ε-Pisot numbers in any real algebraic number field are relatively dense

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Abstract
An algebraic integer is called an ε-Pisot number (ε > 0) if its Galois conjugates have absolute value less than ε. Let K be any real algebraic number field. We prove that the subset of K consisting of ε-Pisot numbers which have the same degree as that of the field is relatively dense in the real line R. This has some applications to non-stationary products of random matrices involving Salem numbers.

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1. Introduction

We are dealing with special algebraic integers, including Pisot numbers and Salem numbers. For basic definitions and results of algebraic numbers we suggest the book [4] and for a survey on Pisot numbers and Salem numbers we suggest the book [6].

Definition 1.1. Let ε > 0 be given. An algebraic integer is called an ε-Pisot number if all its Galois conjugates have modulus less than ε. The usual Pisot numbers correspond to 1-Pisot numbers.
Definition 1.2. An algebraic integer is called a Salem number if all its Galois conjugates have modulus at most 1 and at least one conjugate lies on the unit circle.

Definition 1.3. A subset $E$ of real numbers is said to be relatively dense if there is a number $L$ such that for any $x \in \mathbb{R}$ we have $E \cap [x, x + L] \neq \emptyset$.

Our main result is the following.

Theorem 1.4. Let $K$ be a real algebraic number field. Then for any $\varepsilon > 0$ the set of $\varepsilon$-Pisot numbers in $K$ which have the same degree as that of the field $K$ is relatively dense in $\mathbb{R}$.

Recall that a real algebraic number field $K$ of degree $d$ is an extension of the field $\mathbb{Q}$ of rational numbers by $d$ independent real algebraic numbers. It is also an extension of $\mathbb{Q}$ by a single algebraic number of degree $d$.

A Salem number $\tau$ is always of even degree say $2d$. It has exactly one real Galois conjugate inside the unit circle and all other Galois conjugates are pairwise complex conjugates on the unit circle. We can write them as $\tau, 1/\tau, e^{ia_1}, e^{-ia_1}, \ldots, e^{ia_{d-1}}, e^{-ia_{d-1}}$ where $a_i \in \mathbb{R}$ for $i = 1, \ldots, d - 1$.

Corollary 1.5. Let $\tau$ be a Salem number then for any $\varepsilon > 0$ the set of $\lambda \in \mathbb{R}$ such that $\|\lambda \tau^n\| < \varepsilon$ for all $n \in \mathbb{N}$ is relatively dense in $\mathbb{R}$, where $\|x\|$ denotes the distance to the rational integers of the real number $x$.

Let $\tau > 1$ be a Salem number and let $M : \mathbb{R} \to \text{GL}(\mathbb{R}^d)$ be a 1-periodic matrix-valued Lipschitz function. Consider the matrix products

$$P_n(x) = M(\tau^{n-1} x) \cdots M(\tau x)M(x).$$

Corollary 1.6. Suppose that the entries of $M$ are either identically zero or strictly positive. Then the following limit exists for almost all point $x \in \mathbb{R}$ with respect to the Lebesgue measure and is independent of $x$

$$\lim_{n \to \infty} \frac{1}{n} \log \|P_n(x)\|$$

where $\|A\|$ denotes the norm of a matrix $A$.

Corollary 1.6 is a consequence of Corollary 1.5 and the following Kingman type theorem proved in [1].

Theorem 1.7. Let $(u_n)_{n \geq 0}$ be a totally Bohr ergodic sequence of real numbers and $(f_n)_{n \geq 0}$ be a sequence of uniformly almost periodic functions. Suppose

(i) The sequence $(n^{-1} f_n)$ has joint periods.
(ii) The following subadditivity is fulfilled

\[ f_{n+m}(x) \leq f_n(x) + f_m(u_nx) \quad \text{for a.e. } x \text{ and all } n, m. \]

(iii) For any \( m \geq 1 \) there exists \( g_m(x) \in L^\infty(\mathbb{R}) \) such that

\[ f_n(u_mx) \leq f_n(x) + g_m(x) \quad \text{for a.e. } x \text{ and all } n. \]

Then the following limit exists

\[ \lim_{n \to \infty} \frac{1}{n} f_n(x) = \inf_n \frac{1}{n} \mathbb{M} f_n \quad \text{for a.e. } x. \]

We will explain in the proof the meaning of joint periods. Here we just recall that the sequence \((\beta^k)\) with \( \beta > 1 \) is totally Bohr ergodic and that \( \mathbb{M} f \) denotes the Bohr mean of a uniformly almost periodic function \( f \).

If \( \tau \) is an integer, Corollary 1.6 is well known as a special case of a theorem of Furstenberg–Kesten [2]. If \( \tau \) is a Pisot number, Corollary 1.6 is obtained in [1].

To some extent, the result for Salem numbers in Corollary 1.5 is optimal in the sense that \( \varepsilon \) cannot be replaced by a sequence \( \varepsilon_n \) tending to zero, even for a single \( \lambda \). In fact, otherwise \( \lambda \tau^n \) has zero as limit (on the torus). However \( \lambda \tau^n \) is dense in some interval, so the limit doesn’t exist (see [6, p. 33]). Of course, for a Pisot number \( \theta \), \( \|\lambda \theta^n\| \) is exponentially small as a function of \( n \), for a fixed \( \lambda \) in the field of \( \theta \). The novelty of Corollary 1.5, even for Pisot numbers, is the uniform estimate (uniform for a relatively dense set of \( \lambda \)). Actually, the proof of Corollary 1.5 will show that for any \( \varepsilon > 0 \), the set of \( \lambda \) such that \( \|\lambda \theta^n\| \leq \varepsilon r^n \) (\( \forall n \geq 1 \)) is relatively dense, where \( 0 < r < 1 \).

2. Proofs

Proof of Theorem 1.4. Assume that the degree of \( K \) is \( d \). Thus the field \( K \) admits a basis of algebraic integers

\[ \omega_1, \ldots, \omega_d. \]

We denote their Galois conjugates by (the \( i \)th column corresponding to the conjugates of \( \omega_i \))

\[
\begin{array}{cccc}
\omega_1^{(1)}, & \ldots, & \omega_d^{(1)} \\
\omega_1^{(2)}, & \ldots, & \omega_d^{(2)} \\
\vdots & \ddots & \vdots \\
\omega_1^{(d-1)}, & \ldots, & \omega_d^{(d-1)}
\end{array}
\]

Consider the following system of inequalities whose unknown are rational integers \((m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d\).
\[
|m_1 \omega^{(1)} + \cdots + m_d \omega_d^{(1)}| < \varepsilon,
\]
\[
|m_1 \omega^{(2)} + \cdots + m_d \omega_d^{(2)}| < \varepsilon,
\]
\[
\vdots
\]
\[
|m_1 \omega^{(d-1)} + \cdots + m_d \omega_d^{(d-1)}| < \varepsilon.
\]

To each solution \((m_1, m_2, \ldots, m_d)\), if any, of the system we associate the algebraic integer

\[
\lambda(m_1, m_2, \ldots, m_d) = m_1 \omega^{(1)} + \cdots + m_d \omega_d^{(d)}.
\]

The number \(\lambda(m_1, m_2, \ldots, m_d)\) admits its Galois conjugates

\[
m_1 \omega^{(i)} + \cdots + m_d \omega_d^{(i)},
\]

\(i = 1, \ldots, d - 1\). So, it is an \(\varepsilon\)-Pisot number. We will conclude the proof by showing that

the set of all numbers \(\lambda(m_1, m_2, \ldots, m_d)\) is relatively dense in \(\mathbb{R}\).

Indeed, the set of points \((x_1, \ldots, x_d) \in \mathbb{R}^d\) fulfilling the inequalities

\[
|x_1 \omega^{(1)} + \cdots + x_d \omega_d^{(1)}| < \varepsilon,
\]
\[
|x_1 \omega^{(2)} + \cdots + x_d \omega_d^{(2)}| < \varepsilon,
\]
\[
\vdots
\]
\[
|x_1 \omega^{(d-1)} + \cdots + x_d \omega_d^{(d-1)}| < \varepsilon
\]

is a thin “tube” around the line \(l\) parameterized as \(\{(\alpha_1 t, \ldots, \alpha_d t): t \in \mathbb{R}\}\) whose direction \((\alpha_1, \alpha_2, \ldots, \alpha_d)\) is given by

\[
\alpha_1 \omega^{(i)} + \cdots + \alpha_d \omega_d^{(i)} = 0 \quad (1 \leq i \leq d - 1).
\]

Now the question is how to show that there are \((m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d\) arbitrarily close to the line \(l\) such that \(\lambda(m_1, m_2, \ldots, m_d)\) are relatively dense in the tube.

To this end, we consider the translation \(T: \mathbb{T}^d \to \mathbb{T}^d\) defined by

\[
T(t_1, t_2, \ldots, t_d) = (t_1 + \alpha_1, t_2 + \alpha_2, \ldots, t_d + \alpha_d).
\]

The question will be reformulated in the following way: how to find a relatively dense set of \(n\) such that

\[
T^n \mathbb{Z}^d \cap U_{\varepsilon'}(0)
\]

for any pre-described \(\varepsilon'\), where \(U_{\varepsilon'}(0)\) is the \(\varepsilon'\)-ball of \(\mathbb{T}^d\) centered at 0.

To prove this, we will use a basic fact about translations of the \(d\)-dimensional torus (actually a fact about group translations with respect to the Haar measure).

**Lemma 2.1** (see [3, Proposition 4.3.3]). Let \(T\) be a translation of \(\mathbb{T}^d\) and \(x \in \mathbb{T}^d\). Then the restricted action of \(T\) on the orbit closure \(\overline{T^n x : n \in \mathbb{Z}}\) is uniquely ergodic.
Since the unique ergodicity implies the uniform convergence of the Birkhoff means of continuous functions (see [3, Theorem 4.3.1]), we have for any elementary basic set $B$ (in particular $\epsilon'$-neighborhoods) that the “hitting sequence” \( \{n \in \mathbb{Z} : T^n x \in B \} \) is relatively dense in $\mathbb{Z}$ and hence in $\mathbb{R}$ for an uniquely ergodic action $T$. We may also refer to [5].

Apply this to our case by choosing $B = U_{\epsilon'}(0)$ with
\[
\epsilon' \max_{1 \leq i < d} \sum_{j=1}^{d} |\omega^{(i)}_j| < \epsilon.
\]

Let $Z(\epsilon')$ be the set of $n \in \mathbb{Z}$ having the property that there exists $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$ such that
\[
|\eta_j| < \epsilon', \quad \text{with } \eta_j := n\alpha_j - m_j \ (j = 1, 2, \ldots, d). \tag{2}
\]

On one hand, by (1) and (2), for any $1 \leq i \leq d-1$ we have
\[
\left| \sum_{j=1}^{d} m_j \omega^{(i)}_j \right| = \left| n \sum_{j=1}^{d} \alpha_j \omega^{(i)}_j - \sum_{j=1}^{d} \eta_j \omega^{(i)}_j \right| \leq \epsilon' \sum_{j=1}^{d} |\omega^{(i)}_j| < \epsilon,
\]
and on the other hand we have
\[
\lambda(m_1, m_2, \ldots, m_d) = n \sum_{j=1}^{d} \alpha_j \omega_j + O(\epsilon). \tag{3}
\]

By Lemma 2.1, $Z(\epsilon')$ is relatively dense. Let $a = \sum_{j=1}^{d} \alpha_j \omega_j$. Since $a \neq 0$, the set $a Z(\epsilon')$ is relatively dense and so is the set of the all $\lambda(m_1, m_2, \ldots, m_d)$ because of (3). \(\square\)

**Proof of Corollary 1.5.** Let $\tau$ be a Salem number of degree $2d$ and $K$ its algebraic number field. For fixed $\epsilon > 0$, the set of $\epsilon$-Pisot numbers in $K$ of degree $2d$ is relatively dense in $\mathbb{R}$.

For such an $\epsilon$-Pisot number $\beta$ with its conjugates $\beta^{(j)}$ ($1 \leq j \leq 2d - 1$) we have
\[
\beta \tau^n + \frac{\beta^{(1)}}{\tau^n} + \beta^{(2)} e^{ia_1} + \beta^{(3)} e^{-ia_1} + \cdots + \beta^{(2d-2)} e^{ia_{d-1}} + \beta^{(2d-1)} e^{-ia_{d-1}} \in \mathbb{Z}.
\]

Therefore
\[
\|\beta \tau^n\| < \epsilon(2d - 1) \quad \text{for all } n \in \mathbb{N}.
\]

This proves the corollary. \(\square\)

**Proof of Corollary 1.6.** For a non-negative matrix $A$ and a fixed positive vector $v$, the matrix norm $\|A\|$ has the same size as the vector norm $|Av|$ (one is bounded by the other
up to a multiplicative constant. We will use the norm $|v| = \sum_{j=1}^{d} |v_j|$ for $v = (v_1, \ldots, v_d)$.
So, it suffices to show the limit of $n^{-1} f_n(x)$ is a constant where

$$f_n(x) = \log |M(\tau^{n-1} x) \cdots M(\tau x) M(x) v|.$$  

By Theorem 1.7 (Theorem 2.5 in [1]), we have only to show that for any $\varepsilon > 0$ the set of $\lambda \in \mathbb{R}$ such that

$$n^{-1} \left| f_n(x + \lambda) - f_n(x) \right| \leq \varepsilon \quad (\forall x \in \mathbb{R}, \forall n \geq 1) \quad (4)$$

is relatively dense (such a $\lambda$ is called an $\varepsilon$-period of $n^{-1} f_n$ in [1]). Since $M(x)$ is Lipschitzian, so is $|M(x)v|$. Then, by Corollary 1.5, there is a relatively dense set of $\lambda$ such that

$$\left| M(x + \lambda x^k) v - M(x) v \right| \leq C \varepsilon \quad (5)$$

for some constant $C$, all $x \in \mathbb{R}$ and all $k \geq 1$. On the other hand we have the following distortion lemma, which is easy to prove (see [1])

$$\log \frac{|M(x_1) M(x_2) \cdots M(x_n) v|}{|M(y_1) M(y_2) \cdots M(y_n) v|} \leq \left( \delta^{-1} \sum_{k=1}^{n} |x_k - y_k| \right) \quad (6)$$

where $\delta$ is the minimal value of those strictly positive entries of $M$. Both (5) and (6) together give

$$\left| f_n(x + \lambda) - f_n(x) \right| \leq C \delta^{-1} n \varepsilon.$$ 

This is (4) with $\varepsilon$ replaced by $C \delta^{-1} \varepsilon$. Recall that $\varepsilon > 0$ is arbitrary.  

References