\Big(\delta(\varphi_r(k),\varphi'_{r'}(k)), \hat{d}(\psi^k(a),\psi^k(a'))\Big) \\ &= \max\Big(\max_{0\leq k\leq N}\left(\delta(\varphi_r(k),\varphi'_{r'}(k)\right), \max_{0\leq k\leq N} \hat{d}(\psi^k(a),\psi^k(a'))\Big) \\ &= \max\Big(\max_{0\leq k\leq N}\left(\delta(\varphi_r(k),\varphi'_{r'}(k)\right), \hat{d}_N(a,a')\Big) \qquad (\star) \end{aligned}$$

Let us introduce some notation.
– Given $r \in]0,1]$, we set $C^r := \mathbb{T} \times \{r\} \subset \widehat{\mathscr{A}}$.

– According to remark 3.3.3 one can label the orbits C^r by their period: we write C_q the orbit with period q for $q \in [q^*, +\infty[$. So $C^r = C_{T(r)}$. We write C_∞ for the boundary $\mathbb{T} \times \{0\}$.

– Given two periods $q' \leq q \leq +\infty$, we denote by $\widehat{S}_{q,q'}$ the annulus $\subset \widehat{\mathscr{A}}$ bounded by the curves C_q and $C_{q'}$.

– When a and b are two points on the same curve C_q, we denote by $[a, b]$ the set of all points of C_q located between a and b, relatively to the direct orientation of C_q.

– We denote respectively by ϕ and by ψ the time-one maps of the flows $(\phi^t)_t$ and $(\psi^t)_t$.

a. Proof of $\mathrm{h_{pol}}(\phi) \geq 2$

Given $\varepsilon > 0$, we want to find, for N large enough, a (N, ε)-separated set (relatively to ϕ) with cardinal $\geq c_0 N^2$ for a constant $c_0 > 0$.

It suffices to find a (N, ε)-separated in $\mathbb{T} \times [0, 1]$ with cardinal $\geq c_0 N^2$ relatively to ψ. Indeed assume we are given such a set $A(N, \varepsilon) \in \mathbb{T} \times [0, 1]$. Then by $(\star)$, since $\hat{d}_N(a, a') \geq \varepsilon$, one has $d_N((\varphi, a), (\varphi, a')) \geq \varepsilon$.

Fix a vertical segment $I_{\theta_0} := \{\theta = \theta_0\} \subset \mathbb{T} \times [0, 1]$. For $q \in [q^*, +\infty]$, we set $a_q := I_{\theta_0} \cap C_q$. We choose θ_0 such that a_∞ is not a singular point of V, that is $a_\infty \neq (\frac{k}{p}, 0)$, so $\psi(a_\infty) \neq a_\infty$ and $\psi^{-1}(a_\infty) \neq a_\infty$.

Remark 3.3.4. Due to the torsion condition, for each $q \geq 3$ the projection on C_∞ of the interval $[\psi(a_q), \psi^{[q/2]}(a_q)]$ is contained in $[\psi(a_\infty), \psi^{-1}(a_\infty)]$.

Assume the two (redondant) conditions

$$\varepsilon < \min\Big(\hat{d}(a_\infty, \psi(a_\infty)), \hat{d}(a_\infty, \psi^{-1}(a_\infty))\Big) \quad (1)$$

and that

$$\varepsilon < \min\Big(\hat{d}(a_\infty, \psi^{-1}(a_\infty)), \hat{d}(\psi(a_\infty), \psi^2(a_\infty)\Big) \quad (2).$$

In the following we assume that $q \geq 3$.

Step 1: *If $N \geq q$, C_q contains an (N, ε)-separated set with cardinal* $[q/2]$.

Proof. Fix $N \geq q$. For $k \geq 0$ we set $a^{(k)} = \psi^{-k}(a_q)$. Since $\psi^{k'-k}(a_q) \in [\psi(a_q), \psi^{[q/2]}(a_q)]$, by (1) and remark 3.3.4:

$$\hat{d}_N(a^{(k)}, a^{(k')}) \geq \hat{d}(\psi^{k'}(a^{(k)}), \psi^{k'}(a^{(k')})) = \hat{d}(\psi^{k'-k}(a_q), a_q) > \varepsilon, \quad \forall 0 \leq k < k' \leq [q/2].$$

Therefore, the set $\{a^{(k)} \mid 1 \leq k \leq [q/2]\}$ is (N, ε)-separated ☐

Step 2: *If $N \geq 18$ and $(q, q') \in [\frac{N}{3}, \frac{N}{2}]^2$ with $3 \leq q - q' \leq q - 3$, the pairs of points $(a, a') \in C_q \times C_{q'}$ are (N, ε)-separated.*

Proof. Let us introduce the domains:

$$I_q = [a_q, \psi(a_q)[\subset C_q, \qquad J_{q'} = [\psi^{-1}(a_{q'}), \psi^2(a_{q'})[\subset C_{q'}.$$

Thanks to the torsion condition, by (2), the distance between I_q and the complement $C_{q'} \setminus J_{q'}$ is larger than ε, for each pair (q, q') in $[q^*, +\infty[$.

Now assume that q and q' are contained in the interval $[N/3, N/2]$ and satisfy $q - q' \geq 3$. Consider two points $a \in C_q$ and $a' \in C_{q'}$. There exists a unique integer $n_0 \in \{0, \ldots, q-1\}$ such that $\psi^{n_0}(a) \in I_q$. Thus:

(i) if $\psi^{n_0}(a') \in C_{q'} \setminus J_{q'}$, then $d_N(a, a') \geq d_{n_0}(a, a') > \varepsilon$.

(ii) if $\psi^{n_0}(a') \in J_{q'}$, observe that $\psi^{q-q'}(J_{q'}) \cap J_{q'} = \emptyset$. Indeed,

$$\psi^{q+n_0}(a') = \psi^q(\psi^{n_0}(a')) = \psi^{q-q'}(\psi^{n_0}(a')),$$

with $q' \geq 6$ and $3 \leq q - q' \leq q - 3$ by assumption on N and q, q'. So $\psi^{q+n_0}(a') \notin J_q$ and, by periodicity, $\psi^{q+n_0}(a) \in I_q$.

Therefore, using (2): $\hat{d}_N(a, a') \geq \hat{d}(\psi^{q+n_0}(a), \psi^{q+n_0}(a')) > \varepsilon$. $\square$

Step 3: *If $N \geq 18$, $S_{N/3,N/2}$ contains a (N, ε)-separated set with cardinal $\geq N^2/108$.*

Proof. In the interval $[N/3, N/2]$, there exist at least $[N/18]$ distinct integers (q_i) with $q_j - q_i \geq 3$ if $i \neq j$. On each curve C_{q_i}, one can find an (N, ε)–separated subset with $[q_i/2] \geq [N/6]$ elements by step 1, and the union of all these subsets is still (N, ε)–separated by step 2. Therefore the strip limited by the curves $C_{N/3}$ and $C_{N/2}$ contains a (N, ε)–separated subset $A(N, \varepsilon)$ with more than $N^2/108$ elements. $\square$

Conclusion: Fix $\varphi \in \mathbb{T}$. The set $\{(\varphi, a) \,|\, a \in A(N, \varepsilon)\}$ is (N, ε)-separated with cardinal $\geq N^2/108$. Therefore, for $N > 18$, $S_N^{\phi}(\varepsilon) \geq N^2/108$ and $\mathrm{h_{pol}}(\phi) \geq 2$.

b. Proof of $\mathrm{h_{pol}}(\phi) \leq 2$

The main idea of the construction will be to take advantage of the explicit coverings constructed in [Mar09] for planar p-models and to prove that in the product system on $\mathscr{A}$, the domains involved in this covering are so small that they induce a negligible distorsion in the φ variable. So we will be able to construct a covering for the present p-model simply by taking the product of the domains of the planar p-model with small enough intervals in the φ-direction.

If A is a subset of $\mathscr{A}$ invariant by ϕ, for $n \geq 1$, we denote by $D_n^{\phi}(A, \varepsilon)$ the minimal cardinal of a covering of A by subsets with d_N-diameter ε. We define in the same way, for and $\widehat{A} \subset \widehat{\mathscr{A}}$ invariant by ψ, the number $D_n^{\psi}(\widehat{A}, \varepsilon)$. Given $\varepsilon > 0$, we want to get, for N large enough, a majoration of the form $D_N^{\phi}(\mathscr{A}, \varepsilon) \leq c_0 N^2 + c_1 N + c_2$. To do this, we discriminate between the behavior of the dynamics near $r = 0$ and the behavior of the dynamics near $r = 1$. We split $\widehat{\mathscr{A}}$ into two N-depending sub-annuli $\widehat{\mathscr{A}_N}$ and $\widehat{\mathscr{A}_N^*}$ and we estimate $D_N^{\phi}(\mathscr{A}_N, \varepsilon)$ and $D_n^{\phi}(\mathscr{A}_N^*, \varepsilon)$, where $\mathscr{A}_N := \mathbb{T} \times \widehat{\mathscr{A}_N}$ and $\mathscr{A}_N^* := \mathbb{T} \times \widehat{\mathscr{A}_N^*}$. Fix $\varepsilon > 0$.

b-1. *Construction of the sub-annuli $\widehat{\mathscr{A}_N}$ and $\widehat{\mathscr{A}_n^*}$.* The choice of the cutoff is based on the following lemma.

Lemma 3.3.9. *For $k \in \{1, \ldots, p\}$, let $B_k(\varepsilon)$ be the "block" of $\widehat{\mathscr{A}}$ limited by the vertical segments Δ_k^+ and Δ_k^- of equations $\theta = k/p - \varepsilon/2$ and $\theta = k/p + \varepsilon/2$ respectively. There exists a constant κ and an integer N_0 (both depending on ε) such that if $N \geq N_0$, for each index $k \in \{0, \ldots, p-1\}$:*

$$\psi^n(\Delta_k^-(\kappa N)) \subset B_k(\varepsilon), \qquad \forall n \in \{0, \ldots, N\},$$

where we write $\Delta_k^-(q)$ for the intersection of the left vertical Δ_k^- of $B_k(\varepsilon)$ with the annulus $\widehat{S}_{\infty,q}$.

Proof. With the notation of remark 3.3.3 (2), we set

$$\kappa := \left[2 \max \sum_{\ell=1}^{p-1} \frac{\lambda_\ell(0)}{\lambda_k(0)} \right] + 1$$

Then, by remark 3.3.3, there exists $r_0 > 0$ such that for $r < r_0$, $\tau_k(r, \varepsilon/2) > \frac{1}{\kappa}T(r)$, that is,

$$\psi^{\frac{1}{\kappa}T(r)}(\Delta_k^-(T(r)) \in B_k(\varepsilon).$$

The lemma is proved with $N_0 := \lceil T(r_0) \rceil$. $\square$

For $q \in [q^*, +\infty[$, we write $r(q) := T^{-1}(q) \in]0, 1]$. By remark 3.3.3 (3), there exists $N_1 \in \mathbb{N}^*$ such that if $q > N_1$, $r_q \leq \varepsilon$.

We choose $N \geq \mathrm{Max}\,(N_0, N_1)$ and we set: $\widehat{\mathscr{A}_N} = \widehat{S}_{\infty,\kappa N}$, $\widehat{\mathscr{A}_N^*} = \widehat{\mathscr{A}} \setminus \widehat{\mathscr{A}_N}$, $\mathscr{A}_N := \mathbb{T} \times \widehat{\mathscr{A}_N}$ and $\mathscr{A}_N^* := \mathbb{T} \times \widehat{\mathscr{A}_N^*}$.

We will use twice the following easy remark.

Remark 3.3.5. By remark 3.3.3 (1), for q^* large enough, there exists $\bar{c} \in \mathbb{R}$ such that

$$r'(q) \geq e^{-\bar{c}q}.$$

b-2. *Covering of* $\mathscr{A}_N$. We begin by constructing a covering of $\widehat{\mathscr{A}_N}$. For $k \in \{1, \ldots, p-1\}$ and $r \in [0, 1]$, we write $a_k(r) := (\frac{k}{p} + \frac{\varepsilon}{2}, r) = \Delta_k^+ \cap C^r$. We set

$$\nu_k := \mathrm{Min}\,\{\ell \in \mathbb{N}^* \mid \psi^\ell(a_k(0)) \in B_{k+1}(\varepsilon)\} \quad \text{and} \quad \nu = \mathrm{Max}_{\mathrm{k}}\, \nu_{\mathrm{k}}.$$

By the torsion property, and since $\tau_{k+1}(r, \varepsilon/2) \to \infty$ when $r \to 0$, there exists r^* such that if $r \in [0, r^*]$, $\psi^\nu(a_k(r)) \in B_{k+1}(\varepsilon)$, for all $k \in \{1, \cdots, p\}$. We set $N_2 := \frac{1}{\kappa}\lceil T(r^*) \rceil$. Then for $N \geq N_2$,

$$\psi^\nu(\Delta_k^+(\kappa N)) \subset B_{k+1}(\varepsilon). \tag{**}$$

We set $\mathscr{B} = \cup_{1 \leq k \leq p} B_k(\varepsilon)$. By compactness, there exists a finite covering $B_1, \ldots, B_{i^*}$ of $\widehat{\mathscr{A}_N} \setminus \mathscr{B}$ with $\hat{d}_\nu$–diameter $\leq \varepsilon$. Moreover, one can obviously assume that each B_i is contained in some connected component of $\widehat{\mathscr{A}_N} \setminus \mathscr{B}$.

We claim that, if $N > \mathrm{Max}\,(N_0, N_1, N_2)$, for any n in $\{\nu, \ldots, N\}$, $\psi^n(B_i)$ is contained in some $B_k(\varepsilon)$. Indeed, assume that B_i is contained in the zone limited by the curves $\Delta_k^+(\kappa N)$ and $\Delta_{k+1}^-(\kappa N)$ (according to the direct orientation on $\mathbb{T}$). Then the iterate $\psi^n(B_i)$ is contained in the region limited by $\psi^\nu(\Delta_k^+(\kappa N))$ and $\psi^N(\Delta_{k+1}^+(\kappa N))$. Both of these boundaries are contained in $B_{k+1}(\varepsilon)$, the first one by $(**)$ and the second one by lemma 3.3.9.

By assumption on N, the d–diameter of $B_{k+1}(\varepsilon)$ is ε. Therefore,

$$\mathrm{diam}_N(B_i) \leq \varepsilon$$

where we denote by diam_N the diameter associated with the distance $\hat{d}_N$.

Now, we can assume that ε is small enough and N is large enough so that the intersection $\psi(\Delta_k^-(\kappa N)) \cap B_{k+1}(\varepsilon)$ is empty. Consider the regions $\mathscr{U}_k$ in $\widehat{\mathscr{A}_N}$ bounded by $\Delta_k^+(\kappa N)$ and $\psi(\Delta_k^+(\kappa N))$.

The nonempty intersections $\mathscr{U}_k \cap B_i$ form a finite covering $U_1, \ldots, U_{i^{**}}$ of the union $\cup_{1 \leq k \leq p} \mathscr{U}_k$, with cardinal $i^{**} \leq i^*$. For $1 \leq i \leq i^{**}$, one has:

$$\mathrm{diam}_N(U_i) \leq \varepsilon.$$

Let $\mathscr{V}_k$ be the region bounded by $\psi^{-N}(\Delta_k^+(\kappa N))$ and $\Delta_k^+(\kappa N)$ (relatively to the direct orientation of $\mathbb{T}$). By the same arguments as in the beginning, one sees that $\mathscr{V}_k \subset B_k(\varepsilon)$. Moreover the inverse images

$$B_{n,i} = \psi^{-n}(U_i), \qquad 1 \leq n \leq N,\ 1 \leq i \leq i^{**},$$

form a covering of the union $\mathscr{V} = \cup_{k\in\mathbb{Z}_p}\mathscr{V}_k$ which is contained in $B_k(\varepsilon)$). By construction, each of the $B_{n,i}$ satisfies $\operatorname{diam}_N B_{n,i} \leq \varepsilon$.

Finally, observe that for each k, the complement $B_k(\varepsilon)\setminus\mathscr{V}$ satisfies

$$\psi^n(B_k(\varepsilon)\setminus\mathscr{V}) \subset B_k(\varepsilon)$$

for $0 \leq n \leq N$, and therefore $\operatorname{diam}_N(B_k(\varepsilon)\setminus\mathscr{V}) \leq \varepsilon$. Hence, the subsets

$$(B_i)_{1\leq i\leq i^*},\quad (B_{n,i})_{1\leq n\leq N,\ 1\leq i\leq i^{**}},\quad (B_k(\varepsilon)\setminus\mathscr{V})_{1\leq k\leq p},$$

form a covering $\mathscr{C}(N,\varepsilon)$ of $\widehat{\mathscr{A}_N}$ with subsets of $\hat{d}_N$–diameter $\leq \varepsilon$ and

$$\operatorname{Card}\mathscr{C}(N,\varepsilon) \leq i^* + Ni^{**} + p.$$

We use $(\star)$ to construct a covering of $\mathscr{A}_N$ with d_N-balls of radius ε. Fix an element B of $\mathscr{C}(N,\varepsilon)$. Let $a=(\theta,r), a'=(\theta',r')$ be two elements of B and let $(\varphi,\varphi')\in\mathbb{T}^2$, By $(\star)$, one has

$$d_N((\varphi,a),(\varphi',a')) = \max\left(\max_{0\leq k\leq n}\left(\delta(\varphi_r(k),\varphi'_{r'}(k)\right),\varepsilon\right).$$

Now, for any $1\leq k\leq N$,

$$\delta(\varphi_r(k),\varphi'_{r'}(k)) \leq \delta(\varphi,\varphi') + k|\alpha(r)-\alpha(r')| \leq \delta(\varphi,\varphi') + N\max_{[0,1]}\alpha'(r)|r-r'|.$$

By remark 3.3.5, if q^* is large enough (which is always possible, reducing if necessary the width of the partial neighborhood of the initial polycycle), there exists N_3 such that for all $N\geq N_3$,

$$N\max_{[0,1]}|\alpha'(r)||r-r'| \leq N\max_{[0,1]}|\alpha'(r)|\ r(\kappa N) \leq \varepsilon/2.$$

Therefore, if I is a subset of $\mathbb{T}$ with δ-diameter $\varepsilon/2$, the product $I\times B$ with $B\in\mathscr{C}(N,\varepsilon)$ is a subset with d_N-radius ε as soon as $N\geq\max(N_0,N_1,N_2,N_3)$. This yields a cover of $\mathscr{A}_N$ with subsets with d_N-diameter ε of cardinal $\leq \frac{2}{\varepsilon}(i^*+Ni^{**}+p)$.

b-3. *Covering of* $\mathscr{A}_N^*$. Set $k^*=[\kappa N/q^*]-1$. For $1\leq k\leq k^*$, we set $\widehat{\mathbf{S}}_k := \widehat{S}_{\frac{\kappa N}{k},\frac{\kappa N}{(k+1)}}$. Clearly, the family $(\widehat{\mathbf{S}}_k)$ covers $\widehat{\mathscr{A}_N^*}$. Assume we are given a minimal covering $\mathscr{C}_k(N,\varepsilon)$ of $\widehat{\mathbf{S}}_k$ with subset of $\hat{d}_N$-diameter smaller than ε. To form a covering of the complete annulus $\mathscr{A}$, we will see that it is enough to take the product of the elements of $\mathscr{C}_k(N,\varepsilon)$ with small enough intervals in the φ direction. Let us first construct the covering $\operatorname{Card}\mathscr{C}_k(N,\varepsilon)$, since the form of its elements will play a crucial role.

Lemma 3.3.10. *Consider an integer $m\geq q^*$, and fix $\varepsilon>0$. There exists positive constants c_1 and c_2, depending only on ε, such that if the pair $(q,q')\in[q^*,m]^2$ satisfy*

$$0\leq q'-q\leq \frac{c_1\,\varepsilon}{[m/q]},$$

then the sub-annulus $\widehat{S}_{q,q'}$ satisfies

$$D_m^{\psi}(\widehat{S}_{q,q'},\varepsilon) \leq c_2\,q.$$

Proof. We first construct a covering of a single curve C_q, then we fatten it a little bit to get a covering of a thin strip $S_{q,q'}$. We will use condition (C3). Recall that $\mathscr{K}$ is the fundamental domain for the tameness condition.

Fix $q \in [q^*, m]$. Let λ be the Lipschitz constant of Ψ on the compact set $[-1,1] \times \mathscr{K}$. Let I_q be the interval $C_q \cap \mathscr{K}$. Consider two points $a \leq a'$ contained in I_q. Then by the tameness property the maximum μ of the separation function $E_{a,a'}$ is achieved for t such that $\psi_t(a)$ and $\psi_t(a')$ are located inside I_q. Therefore there exists $t_0 \in [-1,1]$ and $n \in \mathbb{N}$ such that $t = t_0 + nq$. As a consequence, $\mu \leq \lambda\, d(a,a')$.

Hence, for all $k \in \mathbb{N}$, $d_k(a,a') \leq \lambda\, d(a,a')$.

For $q \geq q^*$ we set $j_q^* := [\frac{\operatorname{diam} I_q}{\varepsilon/(2\lambda)}] + 1$ and we cover I_q by consecutive subintervals $J_1, \ldots, J_{j_q^*}$ of d–diameter $\varepsilon/(2\lambda)$. As $I_q, \psi(I_q), \ldots, \psi^{[q]}(I_q)$ is a covering of C_q, one sees that the intervals $I_{ij} = \psi^i(J_j)$, $0 \leq i \leq [q], 1 \leq j \leq j_q^*$ form a covering of C_q by subsets of d_k–diameter $\leq \varepsilon/2$, for *each* integer k. Indeed, if a, a' lie in $I_{ij} \subset I_q$, $d_k(a,a') \leq \lambda\varepsilon/(2\lambda)$.

Due to the torsion property, for $q \leq q^*$, $j_q^* \geq j_{q^*}^*$. We set $c_2 = 2j_{q^*}^*$. Then, for each $q \in [q^*, m[$, each orbit C_q admits a covering by at most $c_2\, q$ subsets whose $\hat{d}_k$–diameter is smaller than $\varepsilon/2$, for any positive integer k.

Fix now an integer $m \geq q^*$. Given the initial period q, we want to find a period $q' \leq q$ such that for any pair of points $a \in C_q$ and $a' \in C_{q'}$ *with the same abscissa* θ, the (maximal) difference of the abscissas of any pair of iterates $\psi^n(a)$ and $\psi^n(a')$, $n \in \{0, \ldots, m\}$, is at most $\varepsilon/2$.

Assume this is done and consider again the covering of C_q by the intervals I_{ij}. Fix such an interval $I_{ij} := [\theta^-, \theta^+]$ and let R_{ij} be the rectangle limited by the curves C_q and $C_{q'}$ and the vertical lines $\theta = \theta^-$ and $\theta = \theta^+$. Fix a lift to the universal covering $\mathbb{R} \times [0,1]$ and consider the associated the lifted flow $\widetilde{\psi}_t$. For $a \in R_{ij}$ and $t > 0$, we set $(x(t), r) = \widetilde{\psi}_t(\tilde{a})$. We set $a^- = (\theta^-, r_q)$ and $a^+ = (\theta^+, r_{q'})$. By torsion property, one has (with obvious notation):

$$x^-(t) \leq x_1(t) < x_2(t) \leq x^+(t), \quad \forall\, a_1, a_2 \in R_{ij}, \quad \forall t > 0.$$

Therefore, $\hat{d}_m(a_1,a_2) \leq \hat{d}_m(a^-,a^+)$ and the rectangles R_{ij} have $\hat{d}_m$-diameter less than ε. They form a covering of the strip $\widehat{S}_{q,q'}$ with at most $c_2\, q$ elements.

So it remains to choose q' close enough to q. Fix two points a, a' located in the same vertical and denote by $\tilde{a}$ and $\tilde{a}'$ two lifts (located in the same vertical). As before, we set $\tilde{\psi}_s(\tilde{a}) = (x(s), r)$, $\tilde{\psi}_s(\tilde{a}') = (x'(s), r')$, so $r' \geq r$ since $q' \leq q$. Given $t \geq 0$, we denote by t' be the time needed for the point a' to reach the vertical through $a(t)$. So t' is defined by the equality

$$x'(t') = x(t).$$

We set $\Delta(a,t) = t - t'$ so, by the torsion property, $D(a,t) \geq 0$. One easily checks that

$$\Delta(a, t_1 + t_2) = \Delta(a, t_1) + \Delta(\psi_{t_1}(a), t_2), \quad \Delta(a, kt) = \sum_{\ell=0}^{k} \Delta(\psi_\ell(a), t).$$

The first equality shows that $t \mapsto \Delta(a, .)$ is an increasing funtion. When $a \in C_q$, the second equality yields $\Delta(a, kq) = k\Delta(a,q)$. It is also easy to see that

$$\Delta(a,q) = q - q', \quad \forall a \in C_q.$$

Now set $\ell' := \max(\ell, \max_{\substack{(\theta,r)\in\mathscr{O}_k \\ 1\leq k\leq p}} V_k(\theta,r))$, where is ℓ defined by Condition (C1). Then:

$$0 \leq x'(t) - x(t) \leq \ell'\, D(a,t).$$

For $a \in C_q$, one has:

$$\Delta(a,m) \leq \Delta(a,([\frac{m}{q}]+1)q) = ([\frac{m}{q}]+1)(q-q').$$

Consequently, for $0 \leq n \leq m$,

$$0 \leq x'(n) - x(n) \leq \ell'\,\Delta(a,n) \leq \ell'\,\Delta(a,m) \leq \ell'\,([\frac{m}{q}]+1)(q-q').$$

which proves our statement for $c_1 = \frac{1}{\ell'}$. □

We want to apply lemma 3.3.10 with $m = \kappa N$ and $q \in]\frac{\kappa N}{(k+1)}, \frac{\kappa N}{k}]$ for $1 \leq k \leq k^*$. Fix k and assume that $q \in]\frac{\kappa N}{(k+1)}, \frac{\kappa N}{k}]$. Then

$$\Big[\frac{\kappa N}{q}\Big] = k.$$

Therefore, by lemma 3.3.10, if $q' - q \leq c_1\varepsilon/k$, the strip $\widehat{S}_{q,q'}$ satisfies

$$D^{\psi}_{\kappa N}(\widehat{S}_{q,q'},\varepsilon) \leq c_2\, q \leq c_2\,\frac{\kappa N}{k}.$$

This upper bound is therefore constant on $\widehat{\mathbf{S}}_k$. Now the strip $\widehat{\mathbf{S}}_k$ is covered by the strips $(\widehat{S}_{q_{i+1},q_i})_{0\leq i\leq i^*(k)}$, with

$$q_i = \frac{\kappa N}{k+1} + i\frac{c_1\varepsilon}{k}, \qquad i^*(k) = \Big[\frac{\kappa\, N}{c_1\,\varepsilon\,(k+1)}\Big] + 1 \leq c_3\frac{\kappa\, N}{c_1\,\varepsilon\, k}$$

for $c_3 > 0$ large enough. So we can choose $\mathscr{C}_k(N,\varepsilon)$ such that

$$\mathrm{Card}\,\mathscr{C}_k(N,\varepsilon) \leq D^{\psi}_N(\widehat{\mathbf{S}}_k,\varepsilon) \leq c_2\,\frac{\kappa N}{k}\, i^*(k) \leq c_\varepsilon\frac{N^2}{k^2}, \qquad c_\varepsilon = \frac{c_2 c_3\kappa^2}{c_1\varepsilon}.$$

Now let us estimate the maximal width Δr of the substrips $(\widehat{S}_{q_{i+1},q_i})_{0\leq i\leq i^*(k)}$ in the r variable. Ther width in the q variable is $c_1\varepsilon/k$ and their are contained in the strip $\widehat{\mathbf{S}}_k$, whose minimal period is $\kappa N/(k+1)$. Therefore, according to the estimate on $r'(q)$ if q^* is assumed to be large enough (which, as observed above is always possible):

$$\Delta r \leq \frac{c_1\varepsilon}{k}\, e^{-\bar{c}\frac{\kappa N}{k+1}} \leq \frac{q^* c_1\varepsilon}{\kappa N} e^{-\bar{c}q^*/2}.$$

For q^* large enough, this width satisfies

$$\Delta r \max_{[0,1]} |\alpha'(r)| \leq \frac{\varepsilon}{2N}.$$

Therefore N iterations of two points (φ,x) and (φ',x') with x and x' in the same domain of the covering $\mathscr{C}_k(N,\varepsilon)$ produce a distorsion of at most $\varepsilon/2$ in the φ-direction. As a consequence, one gets a covering by subsets of d_N-diameter less than ε by taking the products of the elements of $\mathscr{C}_k(N,\varepsilon)$ by intervals of uniform length $\varepsilon/2$ in the φ-direction.

Finally, since the strips $\mathbf{S}_k$ cover $\mathscr{A}^*_N$, one gets

$$D^{\phi}_N(\mathscr{A}^*_N,\varepsilon) \leq \frac{2}{\varepsilon}\sum_{k=1}^{k^*} D^{\psi}_N(\widehat{\mathbf{S}}_k,\varepsilon) \leq \frac{2}{\varepsilon}\sum_{k=1}^{\infty} c_\varepsilon\frac{N^2}{k^2} = \alpha_\varepsilon N^2,$$

with $\alpha_\varepsilon = \frac{2}{\varepsilon} c_\varepsilon \zeta(2)$.

b-4. *Conclusion:* Gathering steps 2 and 3, one has:

$$D_N^\phi(\mathscr{A}, \varepsilon) \leq D_N^\phi(\mathscr{A}_N^*, \varepsilon) + D_N^\phi(\mathscr{A}_N, \varepsilon) \leq \tilde{\alpha}_\varepsilon N^2 + \frac{2}{\varepsilon}(i^* + N i^{**} + p),$$

for any $N \geq \max(N_0, N_1, N_2, N_3, N_4, N_5)$. Therefore $\mathrm{h}_{\mathrm{pol}}(\phi) \geq 2$. $\square$

Chapter 4

When the volume has polynomial growth

Here, we state and prove a polynomial analogue of the classical Manning inequality relating the topological entropy of a geodesic flow with the growth rate of the volume of balls in the universal covering. Let (M, g) be a compact Riemannian manifold and let $\tau(x)$ be defined by

$$\tau(x) = \inf \left\{ s \geq 0 \,|\, \limsup_{r \to \infty} \frac{1}{r^s} \operatorname{Vol} B(x,r) = 0 \right\},$$

where $B(x,r)$ is the ball in the universal cover $\widetilde{M}$ centered at x and of radius r. We will show in section 4.1.2 that $\tau(x)$ is independent of x: it is the degree of growth of the fundamental group $\pi_1(M)$. We denote it by $\tau(M)$. Section 4.1.3 is devoted to the proof of the following result.

Theorem B. *Let ϕ_g be geodesic flow restricted to the unit tangent bundle SM. Then:*

$$\tau(M) \leq \mathrm{h_{pol}}(\phi_g) + 1.$$

As a consequence, since $\tau(\mathbb{T}^n) = n$, the polynomial entropy of a geodesic flow on $\mathbb{T}^n$ is larger than $n - 1$. For a flat metric on $\mathbb{T}^n$, this inequality becomes an equality.

In a second part, we study the particular class of tori of revolution defined as follows:

$$\mathscr{T}_{\mathscr{M}} := \{\Sigma_{x,y}(\mathbb{R}^2) \,|\, (x,y) \in \mathscr{P}^+_{\mathscr{M}} \times \mathscr{P}\} \subset \{\Sigma_{x,y}(\mathbb{R}^2) \,|\, (x,y) \in \mathscr{P}^+ \times \mathscr{P}\} := \mathscr{T} \tag{4.1}$$

where $\mathscr{P}$ is the space of 1–periodic smooth functions $x : \mathbb{R} \to \mathbb{R}$, $\mathscr{P}^+$ the subspace of $\mathscr{P}$ of positive functions, $\mathscr{P}^+_{\mathscr{M}}$ the subset of Morse functions $x \in \mathscr{P}^+$ such that any critical value is reached once, and where $\Sigma_{x,y}$ is defined by

$$\begin{array}{rccl} \Sigma_{x,y} & : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & & (\varphi, s) & \mapsto & (x(s)\cos 2\pi\varphi,\, x(s)\sin 2\pi\varphi,\, y(s)). \end{array}$$

Let $(x, y) \in \mathscr{P}^+ \times \mathscr{P}$. For a sake of lightness, we just denote by Σ the map $\Sigma_{x,y}$. The compact surface $\mathcal{T} := \Sigma(\mathbb{R}^2)$ is homeomorphic to the torus $\mathbb{T}^2 = \mathbb{R}^2 \setminus \mathbb{Z}^2$. We denote by $\widehat{\Sigma}$ the map defined on $\mathbb{T}^2$ such that the following diagramm commutes:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\Sigma"] \arrow[d, "\varpi"'] & \mathscr{T} \\ \mathbb{T}^2. \arrow[ur, "\widehat{\Sigma}"'] & \end{tikzcd}$$

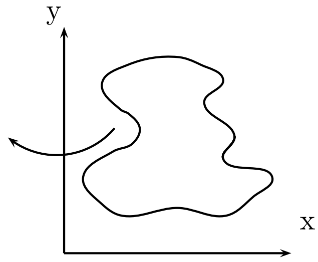


Figure 4.1: A meridian curve of a torus in $\mathscr{T}_{\mathscr{M}}$

Hence

$$\mathscr{T} := \{\widehat{\Sigma}_{x,y}(\mathbb{T}^2) \,|\, (x,y) \in \mathscr{P}^+ \times \mathscr{P}\} \text{ and } \mathscr{T}_{\mathscr{M}} := \{\widehat{\Sigma}_{x,y}(\mathbb{T}^2) \,|\, (x,y) \in \mathscr{P}^+_{\mathscr{M}} \times \mathscr{P}\}.$$

In section 4.2.1, we will show that, for each torus $\mathcal{T} \in \mathscr{T}_{\mathscr{M}}$, the cogeodesic flow ϕ (restricted to the unit cotangent bundle) is a dynamically coherent system that admits a hyperbolic periodic orbit. Therefore, $\mathrm{h_{pol}}(\phi_g) = 2$, and such tori are cases of strict inequality for the theorem B. Notice that since the set $\mathscr{P}_{\mathscr{M}}$ is a G_δ-dense in the Fréchet space $\mathscr{P}$, our result is generic.

Fix a metric g on $\mathbb{T}^n$. As usual, $B(x,r)$ refers to the ball centered at x and of radius r in the universal cover $\mathbb{R}^n$. Burago et Ivanov proved that the limit $\Omega(g) := \lim_{r\to+\infty} \frac{\mathrm{Vol}\, B(x,r)}{r^n}$ exists and is independent of x. It is the *asymptotic volume of* g.

They proved that $\Omega(g)$ is equal, up to a constant υ_g, to the Lebesgue volume $\mathscr{V}_g$ of the unit ball of the *stable norm* associated with g.

We will show that for $\mathcal{T} \in \mathscr{T}_{\mathscr{M}}$, the integrability of the geodesic Hamiltonian permits to compute *explicitely* the volume $\mathscr{V}_g$. Indeed, the stable norm coincide with *Mather's β* function which can be explicitely determinated due to the existence of the first integral.

In section 4.2.2, we recall the definitions of the stable norms and of Mather's functions. In section 4.2.3, we determinate the expression of $\mathscr{V}_g$ for tori in $\mathscr{T}_{\mathscr{M}}$.

If (M,g) is a compact Riemannian manifold, we denote by $\mathcal{M}$ the set of minimizing geodesics of (M,g). For $x \in \widetilde{M}$ and $T > 0$, we introduce the *minimizing ball* centered at x and of radius T:

$$B_{\min}(x,T) = \{\gamma(t) \mid \gamma \in \mathcal{M}, \gamma(0) = x,\, t \in [0,T]\},$$

Clearly $B_{\min}(x,T) \subset B(x,T)$. We say that the geodesic flow of (M,g) has *asymptotically full minimizing domain* when $\mathrm{Vol}\, B(x,T) \simeq_{T\to\infty} \mathrm{Vol}\, B_{\min}(x,T)$.

In section 4.2.5, we see that the geodesic flow of any torus of $\mathscr{T}_{\mathscr{M}}$ has asymptotically full minimizing domain. Therefore, $\Omega(g)$ is actually the limit of the asymptotic volume of the minimizing ball. We give an independent proof of this last result.

4.1 Growth of the volume.

In this section, (M,g) is a compact Riemannian manifold without boundary. We denote by $\widetilde{M}$ its universal covering endowed with the Riemannian metric $\tilde{g} = \varpi^* g$, where ϖ is the covering map $\widetilde{M} \to M$.

4.1.1 The volumic entropy and the Manning inequality.

For $x \in \widetilde{M}$ and $r > 0$ we denote by $B(x,r)$ the open ball (in $\widetilde{M}$) centered at x and of radius r. We set
$$V(x) = \limsup_{r\to\infty} \frac{1}{r}\text{Log Vol}\, B(x,r).$$
It has been proved by Manning ([Man79]) that $V(x)$ is actually a limit, which is independent of x, and moreover uniform with respect to x. This allows one to define the *volumic entropy* of (M,g) as:

$$\mathrm{h}_{\mathrm{vol}}(g) := \limsup_{r\to\infty} \frac{1}{r}\text{Log Vol}\, B(x,r), \quad \forall x \in \widetilde{M}.$$

In all this chapter we will denote by ϕ_g the geodesic flow of (M,g) **restricted to the unit tangent bundle** SM. The following remarkable result is due to Manning and Mañé and Freire.

Theorem 13. (Manning, Mañé-Freire) *For any Riemannian compact manifold* (M,g), $\mathrm{h}_{\mathrm{vol}}(g) \le \mathrm{h}_{\mathrm{top}}(\phi_g)$, *with equality if* (M,g) *has no conjugate points.*

Actually, Manning proved the inequality in the general case in [Man79]. He moreover proved the equality in the case where (M,g) has nonpositive sectional curvature. In [FM82], Mañé and Freire extended the equality cases to manifolds without conjugate points. By Cartan-Hadamard theorem, this includes the previous case.

4.1.2 Polynomial growth of the volume

With the same notation as in the previous paragraph, we set:

$$\tau(x) = \inf\{s \ge 0 \,|\, \limsup_{r\to\infty} \frac{1}{r^s} \text{Vol}\, B(x,r) = 0\} \le +\infty.$$

Proposition 4.1.1. $\tau(x) = \limsup_{r\to\infty} \dfrac{\log \text{Vol}\, B(x,r)}{\log r}$.

Proof. Set $\lambda := \limsup_{r\to\infty} \frac{\log \text{Vol}\, B(x,r)}{\log r}$.

• Proof of $\lambda \le \tau(x)$. It suffices to show that for all $s < \lambda$, $\limsup_{r\to\infty} \frac{1}{r^s} \text{Vol}\, B(x,r) > 0$.
Fix $s < \lambda$. By definition of λ, for all $r > 0$, there exists $r' > r$ such that:

$$\frac{\log \text{Vol}\, B(x,r')}{\log r'} > s.$$

Now for $r' > 0$, one has:

$$\frac{\log \text{Vol}\, B(x,r')}{\log r'} > s \iff \text{Vol}\, B(x,r') > (r')^s \iff \frac{1}{(r')^s} \text{Vol}\, B(x,r') > 1.$$

Therefore, for all $r > 0$ there exists $r' > r$ such that $\dfrac{1}{(r')^s} \text{Vol}\, B(x,r') > 1$ and

$$\limsup_{r\to\infty} \frac{1}{r^s} \text{Vol}\, B(x,r) \ge 1.$$

• Proof of $\lambda \ge \tau(x)$. It suffices to show that for all $s > \lambda$, $\limsup_{r\to\infty} \frac{1}{r^s} \text{Vol}\, B(x,r) = 0$.

Assume that $\lambda < +\infty$ and fix $s > \lambda$. Then $\frac{1}{2}(\lambda + s) > \lambda$, so for r large enough:

$$\frac{\log \operatorname{Vol} B(x,r)}{\log r} < \frac{1}{2}(\lambda + s) \iff \operatorname{Vol} B(x,r) < r^{\frac{1}{2}(\lambda+s)} \iff \frac{\operatorname{Vol} B(x,r)}{r^s} \le r^{\frac{1}{2}(\lambda-s)}.$$

Now $\lim_{r\to\infty} r^{\frac{1}{2}(\lambda-s)} = 0$ so $\limsup_{r\to\infty} \frac{1}{r^s} \operatorname{Vol} B(x,r) = \lim_{r\to\infty} \frac{1}{r^s} \operatorname{Vol} B(x,r) = 0$. □

We will see that $\tau(x)$ is independent of x and of g: it is a *topological* invariant of M.

Growth functions. A *growth function* is a nondecreasing function $\mathbb{R}^+ \to \mathbb{R}^+$. For any $x \in \widetilde{M}$, the map $\nu_x : r \mapsto \operatorname{Vol} B(x,r)$ is a growth function. With any nondecreasing function $\beta : \mathbb{N} \to \mathbb{N}$, one can associate the growth function $\alpha : t \to \beta(\lceil t \rceil)$. Therefore, the considerations below apply to nondecreasing functions $\mathbb{N} \to \mathbb{N}$.

Two growth functions α_1 and α_2 are *weakly equivalent* if there exists $\lambda, \mu > 1$ and $C, C' \ge 0$ such that for all $t \in \mathbb{R}^+$,

$$\alpha_1(t) \le \lambda \alpha_2(\lambda t + C) + C, \quad \alpha_2(t) \le \mu \alpha_1(\mu t + C') + C'.$$

This is denoted by $\alpha_1 \sim \alpha_2$. When the first inequality only is satisfied, we say that α_1 is *dominated* by α_2 and we write $\alpha_1 \prec \alpha_2$.

Let Γ be a finitely generated group and let $S := (s_1, \ldots, s_p)$ be a set of generators. We denote by $\ell_S(\gamma)$ the *word length* of an element $\gamma \in \Gamma$, that is the smallest integer n for which there exists a sequence $(s_1, s_2, \ldots, s_n)$ of elements of $S \cup S^{-1}$ such that $\gamma = s_1 s_2 \cdots s_n$. The *word metric* d_S on Γ is defined by

$$d_S(\gamma_1, \gamma_2) = \ell_S(\gamma_1^{-1}\gamma_2).$$

The group Γ endowed with d_S acts on itself by isometries. The growth function $\beta : \mathbb{N} \to \mathbb{N}$ of the pair (Γ, S) is defined by:

$$\beta(\Gamma, S; k) := \operatorname{Card}\{\gamma \in \Gamma \mid \ell_k(\gamma) \le k\}.$$

The *exponential growth rate* of (Γ, S) is the upper limit :

$$\omega(\Gamma, S) := \limsup_{k\to\infty} \sqrt[k]{\beta(\Gamma, S, k)}$$

Observe that if S' is another finite set of generators of Γ, there exists $\lambda > 0$ such that for all $k \in \mathbb{N}$, $\beta(\Gamma, S', k) \le \beta(\Gamma, S, \lambda k)$. The group Γ is said to be of exponential growth if $\omega(\Gamma, S) > 1$, of subexponential growth if $\omega(\Gamma, S) = 1$ and of polynomial growth if there exists d such that $\beta(\Gamma, S, k) \prec k^d$. These definitions make sense since these properties do not depend on the choice of S.

Quasi-isometries. A *quasi-isometry* between two metric spaces (X, d) and (X, d') is a map $f : X \to X'$ such that there exist constants $\lambda \ge 1$, $C \ge 0$ and $D \ge 0$ such that

- for any $z \in X'$ there exists $x \in X$ such that $d'(z, f(x)) \le D$
- for all $(x, y) \in X^2$, $\frac{1}{\lambda} d(x,y) - C \le d'(f(x), f(y)) \le \lambda d(x,y) + C$

The next theorem is due to Milor and Schwarzc (see [dlH00] or [BH99]).

Theorem 14. *Let (M, g) be a compact Riemannian manifold. We set $R := \operatorname{diam} M := \max\{d(p,q) \mid (p,q) \in M^2\}$. Fix $x \in \widetilde{M}$ and set $B := B(x, R)$. Then*

1. *the set* $S := \{\gamma \in \pi_1(M) \,|\, \gamma \neq 1 \text{ and } \gamma \mathrm{B} \cap \mathrm{B} \neq 0\}$ *is a finite set of generators of* $\pi_1(M)$,
2. *the number* $r := \min\{d(B, \gamma B) \,|\, \gamma \in \pi_1(M),\, \gamma \notin S \cap \{1\}\}$ *is* > 0 *and for all* $\gamma \in \pi_1(M)$,
$$\ell_S(\gamma) \leq \frac{1}{r} d(x, \gamma x) + 1,$$
3. *the map* $\pi_1(M) \to \widetilde{M}, \gamma \mapsto \gamma x$ *is a quasi-isometry.*

With the notation of the theorem above, we denote by β the growth function of the pair $(\pi_1(M), S)$.

Corollary 4.1.1. *For any* $x \in \widetilde{M}$, *the maps* ν_x *and* β *are weakly equivalent.*

Proof. We denote by n_x the order of the isotropy subgroup of x and we set
$$\lambda := \max\{d(x, \gamma x) \ x \in S\}.$$
Fix $k \in \mathbb{N}$. The closed balls $B(y, \frac{1}{3}r)$, for $y \in \pi_1(M)$ are pairwise disjoint, so
$$\frac{1}{n_x} \beta(\pi_1(M), S; k) \nu_x \left(\frac{1}{3} r\right) \leq \nu_x \left(k\lambda + \frac{1}{3} r\right).$$
Conversely, let $y \in B(x, k)$. Since the set $\{\gamma B \,|\, \gamma \in \pi_1(M)\}$ is a covering of $\widetilde{M}$, there exists $\gamma \in \pi_1(M)$, such that $y \in B(\gamma x, R)$. Now by theorem 14 (ii),
$$\ell_S(\gamma) \leq \frac{1}{r} d(x, \gamma x) + 1 \leq \frac{1}{r} d(x, y) + \frac{R}{r}. \tag{*}$$
Let $\delta(x, y) := \frac{1}{r} d(x, y) + \frac{R}{r}$. Then, the set $\{\gamma B(x, R) \,|\, \ell_S(\gamma) \leq \delta(x, y)\}$ cover $B(x, k)$ and
$$\nu_x(k) \leq \beta(\pi_1(M), S; k + \frac{R}{r} + 1) \nu_x(R). \tag{**}$$
Gathering $(*)$ and $(**)$, we conclude the proof. $\square$

Therefore, if $\pi_1(M)$ has exponential or subexponential growth, $\nu(x) = +\infty$. If $\pi_1(M)$ has polynomial growth , $\nu(x) = \inf\{s \geq 0 \,|\, \beta(\pi_1(M), S; k) \prec k^s\}$. We set $\tau(M) := \tau(x)$. It is obviously independent of the metric g.

Example 4.1.1. $\tau(\mathbb{T}^n) = n$.

4.1.3 A polynomial analogue of Manning inequality.

This section is devoted to the proof of Theorem B.

Recall that the Riemannian connexion on M enables one to define a natural connexion on TM in the following way. If $\nu := (x, v) \in TM$, the parallel transports of v along curves starting from x give rise to curves $t \mapsto \gamma(t) = (x(t), v(t)) \in TM$. The *horizontal subspace* $H(\nu)$ generated by the initial conditions $(\gamma(0), \dot{\gamma}(0))$ of these curves is complementary to the *vertical subspace* $V(\nu) := \ker d_\nu \pi$. There exists a natural metric on TM, called the *Sasaki metric* for which $H(\nu)$ and $V(\nu)$ are orthogonal and both isometric to $T_x M$. Let us fix the notation.

- $\pi : TM \to M$ $\tilde{\pi} : T\widetilde{M} \to \widetilde{M}$ and $p : \widetilde{M} \to M$ are the canonical projections,

- d_M and $d_{\widetilde{M}}$ are the Riemannian distances on M and $\widetilde{M}$,
- d_{TM} and $d_{T\widetilde{M}}$ are the Riemannian distances on TM and $T\widetilde{M}$ associated with the Sasaki metric,
- (SM, d_{SM}) and $(S\widetilde{M}, d_{S\widetilde{M}})$ are the unit tangent bundles endowed with their induced metrics,
- $\phi = (\phi^t)_{t\in\mathbb{R}}$ and $\tilde{\phi} = (\tilde{\phi}^t)_{t\in\mathbb{R}}$ are the geodesic flows on SM and $S\widetilde{M}$.

Proof of Theorem B. It suffices to show that if $s > \mathrm{h}_{\mathrm{pol}}(\phi) + 1$, then $s > \tau(M)$. For $\varepsilon > 0$ and $t \geq 0$, we denote by $G_t(\varepsilon)$ and $S_t(\varepsilon)$ the numbers $G_t^{\phi}(\varepsilon)$ and $S_t^{\phi}(\varepsilon)$ Fix $s > \mathrm{h}_{\mathrm{pol}}(\phi) + 1$ and let $\eta > 0$. There exists $t_\eta > 0$ such that for all $t \geq t_\eta$,

$$\frac{1}{t^{s-1}} G_t\left(\frac{\varepsilon}{2}\right) < \eta.$$

Let ρ be such that for all $x \in \widetilde{M}$, the Riemannian projection $p : B(x,\rho) \to M$ is injective. Fix $\varepsilon > 0$ such that $2\varepsilon \leq \rho$. Let $t > 0$. We want to construct a (t,ε)-separated set in SM. Fix $x \in \widetilde{M}$ and let $C(x,t,t+\frac{\varepsilon}{2})$ be the anular zone defined by

$$C\left(x,t,t+\frac{\varepsilon}{2}\right) = B\left(x,t+\frac{\varepsilon}{2}\right) \setminus B(x,t).$$

Let A be a 2ε-separated set in $\tilde{C}(x,t,t+\frac{\varepsilon}{2})$, that is, for any $(a,b) \in A^2$, $d_{\widetilde{M}}(a,b) \geq 2\varepsilon$. For all $a \in A$ there exists a segment of geodesic γ_a with minimal length that joins x and a. Necessarily, $\ell(\gamma_a) \in [t, t+\frac{\varepsilon}{2}]$. Set $v_a := \dot{\gamma}_a(0) \in S_x\widetilde{M}$. Then, for any a and b in A,

$$\begin{aligned} d_{S\widetilde{M}}(\tilde{\phi}^t(v_a), \tilde{\phi}^t(v_b)) &\geq d_{\widetilde{M}}(\tilde{\pi}\circ\tilde{\phi}^t(v_a), \tilde{\pi}\circ\tilde{\phi}^t(v_b)) \\ &\geq d_{\widetilde{M}}(a,b) - d_{\widetilde{M}}(\tilde{\pi}\circ\tilde{\phi}^t(v_a), a) - d_{\widetilde{M}}(\tilde{\pi}\circ\tilde{\phi}^t(v_b), b) \\ &\geq 2\varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So the set $\{v_a \,|\, a \in A\}$ is (t,ε)-separated.

Since the projection T_xp is an isometry, $d_{SM}(d_xp(v_a), d_xp(v_b) = d_{S\widetilde{M}}(v_a, v_b)$, for all $(a,b) \in A^2$. So

$$\sup_{0\leq t'\leq t} d_{SM}(\phi^t(v_a), \phi^t(v_b)) \geq d_{SM}(d_xp(v_a), d_xp(v_b) \geq \varepsilon$$

- Assume that $d_{S\widetilde{M}}(v_a, v_b) \geq \varepsilon$. Then $\sup_{0\leq t'\leq t} d_{SM}(\phi^t(v_a), \phi^t(v_b)) \geq \varepsilon$ and the set $\mathcal{A} = d_xp(\{v_A \,|\, a \in A\})$ is (t,ε)-separated for the flow ϕ_t.
- Assume that $d_{S\widetilde{M}}(v_a, v_b) \leq \varepsilon$, then $d_{\widetilde{M}}(\tilde{\pi}(v_a), \tilde{\pi}(v_b)) \leq \varepsilon$. Now by construction of A, $d_{\widetilde{M}}(\tilde{\pi}\circ\tilde{\phi}^t(v_a), \tilde{\pi}\circ\tilde{\phi}^t(v_b)) \geq \varepsilon$, so there exists $t_0 \in [0,t]$ such that

  $$d_{\widetilde{M}}(\tilde{\pi}\circ\tilde{\phi}^{t_0}(v_a), \tilde{\pi}\circ\tilde{\phi}^{t_0}(v_b)) = \varepsilon.$$

  Therefore, since $2\varepsilon \leq \rho$, one gets $d_M(\pi\circ\phi_{t_0}(d_xp(v_a)), \pi\circ\phi_{t_0}(d_xp(v_b))) = \varepsilon$ and the set $\mathcal{A}$ is again (t,ε)-separated for the flow (ϕ_t).

Now notice that $\sup_{x\in\widetilde{M}} \operatorname{Vol} B(x,2\varepsilon)$ is finite. Indeed, as $2\varepsilon \leq \rho$, $\operatorname{Vol} B(x,2\varepsilon) = \operatorname{Vol}(p(x),2\varepsilon) \leq \operatorname{Vol} M$. Let $v = \operatorname{Vol} M$. For $t \geq t_\eta$,

$$\operatorname{Vol} C\left(x,t,t+\frac{\varepsilon}{2}\right) \leq v \operatorname{Card} \mathcal{A} \leq v \operatorname{Card} A \leq vS_t(\varepsilon) \leq vG_t\left(\frac{\varepsilon}{2}\right) \leq \eta v t^{s-1}.$$

Consequently, since $C(x,t,t+m\frac{\varepsilon}{2}) = \bigcup_{k=0}^{m-1} C(x,t+k\frac{\varepsilon}{2},t+(k+1)\frac{\varepsilon}{2})$, one has, for $m \in \mathbb{N}^*$,

$$\operatorname{Vol} C(x,t,t+m\frac{\varepsilon}{2}) \leq \eta v \left(t^{s-1} + (t+\frac{\varepsilon}{2})^{s-1} + \cdots + (t+m\frac{\varepsilon}{2})^{s-1}\right).$$

Assume that $s \geq 1$. Then for each $k \in \mathbb{N}$, $(t+k\frac{\varepsilon}{2})^{s-1} \leq \dfrac{2}{\varepsilon}\displaystyle\int_{t+k\frac{\varepsilon}{2}}^{t+(k+1)\frac{\varepsilon}{2}} x^{s-1}dx$, which yields

$$\operatorname{Vol} C(x,t,t+m\frac{\varepsilon}{2}) \leq \eta v \frac{2}{\varepsilon}\int_t^{t+m\frac{\varepsilon}{2}} x^{s-1}dx \leq \eta v \frac{2}{s\varepsilon}(t+m\frac{\varepsilon}{2})^s.$$

Assume that $0 < s < 1$. Then for each k, $(t+k\frac{\varepsilon}{2})^{s-1} \leq \dfrac{2}{\varepsilon}\displaystyle\int_{t+(k-1)\frac{\varepsilon}{2}}^{t+k\frac{\varepsilon}{2}} x^{s-1}dx$, which yields

$$\operatorname{Vol} C(x,t,t+m\frac{\varepsilon}{2}) \leq \eta v \frac{2}{\varepsilon}\int_{t-1}^{t+(m-1)\frac{\varepsilon}{2}} x^{s-1}dx \leq \eta v \frac{2}{\varepsilon}\int_{t-1}^{t+m\frac{\varepsilon}{2}} x^{s-1}dx \leq \eta v \frac{2}{s\varepsilon}(t+m\frac{\varepsilon}{2})^s.$$

In each case, on gets, denoting by $\lambda(\varepsilon) = v\dfrac{2}{s\varepsilon}$:

$$\begin{aligned}\operatorname{Vol} B(x,t+m\frac{\varepsilon}{2}) &= \operatorname{Vol} B(x,t) + \operatorname{Vol} C(x,t,t+m\frac{\varepsilon}{2})\\ &\leq \operatorname{Vol} B(x,t) + \eta\lambda(\varepsilon)(t+m\frac{\varepsilon}{2})^s.\end{aligned}$$

Finally

$$\limsup_{r\to+\infty} \frac{\operatorname{Vol} B(x,r)}{r^s} \leq \sup_{t\in[0,\varepsilon/2]} \limsup_{m\to\infty} \frac{\operatorname{Vol} B(x,t+m\frac{\varepsilon}{2})}{(t+m\frac{\varepsilon}{2})^s} \leq \eta\lambda(\varepsilon).$$

Since η is arbitrary, the limit above is zero and $s > \tau(M)$. $\square$

4.1.4 The flat torus $\mathbb{T}^n$.

Let g_b be the flat metric on $\mathbb{T}^n$ defined by a positive definite bilinear form b on $\mathbb{R}^n$. The cogeodesic flow ϕ_H associated with g_b on $S^*\mathbb{T}^n$ is in action-angle form, so by proposition 2.2.3 and example 4.1.1 one has

$$\mathrm{h}_{\mathrm{pol}}(\phi_H) = n - 1 = \tau(\mathbb{T}^n) - 1.$$

The asymptotic equivalent of the volume of balls is easy to determine. Let A be the autoadjoint isomorphism A such that $b(v,v') = \langle Av, v'\rangle$. Then, for all $x \in \mathbb{R}^n$, $B(x,1) := \{x+tv \mid \langle Av,v\rangle \leq 1\}$ and $\operatorname{Vol} B(x,1) = \sqrt{\det A}$. By homogeneity of the Euclidean volume, one gets

$$\operatorname{Vol} B(x,T) = T^n \operatorname{Vol} B(x,1) = T^n\sqrt{\det A}.$$

4.2 Asymptotic volume for tori of revolution.

Fix $\mathcal{T} \in \mathscr{T}$. The Euclidean metric of $\mathbb{R}^3$ induces a Riemannian metric g on $\mathcal{T}$. The pullback $\tilde{g} := \Sigma_{x,y}^* g$ of g is a Riemannian metric on $\mathbb{R}^2$ which reads

$$\tilde{g}_{(\varphi,s)} = \begin{pmatrix} 4\pi^2 x(s)^2 & 0 \\ 0 & r(s)^2 \end{pmatrix}, \quad \forall\,(\varphi, s) \in \mathbb{R}^2,$$

where we denote by $r(s)$ the positive square root of $x'(s)^2 + y'(s)^2$. The Riemannian manifold $(\mathbb{R}^2, \tilde{g})$ is the Riemannian cover of $(\mathcal{T}, g)$.

The projection of $\tilde{g}$ on $\mathbb{T}^2$ is also denoted by g. The quotient map $\widehat{\Sigma}_{x,y}$ becomes an isometry between $(\mathbb{T}^2, g)$ and $(\mathcal{T}, g)$.

Notation 4.2.1. *We set $\bar{m} := (\bar{\varphi}, \bar{s}) \in \mathbb{T}^2$ and $m := (\varphi, s) \in \mathbb{R}^2$. If $\bar{m} = \pi(m)$, the spaces $T^*_{\bar{m}}\mathbb{T}^2$ and $T^*_m\mathbb{R}^2$ are canonically isometric. We denote by $p := (p_\varphi, p_s)$ their elements, so that the Liouville form on $T^*\mathbb{T}^2$ (resp. $T^*\mathbb{R}^2$) reads $\lambda = p_\varphi d\bar{\varphi} + sd\bar{s}$ (resp. $\lambda = p_\varphi d\varphi + sds$). The functions x, y and r induce functions on $\mathbb{T}^2$, also denoted by x, y and r.*

The geodesic Hamiltonian H on $T^*\mathbb{R}^2$ reads

$$H(\varphi, s, p_\varphi, p_s) = \frac{1}{2}\left[\frac{p_\varphi^2}{4\pi^2 x(s)^2} + \frac{p_s^2}{r(s)^2}\right].$$

It is integrable in the Liouville sense, a first integral being the *Clairaut integral* p_φ.

It projects in a natural way on a Hamiltonian function on $T^*\mathbb{T}^2$ also denoted by H. The associated Hamiltonian flows on $T^*\mathbb{T}^2$ and $T^*\mathbb{R}^2$ are respectively denoted by $(\phi_H^t)_{t\in\mathbb{R}}$ and $(\tilde{\phi}_H^t)_{t\in\mathbb{R}}$. Let $\varpi^* : T^*\mathbb{R}^2 \to T^*\mathbb{T}^2$, $\pi : T^*\mathbb{T}^2 \to \mathbb{T}^2$ and $\tilde{\pi} : T^*\mathbb{R}^2 \to \mathbb{R}^2$ be the canonical projections. The following diagram commutes.

$$\begin{tikzcd} T^*\mathbb{R}^2 \arrow[r, "\tilde{\phi}_H^t"] \arrow[d, "\varpi^*"'] & T^*\mathbb{R}^2 \arrow[r, "\tilde{\pi}"] \arrow[d, "\varpi^*"] & \mathbb{R}^2 \arrow[d, "\varpi"] \\ T^*\mathbb{T}^2 \arrow[r, "\phi_H^t"'] & T^*\mathbb{T}^2 \arrow[r, "\pi"'] & \mathbb{T}^2. \end{tikzcd}$$

Remark 4.2.1. If $\widehat{m}$ stands for m or $\bar{m}$, the orbits of $(\widehat{m}, p)$ and $(\widehat{m}, -p)$ project by π or $\tilde{\pi}$ onto the same geodesic which they describe in opposite sense. We set $\zeta : (\widehat{m}, p) \mapsto (\widehat{m}, -p)$.

4.2.1 Bott integrability and dynamical coherence.

We show that the cogeodesic flow (in restriction to any energy level) of a torus of $\mathscr{T}_{\mathscr{M}}$ is dynamically coherent and possesses hyperbolic orbits. We begin with studying the critical set of p_φ.

Every regular energy level $H^{-1}(\{e\})$ is a circle bundle parametrized by

$$\mathbb{T}^3 \ni (\bar{\varphi}, \bar{s}, \theta) \mapsto (\bar{\varphi}, \bar{s}, 2\pi\sqrt{2e}.x(\bar{s})\cos\theta, \sqrt{2e}.r(\bar{s})\sin\theta).$$

Let P_e be the restriction of p_φ to $H^{-1}(\{e\})$. We denote by $\mathcal{R}(e)$ the set of regular values of P_e.

Lemma 4.2.1. *The set $\mathcal{R}(e)$ is a finite union of intervals $-I_k(e), J(e)$ and $I_k(e)$ with $I_k(e) =]2\pi\sqrt{2e}x_k, 2\pi\sqrt{2e}x_{k+1}[$ and $J(e) =]-2\pi\sqrt{2e}x_1, 2\pi\sqrt{2e}x_1[$ where $0 < x_1 < x_2 < \cdots < x_n$ are the critical values of x.*

Proof. Observe that $P_e(\bar\varphi,\bar s,\theta) := 2\pi\sqrt{2e}.x(\bar s)\cos\theta$ does not depend of $\bar\varphi$. We set $\widehat{P}_e$: $(\bar s,\theta)\mapsto P(0,\bar s,\theta)$. The critical points of $\widehat{P}_e$ are the pairs $(\bar s,\theta)$ such that: $\theta = 0\,[\pi]$ and s is a critical point of x. Since x is a Morse function, the set $\overline{S}$ of its critical points is finite and so is its set of critical values. We set $x(\overline{S}) := \{x_1,\cdots,x_n\}$ with $x_1 < x_2 < \cdots < x_n$. One just has to remark that $P_e^{-1}(\{\rho\}) \neq \emptyset$ if and only if $\rho \in [-2\pi\sqrt{2e}x_n, 2\pi\sqrt{2e}x_n]$. □

We denote by $\bar s_i$ the unique critical point in $\mathbb{T}$ such that $x(\bar s_i) = x_i$

Proposition 4.2.1. *The system $(H^{-1}(\{e\}),\phi_H,P_e)$ is dynamically coherent and possesses a hyperbolic orbit.*

Proof. First we note that, in the coordinates $(\bar\varphi,\bar s,\theta)$, X^H reads

$$X^H(\bar\varphi,\bar s,\theta) = \left(\frac{\sqrt{2e}\cos\theta}{2\pi x(\bar s)^2},\frac{\sqrt{2e}\sin\theta}{r(\bar s)},\frac{x'(\bar s)}{r(\bar s)x(\bar s)}\right).$$

By lemma 4.2.1, the critical loci of P_e are the periodic orbits $\mathcal{C}_i^0 := \{(\bar\varphi,\bar s_i,0)\,|\,\bar\varphi\in\mathbb{T}\}$ and $\mathcal{C}_i^\pi := \{(\bar\varphi,\bar s_i,\pi)\,|\,\bar\varphi\in\mathbb{T}\}$ with period $T_i := \frac{\sqrt{2e}}{2\pi x_i^2}$. They are exchanged by the symetry ζ, so we focus on the case where $\theta = 0$. By simple computation, one checks that:
– if x_i is a maximum, there exists $\alpha\in]0,\pi[$ such that the eigenvalues of $D\phi_{T_i}(\bar\varphi,\bar s_i,0)$ are 1, $e^{i\alpha}$ and $e^{-i\alpha}$, and $\mathcal{C}_i^0$ is an elliptic orbit.
– if x_j is a minimum, there exists $\lambda > 0$ such that the eigenvalues of $D\phi_{T_j}(\bar\varphi,\bar s_j,0)$ are 1, e^{λ} and $e^{-\lambda}$ and $\mathcal{C}_j^0$ is a hyperbolic orbit. □

Remark 4.2.2. The level $P_e^{-1}(\{2\pi\sqrt{2e}x_1\})$ is the disjoint union of two ∞-levels $\mathscr{P}_e^0$ and $\mathscr{P}_e^\pi$ exchanged by the symetry ζ. Set $\theta_e : \bar s\mapsto \arccos\frac{x_1}{2\pi\sqrt{2e}x(\bar s)}$. The complementary set in $\mathscr{P}_e^0$ of the circle $\mathcal{C}_1^0$ has the two following connected components:

$$W_e^{0,+} := \{(\bar\varphi,\bar s,\theta_e(\bar s)\,|\,(\bar\varphi,\bar s)\in\mathbb{T}\times\mathbb{T}\setminus\{s_1\}\},$$

and

$$W_e^{0,-} := \{(\bar\varphi,\bar s,-\theta_e(\bar s))\,|\,(\bar\varphi,\bar s)\in\mathbb{T}\times\mathbb{T}\setminus\{s_1\}\}.$$

The unions $\mathcal{G}_e^{0,+}$ of the orbit $\mathcal{C}_1^0$ and the submanifold $W_e^{0,+}$ is a Lagrangian Lipshitz graph over $\mathbb{T}^2$. We define in the same way the graph $\mathcal{G}_e^{0,-}$, and the graphs $\mathcal{G}_e^{\pi,+}$ and $\mathcal{G}_e^{\pi,-}$.

This particular property does not hold when $x_i > x_1$.

4.2.2 Stable norms and Mather's functions

In this paragraph, we briefly define the stable norms for tori and state the result of Burago and Ivanov about the asymptotic volume. We will give no proof of the results, for a more complete introduction to the subject and for proofs, we refer to [Mas96], [BBI01] and [**?**].

Consider a Riemannian metric g on the torus $\mathbb{T}^n$. With an element $\gamma\in H_1(\mathbb{T}^n,\mathbb{Z})$, we associate the set $\mathscr{C}(\gamma)$ of closed piecewise C^1 curves that represent γ. We define a function f on $H_1(\mathbb{T}^n,\mathbb{Z})$ by setting:

$$f(\gamma) := \inf\{\ell_g(c)\,|\,c\in\mathscr{C}(\gamma)\}.$$

The definition of the stable norm is based on the following lemma:

Lemma 4.2.2. *The function $||\cdot||_s : \gamma\mapsto \lim\limits_{n\to\infty}\frac{f(n\gamma)}{n}$ is well defined and satisfies the following properties*

1. *for all* $\gamma \in H_1(\mathbb{T}^n, \mathbb{Z})$, $||\gamma||_s \le f(\gamma)$,
2. *for all* $k \in \mathbb{Z}$ *and all* $\gamma \in H_1(\mathbb{T}^n, \mathbb{Z})$, $||k\gamma||_s = k||\gamma||_s$,
3. *for all* γ_1 *and* γ_2 *in* $H_1(\mathbb{T}^n, \mathbb{Z})$ $||\gamma_1 + \gamma_2||_s \le ||\gamma_1||_s + ||\gamma_2||_s$,
4. *for any* 1*-form* η *and any piecewise* C^1 *curve* c*:*

$$\int_c \eta \le \sup_{x \in \mathbb{T}^n} ||\eta_x||_g \ell_g(c),$$

   where $|| \cdot ||_g$ *denotes the dual norm of the norm of* TM *defined by* g.

As a consequence, one gets the following proposition.

Proposition 4.2.2. *The function* $||\cdot||_s$ *extends to a function on* $H_1(\mathbb{T}^n, \mathbb{R})$ *that is a norm on* $H_1(\mathbb{T}^n, \mathbb{R})$.

Definition 4.2.1. One says that $|| \cdot ||_s$ is the *stable norm associated with* g.

Remark 4.2.3. Here we chosen to focus on tori, but the definitions above also hold for an arbitrary compact connected Riemannian manifold (M, g) (see [Mas96]).

We are now in a position to state the theorem of Burago and Ivanov on the asymptotic volume. We denote by $\tilde{g}$ the lifted metric on the universal cover $\mathbb{R}^n$ of $\mathbb{T}^n$ and by υ_g the Riemannian volume of $[0,1]^n$. Identifying $H_1(\mathbb{T}^n, \mathbb{R})$ with $\mathbb{R}^n$ we denote by $\mathscr{V}_g$ the Lebesgue volume of the unit ball of $|| \cdot ||_s$.

Theorem 15. (Burago-Ivanov.) *For any* x *in the universal cover* $\mathbb{R}^n$ *of* $\mathbb{T}^n$,

$$\lim_{r \to \infty} \frac{\operatorname{Vol} B(x, r)}{r^n} = \upsilon_g \mathscr{V}_g.$$

Let us now recall the foundations of Mather's theory on invariant measures for a Tonelli Lagrangian and its relation with the stable norms in the case of geodesic Lagrangians on tori. As before, we give no proof of the results stated and we refer to [Mat91] or to the very beautiful survey [Sor10].

Consider a compact and connected Riemannian manifold (M, g). As usual, we denote by $||.||_x$ the norm on T_xM induced by the metric g. A C^2 function $L : TM \to \mathbb{R}$ is called a Tonelli Lagrangian if

- L is striclty convex in the fibers, that is, $\frac{\partial^2 L}{\partial v^2}(x, v)$ is positive definite,
- L is superlinear in the fibers, that is, $\lim\limits_{||v||_x \to \infty} \frac{L(x,v)}{||v||_x} = +\infty$,

Obviously, the geodesic Lagrangians are Tonelli Lagrangians.

One can easily prove that the Fenchel-Legendre transform of a Tonelli Lagrangian is also C^2, strictly convex in the fibers and superlinear: such a function $T^*M \to \mathbb{R}$ is called a Tonelli Hamiltonian.

Fix a Tonelli Lagrangian L on TM. We denote by ϕ_L its Euler-Lagrange flow and by H its Fenchel-Legendre transform. The Hamiltonian flow associated with H is denoted by ϕ_H. It follows immediately from the definition of the Fenchel-Legendre transform that for any $(x, v) \in TM$ and any $(x, p) \in T^*M$, the following inequality holds true:

$$\langle p, v \rangle_x \le L(x, v) + H(x, p).$$

This is called the Fenchel-Legendre inequality.

The orbits of ϕ_H are contained in the energy levels $H^{-1}(\{e\})$ and those of ϕ_L in the subsets $\mathcal{L}^{-1}(H^{-1}(\{e\}))$ (where $\mathcal{L}$ is the Legendre transform $TM \to T^*M$). Due to the superlinearity, the sets $H^{-1}(\{e\})$ and $\mathcal{L}^{-1}(H^{-1}(\{e\}))$ are compacts.

Let $\mathscr{M}(L)$ be the set of probability measures μ on TM that are invariant under ϕ_L and such that $\int_{TM} L d\mu < \infty$. The compactness of the sets $\mathcal{L}^{-1}(H^{-1}(\{e\}))$ permits to prove the following proposition.

Proposition 4.2.3. *For any value e of H, the energy level $\mathcal{L}^{-1}(H^{-1}(\{e\}))$ contains at least one ϕ_L-invariant probability measure.*

One defines a function A_L on $\mathscr{M}(L)$, called the *average action*, by

$$A_L(\mu) = \int_{TM} L d\mu.$$

Clearly the measures with support in $\mathcal{L}^{-1}(H^{-1}(\{e\}))$ have finite action. A measure $\mu \in \mathscr{M}_L$ such that $A_L(\mu) = \min_{\mathscr{M}(L)} A_L$ is called an *action-minimizing measure*. The following proposition plays a crucial role in Mather's theory.

Proposition 4.2.4. *The functions A_L is lower semi-continuous. As a consequence, the set of action-minimizing measures is non empty.*

Let $\mu \in \mathscr{M}(L)$. Due to the superlinearity of L, for any closed 1-form η on M, the integral $\int_{TM} \eta d\mu$ is well defined and finite. One can prove that if η is exact, the previous integral vanishes. This allows one to define the following linear functional

$$\begin{array}{ccc} H^1(M,\mathbb{R}) & \to & \mathbb{R} \\ [\eta] & \mapsto & \displaystyle\int_{TM} \eta d\mu, \end{array}$$

where η is any representent of $[\eta]$. By duality, there exists $\omega(\mu) \in H_1(M,\mathbb{R})$ such that

$$\int_{TM} \eta d\mu = \langle \eta, \omega(\mu) \rangle.$$

We say that $\omega(\mu)$ is the rotation vector (or the homology class) of μ.

Proposition 4.2.5. *The map $\omega : \mathscr{M}(L) \to H_1(M,\mathbb{R})$ is continuous affine and surjective.*

Observe that due to semi-continuity of A_L and the compactness of $\omega^{-1}(\mu)$ for any measure μ, the minimum $\min_{\{\mu \in \mathscr{M}(L) \,|\, \omega(\mu)=\omega\}}$ is achieved. This allows us to state the following definition.

Definition 4.2.2. The *Mather's function* β is defined as

$$\begin{array}{cccc} \beta : & H_1(M,\mathbb{R}) & \to & \mathbb{R} \\ & \omega & \mapsto & \min_{\{\mu \in \mathscr{M}(L) \,|\, \omega(\mu)=\omega\}} A_L(\mu) \end{array}$$

One proves that β is a convex function.

The proof of the following proposition is given in [Mas96].

Proposition 4.2.6. *Assume that L is a geodesic Lagrangian on $T\mathbb{T}^n$. Then β coincide with the square of stable norm: for any $\gamma \in H_1(\mathbb{T}^n,\mathbb{R})$, $\beta(\gamma) = ||\gamma||_s^2$.*

Now observe that if η is a 1-form on TM, it defines a new Tonelli Lagrangian on TM by setting:

$$L_\eta(x,v) = L(x,v) - \langle \eta_x, v\rangle.$$

Proposition 4.2.7. *If η is closed, L and L_η have the same Euler-Lagrange flows.*

Actually, changing the Lagrangian L by a closed 1-form does not perturb the dynamics. So one also looks at the minimizing measures for the modified Lagrangians L_η. Fix $c \in H^1(M,\mathbb{R})$, and let η_c be a representant of η. One says a measure $\mu \in \mathscr{M}(L)$ is *c-action-minimizing* if it minimizes $A_{L_{\eta_c}}$ among $\mathscr{M}(L)$.

Definition 4.2.3. The *Mather's function* α is defined as

$$\begin{array}{rlcl} \alpha: & H^1(M,\mathbb{R}) & \to & \mathbb{R} \\ & c & \mapsto & -\min_{\{\mu\in\mathscr{M}(L)\}} A_{L_{\eta_c}}(\mu) \end{array}$$

One checks that this function is well defined, that is, it does not depends on the choice of η_c. One also proves that α is convex.

Proposition 4.2.8. *The functions α and β are convex conjugate, that is, $\alpha^* = \beta$ and $\beta^* = \alpha$.*

For $\omega \in H_1(M,\mathbb{R})$, we denote by $\mathscr{M}^\omega$ the subset of action-minimizing measures with rotation vector ω. For $c \in H^1(M,\mathbb{R})$ we denote by $\mathscr{M}_c$ the subset of c-action-minimizing measures.

Definition 4.2.4. The *Mather set $\widetilde{\mathscr{M}^\omega}$ of a rotation vector* $\omega \in H_1(M,\mathbb{R})$ is defined as:

$$\widetilde{\mathscr{M}^\omega} := \bigcup_{\mu\in\mathscr{M}^\omega} \operatorname{supp}\mu \subset TM.$$

The *Mather set $\widetilde{\mathscr{M}_c}$ of cohomology class* $c \in H^1(M,\mathbb{R})$ is defined as:

$$\widetilde{\mathscr{M}_c} := \bigcup_{\mu\in\mathscr{M}_c} \operatorname{supp}\mu \subset TM.$$

We denote by $\pi : TM \to M$ the canonical projection.

Theorem 16. ***Mather's graph theorem.*** *The sets $\widetilde{\mathscr{M}_c}$ and $\widetilde{\mathscr{M}^\omega}$ are compact, ϕ_L-invariant and the restrictions $\pi_{|\widetilde{\mathscr{M}_c}}$ and $\pi_{|\widetilde{\mathscr{M}^\omega}}$ are injective maps into M whose inverses are Lipschitz.*

If H is a Tonelli Hamiltonian on T^*M, the Fenchel-Legendre inverse transform defines a Tonelli Lagrangian L_H on TM. Therefore, one can associate with H the Mather's functions defined by L_H. We denote them by β_H and α_H.

Theorem 17. *Let H be a Tonelli Hamiltonian. Assume there exists an exact symplectomorphism $\Psi : T^*M \to T^*M$ such that $H \circ \Psi$ is a Tonelli Hamiltonian. Then $\beta_{H\circ\Psi} = \beta_H$ and $\alpha_{H\circ\Psi} = \alpha_H$.*

Tonelli Hamiltonian on $T^*\mathbb{T}^n$ and admissible tori. We now consider a Tonelli Hamiltonian on $T^*\mathbb{T}^n$. We denote by $L : T\mathbb{T}^n \to \mathbb{R}$ its associated Lagrangian. As usual we denote by ϕ_H and ϕ_L their respective flows.

Definition 4.2.5. An *admissible torus with rotation vector* ω is a torus $\mathcal{T} \subset T^*\mathbb{T}^n$ such that

1. $\mathcal{T}$ is a C^1 Lagrangian graph: $\{(x, c + d_x u) \,|\, x \in \mathbb{T}^n\}$ whith $c \in \mathbb{R}^n$ and $u : \mathbb{T}^n \to \mathbb{R}$,
2. $\mathcal{T}$ is ϕ_H-invariant,
3. the restriction of ϕ_H to $\mathcal{T}$ is conjugate to the Kronecker flow ϕ^ω on $\mathbb{T}^n$ defined by $\phi^\omega_t(x) = x + t\omega$.

Remark 4.2.4. Let μ^* be an ergodic invariant probability measure with support in $\mathcal{T}$. We set $\mu := \mathcal{L}^*\mu^*$. One easily checks that:

1. If η is a closed 1-form on M, then $\displaystyle\int_{TM} \eta d\mu = \langle \eta, \omega \rangle$, that is, ω is the rotation vector associated with μ.
2. $\displaystyle\int_{TM} L d\mu = -H(\mathcal{T}) + c \cdot \omega.$

The following proposition is an easy consequence of the Fenchel-Legendre inequality.

Proposition 4.2.9. *1. If $\tilde{\mu}$ is another ϕ_L-invariant probability measure with rotation vector ω, then $A_L(\mu) \leq A_L(\tilde{\mu})$. As a consequence $\mathcal{L}^{-1}(\mathcal{T}) = \widetilde{\mathscr{M}^\omega}$.*

2. If $\tilde{\mu}$ is another ϕ_L-invariant probability measure, then $A_{L_c}(\mu) \leq A_{L_c}(\tilde{\mu})$. As a consequence $\mathcal{L}^{-1}(\mathcal{T}) = \widetilde{\mathscr{M}_c}$.

Assume now that the Hamiltonian H is in action-angle form, that is, $H(x, p) = h(p)$. Its associated Lagrangian is of the form $L(x, v) = \ell(v)$. The cotangent bundle $T^*\mathbb{T}^n$ is globally foliated by admissible tori $\mathbb{T} \times \{c\}$ and the tangent bundle by ϕ_L-invariant tori $\mathbb{T} \times \{\omega\}$. Identifying $H^1(\mathbb{T}^n, \mathbb{R})$ and $H_1(\mathbb{T}^n, \mathbb{R})$ with $\mathbb{R}^n$, one easily deduces from proposition 4.2.9 that:

$$\beta(\rho) = \ell(\rho), \quad \text{and} \quad \alpha(c) = h(c).$$

In particular, one proves that

$$\widetilde{\mathscr{M}_c} = \widetilde{\mathscr{M}^\omega} = \mathcal{L}^{-1}(\mathbb{T}^n \times \{c\}) = \mathbb{T}^n \times \{\omega\},$$

when $\omega = \nabla h(c)$ and $c = \nabla \ell(\omega)$.

4.2.3 The constant $\mathscr{V}_g$ for tori of revolution

We come back to the torus of revolution. We use the notation of lemma 4.2.1. For $e > 0$, we set

$$\begin{aligned}
\mathcal{D}_e &:= P_e^{-1}(\{J(e)\}), \\
\mathcal{Z}_e^- &:= P_e^{-1}([-2\pi\sqrt{2e}.x_n, -2\pi\sqrt{2e}.x_1[), \\
\mathcal{Z}_e^+ &:= P_e^{-1}(]2\pi\sqrt{2e}.x_1, 2\pi\sqrt{2e}.x_n]).
\end{aligned}$$

The domains $\mathcal{D}_e$. For $e > 0$ and $\rho \in J(e)$, we denote by $\theta_{e,\rho}$ the function defined on $\mathbb{T}$ by $\theta_{e,\rho} : \bar{s} \mapsto \arccos \frac{\rho}{2\pi\sqrt{2ex(\bar{s})}}$. The set $\mathcal{D}_e$ has two connected components exchanged by the symetry ζ and foliated by Liouville tori:

$$\mathcal{D}_e^+ := \bigcup_{\rho \in J(e)} \mathcal{T}_{e,\rho}^+ \quad \text{and} \quad \mathcal{D}_e^- := \bigcup_{\rho \in J(e)} \mathcal{T}_{e,\rho}^-$$

where $\mathcal{T}_{e,\rho}^+ := \{(\bar{\varphi}, \bar{s}, \theta_{e,\rho}(s)) \,|\, (\bar{\varphi}, \bar{s}) \in \mathbb{T}^2\}$ and $\mathcal{T}_{e,\rho}^- = \zeta(\mathcal{T}_{e,\rho}^+)$.

The domains $\mathcal{D}_e^+$ and $\mathcal{D}_e^-$ are respectively bounded by $\mathcal{G}_e^{0,+}$ and $\mathcal{G}_e^{\pi,+}$ and by $\mathcal{G}_e^{0,-}$ and $\mathcal{G}_e^{\pi,-}$.

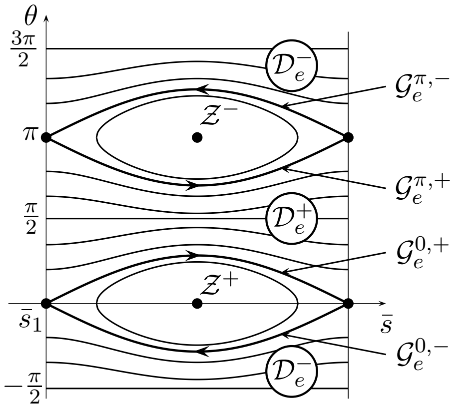


Figure 4.2: The domains $\mathcal{D}_e^+$ and $\mathcal{D}_e^-$.

The domains $\mathcal{Z}_e^\bullet$. Since $P_e^{-1}(\{\cup_{k=1}^{n-1} I_k(e)\})$ and $P_e^{-1}(\{\cup_{k=1}^{n-1} - I_k(e)\})$ are exchanged by ζ, we focus on the first one. For $1 \leq j \leq n$, we denote by $\mathscr{P}_j^e$ the ∞-level defined by $P_e(\mathscr{P}_j^e) = x_j$. It is the complementary set in $\mathcal{Z}_e^+$ of the critical loci of P_e. So it has a finite number of connected components $D_{i,j}$, homotopic to $D^* \times \mathbb{T}$ where D^* is the open pointed disc. Their boundary is either made of piece of a ∞-level $\mathcal{P}_j^e$ and an elliptic orbit $\mathcal{E}_i$ with $x_i < x_j$ or two pieces of ∞-levels $\mathscr{P}_j^e$ and $\mathscr{P}_i^e$ with $x_i < x_j$.

More precisely, $D_{i,j} = \{(\bar{\varphi}, \bar{s}, \theta_{e,\rho}) \,|\, (\bar{\varphi}, \bar{s}) \in \mathbb{T} \times I_s,\ \rho \in]2\pi\sqrt{2e}x_i, 2\pi\sqrt{2e}x_j[\}$, where $\theta_{e,\rho} : s \mapsto \arccos \frac{\rho}{2\pi\sqrt{2ex(s)}}$ and where I_s is a disjoint union of intervals $]\bar{s}_{i_1}, \bar{s}_{j_1}[$ and $]\bar{s}_{j_2}, \bar{s}_{i_2}[$ with

- $\bar{s}_{i_1} < \bar{s}_{i_2}$ in $x^{-1}(x_i)$ and $\bar{s}_{j_1} \leq \bar{s}_{j_2}$ in $x^{-1}(x_j)$,
- $]\bar{s}_{i_k}, \bar{s}_{j_k}[\cap \overline{S} = \emptyset$ and $]\bar{s}_{i_l}, \bar{s}_{j_l}[\cap \overline{S} = \emptyset$.

The Liouville tori contained in $D_{i,j}$ are the connected union

$$\mathcal{T}_{e,\rho} := \{(\bar{\varphi}, \bar{s}, \theta_{e,\rho}(\bar{s})), \,|\, (\bar{\varphi}, \bar{s}) \in \overline{B}_{e,\rho}]\} \cup \{(\bar{\varphi}, \bar{s}, -\theta_{e,\rho}(\bar{s})), \,|\, (\bar{\varphi}, \bar{s}) \in \overline{B}_{e,\rho}]\}.$$

where $\overline{B}_{e,\rho} := \mathbb{T} \times [\underline{c}_{e,\rho}, \overline{c}_{e,\rho}]$ satisfies $[\bar{s}_{j_1}, \bar{s}_{j_2}] \subset [\underline{c}_{e,\rho}, \overline{c}_{e,\rho}] \subset [\bar{s}_{i_1}, \bar{s}_{i_2}]$

Remark 4.2.5. Since x is increasing on $]\bar{s}_{i_k}, \bar{s}_{j_k}[$ and decreasing on $]\bar{s}_{i_l}, \bar{s}_{j_l}[$, if $(\rho, \rho') \in I_{i,j}^2$ with $\rho' < \rho$, then $\overline{B}_{e,\rho} \subset \overline{B}_{e,\rho'}$.

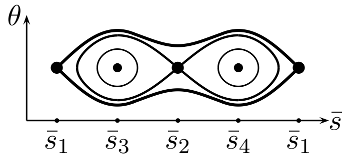


Figure 4.3: The domain $\mathcal{Z}_e^+$ with 4 critical points

The function β. By Mather's graph theorem, none of the levels of p_φ contained in the domains $\mathcal{Z}_e^-$ or $\mathcal{Z}_e^+$ can support a minimizing measure. Let us study the domains $\mathcal{D}_e^+$ and $\mathcal{D}_e^-$.

We set $\mathcal{D}_\infty^+ := \cup_{e>0}\mathcal{D}_e^+$ and $\widetilde{\mathcal{D}}_\infty^+ := (\varpi^*)^{-1}(\mathcal{D}_\infty^+)$. We first remark that the Liouville tori $\mathcal{T}_{e,\rho}$ contained in $\mathcal{D}_\infty^+$ and $\mathcal{D}_\infty^-$ are C^1 graphs over $\mathbb{T}^2$. Due to the symetry ζ, we can focus on $\mathcal{D}_\infty^+$, which admits the following parametrization:

$$\mathcal{D}_\infty^+ = \{(\bar{\varphi}, \bar{s}, e, \rho) \in \mathbb{T}^2 \times D\},$$

where $D := \{(e,\rho) \,|\, e > 0, \rho \in J(e)\}$. By the Arnol'd Liouville Theorem there exist an open domain $B_+ \subset \mathbb{R}^2$ and a symplectic diffeomorphism

$$\begin{array}{rlcl} A_+ : & \mathcal{D}_\infty^+ & \to & \mathbb{T}^2 \times B_+ \\ & (\bar{\varphi}, \bar{s}, e, \rho) & \mapsto & (\alpha^1, \alpha^2, I_1, I_2), \end{array}$$

such that I_1, I_2 depend only on the value (e,ρ) of the moment map $F := (H, p_\varphi)$ and generate 1-periodic Hamiltonian flows. We denote by H_+ the Hamiltonian function on $\mathbb{T}^2 \times B$ defined by $H_+(I) = H \circ A_+^{-1}(I)$.

In the same way, we set $H_- := H \circ A_-^{-1}$, where A_- is the action-angle transformation on $\mathcal{D}_\infty^-$.

Consequence: a) The flow ϕ_H on a torus $\mathcal{T}_{e,\rho}$ is conjugate to a Kronecker flow on the torus $\mathbb{T}^2 \times \{I(e,\rho)\}$, so the torus $\mathcal{T}_{e,\rho}$ is an admissible torus with rotation vector $\nabla H_+(I(e,\rho))$.

b) In the same way, the tori $\mathcal{T}_{e,\rho}$ contained in $\mathcal{D}_\infty^-$ are admissible tori with rotation vectors $\nabla H_-(I(e,\rho))$.

Finally, one proves that the graphs $\mathcal{G}_e^{0,+}$ and $\mathcal{G}_e^{\pi,+}$ and by $\mathcal{G}_e^{0,-}$ and $\mathcal{G}_e^{\pi,-}$ support a minimizing measure, indeed the supports of such measures are contained in the hyperbolic circles $\mathcal{C}_1^0$ and $\mathcal{C}_1^\pi$.

Consequence: Roughly speaking, the function β associated with H is the function β associated with the Hamiltonian H in restriction to $\overline{\mathcal{D}_\infty^+ \bigcup \mathcal{D}_\infty^-}$.

Let us study the action-angle transformations A_+ and A_-. By symetry, we can focus on A_+. Set

$$\tau_{e,\rho} = \int_0^1 \frac{r(t)}{\sqrt{2e - \frac{\rho^2}{4\pi^2 x(t)^2}}}dt \quad \text{and} \quad \varphi_{e,\rho} := \int_0^{\tau_{e,\rho}} \dot{\varphi}(t)dt.$$

In Appendix A, we prove that A_+ can be constructed such that:

- $I_1(e,\rho) = \rho$ and $I_2(e,\rho) = \displaystyle\int_0^1 r(t)\sqrt{(2e - \frac{\rho^2}{4\pi^2 x(t)^2})}dt.$

- $\phi_H^t(m,p) = \phi_{I_2}^{\frac{t}{\tau_{e,\rho}}} \circ \phi_{I_1}^{t\frac{\varphi_{e,\rho}}{\tau_{e,\rho}}}(m,p).$

Moreover, one checks that A_+ preserves the Liouville form. The proof of the following proposition is given in Appendix B.

Proposition 4.2.10. *Let $h_+ : B \to \mathbb{R}$ be such that $H_+(\alpha, I) = h_+(I)$. Then h_+ is convex and superlinear.*

Remark 4.2.6. One checks that the action variables given by A_- are $I_1^-(e,\rho) = I_1(e,\rho)$ and $I_2^-(e,\rho) = -I_2(e,\rho)$.

Corollary 4.2.1. *The function β associated with $H_{|_{\mathcal{D}_\infty^+}}$ coincide with the function β_+ associated with the Hamiltonian H_+.*

Let $\omega_+ : B_+ \to \mathbb{R}^2 : I \mapsto \nabla h_+(I)$. We define in the same way ω_-. By remark 4.2.6, one checks that $\omega_-(I_-)$ is the image of $\omega(I)$ by the map $(\omega_1, \omega_2) \mapsto (\omega_1, -\omega_2)$. We set $\Omega^+ = \omega(h_+^{-1}(\{\frac{1}{2}\}))$ and $J := J(\frac{1}{2}) :=]-\rho_0, \rho_0[$ with $\rho_0 := 2\pi x_1$.

Proposition 4.2.11. *The submanifold Ω^+ is the image of the curve ω parametrized by*

$$\begin{array}{rccl} \omega : & J & \longrightarrow & \mathbb{R}^2 \\ & \rho & \mapsto & \omega(\rho) := (X(\rho), Y(\rho)) = \left(\frac{\varphi_\rho}{\tau_\rho}, \frac{1}{\tau_\rho}\right). \end{array}$$

Proof. Let $(a, I) \in \mathbb{R}^2 \times B_+$. Let (m, p) such that $A(m,p) = (a, I)$. We set

$$(a^1(t), a^2(t)) := (a^1(\phi^t(m,p)), a^2(\phi^t(m,p)).$$

The result comes from: $\phi_{I_1}^{a^1(t)} \circ \phi_{I_2}^{a^2(t)}(\sigma(e,\rho)) = \phi_H^t(m,p) = \phi_{I_2}^{\frac{t}{\tau_{e,\rho}}} \circ \phi_{I_1}^{t\frac{\varphi_{e,\rho}}{\tau_{e,\rho}}}(m,p)$. ☐

Lemma 4.2.3. *Let $\gamma := x''(0)$. We set $\tau_\rho := \tau_{\frac{1}{2},\rho}$, $\varphi_\rho := \varphi_{\frac{1}{2},\rho}$ and $\tau'_\rho := \dfrac{d\tau_\rho}{d\rho}(\rho)$. One has the following asymptotic estimates:*

1. $\tau_\rho \simeq_{\rho \to \rho_0} -\dfrac{\rho_0^{\frac{3}{2}}}{\rho} \dfrac{r(0)}{2\sqrt{\pi\gamma}} \ln(\rho_0 - \rho)$,

2. $\varphi_\rho \simeq_{\rho \to \rho_0} -\dfrac{1}{4\pi^2} \dfrac{1}{\sqrt{\rho_0}} \dfrac{r(0)}{2\sqrt{\pi\gamma}} \ln(\rho_0 - \rho)$,

3. $\tau'_\rho \simeq_{\rho \to \rho_0} \dfrac{1}{4\pi^2} \dfrac{1}{\sqrt{\rho_0}} \dfrac{r(0)}{2\sqrt{\pi\gamma}} \dfrac{1}{\rho_0 - \rho}$.

Corollary 4.2.2. *The curve ω extends by continuity to the closed interval $\bar{J}$ with $\omega(\rho_0) := (\frac{1}{4\pi^2}\frac{1}{\rho_0}, 0)$ and $\omega(-\rho_0) := (-\frac{1}{4\pi^2}\frac{1}{\rho_0}, 0)$. We still denote by Ω^+ the image of ω.*

Recall that $T := \frac{1}{4\pi^2}\frac{1}{\rho_0}$ is the period of the hyperbolic orbits $\mathcal{C}_1^0$ and $\mathcal{C}_1^\pi$. The previous corollary may be interpreted in the following way.

Remark 4.2.7. The invariant measures μ associated with the hyperbolic orbit $\mathcal{C}_1^0$ and $\mathcal{C}_1^\pi$ have respective rotation vectors $\omega(\mu) := (T, 0)$ and $\omega(\mu) := (-T, 0)$.

Obviously, the previous corollary holds for the submanifold Ω^-. We set $\Omega := \Omega^+(\bar{J}) \cup \Omega^-(\bar{J})$.

Corollary 4.2.3. *The closed curve Ω is the unit sphere of the stable norm associated with g. Therefore, the volume $\mathscr{V}_g$ is the volume of the compact convex domain delimited by Ω and*

$$\mathscr{V}_g = 2\int_0^{\rho_0} X'(\rho)Y(\rho)d\rho = 2\int_0^{\rho_0} \frac{\varphi'_\rho\tau_\rho - \varphi_\rho\tau'_\rho}{\tau_\rho^3}d\rho.$$

In the next section we give an elementary proof, that is, without using Mather's theory, of the fact that $\mathscr{V}_g$ is the Lebesgue volume of the domain bounded by Ω.

4.2.4 The minimizing ball

In this section, we see that $\mathscr{V}_g$ is indeed the asymptotic volume of the minimizing balls. We focus on the geodesics with unit speed, that is, we fix $e = \frac{1}{2}$. We write $J(\frac{1}{2}) =]-\rho_0, \rho_0[$. We omit the lower index $\frac{1}{2}$ and set:

$$\begin{aligned}
\mathcal{G}^{0,+} &:= \{(m, p^+_{\rho_0}(m)) \,|\, m \in \mathbb{R}^2\},\\
\mathcal{G}^{\pi,+} &:= \{(m, p^+_{-\rho_0}(m)) \,|\, m \in \mathbb{R}^2\},\\
\mathscr{L}^+_\rho &:= \{(m, p^+_\rho(m)) \,|\, m \in \mathbb{R}^2\}.
\end{aligned}$$

For $m \in \mathbb{R}^2$ and $\bullet = +, -$, we set:

$$\gamma^\bullet_{m,\rho} : \mathbb{R} \to \mathbb{R}^2 : t \mapsto \tilde{\pi} \circ \tilde{\phi}_t(m, p^\bullet_\rho(m)).$$

Due to proposition 5, the geodesics $\gamma^\bullet_{m,\rho}$ are minimizing.

Conjugate points. We show that the geodesics that are the projections of solutions lying in $(\varpi^*)(P^{-1}_{\frac{1}{2}}(I_k(\frac{1}{2})))$ or $(\varpi^*)(P^{-1}_{\frac{1}{2}}(I_k(\frac{1}{2})))$ have conjugate points.

Fix a connected component $D_{i,j}$ of $\mathcal{Z}_{\frac{1}{2}}$ and let $\rho \in I_{i,j} =]2\pi x_i, 2\pi x_j[$. For $\rho \in I_{i,j}$, we denote by $\mathcal{T}_\rho$ the Liouville torus $\mathcal{T}_{\frac{1}{2},\rho}$ and by $\overline{B}_\rho$ the domain $\overline{B}_{e,\rho}$. We also introduce the following sets.

- $\mathscr{L}_\rho := (\varpi^*)^{-1}(\mathcal{T}_\rho)$ and $B_\rho := \varpi^{-1}(\overline{B}_\rho)$,
- $\underline{L}_\rho$ and $\overline{L}_\rho$ are the horizontal lines defined by $\{s = \underline{c}_\rho\}$ and $\{s = \overline{c}_\rho\}$.

Finally, we denote by $\gamma_\rho : t \mapsto (\varphi_\rho(t), s_\rho(t))$ the projection of a solution of X^g lying in $\mathcal{T}_\rho$.

Lemma 4.2.4. *A geodesic γ_ρ joins $\underline{L}_\rho$ to $\overline{L}_\rho$ in a finite time τ_ρ independent of the choice of γ_ρ. Moreover, the function $\rho \mapsto \tau_\rho$ is smooth.*

Proof. The time needed to go from $\underline{L}_\rho$ to $\overline{L}_\rho$ is $\tau_\rho = \displaystyle\int_{\underline{c}_\rho}^{\overline{c}_\rho} \frac{r(s)}{\sqrt{(1 - \frac{\rho^2}{4\pi^2 x(s)^4})}}ds$. Obviously τ_ρ is independent of the choice of γ_ρ. Now since $x'(\underline{c}_\rho) > 0$ and $x'(\overline{c}_\rho) < 0$ (see remark 4.2.5), and denoting by $c = \overline{c}_\rho, \underline{c}_\rho$, one has

$$\frac{r(s)}{\sqrt{(1 - \frac{\rho^2}{4\pi^2 x(s)^4})}} \simeq_{s=c} \frac{r(s)}{\sqrt{\frac{x'(c)}{4\pi^2 x^5(c)}(s-c)}}$$

and the integral above is convergent. Let S be the surface $\{\theta = 0\}$ in the unit sphere $S^*\mathbb{R}^2$. Let ρ_- and ρ_+ in $I_{i,j}$ such that $\rho_- < \rho < \rho_+$ and denote respectively by $\underline{S}$ and $\overline{S}$ the sections $\underline{S} = S \cap \{s \in [\underline{c}_{\rho_-}, \underline{c}_{\rho_+}]\}$ and $\overline{S} = S \cap \{s \in [\overline{c}_{\rho_-}, \overline{c}_{\rho_+}]\}$. One sees that $\underline{S}$ and $\overline{S}$ are transverse to the flow and to the Lagrangian submanifold $\mathscr{L}_\rho$. For any $\varphi \in \mathbb{R}$, we set $x_{\rho,\varphi} := (\underline{c}_\rho, \varphi, 0) \in \underline{S}$. Therefore, $x_{\rho,\varphi} \mapsto \tilde{\phi}_{\tau_\rho}(x_{\rho,\varphi})$ is the Poincaré map between $\underline{S}$ and $\overline{S}$. So it is a smooth map and $x_{\rho,\varphi} \mapsto \tau_\rho$ is also smooth. Since τ_ρ does not depend on the choice of φ, τ_ρ depends smoothly on ρ. □

We now fix $\rho^* \in I_{i,j}$. Let $m \in B_{\rho^*}$ such that $\gamma_{\rho^*}(0) = m$. Since s is not constant along γ_{ρ^*}, we can assume that $m \in \operatorname{Int} B_{\rho^*}$. We denote by $\widehat{\gamma}_{\rho^*}$ the image of γ_{ρ^*}. Let $r < \rho^*$ in $I_{i,j}$. There exists $T > 0$ such that for all $\rho \in [r, \rho^*]$, $\tau_\rho \leq T$.

Lemma 4.2.5. *Let $\rho \in [r, \rho^*[$. Let γ_ρ be such that $\gamma_\rho(0) = m$. There exists $0 < t_\rho \leq T$ such that $\gamma_\rho(t_\rho) \in \widehat{\gamma}_{\rho^*}$. Moreover, there exists $T^* > 0$ such that for any $\rho \in [r, \rho^*[, t_\rho \geq T^*$.*

Proof. Set $m = (\varphi_0, s_0)$. Consider the rectangle $\mathcal{R}$ delimited by the horizontal lines $\overline{L}_{\rho^*}$ and $\underline{L}_{\rho^*}$ and by the vertical lines $\{\varphi = \varphi_0\}$ and $\{\varphi = \varphi_{\rho^*}(T)\}$. According to the previous lemma, and since $\underline{c}_\rho < \underline{c}_{\rho^*} < \overline{c}_{\rho^*} < \overline{c}_\rho$ (remark 4.2.5), the sets $\{t \in [0, T] \,|\, s_\rho(t) = \overline{c}_{\rho^*}\}$ and $\{t \in [0, T] \,|\, s_\rho(t) = \underline{c}_{\rho^*}\}$ are not empty. Let $t_1(\rho) = \min\{t \in [0, T] \,|\, s_\rho(t) = \overline{c}_{\rho^*}\}$ and $t_2(\rho) = \min\{t \in [0, T] \,|\, s_\rho(t) = \underline{c}_{\rho^*}\}$. Without loss of generality, one can suppose that $t_1(\rho) < t_2(\rho)$.

Then, there exists $t_\rho \in [t_1(\rho), t_2(\rho)]$ such that $\gamma_{\rho^*}(t_\rho) \in \widehat{\gamma}_{\rho^*}$. Now the function $\rho \mapsto t_1(\rho) = \displaystyle\int_{s_0}^{\overline{c}_{\rho^*}} \frac{r(u)}{\sqrt{1 - \frac{\rho^2}{x(u)^2}}} du$ is increasing. So for each $\rho \in [r, \rho^*[$, $t_1(\rho) \geq t_1(r)$. Since $t_\rho \geq t_1(\rho)$, the second part of our assertion is proved with $T^* = t_1(r)$. □

Corollary 4.2.4. *There exists $m' \in \widehat{\gamma}_{\rho^*}$ that is conjugate to m along γ_{ρ^*}.*

Proof. We set $\widehat{\gamma}_T = \{\gamma_{\rho^*}(t) \,|\, t \in [0, T]\}$. Consider an increasing sequence $(\rho_n)_{n\in\mathbb{N}}$ in $[r, \rho^*]$ that converges to ρ^*. Then $\{(m_{\rho_n}, t_{\rho_n})_{n\in\mathbb{N}}\}$ is in the compact $\widehat{\gamma}_T \times [0, T]$ and there exists a subsequence $(m_{\rho_{n_k}}, t_{\rho_{n_k}})_{k\in\mathbb{N}}$ which converges to a point $(m', \tau) \in \widehat{\gamma}_T \times [0, T]$. Show that m' is conjugate to m. Notice that $\tau \geq T^* > 0$. For $k \in \mathbb{N}$, let t'_{n_k} such that $\gamma_{\rho^*}(t'_{n_k}) = (m_{\rho_{n_k}})$ and let $p_{n_k} \in S^*_m\mathbb{R}^2$ such that, for all $t, \gamma_{\rho_{n_k}}(t) = \pi \circ \tilde{\phi}_t(m, p_{n_k})$. We denote by $\exp^*_m$ the exponential map $\mathbb{R}_+ \times S^*_m\mathbb{R}^2 \to \mathbb{R}^2 : (t, p) \mapsto \pi \circ \tilde{\phi}^t(m, p)$. One has:

$$\exp{}^*_m(t_{n_k}.p_{n_k}) = m_{\rho_{n_k}} = \tilde{\pi} \circ \tilde{\phi}_{t'_{n_k}}(m, p_{n_k}) = \exp{}^*_m(t'_{n_k}, p).$$

Hence, $\lim_{k\to\infty} \tilde{\phi}^{t_{n_k}}(m, p_{n_k}) = \tilde{\phi}_\tau(m, p) = \lim_{k\to\infty} \tilde{\phi}_{t'_{n_k}}(m, p)$, Therefore, $\exp^*_m$ cannot be a diffeormorphism in a neighborhood of (τ, p), and m' is conjugate to m. □

The minimzing ball. The above observations allow us to defined, for $m \in \mathbb{R}^2$, the *minimizing domain of m* by

$$B_{\min}(m) := \{\gamma^+_{m,\rho}(t) \,|\, (\rho, t) \in \overline{J(\tfrac{1}{2})}) \times \mathbb{R}^+\} \cup \{\gamma^-_{m,\rho}(t) \,|\, (\rho, t) \in \overline{J(\tfrac{1}{2})}) \times \mathbb{R}^+\}.$$

This paragraph is devoted to the description of $B_{\min}(m)$.

Remark 4.2.8. 1) The domains $B^+_{\min}(m) := \{\gamma^+_{m,\rho}(t) \,|\, (\rho, t) \in \overline{J(\tfrac{1}{2})}) \times \mathbb{R}^+\}$ and $B^-_{\min}(m) := \{\gamma^-_{m,\rho}(t) \,|\, (\rho, t) \in \overline{J(\tfrac{1}{2})}) \times \mathbb{R}^+\}$ are symetric with respect to the φ-axis.

2) The geodesics $\gamma^+_{m,\rho}$ and $\gamma^+_{m,-\rho}$ are symetric with respect to the s-axis.

Due to the invariance of the geodesics under φ-translations, we can assume that $m = (0, s_0)$. We set $S_1 := \varpi^{-1}(\{\bar{s}_1\}) \subset \mathbb{R}$.

The description of the geodesics $\gamma^+_{m,\rho}$ is easy. We write $\gamma^+_{m,\rho}(t) = (\varphi_\rho(t), s_\rho(t))$.

• $\rho \in]0, \rho_0[$: $\dot{\varphi_\rho} > 0$ and $\dot{s_\rho} > 0$. Moreover, the time τ_ρ needed to reach the horizontal line $s = s_0 + 1$ is finite, so $t \mapsto s_\rho(t)$ is not bounded. Now, if x_n is the maximal value of x, $\dot{\varphi}_\rho \geq \frac{\rho}{x_n} > 0$, and $t \mapsto \varphi_\rho(t)$ is not bounded. The geodesic $\gamma^+_{m,\rho}$ is the graph of an increasing and unbounded function.

• $\rho = 0$: $\dot{\varphi}_0 \equiv 0$ and $\dot{s}_0 > 0$, $\gamma^+_{m,\rho}$ is the vertical line $\varphi = \varphi_0$.

• $\rho = \rho_0$: there are two cases to distinguish. If $s_0 \in S_1$: $\dot{s}_{\rho_0} \equiv 0$ and $\dot{\varphi}_{\rho_0} \equiv \frac{1}{2\pi x_1}$ so γ^+_{m,ρ_0} is the horizontal line $s = s_0$.

If $s_0 \notin S_1$: $\dot{\varphi}_{\rho_0} > 0, \dot{s}_{\rho_0} > 0$. As before $\dot{\varphi}_{\rho_0} \geq \frac{x_1}{2\pi x_n} > 0$, and φ_ρ is not bounded. Let $s_1 := \min\{s \in S_1 \,|\, s > s_0\}$. Since $x''(s_1) = 0$,

$$\frac{r(s)}{\sqrt{(1 - \frac{x_1^2}{x(s)^2})}} \simeq_{s=s_1} \frac{r(s)}{\sqrt{\frac{x''(1)}{x^3(1)}}(1-s)}.$$

So the integrale

$$\tau = \int_{s_0}^{s_1} \frac{r(s)}{\sqrt{(1 - \frac{\rho_0^2}{4\pi^2 x(s)})}} ds = \int_{s_0}^{s_1} \frac{r(s)}{\sqrt{(1 - \frac{4\pi^2 x_1^2}{4\pi^2 x(s)})}} ds$$

is divergent and s_{ρ_0} is bounded above by s_1. The geodesic γ^+_{m,ρ_0} is the graph of an increasing function bounded by s_1.

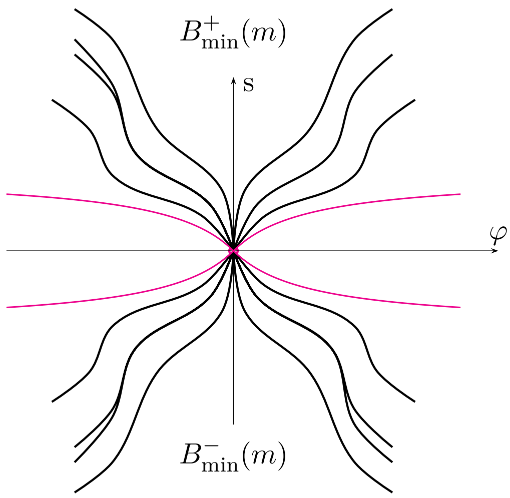


Figure 4.4: The domain $B_{\min}(m)$ when $s_0 \notin S_1$.

Asymptotics of the volume of balls. Set $\widetilde{\mathcal{Z}} := (\varpi^*)^{-1}(\mathcal{Z}^+_{\frac{1}{2}}) \cup (\varpi^*)^{-1}(\mathcal{Z}^-_{\frac{1}{2}})$. For $m \in \mathbb{R}^2$, we set $\mathcal{Z}(m) := \widetilde{\mathcal{Z}} \cup T^*_m\mathbb{R}^2$. Therefore, the domain

$$B_{\widetilde{\mathcal{Z}}}(m) := \{\pi \circ \tilde{\phi}_t(m,p) \,|\, (p,t) \in \widetilde{\mathcal{Z}}(m) \times \mathbb{R}\}$$

is the complementary set in $\mathbb{R}^2$ of the minimizing domain $B_{\min}(m)$.

Remark 4.2.9. The connected components of projection on $\mathcal{T}$ of the non connected subset $\{(\varphi, s) \,|\, \varphi \in \mathbb{R}, s \in S_1\}$ of $\mathbb{R}^2$ are the *inner equators*. When m belongs to an inner equator, $B_{\widetilde{\mathcal{Z}}}(m)$ is empty, so $B_{\min}(m) = \mathbb{R}^2$.

Proposition 4.2.12. *The geodesic flow has asymptotically full minimizing domain.*

Proof. According to remark 4.2.9, one just has to consider the case when $m \notin \mathbb{R} \times S_1$. Fix $m = (\varphi, s) \in \mathbb{R}^2 \setminus (\mathbb{R} \times S_1)$. For $T > 0$, we define the ball $B_{\tilde{\mathcal{Z}}}(m, T) = B_{\tilde{\mathcal{Z}}}(m) \cap B(m, T)$.

Let $s_1 := \min S_1 \cap \{s' > s\}$. Let $\delta > 0$ such that for all $(m, m') \in \mathbb{R}^2$, $d_{\tilde{g}}(m, m') \leq \delta d_\infty(m, m')$, where d_∞ is the distance associated with the Max-norm.
We set $\mathcal{R} := [\lfloor -\delta T \rfloor - 1, \lfloor \delta T \rfloor + 1] \times [-s_1 + s, s_1 - s]$. Then $B_{\mathcal{Z}}(m, T) \subset \mathcal{R}$, and if v' is the Riemannian volume of the rectangle $[0, 1] \times [-s_1 + s, s_1 - s]$, one has

$$\operatorname{Vol} B_{\mathcal{Z}}(m, T) \leq \operatorname{Vol} \mathcal{R} = 2(\lfloor \delta T \rfloor + 1)v'.$$

Therefore

$$\lim_{T\to\infty} \frac{\operatorname{Vol} B_{\mathcal{Z}}(m, T)}{\operatorname{Vol} B_{\min}(m, T)} \leq \lim_{T\to\infty} \frac{2(\lfloor \delta T \rfloor + 1)v'}{\chi(m)T^2} \leq \lim_{T\to\infty} \frac{2(\delta T + 1)v'}{\chi(m)T^2} = 0.$$

Consequently: $\displaystyle\lim_{T\to\infty} \frac{\operatorname{Vol} B(m, T)}{\operatorname{Vol} B_{\min}(m, T)} = \lim_{T\to\infty} \frac{\operatorname{Vol} B_{\min}(m, T)}{\operatorname{Vol} B_{\min}(m, T)} = 1$ □

We conclude this part with an alternative proof of corollary 4.2.3, which does not use Mather's theory. We consider the curves Ω^+, Ω^- and Ω as defined in corollary 4.2.2. We already know that Ω is the boundary of a compact convex domain. We begin with studying geometrical properties of particular hypersurfaces of the Euclidean space. Let $\mathcal{S}$ be a hypersurface of $\mathbb{R}^n$.

Definition 4.2.6. One says that $\mathcal{S}$ satisfies the *transversality property (T)* if for each point $u \in \mathcal{S}$, $\mathbb{R}u$ is transverse to $\mathcal{S}$.

Remark 4.2.10. If h is a strictly convex C^2 function defined on a open domain of $\mathbb{R}^n$ and e is a regular value of h such that $h^{-1}(\{e\})$ is compact and connected, the submanifold $\mathcal{S}_e := dh(h^{-1}(\{e\}))$ satisfies transversality property. As a consequence, Ω obviously satisfies the transversality property.

The following results are proved in Appendix C.

Proposition 4.2.13. *Let $\mathcal{S}$ be a connected hypersurface of $\mathbb{R}^n$ that satisfies (T). Assume moreover that $\mathcal{S}$ is the boundary of a connected compact domain K of $\mathbb{R}^n$ such that $0 \in \overset{\circ}{K}$. Then, the map*

$$\begin{array}{rccc} \Psi : & \mathcal{S} \times \mathbb{R}_+^* & \longrightarrow & (\mathbb{R}^n)^* \setminus \{0\} \\ & (w, t) & \mapsto & tw \end{array}$$

is a diffeomorphism.

Lemma 4.2.6. *Let $\mathcal{S}$ be a compact connected hypersurface that satisfies the hypotheses of propositon 4.2.13. Assume moreover that $\min_{x\in\mathcal{S}} ||x|| > 0$. Then or any $\varepsilon > 0$, there exists t_ε such that for any $T > t_\varepsilon$ if $n(T) := \operatorname{Card} \Psi(\mathcal{S} \times [0, T]) \cap \mathbb{Z}^n$, one has*

$$T^n(1 - \varepsilon)^n \operatorname{Vol_{Leb}} \Psi(\mathcal{S} \times [0, 1]) \leq n(T) \leq T^n(1 + \varepsilon)^n \operatorname{Vol_{Leb}} \Psi(\mathcal{S} \times [0, 1]).$$

This result is easily extended to some subdomains of $\mathcal{S}$. Let $\mathcal{D}$ be a connected open domain of $\mathcal{S}$ such that its boundary $\partial\mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$ is a smooth submanifold of dimension $n - 2$. For $T > 0$, we set $\mathscr{C}_{\mathcal{D},T} := \Psi(\overline{\mathcal{D}} \times [0, T])$ and $n_{\mathcal{D}}(T) := \operatorname{Card} \mathscr{C}_{\mathcal{D},T} \cap \mathbb{Z}^n$.

Corollary 4.2.5. *For any $\varepsilon > 0$, there exists t_ε such that for any $T > t_\varepsilon$*

$$T^n(1 - \varepsilon)^n \operatorname{Vol_{Leb}} \mathscr{C}_{\mathcal{D},1} \leq n_{\mathcal{D}}(T) \leq T^n(1 + \varepsilon)^n \operatorname{Vol_{Leb}} \mathscr{C}_{\mathcal{D},1}.$$

An immediate consequence of proposition 4.2.13 is the following.

Corollary 4.2.6. *We denote by H^+ the upper half-plane. The map*

$$\begin{array}{rccc} \Psi : & J \times \mathbb{R}_+^* & \longrightarrow & H^+ \\ & (t,\rho) & \mapsto & t\Omega^+(\rho) \end{array}$$

is a diffeomorphism.

Remark 4.2.11. The curve Ω^+ can be parametrized with polar coordinates. Let Θ be the homeomorphism defined on $\bar{J}$ by

$$\omega(\rho) = ||\omega(\rho)||e^{i\Theta(\rho)}.$$

Finally, using lemma 4.2.3, one checks the following property.

Property 4.2.1. *Consider the compact sets $\mathscr{C} := \Psi(\bar{J} \times [0,1])$ and for any $\rho \in J \cap \mathbb{R}_+^*$, $\mathscr{C}_\rho := \Psi([-\rho,\rho] \times [0,1])$. Then $\mathscr{C}$ is measurable and one has:*

$$\mathrm{Vol}_{\mathrm{Leb}}\,\mathscr{C} = 2\int_0^{\rho_0} X'(\rho)Y(\rho)d\rho = \sup_{\rho \in J \cap \mathbb{R}_+^*} \mathrm{Vol}_{\mathrm{Leb}}\,\mathscr{C}_\rho < +\infty.$$

For $x \in \mathbb{R}^2$, we denote by C_x the square

$$C_x = \left\{ y \in \mathbb{R}^2, |\, ||x-y||_\infty < \tfrac{1}{2} \right\},$$

where $||\cdot||_\infty$ is the Max-norm in $\mathbb{R}^2$. Since the metric $\tilde{g}$ is periodic, all the squares C_x have the same Riemannian volume v_g.

The end of this section is devoted to the proof of the following:

Claim: *For any m_0 in $\mathbb{R}^2$, $\lim_{t\to+\infty} \mathrm{Vol}\, B_{\min}(m_0,T) \underset{T\to\infty}{\simeq} 2v_g \,\mathrm{Vol}_{\mathrm{Leb}}\,\mathscr{C}T^2$.*

Observe that $\mathrm{Vol}\, B_{\min}(m_0,t) = 2\,\mathrm{Vol}\, B^+_{\min}(m_0,t)$. Using the invariance of the geodesics under φ-translations, we can assume that $m_0 = (0, s_0)$.

Set $\widetilde{\mathcal{D}}^+_\infty := (\varpi^*)^{-1}(\mathcal{D}^+_\infty)$. The action-angle diffeomorphism A_+ lifts to a diffeormorphism $A_+ : \widetilde{\mathcal{D}}^+_\infty \to \mathbb{R}^2, (m,p) \mapsto (a^1, a^2, I_1, I_2)$. If $\mathscr{L}_{e,\rho} := (\varpi^*)^{-1}(\mathcal{T}_{e,\rho})$, we denote by $a_{e,\rho}$ the map $\mathbb{R}^2 \to \mathbb{R}^2$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "a_{e,\rho}"] \arrow[d, "\varpi"'] & \mathbb{R}^2 \arrow[d, "\varpi"] \\ \mathbb{T}^2 \arrow[r, "\alpha_{e,\rho}"'] & \mathbb{T}^2. \end{tikzcd} \tag{4.2}$$

There exists a $\mathbb{Z}^2$-periodic map $q_{e,\rho} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $a_{e,\rho} = \mathrm{Id} + q_{e,\rho}$.

In Appendix A we prove that A_+ can be constructed such that:

- $a^1_{e,\rho}(\varphi, s) = \varphi - \varphi(a^2_{e,\rho}(s)\tau_{e,\rho}) - a^2_{e,\rho}(s)\varphi_{e,\rho} = \varphi + q^1_\rho(s)$

- $a^2_{e,\rho}(s) = \dfrac{\displaystyle\int_{s_0}^{s} \frac{2r(t)dt}{\sqrt{2e - \frac{\rho}{x(t)^2}}}}{\displaystyle\int_0^1 \frac{2r(t)dt}{\sqrt{2e - \frac{\rho}{x(t)^2}}}} = s + q^2_\rho(s)$

Remark that $a_{e,\rho}(0,s_0)=(0,0)$. The restriction of the action-angle map $A_+ : \widetilde{\mathcal{D}}^+_\infty \to \mathbb{R}^2\times B$ to $\widetilde{\mathcal{D}}^+_{\frac{1}{2}}$ reads $(m,p^+_m(\rho))\mapsto(a_\rho(m),I(\rho))$ with

$$a_\rho(\varphi,s)=(\varphi+q^1_\rho(s),s+q^2_\rho(s))$$

where the q^i_ρ are 1-periodic functions.

Remark 4.2.12. For any $s\in\mathbb{R}$, $0\le a^2_\rho(s)\le 1$, so $|q^2_\rho(s)|\le 1$.

Let $(\psi_t)_{t\in\mathbb{R}}$ be the Hamiltonian flow on $\mathbb{R}^2\times B$ associated with $H\circ A_+^{-1}$. Its restriction to the Lagrangian graph $A_+(\{(m,p^+_m(\rho)\,|\,m\in\mathbb{R}^2\})$ reads:

$$\psi_t(a_\rho(m),I(\rho))=(a_\rho(m)+t\omega(\rho),I(\rho)).$$

For $\rho\in J(\frac{1}{2})$, we denote by γ_ρ the geodesic $\gamma_\rho : t\mapsto\tilde\pi\circ\tilde\phi_t(m_0,p^+_{m_0}(\rho))$.

Lemma 4.2.7. *For any $\rho\in J(\frac{1}{2})$, there exists $\beta_\rho\in\mathbb{R}_+$ such that, for all $t\in\mathbb{R}$, $d_2(\gamma_\rho(t),m_0+t\omega(\rho))\le\beta_\rho$, where d_2 is the Euclidean distance.*

Proof. Since $A\circ\tilde\phi_t(m_0,p^+_{m_0}(\rho))=A(\gamma_\rho(t),p_{\gamma^+_\rho(t)}(\rho))$, one has $a_\rho(\gamma_\rho(t))=t\omega(\rho)$. That is,

$$\gamma_\rho(t)+q_\rho(\gamma_\rho(t))=t\omega(\rho). \tag{4.3}$$

Therefore $d_2(\gamma_\rho(t),m_0+t\omega(\rho))=||m_0-q_\rho(\gamma_\rho(t)||^2$. We set $\beta_\rho:=\sqrt{\lambda_\rho^2+(s_0+1)^2}$, where $\lambda_\rho:=\max_{s\in\mathbb{R}}q^1_\rho(s)$. ∎

For $\rho\in J(\frac{1}{2})$ and $t\in\mathbb{R}^*_+$, we introduce the following sets:

- $\mathcal{R}_\rho$ is the rectangle centered at $m_0+t\omega(\rho)$ with horizontal sides of length $2\lambda_\rho$ and vertical sides of length $2(s_0+1)$.
- $\mathcal{R}_{\rho,t}$ is the parallelogram with vertices m_0+B_ρ, m_0-M_ρ, $m_0+t\omega(\rho)+M_\rho$ and $m_0+t\omega(\rho)-B_\rho$, where M_ρ is the point of coordinates $(0,\beta_\rho)$.

If moreover $\rho>0$, we set:

- $\mathscr{C}_{\rho,t}:=\Psi([-\rho,\rho]\times[0,t])$
- $B_\rho(m_0,t):=\{\gamma_\rho(t')\,|\,(\rho,t')\in[-\rho,\rho]\times[0,t]\}$.

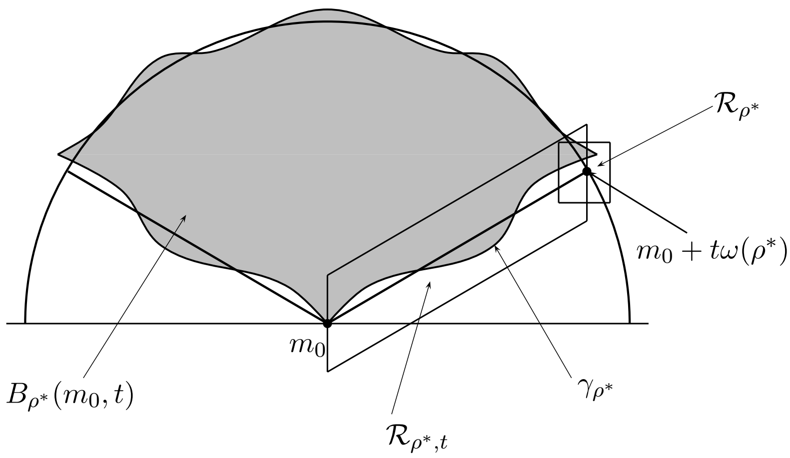


Figure 4.5: The ball $B_{\rho^*}(m,t)$

Lemma 4.2.8. *Fix $\rho^* \in J(\frac{1}{2})$, $\rho^* \geq 0$. There exists a continuous function $\mathbb{R}_+^* \to J : t \mapsto \rho^*(t)$ with $\lim_{t\to\infty} \rho^*(t) = \rho^*$ such that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that if $t \geq t_\varepsilon$:*

$$\mathscr{C}_{\rho^*(t),t(1-\varepsilon)} \setminus (\mathcal{R}_{\rho^*,t} \cup \mathcal{R}_{-\rho^*,t}) \subset B_{\rho_*}(m_0, t),$$
$$B_{\rho_*}(m_0, t) \subset \mathscr{C}_{\rho^*(t),t(1+\varepsilon)} \cup \mathcal{R}_{\rho^*,t} \cup \mathcal{R}_{-\rho^*,t} \cup \mathcal{R}_{\rho^*} \cup \mathcal{R}_{-\rho^*}.$$

Proof. For any (ρ, t) in $[-\rho^*, \rho^*] \times \mathbb{R}_*^+$, there exists a unique pair (ρ_t, t_ρ) in $J(\frac{1}{2}) \times \mathbb{R}^+$ such that $\gamma_\rho(t) = m_0 + t_\rho \omega(\rho_t)$. The map $(\rho, t) \to (t_\rho, \rho_t)$ is continuous. For any $t > 0$, we set $\rho^*(t) := \max_{|\rho| \leq \rho^*} \rho_t$. Now by (4.3), $t_\rho \omega(\rho_t) = t\omega(\rho) - p_\rho(\gamma_\rho(t))$. Then

$$t||\omega(\rho)|| - \lambda_\rho \leq t_\rho ||\omega(\rho_t)|| \leq t||\omega(\rho)|| + \lambda_\rho.$$

Hence,

$$t\frac{||\omega(\rho)||}{||\omega(\rho_t)||} - \frac{\lambda_\rho}{||\omega(\rho_t)||} \leq t_\rho \leq t\frac{||\omega(\rho)||}{||\omega(\rho_t)||} + \frac{\lambda_\rho}{||\omega(\rho_t)||}.$$

So

$$t\frac{||\omega(\rho)||}{||\omega(\rho_t)||} - \frac{\Lambda_{\rho^*}}{||\omega_{\min}||} \leq t_\rho \leq t\frac{||\omega(\rho)||}{||\omega(\rho_t)||} + \frac{\Lambda_{\rho^*}}{||\omega_{\min}||}, \tag{4.4}$$

where $\omega_{\min} = \min_{\bar{J}} ||\omega(\rho)||$. Since $t_\rho \omega(\rho_t) \in t\omega(\rho) + \mathcal{R}_\rho$, one has:

$$|\Theta(\rho) - \Theta(\rho_t)| \leq \arcsin \frac{\sqrt{\lambda_\rho^2 + 1}}{t||\omega(\rho)||} \leq \arcsin \frac{\sqrt{\Lambda_{\rho^*}^2 + 1}}{t||\omega_{\min}||}, \tag{4.5}$$

with Θ defined in remark 4.2.11. Since Θ^{-1} is continuous, for all $\rho \in [-\rho^*, \rho^*]$, $\lim_{t\to\infty} \rho_t = \rho$. Let $\eta > 0$. There exists t_η such that for all $\rho \in [-\rho^*, \rho^*]$ and for all $t \geq t_\eta$, $\rho_t \leq \rho + \eta$. Then for all ρ, $\rho_t \leq \rho^* + \eta$, that is, for all $t \geq t_\eta$, $\rho^*(t) \leq \rho^* + \eta$. Finally, $\limsup_{t\to\infty} \rho^*(t) \leq \rho^* + \eta$. On the other hand, since $\rho_t^* \leq \rho^*(t)$, $\rho^* = \lim_{t\to\infty} \rho_t^* \leq \liminf \rho^*(t)$. Then $\lim_{t\to\infty} \rho^*(t) = \rho^*$.

Let $\varepsilon > 0$. By continuity of $\rho \mapsto ||\omega(\rho)||$, there exists $\eta > 0$ such that

$$|\rho - \rho'| \leq \eta \Rightarrow 1 - \frac{\varepsilon}{2} \leq \frac{||\omega(\rho)||}{||\omega(\rho')||} \leq 1 + \frac{\varepsilon}{2}. \tag{4.6}$$

By continuity of Θ^{-1}, there exists $\alpha > 0$ such that

$$|\Theta(\rho) - \Theta(\rho')| \leq \alpha \Rightarrow |\rho - \rho'| \leq \eta. \tag{4.7}$$

Then by (4.5) there exists T such that for all $t > T$, one has

$$1 - \frac{\varepsilon}{2} \leq \frac{||\omega(\rho)||}{||\omega(\rho_t)||} \leq 1 + \frac{\varepsilon}{2},$$

which yields

$$(1 - \frac{\varepsilon}{2})t - \frac{\Lambda_{\rho^*}}{||\omega_{\max}||} \leq t_\rho \leq (1 + \frac{\varepsilon}{2})t + \frac{\Lambda_{\rho^*}}{||\omega_{\min}||}, \tag{4.8}$$

where $\Lambda_{\rho^*} := \max_{|\rho| \leq \rho^*} \lambda_\rho$. Let T' such that if $t > T'$, $t\frac{\varepsilon}{2} \geq \frac{\Lambda_{\rho^*}}{\omega_{\min}}$. We set $t_\varepsilon = \max(T, T')$.

Finally, notice that $B_{\rho^*}(m_0, t)$ is bounded below by $\gamma_{\rho^*}([0, t])$ and $\gamma_{-\rho^*}([0, t])$. Now, by construction of $\mathcal{R}_{\rho^*,t}$, $\gamma_{\rho^*}([0, t]) \subset \mathcal{R}_{\rho^*,t} \cup \mathcal{R}_{\rho^*}$. Our assertion is then an immediate consequence of (4.8). $\square$

Proof of the claim. **First step**: $\lim_{t\to\infty} \frac{1}{t^2} B_{\rho^*}(m_0, t) = \upsilon \operatorname{Vol} \mathscr{C}_{\rho^*}$, with $\upsilon > 0$.
Recall that for $m \in \mathbb{R}^2$, C_m is the square centered at m with side of length 1. Fix $\rho^* \in J$. For $t, T > 0$, we set:

- $Z_{t,T} := \mathbb{Z}^2 \cap \mathscr{C}_{\rho^*(t),T}$, $\quad U_{t,T} := \bigcup_{k\in Z_{t,T}} C_k$, $\quad n_{t,T} := \operatorname{Card} Z_{t,T}$.
- $R^+_{t,T} := \bigcup_{\lambda\in[0,T]} C_{\lambda\omega(\rho^*(t))}$, $\quad R^-_{t,T} := \bigcup_{\lambda\in[0,T]} C_{\lambda\omega(-\rho^*(t))}$,
- $R_{t,T} = R^+_{t,T} \cup R^-_{t,T}$.

Let $\varepsilon > 0$. One has the following inclusions

$$U_{t,t(1-\varepsilon)-\frac{\sqrt{2}}{\omega_{\min}}} \setminus R_{t,t(1-\varepsilon)} \subset \mathscr{C}_{\rho^*(t),t(1-\varepsilon)}$$

and

$$\mathscr{C}_{\rho^*(t),t(1+\varepsilon)} \subset \left(U_{t,t(1+\varepsilon)+\frac{\sqrt{2}}{\omega_{\min}}} \cup R_{t,t(1+\varepsilon)}\right).$$

Combining with the previous lemma, one gets, for $t > t_\varepsilon$,

$$U_{t,t(1-\varepsilon)-\frac{\sqrt{2}}{\omega_{\min}}} \setminus \left(R_{t,t(1-\varepsilon)} \cup \mathcal{R}_{\rho^*,t} \cup \mathcal{R}_{-\rho^*,t}\right) \subset B_{\rho^*}(m,t)$$

and

$$B_{\rho^*}(m_0,t) \subset U_{t,t(1+\varepsilon)+\frac{\sqrt{2}}{\omega_{\min}}} \cup \left(R_{t,t(1+\varepsilon)} \cup \mathcal{R}_{\rho^*,t} \cup \mathcal{R}_{-\rho^*,t} \cup \mathcal{R}_{\rho^*} \cup \mathcal{R}_{-\rho^*}\right).$$

Since the metric is periodic, all the squares C_m have the same Riemannian volume υ_g, so for $T > 0$, $\operatorname{Vol} R_{t,T} = 2 \operatorname{Vol} R^+_{t,T} \le 2T\upsilon$ and $\operatorname{Vol} U_{t,T} = \upsilon n_{t,T}$. Applying corollary 4.2.5, for t large enough,

$$\left(t(1-\varepsilon) - \frac{\sqrt{2}}{\omega_{\min}}\right)^2 (1-\varepsilon)^2 \operatorname{Vol}_{\mathrm{Leb}} \mathscr{C}_{\rho^*(t)} \le n_{t,t(1-\varepsilon)-\frac{\sqrt{2}}{\omega_{\min}}} \tag{4.9}$$

and

$$n_{t,t(1+\varepsilon)+\frac{\sqrt{2}}{\omega_{\min}}} \le \left(t(1+\varepsilon) + \frac{\sqrt{2}}{\omega_{\min}}\right)^2 (1+\varepsilon)^2 \operatorname{Vol}_{\mathrm{Leb}} \mathscr{C}_{\rho^*(t)} \tag{4.10}$$

Finally, let $b > 0$ such that $\operatorname{Vol} \mathcal{R}_{\rho^*,t} \le b \operatorname{Vol}_{\mathrm{Leb}} \mathcal{R}_{\rho^*,t} = bt\lambda_{\rho^*}||\omega(\rho^*)||$ and let $v := \operatorname{Vol} \mathcal{R}_{\rho^*}$. One gets

$$\begin{aligned} \upsilon_g \left((1-\varepsilon)^4 - \frac{\beta}{t^2}\right) \operatorname{Vol}_{\mathrm{Leb}} \mathscr{C}_{\rho^*(t)} - \frac{\gamma}{t} &\le \frac{1}{t^2} \operatorname{Vol} B_{\rho^*}(m,t) \\ &\le \upsilon_g \left((1+\varepsilon)^4 + \frac{\beta'}{t^2}\right) \operatorname{Vol}_{\mathrm{Leb}} \mathscr{C}_{\rho^*(t)} + \frac{\gamma'}{t} + \frac{v}{t^2}, \end{aligned} \tag{4.11}$$

where

- $\beta = \frac{\sqrt{2}}{\omega_{\min}}(1-\varepsilon)^2$, $\quad \beta' = \frac{\sqrt{2}}{\omega_{\min}}(1+\varepsilon)^2$,
- $\gamma = 2(1-\varepsilon)\upsilon_g + 2b\lambda_{\rho^*}||\omega(\rho^*)||$, and $\gamma' = 2(1+\varepsilon)\upsilon_g + 2b\lambda_{\rho^*}||\omega(\rho^*)||$.

Now

$$\lim_{t\to\infty} \operatorname{Vol}_{\text{Leb}} \mathscr{C}_{\rho^*(t)} = 2\int_0^{\rho^*} Y(\rho)X'(\rho)d\rho = \operatorname{Vol}_{\text{Leb}} \mathscr{C}_{\rho^*},$$

so

$$\upsilon_g(1-\varepsilon)^2 \operatorname{Vol}_{\text{Leb}} \mathscr{C}_{\rho^*} \leq \liminf_{t\to\infty} \frac{1}{t^2} \operatorname{Vol} B_{\rho^*}(m_0,t)$$

and

$$\limsup_{t\to\infty} \operatorname{Vol} B_{\rho^*}(m_0,t) \leq (1+\varepsilon)^2 \operatorname{Vol}_{\text{Leb}} \mathscr{C}_{\rho^*}.$$

Since these inequalities hold for any $\varepsilon > 0$, the first step is proved.

Second step: $\liminf_{t\to\infty} \frac{1}{t^2} \operatorname{Vol} B_{\min}(m_0,t) \geq \upsilon_g \operatorname{Vol}_{\text{Leb}} \mathscr{C}$.
Since for any $\rho \in J$ and any $t > 0$, $B_\rho(m_0,t) \subset B_{\min}(m_0,t)$,

$$\frac{1}{t^2} \operatorname{Vol} B_{\min}(m_0,t) \geq \frac{1}{t^2} \operatorname{Vol} B_\rho(m_0,t).$$

Therefore, for all $\rho \in J$:

$$\liminf_{t\to\infty} \frac{1}{t^2} \operatorname{Vol} B_{\min}(m_0,t) \geq \upsilon \operatorname{Vol}_{\text{Leb}} \mathscr{C}_\rho.$$

To conclude, one just has to remark that $\operatorname{Vol}_{\text{Leb}} \mathscr{C} = \sup_{\rho\in J} \operatorname{Vol}_{\text{Leb}} \mathscr{C}_\rho$.

Last step: $\limsup_{t\to\infty} \frac{1}{t^2} \operatorname{Vol} B_{\min}(m_0,t) \leq \upsilon_g \operatorname{Vol}_{\text{Leb}} \mathscr{C}$.
For $\theta_0 \in [0,\frac{\pi}{2}]$ and $t_0 \in \mathbb{R}^+$, we set $Z(\theta_0,t_0)$ the angular zone defined as

$$Z(\theta_0,t_0) := \{te^{i\theta} \mid (\theta,t) \in [0,\theta_0] \times [0,t_0]\}.$$

If $d_{\tilde{g}}$ and d_2 are respectively the distance in $\mathbb{R}^2$ associated to $\tilde{g}$ and the Euclidean distance, there exists $\beta > 0$ such that $d_2 \leq \beta d_{\tilde{g}}$. Then for all $t > 0$, $B_{\min}(m_0,t) \subset B_{\text{Euc}}(m_0,\beta t)$.
So for any $\rho \in J$ and any $t \in R^+$, $B_{\min}(m,t) \setminus B_\rho(m,t) \subset Z(\Theta(\rho),\beta t)$, that is

$$\frac{1}{t^2} \operatorname{Vol} B_{\min}(m,t) \leq \frac{1}{t^2} \operatorname{Vol} B_\rho(m,t) + \frac{1}{t^2} \operatorname{Vol} Z(\Theta(\rho),\beta t).$$

Now, using again corollary 4.2.5, for $\varepsilon > 0$ and for t large enough:

$$\frac{1}{t^2} \operatorname{Vol} Z(\Theta(\rho),\beta t) \leq \upsilon \frac{1}{t^2}(\beta t + \sqrt{2})^2(1+\varepsilon)^2 \operatorname{Vol}_{\text{Leb}} Z(\Theta(\rho),1) \leq \beta^2(1+\varepsilon)^2\Theta(\rho).$$

To conclude, one just has to notice that $\Theta(\rho)$ can be arbitrarily small. $\square$

Chapter 5

Flat metrics are strict minimizers for $\mathrm{h_{pol}}$

As we have seen in the previous chapter, the geodesic flows associated with the flat metrics on $\mathbb{T}^2$ do minimize $\mathrm{h_{pol}}$. In this chapter, we show that, among the geodesic flows that are dynamically coherent, the geodesic flows associated with flat metrics are local *strict* minima for the strong polynomial entropy. This is a consequence of Theorem A. Our result is the following. We denote by $\mathscr{DC}$ the set of metrics on $\mathbb{T}^2$ with dynamically coherent geodesic flows.

Theorem C. *Let g_0 be a flat metric on $\mathbb{T}^2$. There exists a neighborhood $\mathscr{U}$ of g_0 in the set of C^5 metrics such that, for any $g \in \mathscr{U} \cap \mathscr{DC}$:*
- *either g is flat,*
- *or g possesses a hyperbolic orbit.*

Therefore, due to Theorem A, if $g \in \mathscr{U} \cap \mathscr{DC}$ is not flat, then $\mathrm{h_{pol}}(\phi_g) = 2 > \mathrm{h_{pol}}(\phi_{g_0})$.

The proof of Theorem C is based on the Hopf theorem (Theorem 4) and on a particular property of pertubations of action-angle Hamiltonian systems with two degrees of freedom defined by a quadratic form. A simple but remarkable consequence of the Hopf theorem is that a Riemanniann metric is flat if and only if the unit cotangent bundle is completely foliated by tori that are invariant under the geodesic flow and that are graphs over $\mathbb{T}^2$ (see section 5.2). Now, in section 5.1, we see that if H_ε is a small pertubation of a Hamiltonian system $H : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$ of the form $H(\theta, r) = h(r)$ where h is a positive definite quadratic form, the constant energy levels of H_ε contain tori that are invariant under the flow and that are graphs over $\mathbb{T}^2$. This is the well-known KAM theorem. We prove moreover that if the pertubation is small enough, any torus that is invariant under the flow and that is homotopic to the zero section $\mathbb{T}^2 \times \{0\}$ is a graph over $\mathbb{T}^2$. The proof of theorem C consists in showing that if $g \in \mathscr{DC}$ is close enough to g_0, either the unit tangent bundle (with respect to the metric g) is completely foliated by invariant tori that are graphs over $\mathbb{T}^2$, or the foliation induced by the Bott integral contains a ∞-level.

5.1 A graph property for invariant tori in near-integrable systems.

Consider a positive definite quadratic form $h : \mathbb{R}^2 \to \mathbb{R}, (r_1, r_2) \mapsto ar_1^2 + br_2^2 + cr_1r_2$. Let H be the Hamiltonian function on $\mathbb{T}^2 \times \mathbb{R}^2$ defined by $H(\theta, r) = h(r)$. It is a system in action-angle form. For $t \in \mathbb{R}$,

$$\phi_H^t(\theta, r) = (\theta + t\omega(r)\,[\mathbb{Z}^2], r)$$

where $\omega : \mathbb{R}^2 \to \mathbb{R}^2$, $r \mapsto \nabla h(r)$. Therefore, the whole phase space is completely foliated by the invariant tori $\mathbb{T}^2 \times \{r\}$.

Lemma 5.1.1. *Let $f : \mathbb{T}^2 \times \mathbb{R}^2$ be a C^5 function with $||f||_{C^5} = 1$. For $\varepsilon > 0$, we set $H_\varepsilon : H + \varepsilon f$, and we denote by ϕ_ε the Hamiltonian flow associated to H_ε. There exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$:*

1. *there exist ϕ_ε-invariant tori in $H_\varepsilon^{-1}(\{1\})$ that are graphs of C^1 functions $\mathbb{T}^2 \to \mathbb{R}^2$,*
2. *if $\mathcal{T} \subset H_\varepsilon^{-1}(\{1\})$ is a ϕ_ε-invariant tori that is homotopic to $\mathbb{T}^2 \times \{0\}$, then $\mathcal{T}$ is the graph of a continuous function $\mathbb{T}^2 \to \mathbb{R}^2$,*
3. *there does not exist any ϕ_ε-invariant Klein bottle in $H_\varepsilon^{-1}(\{1\})$.*

The proof of this lemma is based on two results of the theory of dynamical systems. The first one is the KAM theorem, which concerns the behaviour of small perturbation of Hamiltonian systems in action-angle form. The second one is the Birkhoff theorem which concerns particular dynamical systems on the cylinder $\mathbb{T} \times I$ (where I is an interval of $\mathbb{R}$), namely the twist maps. Actually, 1 is exactly the result of the KAM theorem, and 3 is an easy consequence of the particular property for KAM tori to "block" the dynamics in 3-dimensional energy levels and of the form of the Hamiltonian H. The main interest of lemma 5.1.1 is concentrated in 2.

In section 5.1.1 and 5.1.2 we briefly recall the two results mentionned above. The proof of lemma 5.1.1 is given in section 5.1.3.

5.1.1 Basic KAM Theory

In this section, B is a bounded domain of $\mathbb{R}^n$ and $h : B \to \mathbb{R}$ is a smooth function. We consider the Hamiltonian function $H : \mathbb{T}^n \times B \to \mathbb{R}$, $(\theta, r) \mapsto h(r)$ and we denote by ϕ_H the Hamiltonian system associated with H. It is a system in action-angle form.

As before, the whole phase space is completely foliated by the invariant tori $\mathbb{T}^n \times \{r\}$. On each of these tori $\mathbb{T}^n \times \{r\}$ the Hamiltonian system ϕ_H induces a quasi-periodic motion with frequency map $\omega : r \mapsto \omega(r) = \nabla h(r)$.

The dynamics of a quasi-periodic motion $(\phi_H, \mathbb{T}^n \times \{r\})$ is well understood and related to the arithmetic properties of the frequency vector $\omega(r)$. Let us discuss the following example.

Example 5.1.1. Quasi-periodic motion in dimension 2. Consider the Kronecker flow on $\mathbb{T}^2$ defined by

$$\phi_t(\theta_1, \theta_2) = (\theta_1 + tr_1\,[\mathbb{Z}], \theta_2 + tr_2\,[\mathbb{Z}]),$$

where $r_1, r_2 \in \mathbb{R}^* \times \mathbb{R}^*$. One easily sees that only two cases can appear:

- $\frac{r_1}{r_2} = \frac{p}{q} \in \mathbb{Q}$, then *all* the orbits are periodic with the same period,
- $\frac{r_1}{r_2} \notin \mathbb{Q}$, then *all* the orbits are dense.

To generalize this observation in higher dimension, we associate with any $\omega \in \mathbb{R}^n \setminus \{0\}$ the following submodule of $\mathbb{Z}^n$:

$$\mathscr{M}(\omega) := \{k \in \mathbb{Z}^n \mid \langle k, \omega \rangle = 0\} = (\mathbb{R}\omega)^{\perp} \cap \mathbb{Z}^n,$$

where $\langle \, , \rangle$ is the canonical scalar product on $\mathbb{R}^n$. A vector $\omega \in \mathbb{R}^n$ is said to be resonant if $\mathscr{M}(\omega) \neq \{0\}$ and nonresonant otherwise. If ω is nonresonant, all the orbits of the Kronecker flow with frequency ω are dense. One says that the dynamics is *minimal*. Conversely if $\mathscr{M}(\omega)$ has maximal rank $n-1$, all the orbits are periodic, with the same period. Finally, if $\operatorname{rank} \mathscr{M}(\omega) = m \in \{1, \ldots, n-2\}$, the torus $\mathbb{T}^n$ is foliated by a continuous family of invariant tori $\mathbb{T}^{n-m}$ on which the dynamics is minimal.

In an informal way, the KAM theory, named after its founders Kolmogorov, Arnol'd and Moser, is the study of *persistence* of certain of these tori under small perturbations of the system, that is, for systems of the form $H + \varepsilon f$, where f is a "sufficiently regular" bounded function. As already known by Poincaré, the tori with periodic orbits break up under such pertubations. The work of Kolomogorov, Arnol'd and Moser shows that tori corresponding to "strongly nonresonant" frequency vectors persist if the pertubation is small enough and if H satisfies some nondegeneracy conditions. The two following short paragraphs will make this statement more precise. There are a lot of articles, manuals, surveys on the fundamentals of KAM theory. We refer, for example, to [Arn99] and [AKN06].

Isoenergetic nondegeneracy. Recall that we say that a hypersurface of $\mathcal{S} \subset \mathbb{R}^n$ satisfies the transversality property (T) if for every point $u \in \mathcal{S}$, $\mathbb{R}u$ is transverse to $\mathcal{S}$ (definition 4.2.6). For a value e of h, we set $\Omega_e := \omega(H^{-1}(\{e\}))$.

Definition 5.1.1. The Hamiltonian system associated with H is said to be *isoenergetically nondegenerate* in the neighborhood of $H^{-1}(\{e\})$ if there exists a neighborhood V of e in $h(B)$, such that ω never vanishes in $H^{-1}(V)$ and that for any $e \in V$, Ω_e satisfies (T).

In particular, Ω_e is $(n-1)$-dimensional. One easily checks that if h is strictly convex, H is isoenergetically nondegenerate in the neighborhood of its regular energy levels.

This definition can be reformulated in several ways. First it is obviously equivalent to say that for all $r \in B$:

$$\omega(r) \neq 0 \quad \text{and} \quad \lambda d\omega(r)(v) + \ell\omega(r) \neq 0, \quad \forall v \in T_{\omega(r)}\Omega_{h(r)}, \quad \forall \lambda, \ell \in \mathbb{R}_+^*. \tag{5.1}$$

This can be summarized in the following way: for all $r \in B$ and all $\lambda \in \mathbb{R}_+^*$

$$\Delta_{r,\lambda} := \det \begin{pmatrix} \lambda d\omega(r) & \omega(r) \\ {}^t\omega(r) & 0 \end{pmatrix} \neq 0. \tag{5.2}$$

Indeed assume that $\mathbb{R}^n$ is endowed with the canonical scalar product $\langle \, , \rangle$. Then (5.1) is equivalent to

$$\forall r \in B, \quad \omega(r) \neq 0 \quad \text{and} \quad \lambda d\omega(r)(v) + \ell\omega(r) \neq 0, \quad \forall v \in (\mathbb{R}\Omega_{h(r)})^{\perp}, \quad \forall \lambda, \ell \in \mathbb{R}_+^*.$$

That is, for all $(r, \lambda) \in B \times \mathbb{R}_+^*$, the map

$$\begin{array}{rccc} \Psi_{r,\lambda} : & \mathbb{R}^n \times \mathbb{R} & \to & \mathbb{R}^n \times \mathbb{R} \\ & (v, \ell) & \mapsto & (\lambda d\omega(r)(v) + \ell\omega(r), \langle \omega(r), v \rangle) \end{array}$$

is invertible. Finally another way to reformulate definition 5.1.1 is the following: the map

$$\begin{array}{cccc} \Psi : & B \times \mathbb{R}_+^* & \longrightarrow & \mathbb{R}^n \times \mathbb{R} \\ & (r, \lambda) & \mapsto & (\lambda \omega(r), h(r)) \end{array} \tag{5.3}$$

is a local diffeomorphism. Indeed, for $(r, \lambda) \in \mathbb{R}^n \times \mathbb{R}_+^*$

$$D\Psi(r, \lambda) : (v, \ell) \mapsto (\lambda d\omega(r)(v) + \ell\omega(r), \langle \omega(r), v \rangle) = \Psi_{r,\lambda}(v, \ell).$$

KAM Theorem. We start with the following definition which explains what it means for a frequency vector to be“strongly nonresonant”.

Definition 5.1.2. Fix two positive numbers τ, γ. We say that $\omega \in \mathbb{R}^n$ belongs to $\mathscr{D}(\tau, \gamma)$ if

$$|\langle \omega, k \rangle| \geq \frac{\gamma}{||k||^\tau}, \quad \forall\, k \in \mathbb{Z}^n \setminus \{0\}.$$

The set $\mathscr{D}(\tau, \gamma)$ is the set of *Diophantine vectors of type* (τ, γ). The union

$$\mathscr{D}(\tau) := \bigcup_{\gamma > 0} \mathscr{D}(\tau, \gamma)$$

is the set of *Diophantine vectors of type* τ.

It is well known that $\mathscr{D}(\tau)$ has full measure when $\tau > n - 1$.

Theorem 18. ***The KAM Theorem.*** *Let $k > 2n$ and $f : \mathbb{T}^n \times \mathbb{R}^n$ be a C^k function with $||f||_{C^k} = 1$. Fix $\tau \in]n - 1, \frac{1}{2}k - 1[$ and $\gamma > 0$. Finally fix $e \in H(B)$. There exists $\varepsilon_0 > 0$, such that, for all $0 < \varepsilon < \varepsilon_0$ and for all $\omega \in \mathscr{D}(\tau, \gamma) \cap \Omega_e$ such that $\omega(r) \in \mathscr{D}(\tau, \gamma) \cap \Omega_e$, there exists a ϕ_ε-invariant torus $\mathcal{T}_r$ that satisfies:*

- *$\mathcal{T}_r$ is homotopic to $\mathbb{T}^n \times \{r\}$,*
- *there exists $\delta > 0$ independent of ε, such that $\mathcal{T}_r \subset \mathbb{T}^n \times [r - \delta\sqrt{\varepsilon}, r + \delta\sqrt{\varepsilon}]$,*
- *$H_\varepsilon(\mathcal{T}_r) = e$.*

If moreover H is convex, $\mathcal{T}_r$ is the graph of a C^1 function on $\mathbb{T}^2$, with $||f - r||_{C_1} \leq c\sqrt{\varepsilon}$, for a positive number c, independent of ε

5.1.2 Twist maps

In this short section, we consider a particular class of maps on the cylinder $C := \mathbb{T} \times I$ where I is an interval of $\mathbb{R}$. The universal covering of C is the strip $\mathbb{R} \times I$. We denote by π the canonical projection $\pi : \mathbb{R} \times I \to C$. A lift of a map $f : C \to C$ is a map $F : \mathbb{R} \times I \to \mathbb{R} \times I$ such that $\pi \circ F = f \circ \pi$. Therefore, if $F : (x, r) \mapsto (F_1(x, r), F_2(x, r))$, then

- there exists $m \in \mathbb{N}$, such that for all (x, r), $F_1(x + 1, r) = F_1(x, r) + m$
- $F_2(x + 1, r) = F_2(x, r)$, for all (x, r).

One easily checks that the integer m in the first property is independent of the lift.

Now, we assume that I is an interval of the form $[a, b[$ where $a \in \mathbb{R}$ and $b \in]a, +\infty]$. We endow $\mathbb{T} \times I$ with the canonical symplectic form $d\theta \wedge dr$. We could also consider cylinders of the form $C := \{(\theta, r(\theta)) \,|\, a \leq r(\theta) \leq g(\theta)\}$, where $g : \mathbb{T} \to]a, +\infty[$ is a continuous function.

Definition 5.1.3. An *area-preserving twist map* on C is a diffeomorphism $f : C \to C$ such that

1. f preserves the symplectic form,
2. f preserves the boundary $\mathbb{T} \times \{a\}$ in the sense that there exists $\varepsilon > 0$ such that there exists $c \in]a, b[$ such that if $(\theta, r) \in \mathbb{T} \times [0, \varepsilon[$, then $f(\theta, r) \in \mathbb{T} \times [a, c[$,
3. **Torsion condition**: if F is any lift of f to $\mathbb{R} \times [0, 1[$, then, $\frac{\partial F}{\partial r}(x, r) > 0$.

Remark 5.1.1. Let $f_\varepsilon : C \to C$ be such that $f_\varepsilon^* \Omega = \Omega$ and that $||f - f_\varepsilon||_{C^1} \leq \varepsilon$, with $\varepsilon > 0$. Then, for ε small enough, f_ε is a twist map.

Recall that if $f : X \to X$ is a continuous map of a metric space X, a point $x \in X$ is said to be *nonwandering for* f if for any neighborhood U of x, there exists an integer $n \in \mathbb{N}^*$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points is denoted by $NW(f)$. A domain $\mathcal{D} \subset X$ is nonwandering if $\mathcal{D} \subset NW(f)$. If X is compact, $NW(f) \neq \emptyset$.

The following result due to Birkhoff shows that invariant circles for an area-preserving twist map $f : C \to C$ are graphs over $\mathbb{T}$. In particular, they divide C into two domains. We refer to [KH95] for a proof.

Theorem 19. ***Birkhoff's Theorem.*** *Let $f : C \to C$ be an area-preserving twist map. Let $\mathscr{D}$ be an f-invariant relatively compact open domain containing $\mathbb{T} \times \{a\}$ and with connected boundary $\partial\mathcal{D}$. Assume that $\mathcal{D}$ is nonwandering. Then $\partial\mathcal{D}$ is the graph of a Lipschitz function from $\mathbb{T}$ to I.*

5.1.3 Proof of lemma 5.1.1.

Proof. One juste has to prove 2 and 3. We denote respectively by X and X_ε the vector fields associated with the Hamiltonian functions H and H_ε, and by ϕ and ϕ_ε their respective flows.

We start by studying the geometry of $H_\varepsilon^{-1}(\{1\})$. Set $\mathcal{H} := \{(r_1, r_2) \in \mathbb{R}^2 \,|\, h(r_1, r_2) = 1\}$. Then $H^{-1}(\{1\}) := \mathbb{T}^2 \times \mathcal{H}$. The vector field X reads:

$$\dot{\theta}_1 = 2ar_1 + cr_2, \qquad \dot{r_1} = 0$$
$$\dot{\theta}_2 = cr_1 + 2br_2, \qquad \dot{r_2} = 0.$$

We denote by D_1 and D_2 the lines with respective equations $ar_1 + cr_2 = 0$, $br_2 + cr_1 = 0$ in $\mathbb{R}^2$. Then, for $i = 1, 2$, D_i intersects $\mathcal{H}$ at two points A_i, B_i. The connected components of $\mathcal{H} \setminus \{A_1, B_1, A_2, B_2\}$ are

$$\mathscr{D}^{++} := \{r \in \mathcal{H} \,|\, ar_1 + cr_2 > 0,\, cr_1 + br_2 > 0\},$$
$$\mathscr{D}^{+-} := \{r \in \mathcal{H} \,|\, ar_1 + cr_2 > 0,\, cr_1 + br_2 < 0\},$$
$$\mathscr{D}^{--} := \{r \in \mathcal{H} \,|\, ar_1 + cr_2 < 0,\, cr_1 + br_2 < 0\},$$
$$\mathscr{D}^{-+} := \{r \in \mathcal{H} \,|\, ar_1 + cr_2 < 0,\, cr_1 + br_2 > 0\}.$$

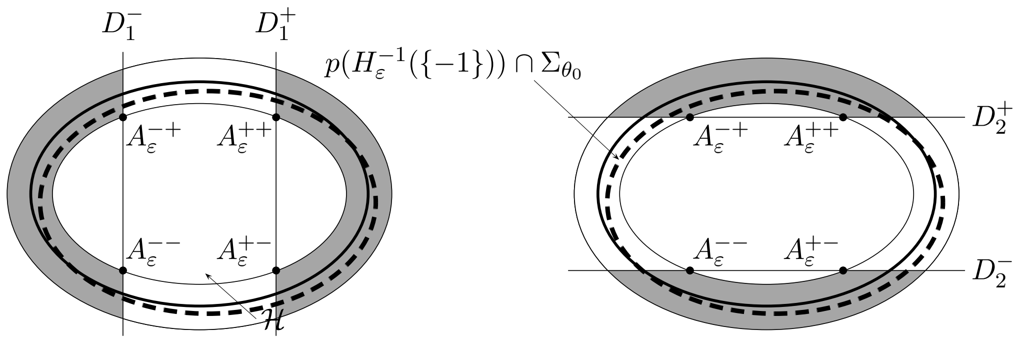
In each of the domains $\mathscr{D}^{**}$, we fix a point A_ε^{**}, where $*$ stand for $+$ or $-$, such that $A_\varepsilon^{**} \in h^{-1}(1 - \varepsilon)$.

Now fix $\theta^0 \in \mathbb{T}^2$. Consider the surface Σ_{θ^0} defined by $\theta = \theta^0$. Since Σ_{θ^0} is transverse to $H^{-1}(\{1\})$, for ε small enough, Σ_{θ^0} is transverse to $H_\varepsilon^{-1}(\{1\})$ and their intersection is a compact submanifold of dimension 1, that is, a circle. Moreover, the projection

$p(H_\varepsilon^{-1}(\{1\}))$ on $\mathbb{R}^2$ of this circle is contained in the annulus delimited by the ellipses with equations $h(r) = 1 + \varepsilon$ and $h(r) = 1 - \varepsilon$.

Consider the four lines $D_1^+ := (A_\varepsilon^{++}, A_\varepsilon^{+-})$, $D_1^- := (A_\varepsilon^{-+}, A_\varepsilon^{--})$, $D_2^+ := (A_\varepsilon^{-+}, A_\varepsilon^{++})$ and $D_2^- = (A_\varepsilon^{+-}, A_\varepsilon^{--})$. We denote by $\mathscr{D}_1^+$ the domain bounded by D_1^+ and the ellipses $h(r) = 1 + \varepsilon$ and $h(r) = 1 - \varepsilon$, which is contained in the set $\{ar_1 + cr_2 \geq 0\}$. We define in the same way the domains $\mathscr{D}_1^-$, $\mathscr{D}_2^+$ and $\mathscr{D}_2^-$ (see the simplified drawing in Figure 1). There exists $\alpha > 0$ such that

- $\mathscr{D}_1^+ \subset \{2ar_1 + cr_2 > \alpha\}$ and $\mathscr{D}_1^- \subset \{2ar_1 + cr_2 < -\alpha\}$
- $\mathscr{D}_2^+ \subset \{cr_1 + 2br_2 > \alpha\}$ and $\mathscr{D}_1^- \subset \{cr_1 + 2br_2 < -\alpha\}$.



The domains $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$ The domains $\mathscr{D}_2^+$ and $\mathscr{D}_2^-$

Figure 5.1: The section Σ_{θ^0}

The four domains $\mathbb{T}^2 \times \mathscr{D}_1^\pm$ and $\mathbb{T}^2 \times \mathscr{D}_2^\pm$ form a covering of $H_\varepsilon^{-1}(\{1\})$. Moreover, since for all $(\theta, r) \in \mathbb{T}^2 \times \mathscr{D}_1^+$, $\frac{\partial H_\varepsilon}{\partial r_1}(\theta, r) \neq 0$, by the implicit function theorem there exist an interval I_2^+ and a function $R_1^+ : \mathbb{T}^2 \times I_2^+ \to \mathbb{R}$ such that:

$$H_\varepsilon^{-1}(\{1\}) \cap \left(\mathbb{T}^2 \times \mathscr{D}_1^+\right) = \{(\theta, R_1^+(\theta, r_2), r_2) \,|\, (\theta, r_2) \in \mathbb{T}^2 \times I_2^+\}. \tag{5.4}$$

In the same way, there exist intervals I_2^-, I_1^+ and I_1^- and functions R_1^-, R_2^+ and R_2^- such that

$$\begin{aligned} H_\varepsilon^{-1}(\{1\}) \cap \left(\mathbb{T}^2 \times \mathscr{D}_1^-\right) &= \{(\theta, R_1^-(\theta, r_2), r_2) \,|\, (\theta, r_2) \in \mathbb{T}^2 \times I_2^-\} \\ H_\varepsilon^{-1}(\{1\}) \cap \left(\mathbb{T}^2 \times \mathscr{D}_2^+\right) &= \{(\theta, r_1, R_2^+(\theta, r_1) \,|\, (\theta, r_1) \in \mathbb{T}^2 \times I_1^+\} \\ H_\varepsilon^{-1}(\{1\}) \cap \left(\mathbb{T}^2 \times \mathscr{D}_2^-\right) &= \{(\theta, r_1, R_2^-(\theta, r_1) \,|\, (\theta, r_2) \in \mathbb{T}^2 \times I_1^-\}. \end{aligned}$$

Since the set of Diophantine numbers $\mathscr{D}(2)$ is dense in $\mathbb{R}^2$ and since $\omega : r \mapsto \omega(r)$ is a diffeomorphism, for any $\delta > 0$, there exists $r^{++} \in B(A^{++}, \delta)$ such that $\omega(r) \in \mathscr{D}(2)$. Moreover, since $A^{++} \in \mathscr{D}_1^+ \cap \mathscr{D}_2^+$, we can assume that δ is small enough so that $r^{++} \in \mathscr{D}_1^+ \cap \mathscr{D}_2^+$. Finally, since $\mathscr{D}(2)$ is stable under multiplication by a real number and since ω is linear, we can assume that $r^{++} \in \mathcal{H}$.

Similarly, there exist r^{+-}, r^{-+} and r^{--} in $\mathcal{H} \cap \mathscr{D}_1^+ \cap \mathscr{D}_2^-$, $\mathcal{H} \cap \mathscr{D}_1^- \cap \mathscr{D}_2^+$ and $\mathcal{H} \cap \mathscr{D}_1^- \cap \mathscr{D}_2^-$ whose images by ω are in $\mathscr{D}(2)$. Let $\gamma > 0$ such that $\{\omega(r^{++}), \omega(r^{-+}), \omega(r^{+-}), \omega(r^{--})\} \subset \mathscr{D}(2, \gamma)$.

By the KAM theorem, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there exist tori $\mathcal{T}^{++}, \mathcal{T}^{+-}, \mathcal{T}^{-+}$ and $\mathcal{T}^{--}$ in $H_\varepsilon^{-1}(\{1\})$, invariant under the flow ϕ_ε, that are the graphs of C^1-functions $g^{**} : \mathbb{T}^2 \to \mathbb{R}^2$ with $||g^{**} - r^{**}||_{C^1} \leq c\sqrt{\varepsilon}$.

We set $\mathcal{T}^{**} := \{(\theta, r^{**}(\theta)) \,|\, \theta \in \mathbb{T}^2\}$. We choose $\varepsilon < \varepsilon_0$ small enough so that

$$\mathcal{T}^{++} \subset \mathbb{T}^2 \times \left(\mathscr{D}_1^+ \cap \mathscr{D}_2^+\right), \quad \mathcal{T}^{+-} \in \mathbb{T}^2 \times \left(\mathscr{D}_1^+ \cap \mathscr{D}_2^-\right),$$
$$\mathcal{T}^{-+} \subset \mathbb{T}^2 \times \left(\mathscr{D}_1^- \cap \mathscr{D}_2^+\right), \quad \mathcal{T}^{--} \in \mathbb{T}^2 \times \left(\mathscr{D}_1^- \cap \mathscr{D}_2^-\right).$$

Fix $\theta^0 \in \mathbb{T}^2$. The intersection $H_\varepsilon^{-1}(\{1\}) \cap \Sigma_{\theta^0}$ is the union of the curves:

- $\mathcal{C}_1^+(\theta^0)$ with endpoints $r^{+-}(\theta_0)$ and $r^{++}(\theta^0)$, which is contained in $\mathbb{T}^2 \times \mathscr{D}_1^+$
- $\mathcal{C}_1^-(\theta^0)$ with endpoints $r^{--}(\theta_0)$ and $r^{-+}(\theta^0)$, which is contained in $\mathbb{T}^2 \times \mathscr{D}_1^-$
- $\mathcal{C}_2^+(\theta^0)$ with endpoints $r^{-+}(\theta_0)$ and $r^{++}(\theta^0)$, which is contained in $\mathbb{T}^2 \times \mathscr{D}_2^+$
- $\mathcal{C}_2^-(\theta^0)$ with endpoints $r^{+-}(\theta_0)$ and $r^{--}(\theta^0)$, which is contained in $\mathbb{T}^2 \times \mathscr{D}_2^-$.

The vector field X^ε reads

$$\begin{aligned} \dot{\theta_1} &= 2ar_1 + cr_2 + \varepsilon\frac{\partial f}{\partial r_1}(\theta, r), &\qquad \dot{r_1} &= -\varepsilon\frac{\partial f}{\partial \theta_1}(\theta, r) \\ \dot{\theta_2} &= cr_1 + 2br_2 + \varepsilon\frac{\partial f}{\partial r_2}(\theta, r), &\qquad \dot{r_2} &= -\varepsilon\frac{\partial f}{\partial \theta_2}(\theta, r). \end{aligned}$$

We assume that ε is small enough so that:

$$2ar_1 + cr_2 + \varepsilon\frac{\partial f}{\partial r_1}(\theta, r) > \frac{1}{2}\alpha, \ \forall(\theta, r) \in \mathcal{C}_1^+ \quad 2ar_1 + cr_2 + \varepsilon\frac{\partial f}{\partial r_1}(\theta, r) < -\frac{1}{2}\alpha, \ \forall(\theta, r) \in \mathcal{C}_1^-, \tag{5.5}$$

$$cr_1 + 2br_2 + \varepsilon\frac{\partial f}{\partial r_2}(\theta, r) > \frac{1}{2}\alpha, \ \forall(\theta, r) \in \mathcal{C}_2^+, \quad cr_1 + 2br_2 + \varepsilon\frac{\partial f}{\partial r_2}(\theta, r) < -\frac{1}{2}\alpha, \ \forall(\theta, r) \in \mathcal{C}_2^-. \tag{5.6}$$

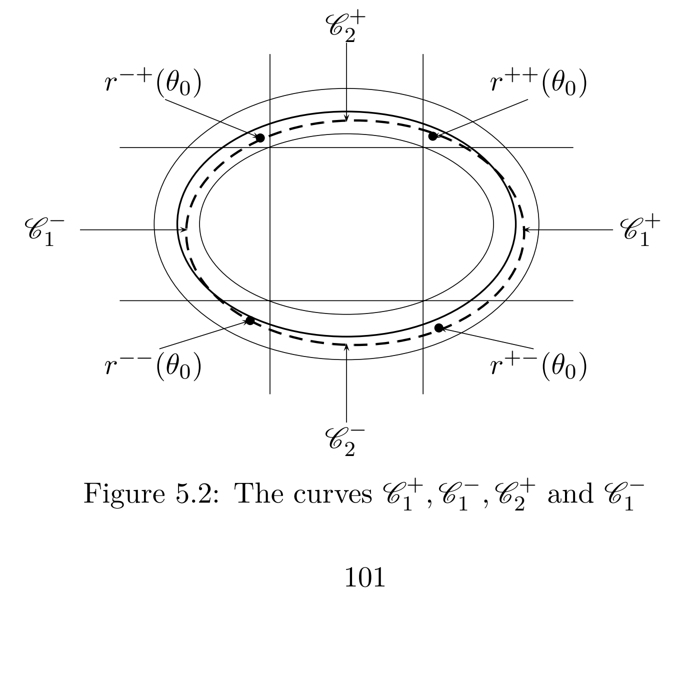


Figure 5.2: The curves $\mathscr{C}_1^+, \mathscr{C}_1^-, \mathscr{C}_2^+$ and $\mathscr{C}_1^-$

We set:

$$\mathcal{H}_{1,\varepsilon}^+ := \bigcup_{\theta\in\mathbb{T}^2} \mathcal{C}_1^+(\theta), \qquad \mathcal{H}_{1,\varepsilon}^- := \bigcup_{\theta\in\mathbb{T}^2} \mathcal{C}_1^-(\theta)$$
$$\mathcal{H}_{2,\varepsilon}^+ := \bigcup_{\theta\in\mathbb{T}^2} \mathcal{C}_2^+(\theta), \qquad \mathcal{H}_{2,\varepsilon}^- := \bigcup_{\theta\in\mathbb{T}^2} \mathcal{C}_2^-(\theta).$$

The four sets above are 3-dimensional manifolds with boundary. They cover $H_\varepsilon^{-1}(\{1\})$. Since the boundary of each of them is the disjoint union of two KAM tori $\mathcal{T}^{**}$, they are invariant under the flow ϕ_ε. Therefore a ϕ_ε-invariant surface contained in $H_\varepsilon^{-1}(\{1\})$ is necessarily contained in one of these submanifolds.

Let $\mathscr{L}$ be a ϕ_ε-invariant surface and let us see that $\mathscr{L}$ must be a torus, which will prove 3. Assume that $\mathscr{L} \subset \mathcal{H}_{1,\varepsilon}^+$. By (5.5), for any $\theta_1^0 \in \mathbb{T}$, the 3-dimensional submanifold $\widehat{S}_{\theta_1^0} := \{\theta_1 = \theta_1^0\}$ is transverse to $\mathcal{H}_{1,\varepsilon}^+$. We denote by $S_{\theta_1^0}^+$ the symplectic surface $S_{\theta_1^0,\varepsilon}^+ := \widehat{S}_{\theta_1^0} \cap \mathcal{H}_{1,\varepsilon}^+$. We can assume without loss of generality that $r_2^{+-} < r_2^{++}$.

Notation 5.1.1. *1) In what follows, we will only work in $\mathcal{H}_{1,\varepsilon}^+$. We will omit the subscript $+$ and will write $\mathcal{H}_{1,\varepsilon}$. In the same way, we set $S_{\theta_1^0,\varepsilon} := S_{\theta_1^0,\varepsilon}^+$ and $r_2^+ := r_2^{++}$, $r_2^- := r_2^{+-}$.*

2) We denote by $\mathcal{H}_1$ the intersection $H^{-1}(\{1\}) \cap (\mathbb{T}^2 \times \{2ar_1 + br_2 > \frac{1}{4}\alpha\})$. There exist an interval I_2 and a function $R_1 : I_2 \to \mathbb{R}$ such that:

$$\mathcal{H}_1 = \{(\theta, R_1(r_2), r_2) \,|\, (\theta, r_2) \in \mathbb{T}^2 \times I_2\}. \tag{5.7}$$

Obviously, $I_2 \supset [r_2^-(\theta), r_2^+(\theta)]$ for all $\theta \in \mathbb{T}^2$.

One has:

$$S_{\theta_1^0,\varepsilon} := \{(\theta_1^0, \theta_2, R_1^+(\theta_1^0, \theta_2, r_2), r_2) \,|\, r_2 \in [r_2^-(\theta_1^0, \theta_2), r_2^+(\theta_1^0, \theta_2)]\},$$

that is, $S_{\theta_1^0,\varepsilon}$ is parametrized by $(\theta_2, r_2) \in \mathbb{T} \times [r_2^-(\theta_1^0, \theta_2), r_2^+(\theta_1^0, \theta_2)]$.

Since for all $\theta_1^0 \in \mathbb{T}$, $\mathscr{L}$ is transverse to $S_{\theta_1^0,\varepsilon}$ in $\mathcal{H}_{1,\varepsilon}$, the intersection $\mathscr{L} \cap S_{\theta_1^0,\varepsilon}$ is a closed 1-dimensional submanifold $\mathscr{C}(\theta_1^0)$ possibly non connected. Assume that $\mathscr{C}(\theta_1^0)$ is a finite union of circles $\Gamma_1, \dots, \Gamma_m$. The Poincaré return map $\wp_\varepsilon : S_{\theta_1^0,\varepsilon} \to S_{\theta_1^0,\varepsilon}$ with respect to the flow ϕ_ε is well defined by (5.5). Necessarily, for any $q \in \{1, \dots, m\}$, there exists $p \neq q$ in $\{1, \dots, m\}$ such that $\wp_\varepsilon(\Gamma_q) \subset \Gamma_p$. Since conversely, $\wp_\varepsilon^{-1}(\Gamma_p) \subset \Gamma_q$, one has $\wp_\varepsilon(\Gamma_p) = \Gamma_q$. We set

$$\wp_\varepsilon(x) = \phi_\varepsilon^{\tau(x)}(x).$$

Observe that the map

$$\begin{array}{lcl} [0,1] \times \Gamma_q & \to & \mathscr{L} \\ (t, z) & \mapsto & \phi_\varepsilon^{t\tau(z)}(z) \end{array}$$

is a homotopy.

Now, for any $z \in \Gamma_1$, $\phi_\varepsilon(m\tau(z), z) \in \Gamma_1$. Hence, the map

$$\begin{array}{lcl} [0,1] \times \Gamma_1 & \to & \mathscr{L} \\ (t, z) & \mapsto & \phi_\varepsilon^{tm\tau(z)}(z) \end{array}$$

is surjective and $\mathscr{L}$ is diffeomorphic to the quotient $[0,1] \times \Gamma_1 / \{(0, \phi_\varepsilon(m\tau(z), z) = (1, z)\}$. Since the diffeomorphism $z \mapsto \phi_\varepsilon(m\tau(z), z)$ is homotopic to the identity, $\mathscr{L}$ is a torus and 3 is proved.

In remains to prove 2. Fix a ϕ_ε-invariant torus $\mathcal{T}$ in $\mathcal{H}_{1,\varepsilon}$. Assume that $\mathcal{T}$ is homotopic to $\mathbb{T}^2 \times \{0\}$. For $\theta_1 \in \mathbb{T}$, we set $\mathscr{C}(\theta_1) = \mathcal{T} \cap S_{\theta_1,\varepsilon}$. We have already seen that $\mathscr{C}(\theta_1)$ is a finite union of circles $\Gamma_1, \ldots, \Gamma_m$.

Observe that for all θ_1^0, θ_1^1 in $\mathbb{T}$, there exists a Poincaré map $P_{\theta_1^0,\theta_1^1}$ between the surfaces $S_{\theta_1^0,\varepsilon}$ and $S_{\theta_1^1,\varepsilon}$. As before, one checks that $P_{\theta_1^0,\theta_1^1}(\mathscr{C}(\theta_0^1)) = \mathscr{C}(\theta_1^1)$ and that $P_{\theta_0^1,\theta_1^1}$ leads to a homotopy between $\mathscr{C}(\theta_0^1)$ and $\mathscr{C}(\theta_1^1)$. Thus, all the submanifolds $\mathscr{C}(\theta_1^0)$ are homotopic (and in particular homologous) in $\mathcal{T}$. Let us denote by $[\Gamma]$ the common homology class (in $\mathcal{T}$) of the circles Γ_k. Set $\mathscr{C} := \mathbb{T} \times \{0\} \subset \mathbb{T}^2$. Clearly, $[\Gamma]$ and $[\mathscr{C}]$ are independent in $H_1(\mathcal{T}, \mathbb{Z})$. Since $\mathcal{T}$ is homotopic to $\mathbb{T}^2 \times \{0\}$, the circles Γ_k must be essential in the cylinder $(\theta_2, r_2) \in \mathbb{T}^2 \times [r_2^-(\theta_1^0, \theta_2), r_2^+(\theta_1^0, \theta_2)]$, otherwise $\mathcal{T}$ would be homotopic to the curve $\{(m\theta_1, 0) \,|\, \theta_1 \in \mathbb{T}\} \times \{0\} \subset \mathbb{T}^2 \times \{0\}$.

For $1 \leq k \leq m$, we denote by $\mathcal{I}_k$ the domain in $S_{\theta_1^0,\varepsilon}$ bounded Γ_k and the lower boundary $\{(\theta_2, r^-(\theta_1^0, \theta_2) \,|\, \theta_2 \in \mathbb{T}\}$. Now $\wp_\varepsilon$ is symplectic and in particular preserves the areas, so all the $\mathcal{I}_k$ have the same area, that is, all the Γ_k coincide and the intersection $\mathscr{C}(\theta_1^0)$ between $\mathcal{T}$ with $S_{\theta_1^0,\varepsilon}$ is a single essential circle.

Let $L_\varepsilon : S_{\theta_1^0,\varepsilon} \to \mathbb{T} \times I : (\theta_2, r_2) \mapsto (\theta_2, r_2 - r_2^-(\theta_1^0, \theta_2) + r_2^-)$ and set

$$\begin{array}{llll} \widehat{\wp}_\varepsilon & : \mathbb{T} \times \mathscr{I} & \longrightarrow & \mathbb{T} \times \mathscr{I} \\ & (\theta_2, r_2) & \longmapsto & L_\varepsilon^{-1} \circ \wp_\varepsilon \circ L_\varepsilon(\theta_2, r_2), \end{array}$$

where $\mathbb{T} \times \mathscr{I}$ is the cylinder contained in $\mathbb{T} \times I$ whose boundaries are $\mathbb{T} \times \{r_2^-\}$ and the graph of the function $\theta_2 \mapsto r_2^+(\theta_1^0, \theta_2) - r_2^-(\theta_1^0, \theta_2) + r_2^-$.

We will apply Birkhoff's Theorem to $\widehat{\wp}_\varepsilon$ to see that $\mathscr{C}(\theta_1^0)$ is the graph of a function $\mathbb{T} \to \mathscr{I}$. Obviously, $\widehat{\wp}_\varepsilon$ preserves the symplectic form $d\theta_2 \wedge dr_2$ and the boundary $\mathbb{T} \times \{r_2^-\}$. One just has to check the torsion condition. To do this, we will see that $\widehat{\wp}_\varepsilon$ is $\sqrt{\varepsilon}$-close in C^1-topology to the twist map defined by the Poincaré return map $\wp$ (with respect to ϕ) associated with the surface $S_{\theta_1^0} = \widehat{S}_{\theta_1^0} \cap \mathcal{H}_1$. We first note that there exists $\delta > 0$, independent of ε, such that $|L_\varepsilon - \mathrm{Id}\,|_{C^1, \mathbb{T}^2 \times \mathscr{I}} \leq \delta\sqrt{\varepsilon}$. One has:

$$S_{\theta_1^0} := \{(\theta_1^0, \theta_2, R_1(r_2), r_2) \,|\, (\theta_2, r_2) \in \mathbb{T} \times I_2\}.$$

The Poincaré map $\wp$ reads:

$$\wp(\theta_2, r_2) = \left(\theta_2 + \frac{cR(r_2) + 2br_2}{2aR(r_2) + cr_2}, r_2\right) = (\wp_1(\theta_2, r_2), \wp_2(\theta_2, r_2)).$$

Hence:

$$\frac{\partial \wp_1}{\partial r_2} = \frac{(4ab - c^2)(R(r_2) - R'(r_2)r_2)}{(2aR(r_2) + cr_2)^2}.$$

Using the fact that $r_2 \mapsto aR(r_2)^2 + br_2^2 + cR(r_2)r_2$ is a constant function, one gets:

$$R'(r_2) = -\frac{2br_2 + cR(r_2)}{2aR(r_2) + cr_2} = -\frac{2br_2 + cr_1}{2ar_1 + cr_2}.$$

Thus:

$$\frac{\partial \wp_1}{\partial r_2} = -\frac{4ab - c^2}{(2aR(r_2) + cr_2)^2} \frac{r_1(2br_2 + cr_1) + r_2(2ar_1 + cr_2)}{2ar_1 + cr_2}.$$

Since $(r_1, r_2) \in \mathscr{D}_1^+$, $2ar_1 + cr_2 > 0$. Now $4ab - c^2 = 4 \det h > 0$. Finally, $r_1(2br_2 + cr_1) + r_2(2ar_1 + cr_2) = \langle r, n(r) \rangle$ where $n(r)$ is the normal vector pointing outwards the ellipse $\mathcal{H}$. Since $\mathcal{H}$ is convex, this scalar product has constant sign. We can assume without loss of generality, that $\langle r, n(r)\rangle > 0$. Thus, $\wp$ is an area-preserving twist map. Its return-time map $\tau : (\theta_2, r_2) \mapsto 2aR_1(r_2) + cr_2$ only depends on r_2.

For $(\theta_2, r_2) \in S_{\theta_1^0,\varepsilon}$, the return-time map $\tau_\varepsilon(\theta_2, r_2)$ is defined by:

$$\int_0^{\tau_\varepsilon(\theta_2,r_2)} \dot{\theta}_1(s)ds = \int_0^{\tau(\theta_2,r_2)} 2ar_1(s) + cr_2(s) + \varepsilon\frac{\partial f}{\partial r_1}(\theta(s), r(s))ds = 1.$$

One easily checks that $||\tau - \tau_\varepsilon||_{C^1(\mathbb{T}\times\mathscr{I})} \leq c\varepsilon$ for a suitable constant $c > 0$, independent of ε.

Set $J := [0, \max_{\mathbb{T}^2\times\mathscr{I}}(\tau, \tau_\varepsilon)]$ and $K := J \times (\mathbb{T} \times \mathscr{I})$. We denote by Φ and Φ_ε the maps defined on K by $\Phi(t, (\theta, r)) = \phi^t(\theta, r)$ and $\Phi_\varepsilon(t, (\theta, r)) = \phi_\varepsilon^t(\theta, r)$. By the Gronwall lemma, there exists $k > 0$ such that:

$$||\Phi - \Phi_\varepsilon||_{C^1(K)} \leq k||X - X_\varepsilon||_{C^1(K)}.$$

Therefore there exists $\gamma > 0$ independent of ε, such that:

$$||\wp-\widehat{\wp}_\varepsilon||_{C^1(\mathbb{T}\times\mathscr{I})} \leq \gamma \sup\Big(||X-X_\varepsilon||_{C^1(K)}+||\tau-\tau_\varepsilon||_{C^1(J)}+||\operatorname{Id}-L_\varepsilon^{-1}||_{C^1(\mathbb{T}\times\mathscr{I})}+||\operatorname{Id}-L_\varepsilon||_{C^1(\mathbb{T}\times\mathscr{I})}\Big),$$

that is, there exists γ' independent of ε such that:

$$||\wp - \widehat{\wp}_\varepsilon||_{C^1} \leq \gamma'\sqrt{\varepsilon}.$$

Then for ε small enough, $\widehat{\wp}_\varepsilon$ satisfies the torsion condition by remark 5.1.1. Therefore we can apply Birkhoff's theorem and $\mathscr{C}(\theta_1^0)$ is the graph of a Lipschitz fonction $R_{\theta_1^0} : \mathbb{T} \to \mathbb{R}^2$. As a consequence,

$$\mathscr{L} := \bigcup_{\theta_1\in\mathbb{T}} \mathscr{C}(\theta_1) = \bigcup_{\theta_1\in\mathbb{T}} \{(\theta_2, R_{\theta_1}(\theta_2)) \,|\, \theta_2 \in \mathbb{T}\} = \{(\theta_1, \theta_2, R(\theta_1, \theta_2)) \,|\, (\theta_1, \theta_2) \in \mathbb{T}^2\}.$$

and $R : (\theta_1, \theta_2) \mapsto R(\theta_1, \theta_2)$ is continuous. The same argument holds true in each of the domains $\mathcal{H}^-_{1,\varepsilon}, \mathcal{H}^-_{2,\varepsilon}$ and $\mathcal{H}^+_{2,\varepsilon}$, which concludes the proof. □

5.2 Proof of Theorem C

Consider a C^2 Riemannian metric g on $\mathbb{T}^2$. We denote by H_g the geodesic Hamiltonian function on $T^*\mathbb{T}^2$ and by ϕ_g its associated Hamiltonian flow. Finally, we denote by $\mathscr{E}_g$ the compact energy level $\mathscr{E}_g := H_g^{-1}(\{1\})$. Then one has the following equivalence that plays an important role in the proof of theorem C:

($\star$) *g is flat $\Longleftrightarrow$ $\mathscr{E}_g$ is foliated by ϕ_g-invariant tori that are C^1 graphs over $\mathbb{T}^2$.*

Indeed, ($\Longrightarrow$) is obvious. Conversely, assume that $\mathscr{E}_g$ is foliated by ϕ_g-invariant tori that are graphs over the base $\mathbb{T}^2$. Then, by Theorem 5 all the geodesics are minimizing. In particular, they do not have conjugate points, and by the Hopf theorem, g is flat.

Proof of Theorem C. We denote by H_{g_0} the geodesic Hamiltonian function on $T^*\mathbb{T}^2$ defined by g_0. For $\varepsilon > 0$, we denote by $\mathscr{U}_\varepsilon$ the set of C^5 Riemannian metrics g on $\mathbb{T}^2$, such that $||g - g_0||_{C^5} \leq \varepsilon$ (where $||\cdot||_{C^5}$ is the C^5-norm on the space of metrics on $\mathbb{T}^2$). For $g \in \mathscr{U}_\varepsilon$, we denote by H_g the geodesic Hamiltonian function on $T^*\mathbb{T}^2$ defined by g. Fix a compact neighborhood K of $H_{g_0}^{-1}(\{1\})$. There exists $c > 0$, independent of ε, such that

$||H_{g_0} - H_g||_{K,C^5} \leq c\varepsilon$ (where here, $||\cdot||_{K,C^5}$ is the C^5-norm on the space of functions $H : K \to \mathbb{R}$).

By lemma 5.1.1 (1), if ε is small enough, there exist invariant tori in $H_g^{-1}(\{1\})$ that are the graphs of C^1 functions: $\mathbb{T}^2 \to \mathbb{R}^2$.

Assume now that $g \in \mathscr{DC}$ and that g is not flat. We denote by f a nondegenerate Bott integral for ϕ_g in restriction to the unit cotangent bundle $\mathscr{E}_g$ of $\mathbb{T}^2$. We want to see that $\mathscr{E}_g$ contains a hyperbolic orbit. Since H_g is dynamically coherent, it suffices to show that $\mathscr{E}_g$ contains a ∞-level.

By $(\star)$, at least one leaf of the foliation induced by f in $\mathscr{E}_g$ is not a C^1 graph over $\mathbb{T}^2$. Let $\mathscr{L}$ be such a leaf. By lemma 5.1.1 (3), $\mathscr{L}$ is either an elliptic orbit, or a torus, or an ∞-level. Note that such tori are C^1 submanifolds.

If $\mathscr{L}$ is an ∞-level, the proof is complete. If $\mathscr{L}$ is an elliptic orbit, there exists a neighborhood U of $\mathscr{L}$, saturated for f, such that $\mathcal{A} = U \setminus \mathscr{L}$ is a maximal action-angle domain. The domain $\mathcal{A}$ is foliated by tori homotopic to $\mathscr{L}$, these tori are obviously non homotopic to $\mathbb{T}^2 \times \{0\}$. Now if $\mathscr{L}$ is a torus, by lemma 5.1.1 (2), this torus is not homotopic to $\mathbb{T}^2 \times \{0\}$. So $\mathscr{L}$ is contained in a maximal action-angle domain $\mathcal{A}$ that is foliated by tori non homotopic to $\mathbb{T}^2 \times \{0\}$.

Now, each torus $\mathcal{T}$ that is a C^1 graph over $\mathbb{T}^2$ is contained in an action-angle domain $\mathcal{A}'$ in $\mathscr{E}_g$. Such a domain $\mathcal{A}'$ is foliated by tori homotopic to $\mathcal{T}$ (indeed, by lemma 5.1.1 (2), these tori are C^1 graph over $\mathbb{T}^2$). Therefore the boundary of one of the domains $\mathcal{A}'$ must intersect the boundary of one of the previous domains $\mathcal{A}$ and this intersection must be contained in an ∞-level. $\square$

Appendix A

Action-angle variables

In the first section we give a complete proof of Arnol'd-Liouville's Theorem and of its consequence to Hamiltonian systems. In the second section we explicit the construction of Arnol'd of the action variables by "quadratures". These two sections are inspired in a large way of [Aud01] and [Dui80]. In the last section we explicit the construction of the action-angle for the domain $\mathcal{D}_\infty^+$ defined in chapter 4.

A.1 A proof of Arnol'd-Liouville Theorem.

We use the notation of the section 1.1.2. We begin with the proof corollary 1.2.1

Proof of corollary 1.2.1. Set $\widehat{H} = H \circ \Psi^{-1}$. We consider I_k as a function on $\mathbb{T}^n \times B$. Then for $1 \leq k \leq n$, $\frac{\partial \widehat{H}}{\partial \alpha_k}(\varphi, I) = X^{I_k}(\alpha, I)(\widehat{H}) = \{I_k, \widehat{H}\}$. Now, by theorem 8, there exists a function β_k such that $I_k = \beta_k \circ F \circ \Psi^{-1}$. Then, since Ψ is symplectic, one gets :

$$\{I_k, \widehat{H}\}(\alpha, I) = \{I_k \circ \Psi, \widehat{H} \circ \Psi\} \circ \Psi^{-1}(\alpha, I) = \{\beta_k \circ F, H\} \circ \Psi^{-1}(\alpha, I), \quad \forall (\alpha, I) \in \mathbb{T}^n \times B.$$

Set $x = \Psi^{-1}(\alpha, I)$. Then:

$$\{\alpha_k \circ F, H\} \circ \Psi^{-1}(\alpha, I) = \{\beta_k \circ F, H\}(x) = d_{F(x)}\beta_k(d_x F(X^H(x))) = 0,$$

the last equality coming from the identity: $d_x F(X^H(x)) = 0$. Thus, for all $(\alpha, I) \in \mathbb{T}^n \times B$, $\frac{\partial \widehat{H}}{\partial \alpha_k}(\varphi, I) = 0$, that is, the Hamiltonian $\widehat{H}$ only depends on the variable I. $\square$

We will now give the proof theorem 8. It is essentially a detailed version of Duistermaat's proof. We set $M_c := \{x \in M \mid F^{(x)} \text{ is compact}\}$. By assumption, it is a non empty set. For $x \in M_c$, we denote by $F^{(x)}$ the connected component of x in the set $\{y \in M \mid F(y) = F(x)\}$.

• Step 1 : M_c is an open domain on which F is a locally trivial fibration.

We endow M with a Riemannian metric. For $x \in M$, we set $H_x = (\operatorname{Ker} d_x F)^\perp$. By assumption on F, the map $d_x F$ is an isomorphism from H_x onto $T_{F(x)}\mathbb{R}^n \simeq \mathbb{R}^n$. We set $L_x := d_x F$. Fix $x_0 \in M_c$. Set $b_0 = F(x_0)$ and $\mathcal{S} := \{b_0 + u \mid u \in \mathbb{S}^{n-1}\}$.

For $b \in \mathcal{S}$, we introduce the curve $s_b : [0, 1] \to \mathbb{R}^n$, $t \mapsto tb + (1 - t)b_0$ and we set $S_b := s_b([0, 1])$. We denote by X the vector field along S_b defined by $X(s_b(t)) = s_b'(t) = b - b_0$. It lifts on a vector field $\widetilde{X}(x) = L_x^{-1}(b - b_0)$ on $F^{-1}(S_b)$. Finally, for $x \in F^{(x_0)}$, we denote by $\gamma_{x,b}$ the integral curve for $\widetilde{X}$ that satisfies $\gamma_{x,b}(0) = x$.

By the Cauchy-Lipshitz Theorem there exists a neighborhood U_x of x in $F^{(x_0)}$ such that there exists $\varepsilon_x > 0$ such that for all $y \in U_x$, $t \mapsto \gamma_{y,b}$ is defined on $[0, \frac{\varepsilon_x}{2}]$. The open domains U_x form a covering of $F^{(x_0)}$. By taking a finite subcovering, we find $\varepsilon'_b > 0$ such that for all $x \in F^{(x_0)}$, $\gamma_{x,b}$ is defined on $[0, \varepsilon'_b]$. Set $\varepsilon_b = \frac{\varepsilon'_b}{2}$. There exists a neighborhood $F^{(x_0)}$ such that the flow (ϕ_b^t) associated with $\widetilde{X}$ is defined for all $t \in [0, \varepsilon_b]$. Note that, for every $t \in [0, \varepsilon_b]$, ϕ_b^t is a local diffeomorphism and show that it realize a diffeomorphism between $F^{(x_0)}$ and $F^{(\phi_b^t(x_0))}$.

•$\phi_b^t(F^{(x_0)}) \subset F^{(\phi_b^t(x_0))}$. Let $x \in F^{(x_0)}$. The curve $\gamma = F \circ \gamma_{x,b} : [0, \varepsilon_b] \to \mathbb{R}^n$ is an integral curve for $\widetilde{X}$. Indeed, $\gamma'(t) = d_{\gamma_{x,b}(t)} F(\tilde{X}(\gamma_{x,b}(t)) = X(F(\gamma_{x,b}(t)) = X(\gamma(t))$. Then $\gamma_{x,b}(t) = s_b(t)$ and $\phi_b^t(x) \in F^{-1}(\{s_b(t)\})$. By connexity, $\phi_b^t(F^{(x_0)})$ is contained in the connected component of $F^{-1}(s_b(t))$ that contains $\phi_b^t(x_0)$, that is, $F^{(\phi_b^t(x_0))}$.

•ϕ_b^t is clearly injective.

•ϕ_b^t is surjective. Show that $\phi_b^t(F^{(x_0)}) = F^{(\phi_b^t(x_0))}$. Let $y \in \phi_b^t(F^{(x_0)})$ and set $x := \phi_b^{-t}(y) \in F^{(x_0)}$. Fix an open neighborhood U_x of x in $F^{(x_0)}$. Then $\phi_b^t(U_x)$ is a neighborhood of y in $\phi_b^t(F^{(x_0)})$ and $\phi_b^t(F^{(x_0)})$ is open in in $F^{(\phi_b^t(x_0))}$. Now, since $F^{(x_0)}$ is compact, $\phi_b^t(F^{(x_0)})$ is compact, and therefore closed. By connexity $\phi_b^t(F^{(x_0)}) = F^{(\phi_b^t(x_0))}$ and ϕ_b^t is surjective.

Fix $b \in \mathcal{S}$. There exists a neighborhood U_b of b in $\mathcal{S}$ such that for all $b' \in U_b$, $\varepsilon_{b'} \geq \frac{\varepsilon_b}{2}$. We extract a finite subcovering $(U_{b_i})_{1 \leq i \leq m}$ of $\mathcal{S}$ U_b and we set $\varepsilon = \min\{\frac{\varepsilon_{b_i}}{2} \,|\, 1 \leq i \leq m\}$. For all $(b,t) \in \mathcal{S} \times [0, \varepsilon]$, the map ϕ_b^t is well defined. Set $\mathcal{B} := B(b_0, \varepsilon)$. For $b \in \mathcal{B}$, we denote by D_b the half-line starting from b_0 and going through b and we set $\sigma(b) = D_b \cap \mathcal{S}$. Let τ be the function on $B(b_0, \varepsilon)$ defined by $\tau(b) = \|b - b_0\|$. By construction, for all $b \in B(b_0, \varepsilon)$, $s_{\sigma(b)}(\tau(b)) = b$. Then, the map

$$\begin{array}{rlcl} \Phi : & F^{(x_0)} \times \mathcal{B} & \to & F^{-1}(\mathcal{B}) \\ & (x, b) & \mapsto & \phi_{\sigma(b)}^{\tau(b)}(x) \end{array}$$

is continuous and injective and satisfies

$$F(\Phi(x,b)) = b, \quad \forall (x,b) \in F^{(b_0)} \times \mathcal{B}$$

Then $\mathcal{U} := \Phi(F^{(x_0)} \times \mathcal{B}) \subset F^{-1}(\mathcal{B})$ is a neighborhood of x_0 in M_c and the restriction of F to $\mathcal{U}$ is a trivial fibration trivial whose each fiber is diffeomorphic to $F^{(x_0)}$.

In the following, we will work on $\mathcal{U}$ and we still denote by F the restriction of F to $\mathcal{U}$. If $b \in \mathcal{B}$, we denote by F_b the fiber $F^{-1}(\{b\}) \subset \mathcal{U}$. Then $\mathcal{U} \simeq F_b \times \mathcal{B}$, for an arbitrary $b \in \mathcal{B}$.

•Step 2 : $F_b \simeq \mathbb{R}^n \setminus \Gamma_b$ where Γ_b is a maximal lattice of $\mathbb{R}^n$.

For $1 \leq i \leq n$, we denote by ϕ_i^t the flow associated with the Hamiltonian vector field X^{f_i}. These flows are complete since each orbits is contained in a fiber F_b and they commute. One can define an action Φ of $\mathbb{R}^n$ on $\mathcal{U}$ by:

$$\begin{array}{rlcl} \Phi : & \mathbb{R}^n \times \mathcal{U} & \to & \mathcal{U} \\ & (T, x) & \mapsto & \Phi^T(x) = \phi_1^{t_1} \circ \cdots \circ \phi_n^{t_n}(x), \quad \text{if} \quad T = (t_1, ..., t_n). \end{array}$$

By construction, for $T \in \mathbb{R}^n$, Φ^T is symplectic and $\Phi^T(F_b) \subset F_b$. Let us emphasize the following fact:

Remark A.1. For all $x \in \mathcal{U}$, the map $\Phi(x) : \mathbb{R}^n \to \mathcal{U}, : T \mapsto \Phi^T(x)$ is a local diffeomorphim. Indeed fix $T = (t_1, ..., t_n) \in \mathbb{R}^n$. Then $\frac{\partial \Phi(x)}{\partial t_i}(T) = X^{f_i}(\Phi^T(x))$, so $\mathrm{Jac}(\Phi(x))(T)$ is invertible.

Show that, for any b, F_b is an orbit of Φ. Fix $b \in B_0$ and $x \in F_b$. We denote by $\mathcal{O}_x$ the orbit of x under Φ. Show that $\mathcal{O}_x$ is open in F_b. Fix $y \in \mathcal{O}_x$ and let $\tau \in \mathbb{R}^n$ be such that $y = \Phi^\tau(x)$. By remark A.1, there exists an open neighborhood Δ of 0 in $\mathbb{R}^n$ such that the map $\Delta \to F_b$, $T \mapsto \Phi^{T+\tau}(x) = \Phi^T(y)$ is a diffeomorphisme onto its image. Then $\{\Phi^T(y) | T \in \Delta\}$ is an open neighborhood of y in F_b. Now, $F_b := \bigcup_{x' \in F_b} \mathcal{O}(x')$, so $F_b \setminus \mathcal{O}(x)$ is open in F_b and $\mathcal{O}_x$ is closed in F_b. By connexity, $F_b = \mathcal{O}_x$.

We denote by Γ_b the isotropy sub-group of F_b. By remark A.1, Γ_b is discrete, so it is a lattice of $\mathbb{R}^n$. The quotient $\mathbb{R}^n / \Gamma_b$ is diffeomorphic to F_b, so it is compact and Γ_b has maximal rank.

•Step 3 : Γ_b depends smoothly on b.

We denote by σ the diffeomorphism $\sigma : \mathcal{U} \to F_b \times \mathcal{B}$. Fix $x \in F_b$ and consider the section s of F defined by

$$s(c) = \sigma^{-1}(x, c).$$

The n-dimensional submanifold $S := \sigma^{-1}(\{x\} \times \mathcal{B})$ is transverse to the fibers F_b.

Lemma A.1. *: There exists an open domain Δ of $\mathbb{R}^n$, a neighborhood $B_0 \subset \mathcal{B}$ of b such that the map $\psi : B_0 \times \Delta \to M$ defined by $\psi(c, T) = \Phi^T(s(c))$ is a local diffeomorphism.*

Proof. Let us compute $d_{(b,0)}\psi$. We denote by $V_1, ..., V_n, W_1, ..., W_n$ the column vectors of $\mathrm{Jac}\,\psi(b, 0)$. Then $\frac{\partial \psi}{\partial b_i}(b, T) = T_{\Phi^T(s(b))}\Phi^T(\frac{\partial s(b)}{\partial b_i})$. So $V_i = \frac{\partial s(b)}{\partial b_i}$ and $(V_1 \cdots, V_n)$ is a basis of $T_{s(b)}S$. Similarly, $\frac{\partial \psi}{\partial t_i}(b, t) = X^{J_i}(\Phi^T(s(b)))$. So $W_i = X^{f_i}(s(b))$ and $(W_1, \ldots, W_n)$ is a basis of $T_{s(b)}F_b$. Then, $\mathrm{Jac}\,\psi(b, 0)$ is invertible, so if B_0 is small enough, ψ is a diffeomorphism. □

Let $(e_1, ..., e_n)$ be a basis of Γ_b. We show that, on a neighborhood $B \subset B_0$ of b, there exist n maps $T^1, \cdots, T^n : B \to \mathbb{R}^n$ such that:

- $T^i(b) = e_i$ for all $1 \leq i \leq n$
- $T(c) = (T^1(c), ..., T^n(c))$ is a basis of Γ_c for all $c \in B$

Fix i. We want to show that there exists a neighborhood B_i of b and a map T^i such that

$$\forall c \in B_i,\ \Phi^{T^i(c)}(s(c)) = s(c), \quad \forall c \in B_i.$$

Let $\psi_i : \mathcal{B} \times \mathbb{R}^n \to M : (c, T) \mapsto \Phi^{e_i} \circ \psi \circ h_{e_i}(c, T)$ where $h_{e_i}(c, T) = (c, T - e_i)$. Then $\psi_i(b, e_i) = s(b)$. Since S is transverse to each orbite of Φ^T there exists a neighborhood B_i of b, such that $\psi_i^{-1}(S)$ is the graph of a map $T^i : B_i \to \mathbb{R}^n$. Such a map T^i satisfies $\Phi^{T^i(c)}(s(c)) \in S$, for all $c \in B_i$. Now if $c \in B_i$, $\Phi^{T_i}(s(c)) \in F_c$, that is, $\Phi^{T^i}(s(c)) = s(c)$.

Let $B := \cap_{1 \leq i \leq n} B_i$. Shrinking B if necessary, for all $c \in B$, the vectors $T^i(c)$ form a basis of Γ_c and the map $T : B \to \mathbb{R}^n : c \mapsto (T^1(c), ..., T^n(c))$ is smooth. We can moreover assume that B is convex

•Step 4 : Existence of the variables I.

We set $V = F^{-1}(B)$. We look for functions $I_1, ..., I_n$ defined on V that depend only of $f_1, ..., f_n$ and that generate 1-periodic flows. We denote by X^{I_k} and $(\phi^t_{I_k})$ their associated

Hamiltonian vector field and Hamiltonian flow. Show that we can construct the functions I_k such that

$$\phi^t_{I_k}(x) = \Phi^{t.T^k(c)}(x). \tag{$*$}$$

The second condition is then obvioulsy satisfied. Assume that I_k is constructed such that $(*)$ is satisfied. Then

$$X^{I_k}(x) = \frac{\partial}{\partial t}\Phi^{t.T^k(c)}(x)_{|0} = \sum_i \frac{\partial}{\partial t_i}\Phi^{T^k(c)}(x)_{|0}.T^k_i(c).$$

Since the flow ϕ_i pairwise commute, one gets

$$\frac{\partial}{\partial t_i}\Phi^{T^k(c)}(x)_{|0} = X^{f_i}(x),$$

Which yields

$$X^{I_k}(x) = \sum_i T^k_i(F(x))X^{f_i}(x). \tag{1.1}$$

Now the first condition reads $I_k = \beta_k \circ F$, where $\beta_k : B \to \mathbb{R}$. That is, $dI_k = \sum_j (\frac{\partial \beta_k}{\partial c_j} \circ F) \circ df_j$ or equivalently

$$X^{I_k} = \sum_j \left(\frac{\partial \beta_k}{\partial c_j} \circ F\right) \circ X^{f_j}. \tag{$**$}$$

The condition $(**)$ therefore reads $d\beta_k = T^k$. By Poincaré lemma, since B is convex, this is possible if and only if T^k (seen as a 1-form $\sum_j T^k_j(c)dc_j$) is closed, that is,

$$\frac{\partial T^k_j}{\partial c_i} = \frac{\partial T^k_i}{\partial c_j},\ \forall i,j = 1,...,n. \tag{$\star$}$$

Assume that condition $(\star)$ is checked. Then β_k is uniquely defined up to a constant. By construction, the functions $I_1, \cdots I_n$ are linearly independent and involution. The map $I = (I_1,...,I_n) : V \to \mathbb{R}^n$ is a trivial fibration with fibers the F_c and I induces an action $\Phi_I : \mathbb{R}^n \times V \to V, (T = (t_1,...,t_n), x) \mapsto \phi^{t_1}_{I_1} \circ \cdots \circ \phi^{t_n}_{I_n}(x)$. For all T, $\Phi^T_I : x \mapsto \Phi_I(T,x)$ is a symplectic diffeomorphism.

Let us now check the condition $(\star)$. In any small enough open domain of M, there exist functions $g_1,...,g_n$ such that

$$\{f_j, g_i\} = \delta^j_i, \quad \{g_j, g_i\} = 0.$$

This implies that $\frac{\partial}{\partial g_I} = X^{f_i}$. We set $G : x \mapsto (g_1(x),...,g_n(x))$. In the following, we identify a point $x \in V$ with its coordinates (F,G).

By definition of T^k, for all $(F,G) \in \mathcal{U}$:

$$\Phi^{T^k(F)}(F,G) = (F,G).$$

Let us derive this equation in the direction f_j. We set $\Phi_{(F,G)} : \mathbb{R}^n \to M$, $T \mapsto \Phi^T(F,G)$. :

$$\partial_{f_j}\Phi^{T^k(F)}(F,G) = D\Phi_{(F,G)}(T^k(F))\partial_{f_j}T^k(F) + D\Phi^{T^k(F)}(F,G)\partial_{f_j}(F,G).$$

One has:

$$D\Phi_{(F,G)}(T^k(F)) = \left(\frac{\partial \Phi_{(F,G)}}{\partial t_1}(T^k(F)),...,\frac{\partial \Phi_{(F,G)}}{\partial t_n}(T^k(F))\right)$$

and

$$\partial_{f_j} T^k(F) = \begin{pmatrix} \frac{\partial T_1^k}{\partial f_j}(F) \\ \vdots \\ \frac{\partial T_n^k}{\partial f_j}(F) \end{pmatrix}.$$

Now since the flows ϕ_i commute, one has

$$\frac{\partial \Phi_{(F,G)}}{\partial t_i}(T^k(F)) = X^{f_i}((F,G)).$$

Finally, since $\partial_{f_j}(F,G) = \frac{\partial}{\partial f_k}$, one gets :

$$\sum_i \frac{\partial T_i^k(F)}{\partial f_j} X^{f_i}(F,G) + D(\Phi^{T^k(F)})(F,G)(\frac{\partial}{\partial f_j}) = \frac{\partial}{\partial f_j}. \tag{$\star\star$}$$

Let us study $D(\Phi^{T^k(F)})(F,G)$. Fix $F = F_0$. For all x such that $F(x) = F_0$, $\Phi^{T^k(F_0)}(x) = x$. That is, for all G, $\Phi_{T^k(F_0)}(F_0, G) = (F_0, G)$. Then if P is the matrix of $D\Phi^{T^k(F)}(F,G)$ in the basis $(\frac{\partial}{\partial f_1}, ..., \frac{\partial}{\partial f_n}, \frac{\partial}{\partial g_1}, ..., \frac{\partial}{\partial g_n})$, P has the following form:

$$P = \begin{pmatrix} R & 0 \\ Q & I \end{pmatrix}$$

Now, since P is symplectic:

$$P^{-1} = -J\,{}^tPJ = \begin{pmatrix} -I & 0 \\ {}^tQ & {}^tR \end{pmatrix}.$$

Computing PP^{-1}, one gets

$$R = I, \quad \text{and} \quad Q = {}^tQ.$$

We set $Q := (Q_{ij})$. Then by $(\star)$, one has since $X^{f_i} = \frac{\partial}{\partial g_i}$:

$$\begin{aligned} \Omega\left(\frac{\partial}{\partial f_k}, \frac{\partial}{\partial f_j}\right) &= \Omega\left(\sum_i \frac{\partial T_i^k}{\partial f_j}(F) X^{f_i}(F,G) + D(\Phi^{T^k(F)})(F,G)(\frac{\partial}{\partial f_j}), \frac{\partial}{\partial f_j}\right) \\ 0 &= \frac{\partial T_j^k}{\partial f_k}(F) + Q_{ij} \end{aligned}$$

The result comes from the symetry of Q.

•Step 5 : Existence of the variables a**.**

The existence of the angle variables is based on the following lemma.

Lemma A.2. *Let $g_1, \cdots, g_n$ be n functions $M \to \mathbb{R}$. Assume that $G := (g_1, \cdots, g_n)$ is a submersion on $\mathbb{R}^n$. For any $b \in G(M) \subset \mathbb{R}^n$, we set $\mathscr{L}_b := G^{-1}(\{b\})$. The submanifolds $\mathscr{L}_b$ are all Lagrangian if and only if the functions g_i are pairwise in involution.*

Proof. Fix $b \in \mathbb{R}^n$, $x \in \mathscr{L}_b$ and $v \in T_xM$. Then

$$\begin{aligned} v \in T_x\mathscr{L}_b \iff d_xG(v) = 0 &\iff d_x g_i(v) = 0 && \forall i \\ &\iff \Omega(X^{g_i}(x), v) = 0 && \forall i \\ &\iff v \perp X^{g_i}(x) && \forall i. \end{aligned}$$

Assume that $\mathscr{L}_b$ is Lagrangian. Then $X^{g_i}(x) \in T_x\mathscr{L}_b$ et $\Omega(X^{g_i}(x), X^{g_k}(x)) = 0$, that is, $d_x f_i(X^{f_k}) = 0$.

On the other hand, assume that the functions g_i are in involution. Then $d_x g_i(X^{f_k}) = 0$, that is, $X^{g_i}(x) \in T_x F_b$ and the vectors $X^{g_i}(x)$ form a basis of $T_x\mathscr{L}_b$ which is therefore Lagrangian. □

Remark A.2. The fibers F_b are Lagrangian submanifolds. Observe that we did not need this property until now.

In any small enough open domain of $\mathcal{U}$ where $I_1, \dots, I_n$ are defined, there exist n functions $\alpha_1, \dots, \alpha_n$ such that

$$\{\alpha_i, \alpha_j\} = 0 \quad \text{and} \quad \{\alpha_i, I_j\} = \delta_i^j, \ \forall\, i, j.$$

In this coordinates the symplectic form Ω reads $\Omega = \sum_j d\varphi_j \wedge dI_j$.

Fix $(\alpha_1^0, \dots \alpha_n^0) \in \alpha_1(\mathcal{U}) \times \dots \times \alpha_n(\mathcal{U})$. The submanifold $\{\alpha_i = \alpha_i^0,\ 1 \leq i \leq n\}$ is a local Lagrangian section of F. Fix $x \in M$, $T = (t_1, ..., t_n) \in \mathbb{R}^n$ and $y = \Phi_I^T(x)$. For T small enough

$$y = \phi_{I^1}^{t_1} \circ \dots \circ \phi_{I^n}^{t_n}(x) = (\alpha(x) + T, I).$$

That is,

$$\alpha_k(y) = \alpha_k(x) - t_k, \quad k \in \{1, \cdots, n\}. \tag{A.1}$$

Now, we will see that the 1-periodicity of the flows ϕ_{I_k} allows us to define the α_k as globally $\mathbb{T}$-values functions.

Fix $x_0 \in \mathcal{U}$ and let U be a neighborhood of x_0 such that (α, I) define coordinates on U. We can assume that $\alpha(x_0) = 0$. Shrinking U if necessary, we can assume that S is a section of I.

Fix $b \in F(U)$. For all k, we define α_k on F_b by

$$x = \phi_{I^1}^{\alpha_k(x)} \circ \dots \circ \phi_{I^n}^{\alpha_n(x)}(x_0).$$

The function α_k thus constructed coincide with the initial function α_k in U.

Since the vectors $X^{\alpha_1}(x_0), ..., X^{\alpha_n}(x_0)$ are independent and since for $T \in \mathbb{T}^n$, $D\Phi_I^T(x_0)$ is an isomorphism that sends $X^{\alpha_k}(x_0)$ on $X^{\alpha_k}(\Phi_I^T(x_0))$, the functions $\alpha_1, \cdots, \alpha_n$ are independent. Moreover, since Φ_I^T is symplectic $\Phi_I^T(S)$ is Lagrangian. Now $\Phi_I^T(S) := \alpha^{-1}(T)$. By lemma A.2, $\alpha_1, \cdots, \alpha_n$ are pairwise in involution.

• **Conclusion.** The domain $W := F^{-1}(F(U)) \cap \mathcal{U}$ is a neigborhood of $F^{(x_0)}$ foliated by tori homotopic to $F^{(x_0)}$. The map $(\alpha, I) : W \to \mathbb{T}^n \times F(U)$ is a symplectic diffeomorphism.

A.2 Arnol'd's construction by "quadratures"

Here we give the construction of the action variable due to Arnol'd. Recall that if a symplectic form Ω is exact on a open domain $\mathcal{O} \subset M$, we call a *Liouville form*, a 1-form such that $\Omega = d\lambda$. The construction is based on the following lemma (see [Aud01] for a detailed proof).

Lemma A.1. *Consider a smooth vector field X on an open domain $\mathcal{O} \subset M$ that generate a 1-periodic flow $(\phi_t)_t$. Assume that Ω is exact on $\mathcal{O}$. Assume moreover that $\iota_X \mathscr{L}_X \Omega = 0$. Then X is a Hamiltonian vector field with Hamiltonian*

$$H : x \mapsto \int_{\mathscr{O}(x)} \lambda.$$

where λ is a Liouville form on $\mathcal{O}$ and $\mathscr{O}(x)$ is the closed orbit of x.

If γ is a closed curve on a compact submanifold N, we denote by $[\gamma]$ its homology class in $H_1(N, \mathbb{Z})$.

Theorem A.20. *Let $\gamma_1^b, ..., \gamma_n^b$ be closed curves of F_b depending smoothly of b such that $([\gamma_1^b], ..., [\gamma_n^b])$ is a basis of $H_1(F_b, \mathbb{Z})$. There exists a neighborhood U of F_b, such that Ω is exact on U. Then the maps $\widehat{I}_k$ defined by*

$$\widehat{I}_k(x) = \psi_k \circ F(x), \text{ avec } \psi(b) = \int_{\gamma_k^b} \lambda,$$

are action coordinates on U.

Remark A.1. The existence of U is obvious with Arnol'd-Liouville Theorem (the symplectic form is $\Omega = da \wedge dI$!). Nevertheless by the Weinstein Lagrangian neighborhood theorem such a neighborhood exists for any Lagrangian submanifold.

Proof. Fix $T \in \Gamma_b$ and denote by $[\gamma_T]$ the homology class of the curves $\gamma_T(x) : t \mapsto \Phi_I^{tT}(x)$. The map $T \mapsto [\gamma_T]$ is an isomorphism between Γ_b and $H_1(F_b, \mathbb{Z})$. Fix a smooth basis $(T_b^1, ..., T_b^n)$ of Γ_b such that $[\gamma_{T^i}] = [\gamma_i^b]$ for all i. The flows $\phi_i^t := \Phi^{tT_i}$ are 1-periodic flow with vector fields

$$X^i = \sum_k T_k^i(F) X^{f_i}.$$

Then

$$\iota_{X^i} \mathscr{L}_{X^i} \Omega = d\iota_{X^i} \Omega = \iota_{X^i} d \sum_k T_k^i \iota_{X^{f_i}} \Omega = \iota_{X^i} d \sum_k T_k^i df_i = \iota_{X^i} \sum_k dT_k^i \wedge df_i,$$

the first equality coming from Cartan's formula. Now for any i, j, $X^{f_i} \in \ker df_j$ so $X^i \in \ker df_j$. Similarly, since T_j^i are first integral of X^{f_i}, $X^i \in \ker dT_j^i$, that is $\iota_{X^i} \mathscr{L}_{X^i} \Omega = 0$ Applying lemma A.1, one gets n functions $I_1, \cdots, I_n$ that generate 1-periodic Hamiltonian flows. Obvioulsy, by construction these functions are independant since the vector fiels X^i are so. □

A.3 The action-angle variables for the torus of revolution

We focus on the domain $\mathcal{D}_\infty^+$ and we use Arnol'd's method "by quadrature".

For $T = (a, b) \in \mathbb{R}^2$ we denote by Φ^T the joint flow of the moment map (H, p_φ), that is, $\Phi^{(a,b)}(m, p) := \phi_H^a \circ \phi_{p_\varphi}^b(m, p)$.

For any $\rho \in \mathcal{R}(e)$ the Liouville torus $\mathcal{T}_{e,\rho}$ is parametrized by $(\bar{\varphi}, \bar{s})$ and a basis of $H_1(\mathcal{T}_{e,\rho}, \mathbb{Z})$ is given by $([\gamma_1], [\gamma_2])$ where

$$\gamma_1(t) = (t, 0) \text{ and } \gamma_2(t) = (0, t).$$

Let now $\bar{\sigma}$ be any Lagrangian section of F with equation $(\bar{\varphi}, \bar{s}) = (0, \bar{s}_0)$. We set $\bar{\sigma}_{e,\rho} = \bar{\sigma} \cap \mathcal{T}_{e,\rho}$. We look for a basis (T_1, T_2) of the isotropy subgroup of $\mathcal{T}_{e,\rho}$ (that depends smoothly on (e, ρ)) such that:

$$(t \mapsto \Phi^{tT_\ell}(\bar{\sigma}_{e,\rho})) \in [\gamma_\ell] \quad \ell = 1, 2,$$

where $[\gamma_\ell]$ denotes the rationnally homology class of the curve γ_ℓ. Denoting by λ the Liouville form on $T^*\mathbb{T}^2$, the action variable I_1, I_2 will be defined as

$$I_\ell := \int_{\gamma_\ell} \lambda.$$

One checks that we can choose $T_1 = (0,1)$ and $T_2 = (\tau_{e,\rho}, -\varphi_{e,\rho})$ with

$$\tau_{e,\rho} = \int_0^1 \frac{r(t)}{\sqrt{2e - \frac{\rho^2}{4\pi^2 x(t)^2}}} dt \quad \text{and} \quad \varphi_{e,\rho} := \int_0^{\tau_{e,\rho}} \dot{\varphi}(t) dt.$$

This yields:

$$I_1(e,\rho) = \rho \quad \text{and} \quad I_2(e,\rho) = \int_0^1 r(t) \sqrt{(2e - \frac{\rho^2}{4\pi^2 x(t)^2})} dt.$$

From $\phi^t_{I_2} := \phi_H^{t\tau_{e,\rho}} \circ \phi_\varphi^{-t\varphi_{e,\rho}} = \phi_H^{t\tau_{e,\rho}} \circ \phi_{I_1}^{-t\varphi_{e,\rho}}$, one immediately deduces

$$\phi_H^t = \phi_{I_2}^{\frac{t}{\tau_{e,\rho}}} \circ \phi_{I_1}^{\frac{\varphi_{e,\rho}}{\tau_{e,\rho}}}.$$

The associated angle variables $(\alpha_1, \alpha_2) \in \mathbb{T}^2$ are defined as $(\mathbb{R}/\mathbb{Z})$-valued functions such that

$$\forall\, (\bar{m}, p) \in \mathcal{T}_{e,\rho},\ \phi_{I_1}^{\alpha_1(\bar{m},p)} \circ \phi_{I_2}^{\alpha_2(\bar{m},p)}(\bar{\sigma}_{e,\rho}) = (\bar{m}, p),$$

where $\bar{\sigma}_{e,\rho} = \bar{\sigma} \cap \mathcal{T}^+_{e,\rho}$. Fix a Liouville torus $\mathcal{T}_{e,\rho}$ in $\mathcal{D}^+_\infty$. The angle variables (α_1, α_2) yield a diffeomorphism:

$$\begin{array}{rccl} \alpha_{e,\rho}: & \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & (\bar{\varphi}, \bar{s}) & \mapsto & (\alpha^1_{e\,\rho}(\bar{\varphi}, \bar{s}), \alpha^2_{e,\rho}(\bar{\varphi}, \bar{s})) := (\alpha^1(\bar{\varphi}, \bar{s}, e, \rho), \alpha^2(\bar{\varphi}, \bar{s}, e, \rho)). \end{array}$$

Set $\widetilde{\mathcal{D}}^+_\infty := (\varpi^*)^{-1}(\mathcal{D}^+_\infty)$. The diffeomorphism A lifts to a diffeormorphism

$$\begin{array}{rccl} A: & \widetilde{\mathcal{D}}^+_\infty & \to & \mathbb{R}^2 \times D \\ & (m,p) & \mapsto & (a^1, a^2, I_1, I_2) \end{array}$$

If $\mathscr{L}_{e,\rho} := (\varpi^*)^{-1}(\mathcal{T}_{e,\rho})$, we denote by $a_{e,\rho} : \mathbb{R}^2 \to \mathbb{R}^2$ the map such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "a_{e,\rho}"] \arrow[d, "\varpi"'] & \mathbb{R}^2 \arrow[d, "\varpi"] \\ \mathbb{T}^2 \arrow[r, "\alpha_{e,\rho}"'] & \mathbb{T}^2. \end{tikzcd} \tag{A.2}$$

There exists a $\mathbb{Z}^2$-periodic map $q_{e,\rho} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $a_{e,\rho} = \mathrm{Id} + q_{e,\rho}$.

Fix $(\varphi, s) \in \mathbb{R}^2$ and look for $(a^1_{e,\rho}(\varphi, s), a^2_{e,\rho}(\varphi, s))$. We choose $a^2_{e,\rho}(\varphi, s)$ to be the time needed to reach s following the orbit $t \mapsto \widetilde{\Phi}^{tT_2}(\sigma_{e,\rho})$. On gets:

$$a^2_{e,\rho}(\varphi, s) = \frac{\displaystyle\int_{s_0}^s \frac{r(t)dt}{\sqrt{2e - \frac{\rho}{4\pi^2 x(t)^2}}}}{\displaystyle\int_0^1 \frac{r(t)dt}{\sqrt{2e - \frac{\rho}{4\pi^2 x(t)^2}}}}$$

Then, the angle variable $a^1_{e,\rho}$ is defined as the time needed to reach (φ, s) following the flow $t \mapsto \tilde{\phi}^t_{I_2}(\varphi', s)$ where $(\varphi', s) = \Phi^{tT_2}(\sigma_{e,\rho})$.

We denote by $(\varphi(t)$ the φ-coordinate of of $\phi^t_H(\sigma_{e,\rho})$. Since $\tilde{\phi}^t_{I_2} = \tilde{\phi}_{p_\varphi}^{-t(\varphi_{e,\rho})} \circ \phi_H^{t\tau_{e,\rho}}$, one has $\varphi' = \varphi(a_2(s)\tau_{e,\rho}) - a_2(s)(\varphi_{e,\rho})$. So one gets:

$$a^1_{e,\rho}(\varphi, s) = \varphi - \varphi(a^2_{e,\rho}(s)\tau_{e,\rho}) + a^2_{e,\rho}(s)\varphi_{e,\rho}.$$

Appendix B

Properties of h and asymptotics equivalents of chapter 4

B.1 Convexity and superlinearity of h.

Recall that the action variables are given by:

$$I_1(e,\rho) = \rho \quad \text{and} \quad I_2(e,\rho) = \int_0^1 r(t)\sqrt{(2e - \frac{\rho^2}{4\pi^2 x(t)^2})}dt,$$

where $(e,\rho) \in D := \{(e,\rho)\,|\,e>0, \rho \in J(e)\} := \{(e,\rho)\,|\,e>0, |\rho| \leq 2\pi\sqrt{2e}x_1\}$.

Let f be the function defined on $]\frac{1}{x(1)^2}, +\infty[$ by :

$$f : u \mapsto \int_0^1 r(t)\sqrt{(u - \frac{1}{x(r)^2})}dt.$$

It is an increasing bijection. Denoting by g its inverse, one has $\frac{I_2}{I_1} = f\left(\frac{2h}{I_1^2}\right)$, that is,

$$h(I_2, I_1) = \frac{I_1^2}{2} g\left(\frac{I_2}{I_1}\right).$$

Convexity of h. One has: $D^2h(I_1, I_2) = \frac{1}{2}G$ where G has the following form:

$$G = \begin{pmatrix} g''\left(\frac{I_2}{I_1}\right) & g'\left(\frac{I_2}{I_1}\right) - \left(\frac{I_2}{I_1}\right)g''\left(\frac{I_2}{I_1}\right) \\ g'\left(\frac{I_2}{I_1}\right) - \left(\frac{I_2}{I_1}\right)g''\left(\frac{I_2}{I_1}\right) & 2g\left(\frac{I_2}{I_1}\right) - 2\left(\frac{I_2}{I_1}\right)g'\left(\frac{I_2}{I_1}\right) + \left(\frac{I_2}{I_1}\right)^2 g''\left(\frac{I_2}{I_1}\right) \end{pmatrix}$$

It suffices to show that the principal minors of this matrix are positive, that is, $g''\left(\frac{I_2}{I_1}\right) > 0$ and $\det D^2\mathcal{H}(I_2, I_1) > 0$.
Since f is strictly concave and increasing, g est strictly convex, thus $g'' > 0$. On the other hand,

$$\begin{aligned} \det D^2h(I_1, I_2) &= 2g\left(\frac{I_2}{I_1}\right)g''\left(\frac{I_2}{I_1}\right) - g'^2\left(\frac{I_2}{I_1}\right) \\ &= \frac{1}{4}(g^2)''\left(\frac{I_2}{I_1}\right) + \frac{3}{2}g^2\left(\frac{I_2}{I_1}\right)(\log g)''\left(\frac{I_2}{I_1}\right). \end{aligned}$$

Since g is convex, increasing and positive, g^2 is still convex, thus $(g^2)'' > 0$. Let us show that $\log g$ is convex. Since $\log g$ is an increasing bijection, it suffices to show that its inverse $\tilde{f}$ is concave. One has

$$\tilde{f}(u) = f(e^u) = \int_0^1 \sqrt{r(t)(e^u - \frac{1}{x(r)^2})}dt.$$

Then

$$\tilde{f}'(u) = \int_0^1 \frac{r(t)e^u}{\sqrt{r(t)(e^u - \frac{1}{x(r)^2})}}dt,$$

and

$$\tilde{f}''(u) = \int_0^1 \frac{-1}{x(t)^2} \frac{r(t)e^u}{(r(t)(e^u - \frac{1}{x(r)^2}))^{\frac{3}{2}}}dt < 0.$$

Superlinearity of h. Set $k := \max\{2\sqrt{2}x_1, \int_0^1 r(t)dt\}$. Then

$$\max(|I_1(e,\rho)|, |I_2(e,\rho)|) \leq k\sqrt{e},$$

that is,

$$\max\{|I_1|, |I_2|\} \leq k\sqrt{h(I_1, I_2)},$$

from which one immediately deduces the superlinearity.

B.2 Asymptotic estimates for φ_ρ, τ_ρ and τ'_ρ

Since $x'(0) = 0$, one has

$$\frac{1}{4\pi^2 x(s)^2} \simeq_{s\to 0} \frac{1}{\rho_0^2}\left(1 - \frac{4\pi\gamma}{\rho_0}s^2\right).$$

Let $\alpha < \gamma < \beta$. There exists $\delta_1 > 0$ such that forall $s \in]-\delta, \delta[$,

$$\begin{array}{ccccc}
\frac{1}{\rho_0^2}\left(1 - \frac{4\pi\beta}{\rho_0}s^2\right) & \leq & \frac{1}{4\pi^2 x(s)^2} & \leq & \frac{1}{\rho_0^2}\left(1 - \frac{4\pi\alpha}{\rho_0}s^2\right) \\
1 - \frac{\rho^2}{\rho_0^2}\left(1 - \frac{4\pi\alpha}{\rho_0}s^2\right) & \leq & 1 - \frac{\rho^2}{4\pi^2 x(s)^2} & \leq & 1 - \frac{\rho^2}{\rho_0^2}\left(1 - \frac{4\pi\beta}{\rho_0}s^2\right) \\
\frac{1}{\sqrt{1 - \frac{\rho^2}{\rho_0^2}\left(1 - \frac{4\pi\beta}{\rho_0}s^2\right)}} & \leq & \frac{1}{\sqrt{1 - \frac{\rho^2}{4\pi^2 x(s)^2}}} & \leq & \frac{1}{\sqrt{1 - \frac{\rho^2}{\rho_0^2}\left(1 - \frac{4\pi\alpha}{\rho_0}s^2\right)}} \\
\frac{\rho_0}{\rho_0^2 - \rho^2}\frac{1}{\sqrt{1 + \frac{\rho^2}{\rho_0^2 - \rho^2}\frac{4\pi\beta}{\rho_0}s^2}} & \leq & \frac{1}{\sqrt{1 - \frac{\rho^2}{4\pi^2 x(s)^2}}} & \leq & \frac{\rho_0}{\rho_0^2 - \rho^2}\frac{1}{\sqrt{1 + \frac{\rho^2}{\rho_0^2 - \rho^2}\frac{4\pi\alpha}{\rho_0}s^2}}
\end{array}$$

1)$\tau_\rho = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{r(s)ds}{\sqrt{1 - \frac{\rho^2}{4\pi^2 x(s)^2}}}$. Let $a < 1 < b$. There exists δ_2 such that for all $s \in]-\delta_2, \delta_2[$, $ar(0) \leq r(s) \leq br(0)$. Let $\delta := \min(\delta_1, \delta_2)$. For all $s \in]-\delta, \delta[$, one has:

$$\frac{\rho_0}{\rho_0^2 - \rho^2}\frac{ar(0)}{\sqrt{1 + \frac{\rho^2}{\rho_0^2 - \rho^2}\frac{4\pi\beta s^2}{\rho_0}}} \leq \frac{r(s)}{\sqrt{1 - \frac{\rho^2}{4\pi^2 x(s)^2}}} \leq \frac{\rho_0}{\rho_0^2 - \rho^2}\frac{br(0)}{\sqrt{1 + \frac{\rho^2}{\rho_0^2 - \rho^2}\frac{4\pi\alpha s^2}{\rho_0}}}$$

Hence

$$\frac{ar(0)\rho_0}{\rho_0^2-\rho^2}\int_{-\delta}^{\delta}\frac{ds}{\sqrt{1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\beta s^2}{\rho_0}}}\leq\int_{-\delta}^{\delta}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}}$$
$$\leq\frac{br(0)\rho_0}{\rho_0^2-\rho^2}\int_{-\delta}^{\delta}\frac{ds}{\sqrt{1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\alpha s^2}{\rho_0}}}. \quad \text{(B.1)}$$

Let $\zeta := \frac{\sqrt{\rho_0^2-\rho^2}}{\rho}$, let k stands for $4\pi\beta$ or $4\pi\alpha$ and c stands for a or b. One has using the change of variable $u=\sqrt{\frac{k}{\rho_0}}\frac{1}{\zeta}s$:

$$\begin{aligned}\frac{c\rho_0 r(0)}{\rho\zeta}\int_{-\delta}^{\delta}\frac{ds}{\sqrt{1+\frac{k}{\rho_0}\frac{s^2}{\zeta^2}}} &= \frac{c\rho_0 r(0)}{\rho\zeta}\frac{\zeta\sqrt{\rho_0}}{\sqrt{k}}\int_{-\sqrt{\frac{k}{\rho_0}}\frac{\delta}{\zeta}}^{\sqrt{\frac{k}{\rho_0}}\frac{\delta}{\zeta}}\frac{du}{\sqrt{1+u^2}}\\ &= 2\frac{\rho_0^{\frac{3}{2}}}{\rho}\frac{cr(0)}{\sqrt{k}}\operatorname{argsh}\left(\sqrt{\frac{k}{\rho_0}}\frac{\delta}{\zeta}\right)\\ &= -\frac{1}{2}\ln(\rho_0-\rho)+\ln\left(\frac{\rho\sqrt{\frac{k}{\rho_0}}\delta}{\sqrt{\rho_0+\rho}}\left(1+\sqrt{1+\frac{\rho_0^2-\rho^2}{\rho^2\frac{k}{\rho_0}\delta^2}}\right)\right).\end{aligned}$$

If $f(\rho)=\ln\left(\frac{\rho\sqrt{\frac{k}{\rho_0}}\delta}{\sqrt{\rho_0+\rho}}\left(1+\sqrt{1+\frac{\rho_0^2-\rho^2}{\rho^2\frac{k}{\rho_0}\delta^2}}\right)\right)$, f is bounded over $[0,\rho_0]$. Then:

$$\frac{\rho_0^{\frac{3}{2}}}{\rho}\frac{ar(0)}{2\sqrt{\pi\beta}}\leq\liminf_{\rho\to\rho_0}\frac{1}{\ln(\rho_0-\rho)}\leq\limsup_{\rho\to\rho_0}\frac{1}{\ln(\rho_0-\rho)}\leq\frac{\rho_0^{\frac{3}{2}}}{\rho}\frac{br(0)}{2\sqrt{\pi\alpha}}.$$

Since these inequalities holds for any $a<1<b$ and any $\alpha<\gamma<\beta$, one has

$$\int_{-\delta}^{\delta}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}}\simeq_{\rho\to\rho_0}-\frac{\rho_0^{\frac{3}{2}}}{\rho}\frac{r(0)}{2\sqrt{\pi\gamma}}.$$

Now since $\mathcal{T}_\rho=\int_{-\delta}^{\delta}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}}+\int_{\delta}^{\frac{1}{2}}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}}+\int_{-\frac{1}{2}}^{-\delta}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}}$ and since the two last integrals are uniformly bounded on $[0,\rho_0]$, one gets the first equivalent.

2)$\varphi_\rho=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\rho}{x(s)^2}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}}$. Let $a<1<b$. There exists δ'_2 such that for all $s\in]-\delta'_2,\delta'_2[$,

$$\frac{ar(0)}{4\pi^2\rho_0^2}\leq\frac{r(s)}{x(s)^2}\leq\frac{br(0)}{4\pi^2\rho_0^2}.$$

Let $\delta' := \min(\delta_1, \delta_2')$. One has:

$$\begin{aligned}\frac{\rho}{4\pi^2\rho_0^2}\frac{ar(0)\rho_0}{\rho_0^2-\rho^2}\int_{-\delta}^{\delta}\frac{ds}{\sqrt{1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\beta s^2}{\rho_0}}} &\\ \leq \int_{-\delta'}^{\delta'}\frac{\rho}{x(s)^2}\frac{r(s)ds}{\sqrt{1-\frac{\rho^2}{4\pi^2x(s)^2}}} &\\ \leq \frac{\rho}{4\pi^2\rho_0^2}\frac{br(0)\rho_0}{\rho_0^2-\rho^2}\int_{-\delta}^{\delta}\frac{ds}{\sqrt{1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\alpha s^2}{\rho_0}}}. &\quad \text{(B.2)}\end{aligned}$$

The end of the calculus is similar to the previous one and one gets the second equivalent.

3)$\tau_\rho' = \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\rho}{4\pi x(s)^2}\frac{r(s)ds}{(1-\frac{\rho^2}{4\pi^2x(s)^2})^{\frac{3}{2}}}$. Let $a<1<b$. There exists δ_2'' such that for all $s\in]-\delta_2'',\delta_2''[$,

$$\frac{ar(0)}{16\pi^4\rho_0^2}\leq\frac{r(s)}{4\pi^2x(s)^2}\leq\frac{br(0)}{16\pi^4\rho_0^2}.$$

Let $\delta'' := \min(\delta_1, \delta_2'')$. One has:

$$\begin{aligned}\frac{ar(0)}{16\pi^4}\frac{\rho\rho_0}{(\rho_0^2-\rho^2)^{\frac{3}{2}}}\int_{-\delta''}^{\delta''}\frac{ds}{(1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\beta s^2}{\rho_0})^{\frac{3}{2}}} &\\ \leq \int_{-\delta''}^{\delta''}\frac{\rho}{4\pi^2x(s)^2}\frac{r(s)ds}{(1-\frac{\rho^2}{4\pi^2x(s)^2})^{\frac{3}{2}}} &\\ \leq \frac{br(0)}{16\pi^4}\frac{\rho\rho_0^2}{(\rho_0^2-\rho^2)^{\frac{3}{2}}}\int_{-\delta''}^{\delta''}\frac{ds}{(1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\alpha s^2}{\rho_0})^{\frac{3}{2}}}. &\quad \text{(B.3)}\end{aligned}$$

Using the same change of variables $u=\sqrt{\frac{k}{\rho_0}}\frac{1}{\zeta}s$, one gets if k stands for $4\pi\beta$ or $4\pi\alpha$:

$$\begin{aligned}\int_{-\delta''}^{\delta''}\frac{ds}{(1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\beta s^2}{\rho})^{\frac{3}{2}}} &= \zeta\sqrt{\frac{\rho_0}{k}}\int_{-\sqrt{\frac{k}{\rho_0}}\frac{\delta}{\zeta}}^{\sqrt{\frac{k}{\rho_0}}\frac{\delta}{\zeta}}\frac{du}{(1+u^2)^{\frac{3}{2}}}\\ &= 2\zeta\sqrt{\frac{\rho_0}{k}}\sqrt{\frac{k}{\rho_0}}\frac{\delta}{\zeta}\left(\sqrt{1+\frac{k}{\rho_0}\frac{\delta''^2}{\zeta^2}}\right)^{-1}\\ &= 2\zeta\sqrt{\frac{\rho_0}{k}}\left(\sqrt{1+\frac{\rho_0}{k}\frac{\zeta^2}{\delta''^2}}\right)^{-1}\\ &= \frac{\sqrt{\rho_0^2-\rho^2}}{\rho}\frac{\sqrt{\rho_0}}{k}\left(\sqrt{1+\frac{\rho_0}{k}\frac{\zeta^2}{\delta''^2}}\right)^{-1}.\end{aligned}$$

Hence if c stands for a or b:

$$\frac{cr(0)}{16\pi^4}\frac{\rho\rho_0}{(\rho_0^2-\rho^2)^{\frac{3}{2}}}\int_{-\delta''}^{\delta''}\frac{ds}{(1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\beta s^2}{\rho})^{\frac{3}{2}}} \quad = \quad \frac{cr(0)}{16\pi^4}\frac{1}{\sqrt{k\rho_0}}\frac{\rho_0^2}{\rho_0^2-\rho^2}\left(\sqrt{1+\frac{\rho_0}{k}\frac{\zeta^2}{\delta''^2}}\right)^{-1}.$$

Now since $\frac{\rho_0^2}{\rho_0^2-\rho^2} = \frac{1}{2}\left(\frac{\rho_0}{\rho_0-\rho} + \frac{\rho_0}{\rho_0+\rho}\right)$, one has the following equivalent:

$$\frac{cr(0)}{16\pi^4}\frac{\rho\rho_0}{(\rho_0^2-\rho^2)^{\frac{3}{2}}}\int_{-\delta''}^{\delta''}\frac{ds}{(1+\frac{\rho^2}{\rho_0^2-\rho^2}\frac{4\pi\beta s^2}{\rho})^{\frac{3}{2}}} \simeq_{\rho_0\to\rho} \frac{cr(0)}{16\pi^4}\frac{\sqrt{\rho_0}}{\sqrt{k}}\frac{1}{\rho_0-\rho}.$$

As before, since $a < 1 < b$ and $\alpha < \gamma < \beta$ are arbitrary, one gets

$$\int_{-\delta''}^{\delta''}\frac{\rho}{4\pi^2 x(s)^2}\frac{r(s)ds}{(1-\frac{\rho^2}{4\pi^2 x(s)^2})^{\frac{3}{2}}} \simeq_{\rho_0\to\rho} \frac{r(0)}{32\pi^4}\frac{\sqrt{\rho_0}}{\sqrt{\pi\gamma}}\frac{1}{\rho_0-\rho}.$$

To conclude, one juste has to observe that the two integrals

$$\int_{-\frac{1}{2}}^{-\delta''}\frac{\rho}{4\pi x(s)^2}\frac{r(s)ds}{(1-\frac{\rho^2}{4\pi^2 x(s)^2})^{\frac{3}{2}}} \quad\text{and}\quad \int_{\delta''}^{\frac{1}{2}}\frac{\rho}{4\pi x(s)^2}\frac{r(s)ds}{(1-\frac{\rho^2}{4\pi^2 x(s)^2})^{\frac{3}{2}}}$$

are bounded on $[0, \rho_0]$.

Appendix C

Proofs of proposition 4.2.13, lemma 4.2.6 and corollary 4.2.5

C.1 Proof of proposition 4.2.13

Lemma C.1. *Let $\mathcal{S}$ be a hypersurface of $\mathbb{R}^n$ that satisfies the hypotheses of proposition 4.2.13. The map F defined on $\mathcal{S}$ by $F : w \mapsto \frac{w}{||w||}$ is a diffeomorphism onto $\mathbb{S}^{n-1}$.*

Proof. We first show that F is a covering map. Denote by f the map defined on $\mathcal{S}$ by $f(w) = ||w||^{-1}$. Let $w_0 \in \mathcal{S}$. Then

$$\begin{array}{rccc} DF(w_0) : & T_{w_0}\mathcal{S} & \longrightarrow & T_{F(w_0)}\mathbb{S}^{n-1} \\ & v & \mapsto & f(w_0)v + D_{w_0}f(v)w_0 \end{array}$$

So $\ker(DF(w_0)) \subset \mathbb{R}w_0$ and by (T) $DF(w_0)$ is inversible. By the local inverse theorem F is a local diffeomorphism.
Let $\theta \in \mathbb{S}^{n-1}$. We denote by D_θ the half line $\mathbb{R}_+^*\theta$, so $F^{-1}(\theta) = D_\theta \cap \mathcal{S}$. Since $0 \in \overset{\circ}{K}$, there exists $\varepsilon > 0$ such that $\varepsilon\theta \in \overset{\circ}{K}$. On the other hand, since K is compact, there exists $R > 0$ such that $R\theta \notin K$. Then $D_\theta \cap \mathcal{S}$ is not empty and F is surjective.
Moreover (T) implies that $D_\theta \cap \mathcal{S}$ is discrete. Since $D_\theta \cap \mathcal{S}$ is closed by continuity of F and therefore compact, $D_\theta \cap \mathcal{S}$ is a finite set.
It suffices to show that the fibers $F^{-1}(\{\theta\})$ have the same cardinal. Let n be the map defined on $\mathbb{S}^{n-1}$ by $n(\theta) = \operatorname{Card} F^{-1}(\{\theta\})$. We will show that n is locally constant and the connexity of $\mathbb{S}^{n-1}$ will allows us to conclude.
Fix $\theta \in \mathbb{S}^{n-1}$ and denote by $\theta_1, ..., \theta_k$ its preimages under F. For $1 \leq i \leq k$, there exists an open neighborhood V_i of w_i in $\mathcal{S}$ such that $F_{|_{V_i}}$ is a diffeormorphism. The set $W = \mathcal{S} \setminus (\cup_i Vi)$ is compact and so is $F(W)$. Moreover, since $\theta \notin F(W)$, there exists an open neighborhood U of θ such that $U \cap F(W) = \emptyset$. Set $U' = U \cap \left(\cap_{i=1}^k F(V_i)\right)$. Then for all $\theta' \in U'$, $n(\theta') = n(\theta) = k$ and F is a covering map.
Denote by k the degree of the covering. Show that $k = 1$. For $\theta \in \mathbb{S}^{n-1}$, we order the preimages $\theta_1, \ldots, \theta_k$ of θ such that $||\theta_i|| \leq ||\theta_{i+1}||$ for all $i \in [\![1, k-1]\!]$. If $w \in \mathcal{S}$, there exists a unique $\theta \in \mathbb{S}^{n-1}$ and a unique $i \in [\![1, k]\!]$ such that $w = \theta_i$. It sufficies to show that the map $\mathcal{I}$ which with w associate the index i is locally constant. Indeed, assume this is down, then k is equal to the number of connected components of $\mathcal{S}$. Since $\mathcal{S}$ is connected $k = 1$ and F is a diffeormorphism.
Let $w \in \mathcal{S}$, and let $\theta = F(w)$. There exists an open neighborhood U of θ and pairwise disjoint open domains $U_1, ..., U_k$ of $\mathcal{S}$, such that for all $1 \leq i \leq k$, $F_i = F_{|_{U_i}}$ is a diffeomorphism onto U. Assume that for all i, $\theta_i = F_i^{-1}(\theta)$. Show that there exists a neighborhood

V of θ in $\mathbb{S}^{n-1}$ contained in U such that for all $\theta' \in V$, $F_i^{-1}(\theta') = \theta'_i$.
Let $\varepsilon = \min_{1\le i\le k-1}(||\theta_{i+1}|| - ||\theta_i||)$. For all i, there exists η_i such that for all $\theta' \in U$ that satisfies $||\theta' - \theta|| \le \eta_i$, one has $||F_i^{-1}(\theta') - F_i^{-1}(\theta)|| \le \frac{\varepsilon}{3}$. Let $\eta = \min_i \eta_i$ and set $V = B(\theta,\eta) \cap \mathbb{S}^{n-1}$. Then for all i

$$||F_{i+1}(\theta')|| \ge ||F_{i+1}(\theta)|| - \frac{\varepsilon}{3} \quad \text{and} \quad ||F_i(\theta)|| \ge ||F_i(\theta')|| - \frac{\varepsilon}{3}$$

Hence

$$||F_{i+1}(\theta')|| - ||F_i(\theta')|| \ge ||F_{i+1}(\theta)|| - ||F_i(\theta)|| - \frac{2\varepsilon}{3} = ||\theta_{i+1}|| - ||\theta_i|| - 2\frac{\varepsilon}{3} \ge \frac{\varepsilon}{3}.$$

One deduces immediately that for all $\theta' \in V$, $F_i(\theta') = \theta'_i$. Then, if $\mathcal{I}(w) = i$, $w \in U_i$ and for all $w' \in U_i$, $\mathcal{I}(w') = i$, that is $\mathcal{I}$ is locally constant. $\square$

Proof of proposition 4.2.13. Obviously, Ψ is smooth. The surjectivity of Ψ is proved with the same connexity argument as the one for the surjectivity of F. The injectivity is a consequence of the previous lemma. $\square$

C.2 Proofs of lemma 4.2.6 and corollary 4.2.5

Proof of lemma 4.2.6. We denote respectively by $||\cdot||$ and by $||\cdot||_\infty$ the Euclidean norm and Max-norm in $\mathbb{R}^n$ and by d_∞ the distance associated with the Max-norm. For $t > 0$, the balls $B(\cdot, t)$ will refer to the balls with respect to the Euclidean distance. We introduce the following sets

$$K := \Psi(\mathcal{S} \times [0,1]), \quad K_T := \Psi(\mathcal{S} \times [0,T]), \quad \mathcal{S}_T := \Psi(\mathcal{S} \times \{T\}) = \partial K_T, \quad T > 0.$$

For $x \in \mathbb{R}^n$, we introduce the hypercube

$$C_x = \{y \in \mathbb{R}^n \mid ||y - x||_\infty \le \tfrac{1}{2}\}.$$

Then for all $T > 0$,

$$\bigcup_{x\in K_T} C_x = K_T \cup \left(\bigcup_{x\in\mathcal{S}_T} C_x \right).$$

Indeed, assume that $y \notin K_T \cup \left(\bigcup_{x\in\mathcal{S}_T} C_x \right)$. Then $d_\infty(y, \mathcal{S}_T) \ge \frac{1}{2}$. So for any $x \in K_T$, the segment $s_{x,y}$ with endpoints x and y meets $\mathcal{S}_T$ at least once. Let z be such an intersection point. One has $d_\infty(x,y) \ge d_\infty(x,y) \ge \frac{1}{2}$, that is $y \notin C_x$ and the first inclusion is proved. The second is obvious. Therefore

$$K_T \setminus \left(\bigcup_{x\in\mathcal{S}_T} C_x \right) \subset \bigcup_{k\in K_T\cap\mathbb{Z}^n} C_k \subset K_T \cup \left(\bigcup_{x\in\mathcal{S}_T} C_x \right).$$

Fix $T > 0$ and $w_0 \in \mathcal{S}$. By proposition 4.2.13, there exists $t \in \mathbb{R}$ and $w_1 \in \mathcal{S}$ such that if $y \in C_{Tw}$, $y = tw_1$. Then $tw' \in B(Tw, \sqrt{n})$, that is,

$$T||w_0|| - \sqrt{n} \le t||w_1|| \le T||w_0|| + \sqrt{n}.$$

Hence,

$$T\frac{||w_0||}{||w_1||} - \frac{\sqrt{n}}{||w_1||} \le t \le T\frac{||w_0||}{||w_1||} + \frac{\sqrt{n}}{||w_1||}.$$

Thus

$$T\frac{||w_0||}{||w_1||}-\frac{\sqrt{n}}{w_{\min}}\leq t\leq T\frac{||w_0||}{||w_1||}+\frac{\sqrt{n}}{w_{\min}}. \tag{C.1}$$

where $w_{\min}=\min_{w\in\mathcal{S}}||w||$.

Fix $\varepsilon>0$. Since $w\mapsto||w||$ is continuous on $\mathcal{S}$, there exists $\eta>0$ such that

$$||w-w'||\leq\eta\Rightarrow 1-\frac{\varepsilon}{2}\leq\frac{||w||}{||w'||}\leq 1+\frac{\varepsilon}{2}. \tag{C.2}$$

If F is the diffeomorphism defined in lemma C.1, there exists $\alpha>0$ such that

$$||F(w)-F(w')||_\infty\leq\alpha\Rightarrow||w-w'||\leq\eta. \tag{C.3}$$

Finally, there exists $\beta>0$ such that

$$||y-w||_\infty\leq\beta\Rightarrow||F(y)-F(w)||\leq\alpha. \tag{C.4}$$

Since C_{Tw} is the homothetic transform of the square $\{y\in\mathbb{R}^n\mid||y-w_0||_\infty\leq\frac{1}{2T}\}$, one has $||\frac{t'w'}{T}-w||_\infty\leq\frac{1}{2T}$. Now $F(w')=F(\frac{tw'}{T})$, so for any $T\geq(2\beta)^{-1}$:

$$T(1-\frac{\varepsilon}{2})-\frac{\sqrt{n}}{w_{\min}}\leq t\leq T(1+\frac{\varepsilon}{2})+\frac{\sqrt{n}}{w_{\min}}.$$

Now there exits t such that if $T>t$, $T\frac{\varepsilon}{2}\geq\frac{\sqrt{n}}{w_{\min}}$. Then if $T\geq\max(t,(2\beta)^{-1})$, one gets

$$K_{T(1-\varepsilon)}\subset\bigcup_{k\in K_T\cap\mathbb{Z}^n}C_k\subset K_{T(1+\varepsilon)}.$$

The inclusions above immediately yield

$$T^n(1-\varepsilon)^n\operatorname{Vol}_{\text{Leb}}\Psi(\mathcal{S}\times[0,1])\leq n(T)\leq T^n(1+\varepsilon)^n\operatorname{Vol}_{\text{Leb}}\Psi(\mathcal{S}\times[0,1]).$$

with $t_\varepsilon=\max(t,(2\beta)^{-1})$. □

Proof of corollary 4.2.5. As before, one has the following inclusions:

$$\mathscr{C}_{\mathcal{D},T}\setminus\left(\bigcup_{x\in\Psi(\partial\mathcal{D}\times[0,T])}C_x\right)\subset\bigcup_{k\in\mathscr{C}_{\mathcal{D},T}\cap\mathbb{Z}^n}C_k\subset\mathscr{C}_{\mathcal{D},T}\cup\left(\bigcup_{x\in\Psi(\partial\mathcal{D}\times[0,T])}C_x\right) \tag{C.5}$$

with

$$\bigcup_{x\in\Psi(\partial\mathcal{D}\times[0,T])}C_x=\left(\bigcup_{x\in\Psi(\partial\mathcal{D}\times\{T\})}C_x\right)\cup\left(\bigcup_{x\in\Psi(\partial\mathcal{D}\times[0,T[)}C_x\right)$$

Now

$$\operatorname{Vol}_{\text{Leb}}\bigcup_{x\in\Psi(\partial\mathcal{D}\times[0,T[)}C_x\leq T\operatorname{Vol}^{2n-2}_{\text{Leb}}\partial\mathcal{D},$$

where, for $p \leq n$, $\mathrm{Vol}^{p}_{\mathrm{Leb}}$ is the canonical p-volume of a submaniofd of $\mathbb{R}^n$ of dimension $p \leq n$. Let $\varepsilon > 0$ and let $\varepsilon' < \varepsilon$. With the same argument as for the previous lemma, there exists $t > 0$ such that if $T > t$,

$$\mathscr{C}_{\mathcal{D},T(1-\varepsilon')} \subset \mathscr{C}_{\mathcal{D},T} \cup \left(\bigcup_{x \in \partial\mathcal{D} \times \{T\}} C_x \right) \subset \mathscr{C}_{\mathcal{D},T(1+\varepsilon')}$$

Then

$$\mathrm{Vol}\,\mathscr{C}_{\mathcal{D},T(1-\varepsilon')} - T\,\mathrm{Vol}^{2n-2}_{\mathrm{Leb}}\,\partial\mathcal{D} \leq n_{\mathcal{D}}(T) \leq \mathrm{Vol}\,\mathscr{C}_{\mathcal{D},T(1+\varepsilon')} + T\,\mathrm{Vol}^{2n-2}_{\mathrm{Leb}}\,\partial\mathcal{D}$$

That is, using again the homogeneity of the Lebesque volume:

$$T^n(1-\varepsilon')^n\,\mathrm{Vol}\,\mathscr{C}_{\mathcal{D}} - T\,\mathrm{Vol}^{2n-2}_{\mathrm{Leb}}\,\partial\mathcal{D} \leq n_{\mathcal{D}}(T) \leq T^n(1+\varepsilon')^n\,\mathrm{Vol}\,\mathscr{C}_{\mathcal{D}} + T\,\mathrm{Vol}^{2n-2}_{\mathrm{Leb}}\,\partial\mathcal{D}$$

Now there exists t' such that if $T > t'$,

$$\frac{1}{T^{n-1}}\,\mathrm{Vol}^{2n-2}_{\mathrm{Leb}}\,\partial\mathcal{D} \leq \max((1-\varepsilon')^n - (1-\varepsilon)^n, (1+\varepsilon)^n - (1+\varepsilon')^n).$$

One can choose $t_\varepsilon = \max(t, t')$. □

Notation

Chapitre 3

Chapitre 4

Chapitre 5

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