The Lawrence-Krammer-Bigelow representation detects the dual braid length

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Joint work with Tetsuya Ito (UBC, Vancouver)

1. Classical and dual Garside structure on braid groups
2. Labellings of curve diagrams
3. The Lawrence-Krammer-Bigelow representation
4. The LKB representation detects dual braid length

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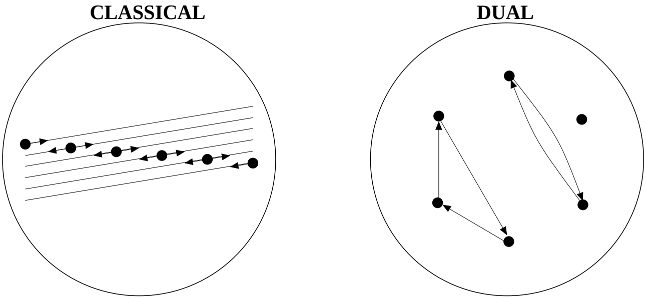
2. Labellings of curve diagrams
3. The Lawrence-Krammer-Bigelow representation
4. The LKB representation detects dual braid length

There are two Garside structures on the braid group B_n

- **Classical** Punctures lined up horizontally. $\Delta = $ half-twist. Divisors of $\Delta \leftrightarrow$ permutations of the n punctures
- **Dual** Punctures on a circle. $\delta = \frac{2\pi}{n}$ turn. Divisors of $\delta \leftrightarrow$ disjoint, non-nested polygons, (possibly degenerate, i.e. having only two vertices)



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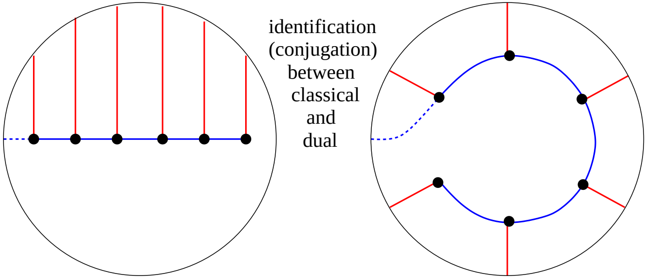
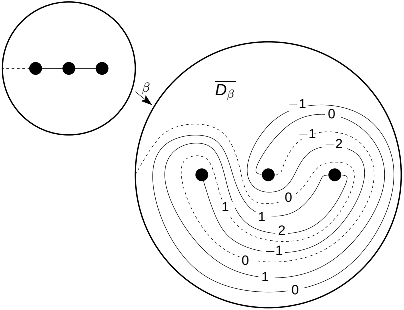


For any braid $\beta \in B_n$, consider its *curve diagram* $\overline{D_\beta}$ with

<u>Winding number labeling (WNu)</u>



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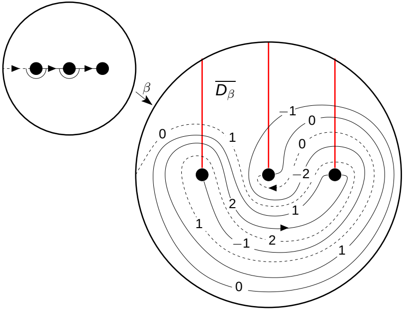

Look at the maximal and minimal labels of the <u>solid</u> arcs.

Thm 1 [W, 2010]

$Min_{WNu}(\beta) = \inf_{Class.Garside}(\beta)$

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Thm 2 [Ito & W, 2011]

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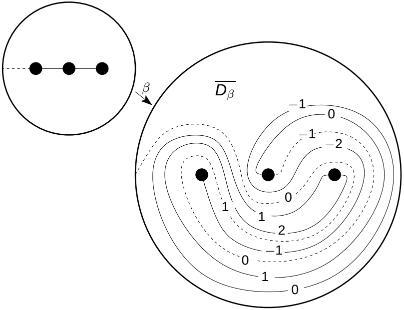

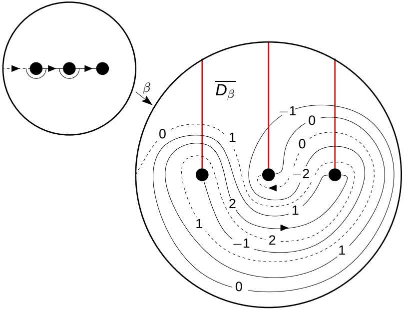

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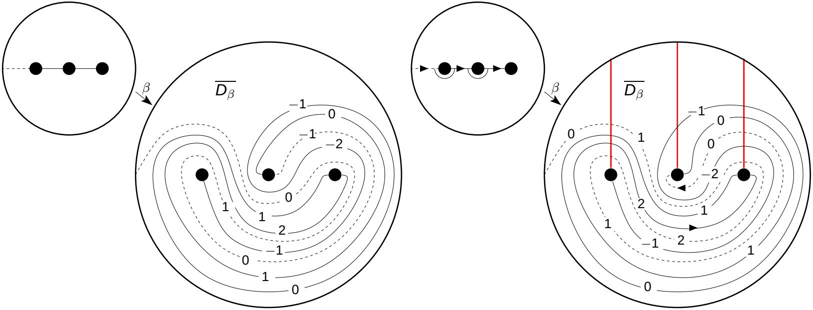

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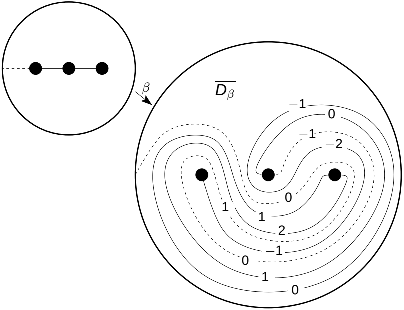

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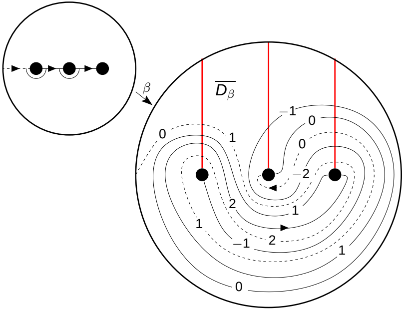

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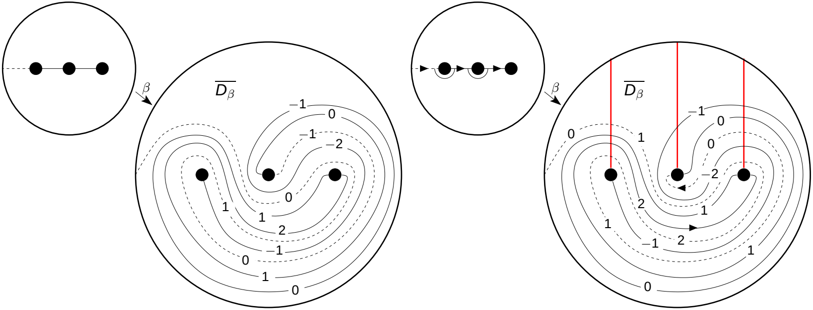
Check the theorems in this special case :



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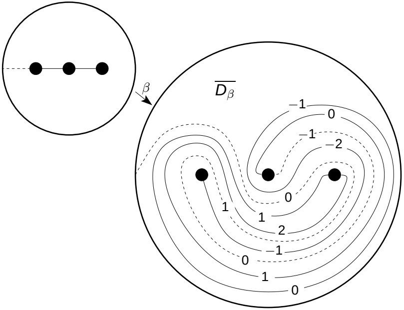

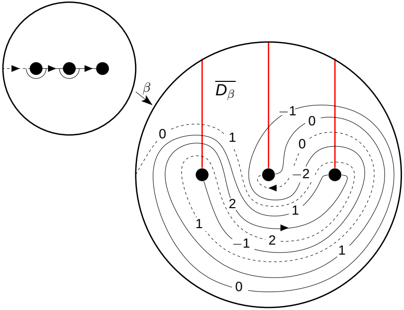

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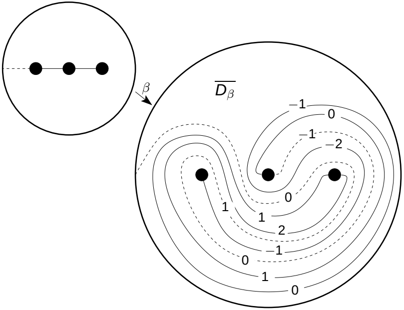

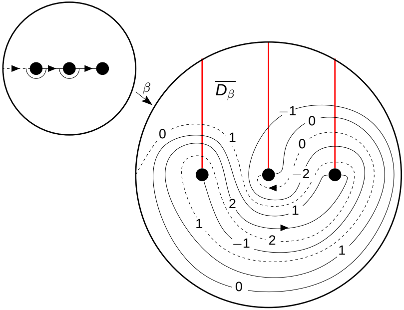

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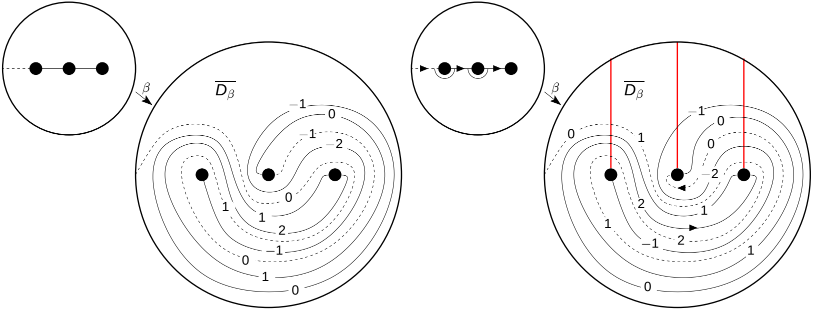

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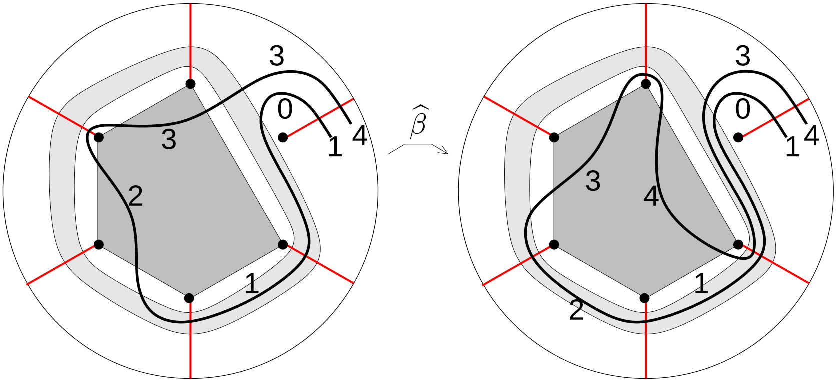
Proof of Theorem 2 (beginning)

Lemma Let $\beta \in B_n$, and $\widehat{\beta}$ a divisor of δ. Action of $\widehat{\beta}$ on D_β : an arc in D_β labelled k gives rise to one or several arcs in $D_{\widehat{\beta}\cdot\beta}$, labelled k or $k+1$.



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For simplicity, suppose $Min_{WCr}(\beta) = 0$.

Need to prove : $Max_{WCr}(\beta) = \sup_{Dual}(\beta)$

Proof of "$\leqslant$" : follows from Lemma (acting by a divisor of δ can only increase maximal label by 1).

Proof of "$\geqslant$" : by induction on $Max_{WCr}(\beta)$.

1. Construct a collection P of disjoint polygons intersecting all maximally labelled arcs, but none of the minimally labelled ones. Then let $\widehat{\beta}$ be the divisor of δ corresponding to P.
2. Prove that acting on D_β by $\widehat{\beta}^{-1}$ decreases the maximal label without decreasing the minimal label :
   - $Max_{WCr}(\widehat{\beta}^{-1}\beta) = Max_{WCr}(\beta) - 1$
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Historical reminder

Question : Is B_n linear, i.e. the subgroup of a matrix group ?

The *Burau* representation is *not* faithful for $n \geqslant 5$ [Bigelow, Long, Moody], it *is* faithful for $n = 2, 3$, and for $n = 4$ the question is open.

The first representation for which faithfulness for all n was proven was the representation of Ruth Lawrence

$$B_n \xrightarrow{\mathcal{L}} GL\left(\mathbb{R}\left[q^{\pm 1}, t^{\pm 1}\right], \frac{n(n-1)}{2}\right)$$

(Actually, the matrix coefficients happen to lie in $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$.)

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Explicit formula for the representation $\mathcal{L}$

$$B_n \xrightarrow{\mathcal{L}} GL\left(\mathbb{R}\left[q^{\pm 1}, t^{\pm 1}\right], \frac{n(n-1)}{2}\right)$$

Denote the basis vectors of $\mathbb{R}^{\frac{n(n-1)}{2}}$ by $F_{i,j}$ (for $1 \leqslant i < j \leqslant n$).
Then $\mathcal{L}(\sigma_k)$ sends

$$F_{i,j} \mapsto \begin{cases} F_{i,j} & k \notin \{i-1, i, j-1, j\} \\ qF_{k,j} + (q^2 - q)F_{k,i} + (1-q)F_{i,j} & k = i-1 \\ F_{i+1,j} & k = i \neq j-1 \\ qF_{i,k} + (1-q)F_{i,j} + (q-q^2)tF_{k,j} & k = j-1 \neq i \\ F_{i,j+1} & k = j \\ -q^2tF_{i,j} & k = i = j-1 \end{cases}$$

Krammer's proof that $\mathcal{L}$ is faithful

For any $\beta \in B_n$, consider the maximal and minimal powers of t occurring in the matrix $\mathcal{L}(\beta)$.

Krammer's main lemma (which implies faithfulness) :

$$\text{maximal power of } t \text{ in } \mathcal{L}(\beta) = \sup_{ClassicalGarside}(\beta)$$

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Theorem (Ito,W) Krammer's conjecture is true.

We unsuccessfully tried to also reprove Krammer's result using our techniques.

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Bigelow's proof that $\mathcal{L}$ is faithful

Let X be the configuration space of unordered pairs of points in the n-times punctured disk D_n :

$$X = \Big\{ \{x_1, x_2\} \mid x_1, x_2 \in D_n \ , \ x_1 \neq x_2 \Big\}$$

equipped with a basepoint $\{d_1, d_2\}$ (see blackboard).
There is a homomorphism

$$\pi_1(X) \to \mathbb{Z}^2 = \langle q, t \rangle, \ \ \gamma = \{\gamma_1, \gamma_2\} \mapsto q^a t^b$$

where

$a =$ sum of the winding numbers of γ_1 and γ_2 around all n punctures

$b = 2 \cdot$ (relative winding number of γ_1 and γ_2)

Let $\widetilde{X}$ be the cover corresponding to $\ker\left(\pi_1(X) \to \mathbb{Z}^2\right)$.
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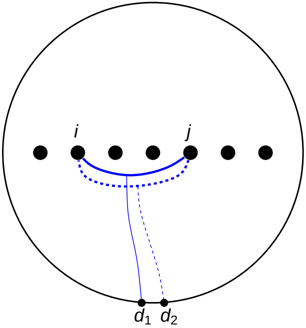
Note : the B_n-action on X by homeos lifts to a B_n-action on $\widetilde{X}$. Hence B_n acts on the second homology $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$ by automorphisms.

Proposition (Bigelow) The second homology

$$H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}]) \text{ is of dimension } \frac{n(n-1)}{2}$$

with generators $F_{i,j}$ ("forks") as in the following figure.



Moreover, the B_n-action on $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$ coincides with the representation $\mathcal{L}$.

Bigelow's Key Lemma

- Recall $\widetilde{X}$ is a 4-dim. mfd. with covering action by $\mathbb{Z}^2 = \langle q, t \rangle$.
- Recall the "fork" $F_{i,j}$ is a surface (more precisely a square) in $\widetilde{X}$ representing a generator of $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$.
- Let $\widetilde{X}_{0,0}$ be a fund. domain of the $\mathbb{Z}^2$-action containing the basept. $\{\widetilde{d}_1, \widetilde{d}_2\}$ and all forks $F_{i,j}$. Let $\widetilde{X}_{a,b} = q^a t^b . X_{0,0}$.

Handwaving version of Bigelow's "Key Lemma" Let $\beta \in B_n$ and $1 \leqslant i < j \leqslant n$, and consider $\beta(F_{i,j}) \subset \widetilde{X}$. Among the fundamental domains $\widetilde{X}_{a,b}$ intersected by $\beta(F_{i,j}) \subset \widetilde{X}$, select the one $\widetilde{X}_{a_{\max}, b_{\max}}$ with maximal (a, b) (lexicographically). Now in $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$ we can write uniquely

$$\beta(F_{i,j}) = \sum_{1 \leqslant i' < j' \leqslant n} P_{i',j'}(q,t) F_{i',j'} \quad (\text{with } P_{i',j'} \in \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$$

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The LKB representation detects $\sup_{DualGarside}(\beta)$

Proof that $2 \cdot \sup_{DualGarside}(\beta) =$ **maximal power of q in $\mathcal{L}(\beta)$**

Proof of "$\geqslant$" is easy ($\mathcal{L}$: divisors of $\delta \mapsto$ matrix of q-degree 2).

Proof of "$\leqslant$" :

$2 \cdot \sup_{Dual}(\beta) \overset{\text{Thm } 2}{=} 2 \cdot Max_{WCr}(\beta)$, the max. wall crossing labeling

$\overset{!!!}{=}$ the maximal number a such that $\beta.F_{i,i+1}$ intersects $q^a t^b.X_{0,0}$ for some i, b

Now according to Bigelow's Key Lemma, the monomial $q^a t^b$ occurs somewhere in the matrix $\mathcal{L}(\beta)$. □

Questions

1. Recall Theorem 2 : for a braid β, we can read the length of β from the wall crossing labellings occurring in the curve diagram of β.

   Question : is there some analogue for $Out(F_n)$, with the sphere system in $(S^1 \times S^2)\# \ldots \#(S^1 \times S^2)$ playing the role of the curve diagram ?

2. Is there a generalization of our main theorem to Artin-Tits groups of finite type ? The question makes sense, as
   - there are dual Garside structures on such groups (Bessis, T.Brady-Watt)
   - there are Lawrence-Krammer-Bigelow type representations on such groups (Digne, Cohen-Wales).

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1. Classical and dual Garside structure on braid groups
2. Labellings of curve diagrams
3. The Lawrence-Krammer-Bigelow representation
4. The LKB representation detects dual braid length