Garside structures of braid groups

Sang-Jin Lee

Garside Theory

May 31, 2012

Plan

① Brief survey

- Garside's approach to Artin braid groups
- Garside groups
- Dehornoy's criterion for Garside structure

② Braid groups of complex reflection groups

- Garside structure of $B(e,e,r)$
- quasi-Garside structure of $B(de,e,r)$

(Jointwork with Eon-Kyung LEE
Same quasi-Garside structure as Ruth Corran's)

◎ Artin braid group

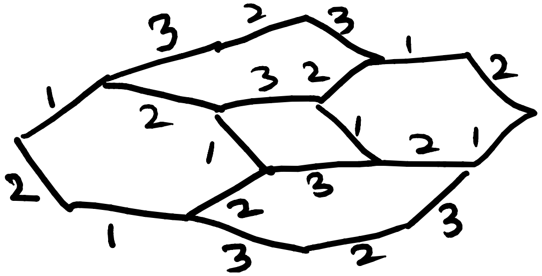
$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \end{array} \right\rangle$$

$B_n^+ = \langle \sigma_1, \cdots, \sigma_{n-1} \mid \quad " \quad \rangle_{\text{monoid}}$: positive braid monoid

$\equiv_+$: positive equivalence

e.g.) $\sigma_1\underline{\sigma_3\sigma_2\sigma_3}\sigma_1\sigma_2 \equiv_+ \sigma_1\sigma_2\sigma_3\underline{\sigma_2\sigma_1\sigma_2} \equiv_+ \sigma_1\sigma_2\underline{\sigma_3\sigma_1}\sigma_2\sigma_1$

$\equiv_+ \underline{\sigma_1\sigma_2\sigma_1}\sigma_3\sigma_2\sigma_1 \equiv_+ \sigma_2\sigma_1\underline{\sigma_2\sigma_3\sigma_2}\sigma_1 \equiv_+ \sigma_2\sigma_1\sigma_3\sigma_2\sigma_3\sigma_1$



van Kampen diagram

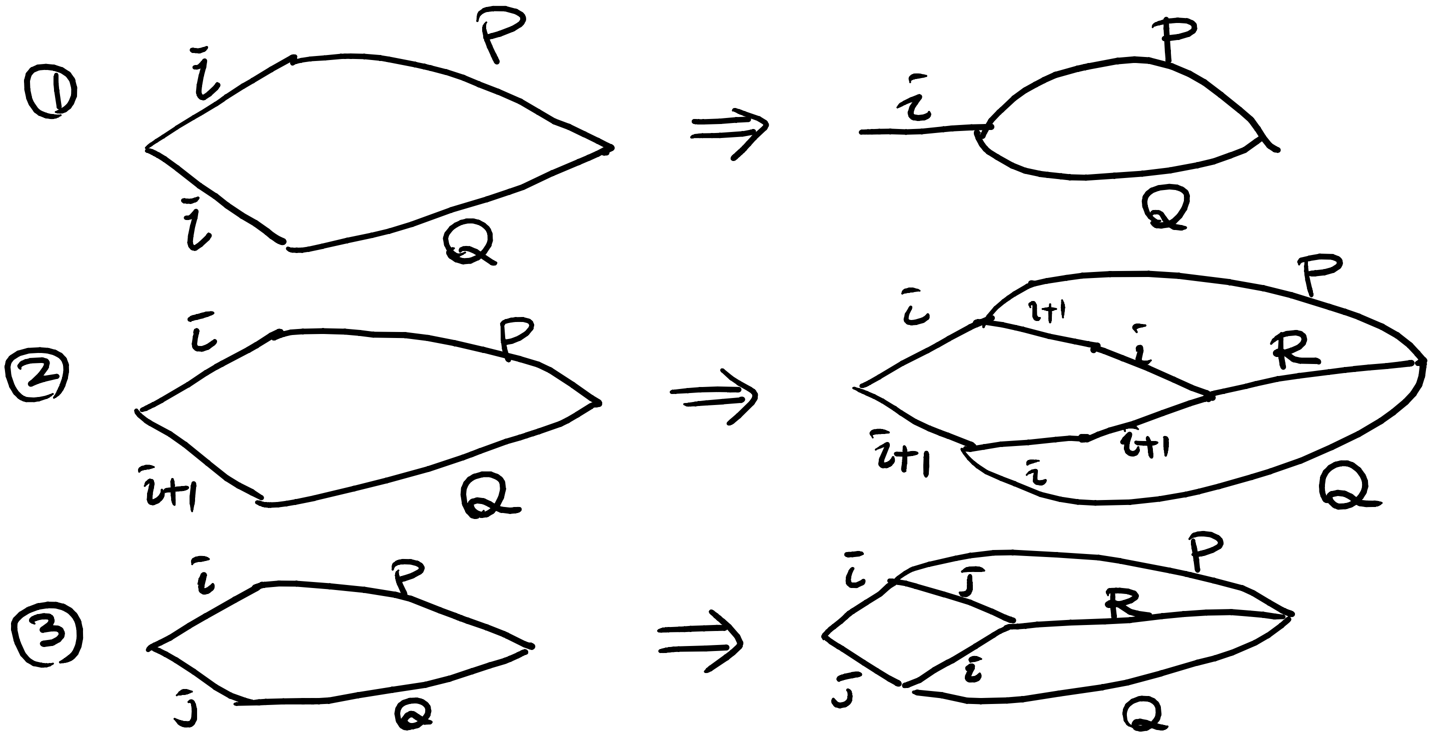
Thm (Garside '69)

Let $\sigma_i P \equiv_+ \sigma_j Q$ for $P, Q \in B_n^+$. Then

① if $i=j$, $P \equiv_+ Q$

② if $|i-j|=1$, $P \equiv_+ \sigma_j \sigma_i R$ and $Q \equiv_+ \sigma_i \sigma_j R$ for some $R \in B_n^+$

③ if $|i-j| \geq 2$, $P \equiv_+ \sigma_j R$ and $Q \equiv_+ \sigma_i R$ for some $R \in B_n^+$



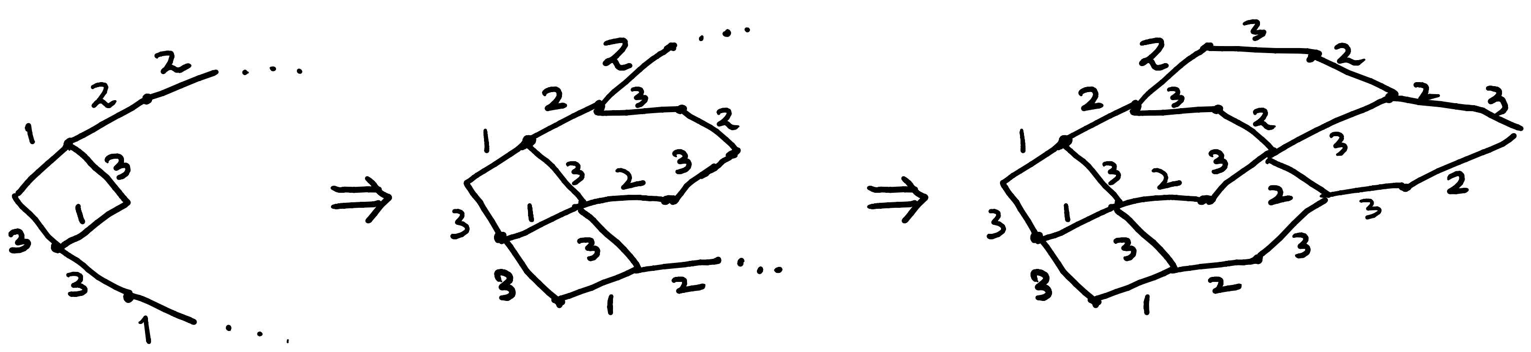
⓪ Some consequences

① B_n^+ is left-cancellative: $PQ \equiv_+ PR \Rightarrow Q \equiv_+ R$

② Word problem in B_n^+ is solvable.

- construct van Kampen diagram

e.g.) $1223223 \overset{?}{\equiv}_+ 3312332$



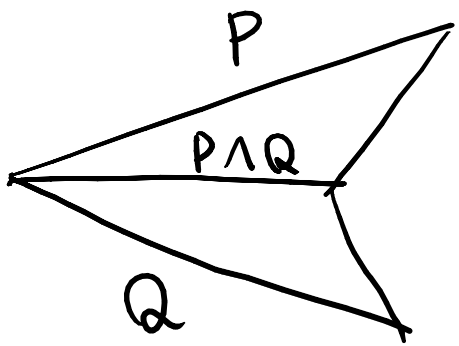
③ $(B_n^+, \preccurlyeq)$ is a lattice. (not a direct consequence)

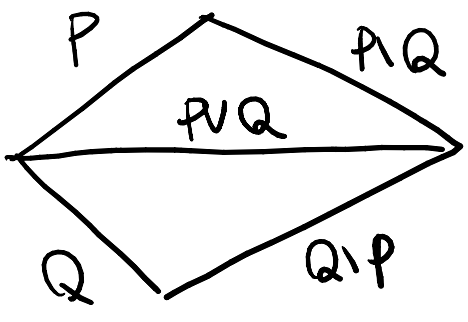
$P \preccurlyeq Q$ if $Q \equiv_+ PR$ for some $R \in B_n^+$.

(P is a left-divisor of Q
Q is a right multiple of P.)

$\preccurlyeq$ is a partial order on B_n^+ such that there exist left-gcd $P \wedge Q$ and right-lcm $P \vee Q$ for any $P, Q \in B_n^+$

$P \wedge Q \overset{\text{def}}{=}$ largest common left-divisor of P and Q
$P \vee Q \overset{\text{def}}{=}$ smallest common right-multiple of P and Q.





Garside monoid (Dehornoy - Paris 1999)

A Garside monoid is a pair (M, Δ), where

① M is a left- and right- cancellative monoid

② $\exists \lambda : M \to \mathbb{Z}_{\geq 0}$ s.t. $\begin{cases} \lambda(g_1 g_2) \geq \lambda(g_1) + \lambda(g_2) \\ \lambda(g) = 0 \iff g = 1 \end{cases}$

③ $(M, \leq)$ and $(M, \geq)$ are lattices

④ Δ is a Garside element :

- $\{$left-divisors of $\Delta\} = \{$right divisors of $\Delta\}$
- $\mathrm{Div}(\Delta) = \{$left-divisors of $\Delta\}$ generates M
- $|\mathrm{Div}(\Delta)| < \infty$

A Garside group is a group of fractions of a Garside monoid.

If we allow $|\mathrm{Div}(\Delta)| = \infty$, (M, Δ) is called a quasi-Garside monoid, or a Garside monoid of infinite type.

Properties

① Garside monoid embeds in its group of fractions

② torsion-free

③ solvable word problem : $\exists$ Normal Form

...

④ Solvable conjugacy problem

⑤ biautomatic

⑥ finite $K(\pi, 1)$

⑦ Strongly translation discrete.

Important Examples

① Artin braid group [Garside, Elrifai-Morton, Thurston, ...]

② Artin groups of finite type [Deligne, Brieskorn-Saito, Bessis, ...]

③ braid groups of some complex reflection groups.

[Bessis-Corran, Corran-Picantin, Bessis, ...]

(quasi-Garside groups)

④ Free groups [Bessis, ...]

⑤ Artin groups of type $\tilde{A}$ and $\tilde{C}$ [Digne]

Garside structures given by positive presentations

① G = free abelian group $\mathbb{Z}^n$

$$= \langle a_1, \cdots, a_n \mid a_i a_j = a_j a_i \quad \forall i,j \rangle$$

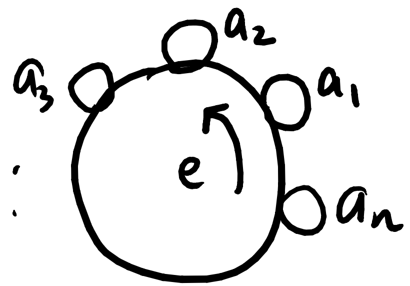
$$\Delta = a_1 a_2 \cdots a_n$$

② G = torus-like group

$$\langle a_1, \cdots, a_n \mid a_1^{p_1} = a_2^{p_2} = \cdots = a_n^{p_n} \rangle, \quad p_i \geq 1$$

$$\Delta = a_1^{p_1}$$

③ $G = \langle a_1, \cdots, a_n \mid \underbrace{a_1 a_2 \cdots a_e}_{e} = \underbrace{a_2 a_3 \cdots a_{e+1}}_{e} = \cdots \rangle$

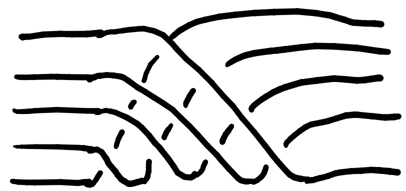


e.g.) $\langle a_1, a_2, a_3 \mid a_1 a_2 = a_2 a_3 = a_3 a_1 \rangle \cong B_3$

Artin braid group B_n

① classical Garside structure

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & |i-j| = 1 \end{array} \right\rangle \qquad \Delta =$$



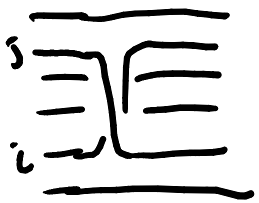
$$\Delta = \sigma_1 (\sigma_2 \sigma_1) \cdots (\sigma_{n-1} \sigma_{n-2} \cdots \sigma_1)$$

② dual Garside structure

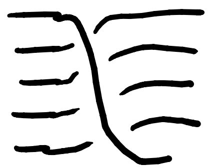
$$B_n = \left\langle a_{ij},\ 1 \leq i \neq j \leq n \;\middle|\; \begin{array}{l} a_{ij} = a_{ji} \\ a_{ij} a_{kl} = a_{kl} a_{ij} \\ a_{ij} a_{jk} = a_{jk} a_{ki} = a_{ki} a_{ij} \end{array} \right\rangle$$

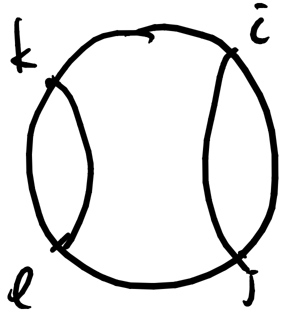
$$\Delta = a_{n(n-1)} a_{(n-1)(n-2)} \cdots a_{21}$$

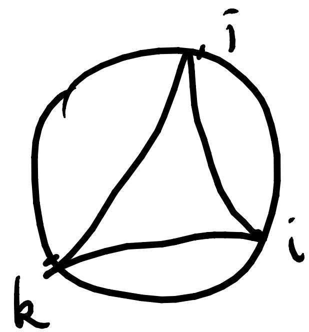
$a_{ij} =$



$\Delta =$







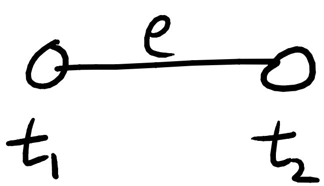
Artin group of type $I_2(e)$

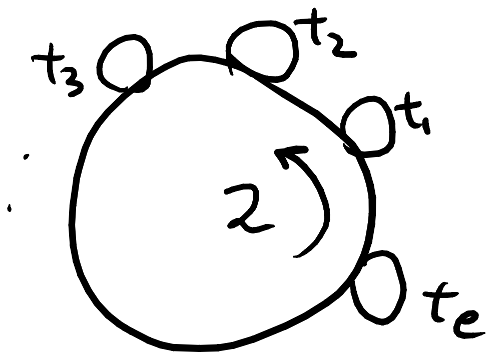
① Classical Garside structure

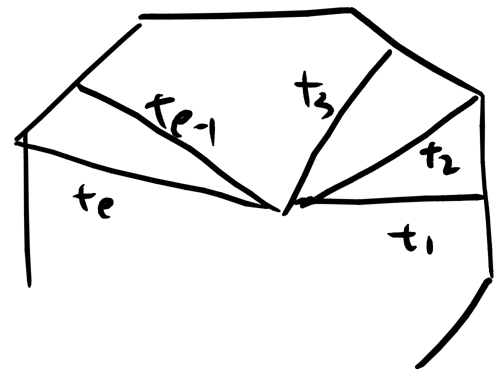
$B(I_2(e)) = \langle t_1, t_2 \mid \underbrace{t_1 t_2 t_1 \cdots}_{e} = \underbrace{t_2 t_1 t_1 \cdots}_{e} \rangle \qquad \Delta = \underbrace{t_1 t_2 t_1 \cdots}_{e}$

② dual Garside structure

$B(I_2(e)) = \langle t_1, t_2, \cdots, t_e \mid t_1 t_2 = t_2 t_3 = \cdots = t_{e-1} t_e = t_e t_1 \rangle \qquad \Delta = t_1 t_2$



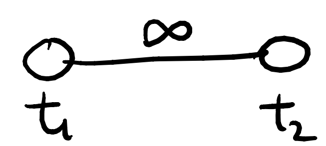




Free group of rank 2

① Standard presentation does not give a Garside structure.

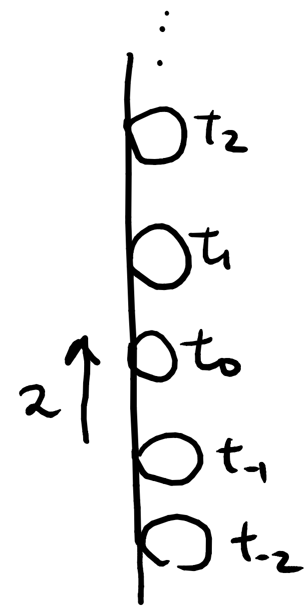
$$F_2 = \langle t_1, t_2 \mid \ \rangle = B(I_2(\infty))$$



② (dual) presentation gives a quasi-Garside structure

$$F_2 = \langle t_i,\ i \in \mathbb{Z} \mid t_i t_{i+1} = t_j t_{j+1} \ \forall i,j \rangle$$

$$\Delta = t_1 t_2$$

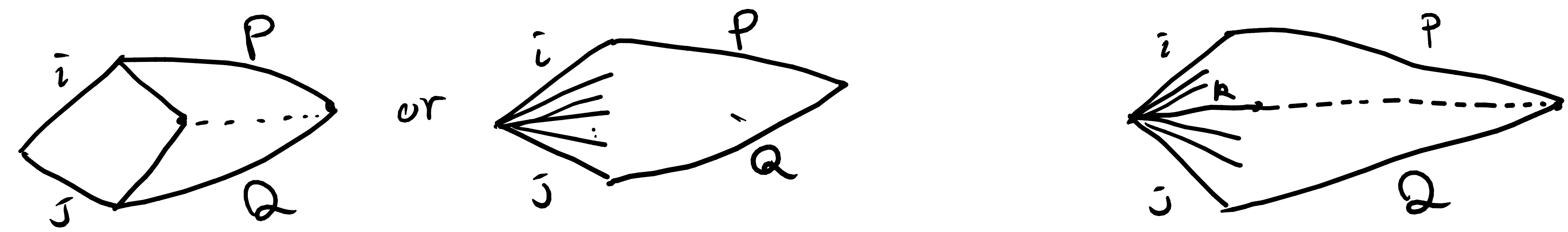


$$\left(\because F_2 = \langle t_1, t_2 \mid \ \rangle = \langle t_1, t_2, t_3 \mid t_1 t_2 = t_2 t_3 \rangle \right.$$
$$= \langle t_1, t_2, t_3, t_4 \mid t_1 t_2 = t_2 t_3 = t_3 t_4 \rangle$$
$$\left. = \langle t_0, t_1, t_2, t_3, t_4 \mid t_0 t_1 = t_1 t_2 = t_2 t_3 = t_3 t_4 \rangle = \cdots \right)$$

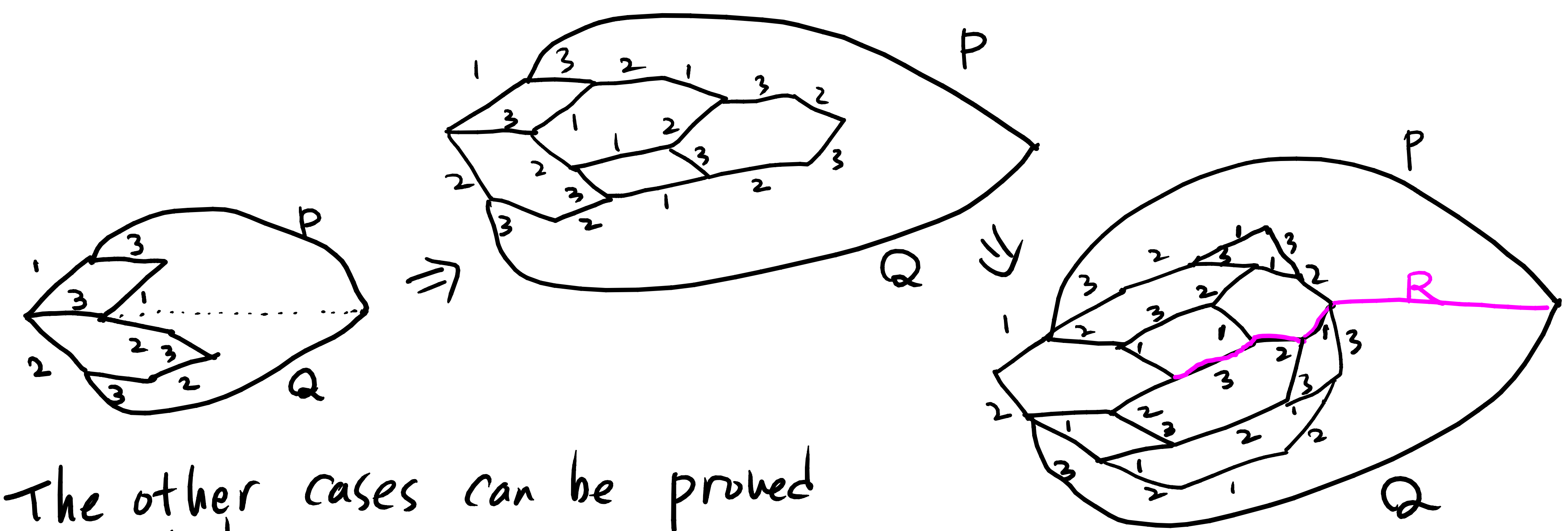
* Similarly, free group F_e is a quasi-Garside group $\forall e \in \mathbb{N}$

Garside's Proof of the thm

Suppose $\sigma_i P \equiv_+ \sigma_j Q$. Consider van Kampen diagram.

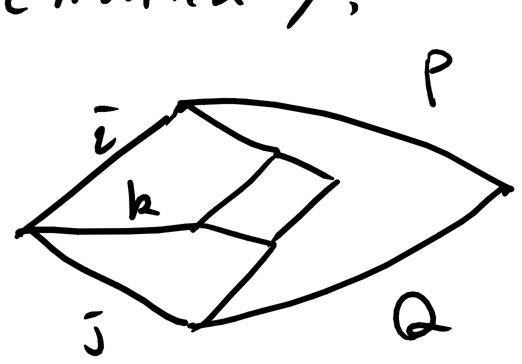


Let $(i,j,k) = (1,2,3)$. Using induction, we modify the diagram.

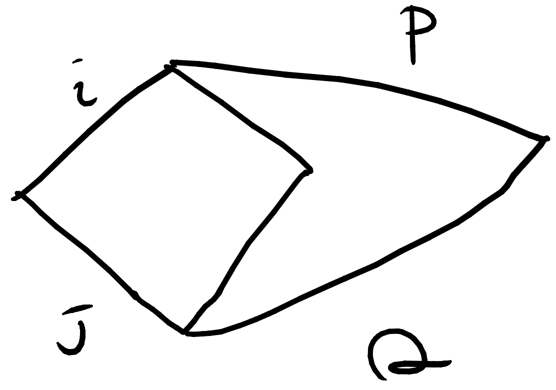


The other cases can be proved similarly.

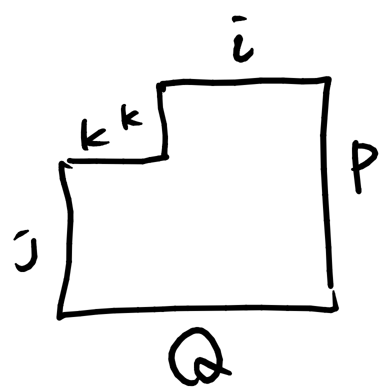
Skematically,



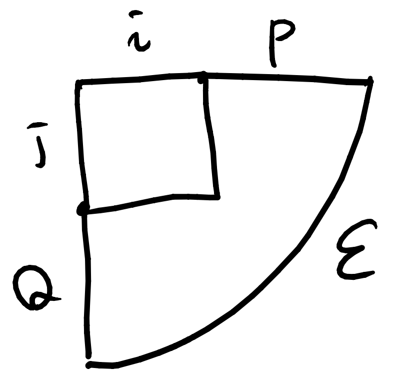
modify $\Longrightarrow$



van Kampen diagram



modify $\Longrightarrow$



word reversing diagram.

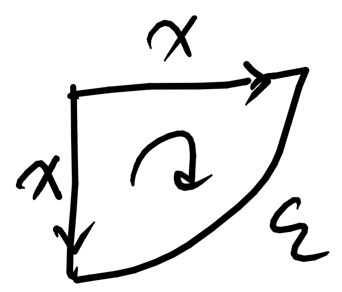
⊙ Word reversing

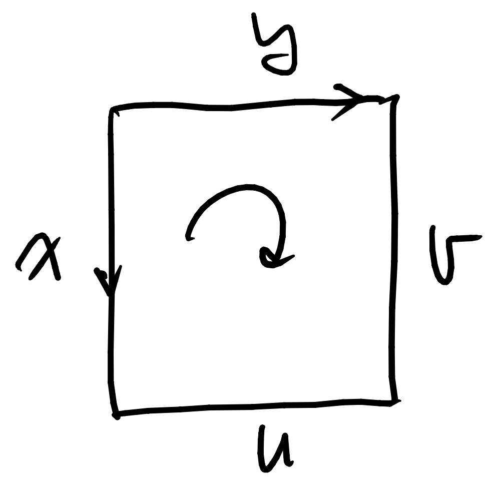
$\langle \mathcal{A} \mid \mathcal{R} \rangle$ monoid presentation

w, w' : words in $\mathcal{A} \cup \mathcal{A}^{-1}$

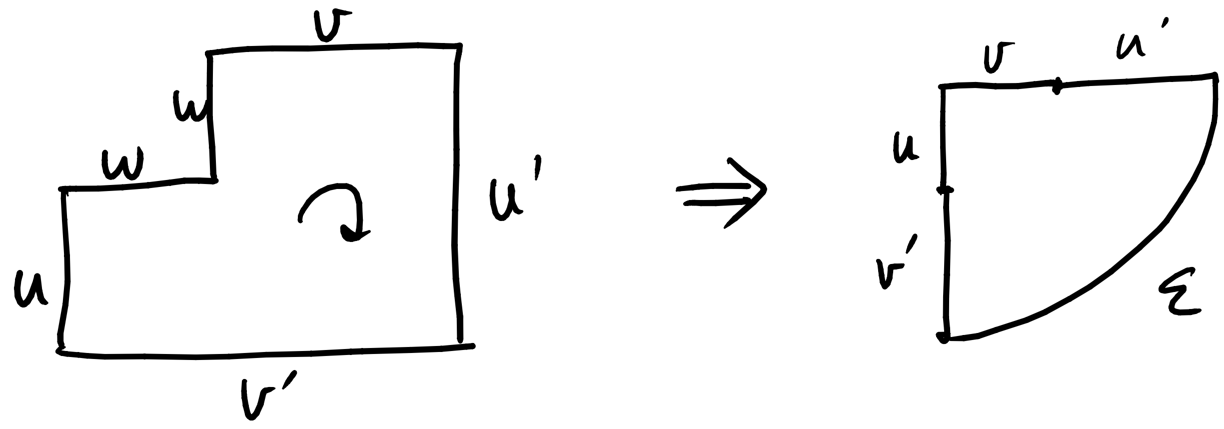
w <u>reverses</u> to w', written $w \curvearrowright w'$ if w' is obtained from w by iteratively

- deleting $x^{-1}x$ for $x \in \mathcal{A}$
- replacing $x^{-1}y$ with uv^{-1} for $xu = yv$ in $\mathcal{R}$

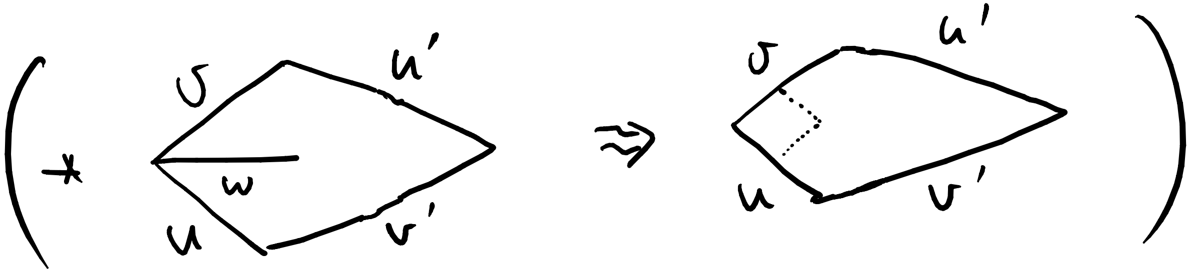




Cube condition for (u, v, w)



$$u^{-1} w\, w^{-1} v \curvearrowright v' u'^{-1} \Rightarrow (u v')^{-1} v u' \curvearrowright \varepsilon$$



Thm (Dehornoy) If cube condition holds for every triple (u, v, w), then $\langle A \mid R \rangle$ is a "complete" presentation, i.e. $u \equiv_+ v \iff u^{-1} v \curvearrowright \varepsilon$.

A presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is <u>complemented</u> if
$\forall x, y \in \mathcal{A}$, $\exists$ at most one relation of type $xP = yQ$
$\nexists$ no relation of type $xP = xQ$.

<u>Thm</u> (Dehornoy)

If a monoid is defined by a "complemented" and "complete" presentation, and if it admits a Garside element, then it is a Garside monoid.

<u>Complex Reflection Group</u>

V : $\mathbb{C}$-vector space, $\dim V < \infty$

$s \in GL(V)$ is a <u>(pseudo-)reflection</u> if there is a hyperplane H on which s acts trivially.

$W \leq GL(V)$ is a <u>finite complex reflection group</u> if it is a finite subgroup of $GL(V)$ generated by pseudo-reflections

$$M = V - \bigcup_{H:\ \text{reflecting hyperplanes of reflections in } W} H$$

$B(W) \overset{\text{def}}{=} \pi_1(W \backslash M)$: the associated <u>braid group</u>.

Shephard-Todd Classification of irreducible finite complex reflection groups.

- an infinite family $G(de, e, r)$, $d, e, r \geq 1$
- 34 exceptions $G_4, G_5, \cdots, G_{37}$

[Bessis-Corran 2006] $B(e,e,r)$ admits a dual Garside structure.

[Corran-Picantin 2011] $B(e,e,r)$ admits a post-classical Garside structure.

[Bessis 2007] Braid groups of "well-generated" complex reflection groups are Garside groups.

$d_1 \leq d_2 \leq \cdots \leq d_n$, $d_1^* \geq d_2^* \geq \cdots \geq d_n^*$: degrees and codegrees of n.
W is "well-generated" if W can be generated by n reflections.
($\Leftrightarrow$ W is a "duality group", i.e. $d_i + d_i^* = d_n$.)

$G(e,e,r)$ is well-generated
Many exceptional complex reflection groups are well-generated.

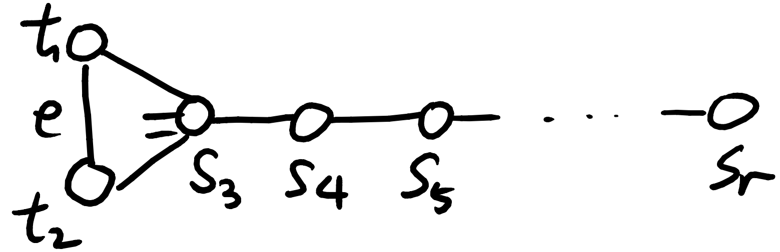
Up to now, all the braid groups of complex reflection groups except G_{31} and $G(de,e,r)$ are all Garside groups.

Broué-Malle-Rouquier presentation of $B(e,e,r)$

generators : $\{t_1, t_2\} \cup \{s_3, s_4, \cdots, s_r\}$

relations : $\begin{cases} s_i s_j = s_j s_i, & |i-j| \geq 2 \\ s_i s_j s_i = s_j s_i s_j, & |i-j| = 1 \end{cases}$

$$\Big\{ \underbrace{t_1 t_2 t_1 \cdots}_{e} = \underbrace{t_2 t_1 t_2 \cdots}_{e}$$



$$\begin{cases} s_j t_i = t_i s_j & j \geq 4, \ i = 1,2 \\ s_3 t_i s_3 = t_i s_3 t_i & i = 1,2 \\ s_3 t_1 t_2 s_3 t_1 t_2 = t_1 t_2 s_3 t_1 t_2 s_3 \end{cases}$$

Rmk) BMR-presentation is <u>not</u> complemented

[Bessis-Corran 2006] BMR-presentation does not give a Garside structure.

Corran-Picantin presentation of $B(e,e,r)$

generators : $\{t_1, t_2, \cdots, t_e\} \cup \{s_3, s_4, \cdots, s_r\}$

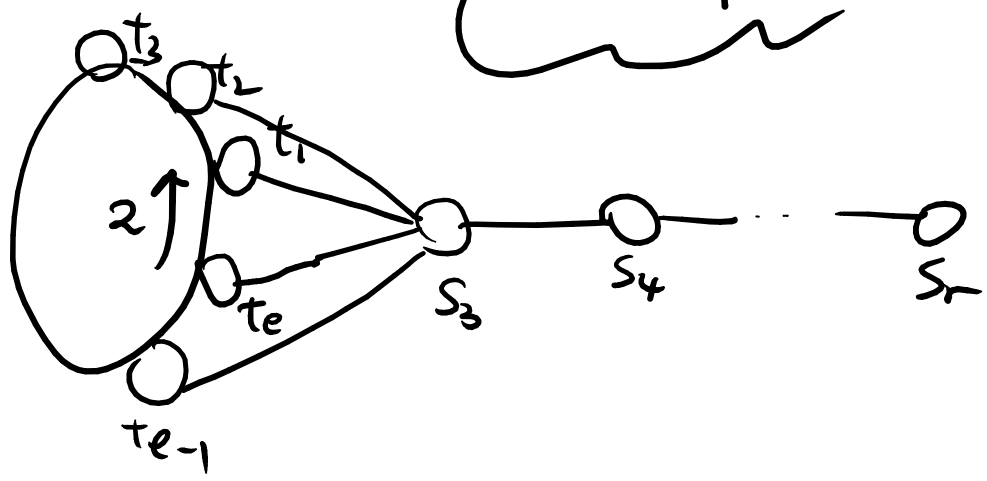
dualize the $I_2(e)$ part

relations

$$\begin{cases} s_i s_j = s_j s_i, & |i-j| \geq 2 \\ s_i s_j s_i = s_j s_i s_j & |i-j| = 1 \end{cases}$$

$$\{ t_1 t_2 = t_2 t_3 = \cdots = t_{e-1} t_e = t_e t_1$$

$$\begin{cases} s_j t_i = t_i s_j & j \geq 4, \quad i = 1, \cdots, e \\ s_3 t_i s_3 = t_i s_3 t_i & i = 1, \cdots, e \end{cases}$$



[Corran-Picantin 2011] This presentation gives a finite type Garside structure on $B(e,e,r)$

◎ Periodic elements

Let $S = s_r s_{r-1} \cdots s_3$.

$$\begin{cases} \Delta = (S t_1 t_2)^{r-1} \\ \delta = S t_1 t_2 \\ \varepsilon = S t_1 S t_2 \cdots S t_e \end{cases}$$

Then $\delta^{\frac{e(r-1)}{e \wedge r}} = \Delta^{\frac{e}{e \wedge r}} = \varepsilon^{\frac{r}{e \wedge r}}$ is the generator of Center of $B(e,e,r)$.

Proof of the identity $\delta^{\frac{e(r-1)}{e\wedge r}} = \Delta^{\frac{e}{e\wedge r}} = \varepsilon^{\frac{r}{e\wedge r}}$

- $$\delta^{r-1} = (St_1t_2)^{r-1} = St_1 St_2 \cdots St_r$$
$$= St_{j+1} St_{j+2} \cdots St_{j+r} \quad \forall j \in \mathbb{Z}$$

- $$\varepsilon^{\frac{r}{e\wedge r}} = \underbrace{(St_1 St_2 \cdots St_e)(St_{e+1} \cdots St_{2e}) \cdots (\cdots St_{\frac{r}{e\wedge r}e})}_{\frac{r}{e\wedge r}\cdot e = r\cdot\frac{e}{e\wedge r} \text{ terms of } St_i}$$
$$= (St_1 St_2 \cdots St_r)^{\frac{e}{e\wedge r}}$$
$$= (St_1t_2)^{\frac{(r-1)e}{e\wedge r}}$$

Every periodic element of $B(e,e,r)$ is a power of δ or ε.

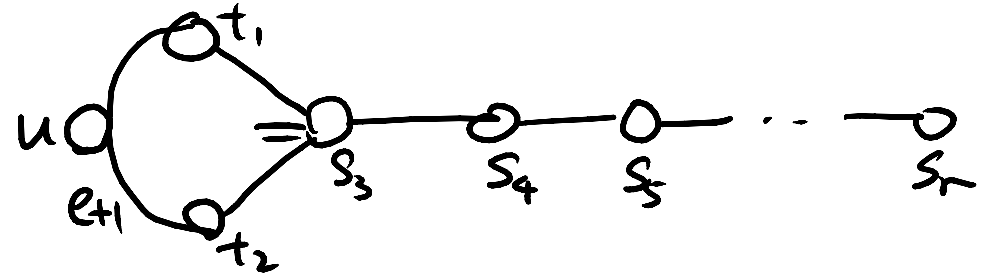
(Bessis 2007)

W : well-generated complex reflection group

$g, h \in B(W)$ s.t. $g^k = h^k \in \mathrm{Center}(B(W))$, $k \neq 0$

$\Rightarrow$ g is conjugate to h.

Broué-Malle-Rouquier presentation of $B(de,e,r)$



generators : $\{u\} \cup \{t_1, t_2\} \cup \{s_3, s_4, \cdots, s_r\}$

relations : $\begin{cases} s_i s_j = s_j s_i & |i-j| \geq 2 \\ s_i s_j s_i = s_j s_i s_j & |i-j| = 1 \end{cases}$

$$\begin{cases} u t_1 t_2 = t_1 t_2 u \\ \underbrace{u t_1 t_2 t_1 \cdots}_{e+1} = \underbrace{t_2 u t_1 t_2 \cdots}_{e+1} \end{cases}$$

$$\begin{cases} s_3 t_1 t_2 s_3 t_1 t_2 = t_1 t_2 s_3 t_1 t_2 s_3 \\ s_j t_i = t_i s_j & j \geq 4, \ i = 1,2 \\ s_3 t_i s_3 = t_i s_3 t_i & i = 1,2 \\ u s_j = s_j u & j \geq 3 \end{cases}$$

Rmk

① no relation between t_1 and t_2

② $B(de,e,r)/\langle\langle u \rangle\rangle \simeq B(e,e,r)$

③ independent of d

④ does not give a Garside structure

New presentation of $B(de,e,r)$

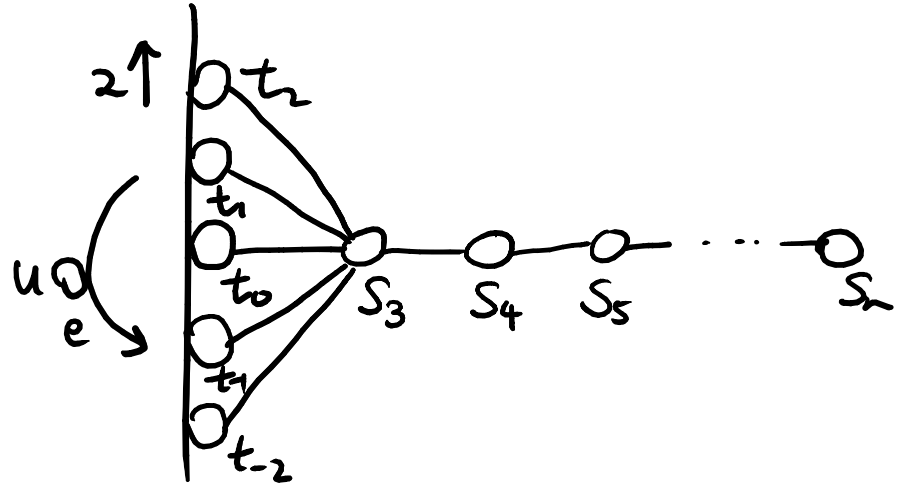
generators: $\{u\} \cup \{t_i \mid i \in \mathbb{Z}\} \cup \{s_3, s_4, \cdots, s_r\}$

relations:

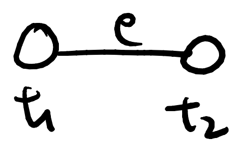
$$\begin{cases} s_i s_j = s_j s_i & |i-j| \geq 2 \\ s_i s_j s_i = s_j s_i s_j & |i-j| = 1 \end{cases}$$

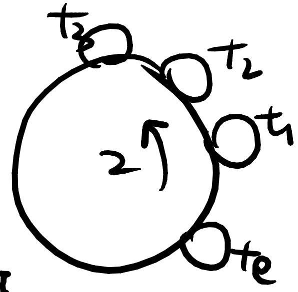
$$\begin{cases} t_i t_{i+1} = t_j t_{j+1}, & \forall i,j \in \mathbb{Z} \\ u t_i = t_{i+e} u \end{cases}$$

$$\begin{cases} s_3 t_i s_3 = t_i s_3 t_i, & i \in \mathbb{Z} \\ s_j t_i = t_i s_j & i \in \mathbb{Z},\ j \geq 4 \\ s_j u = u s_j & j \geq 3 \end{cases}$$

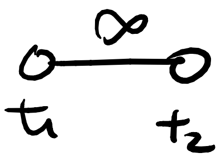


$$\boxed{\text{Thm (Lee-L, 2012) This new presentation gives an infinite type Garside structure on } B(de,e,r).}$$

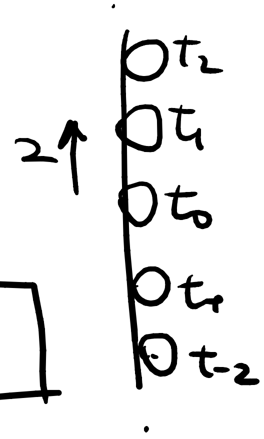


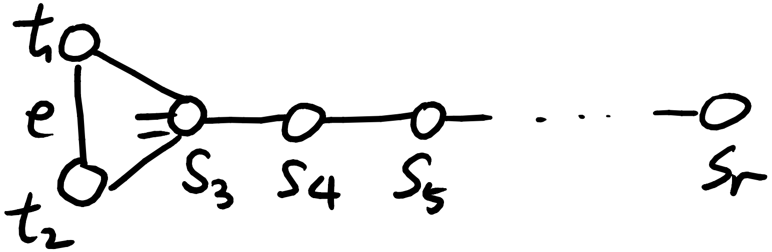


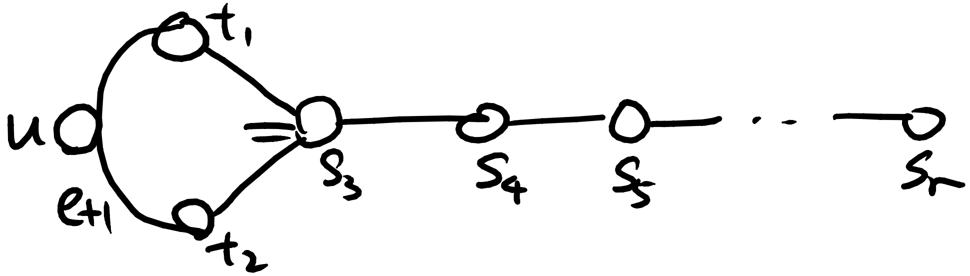
$I_2(e)$



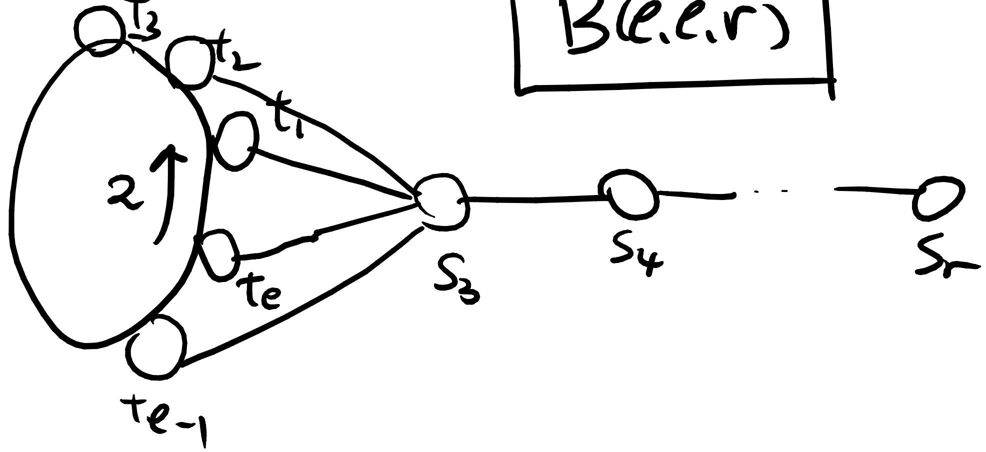
F_2



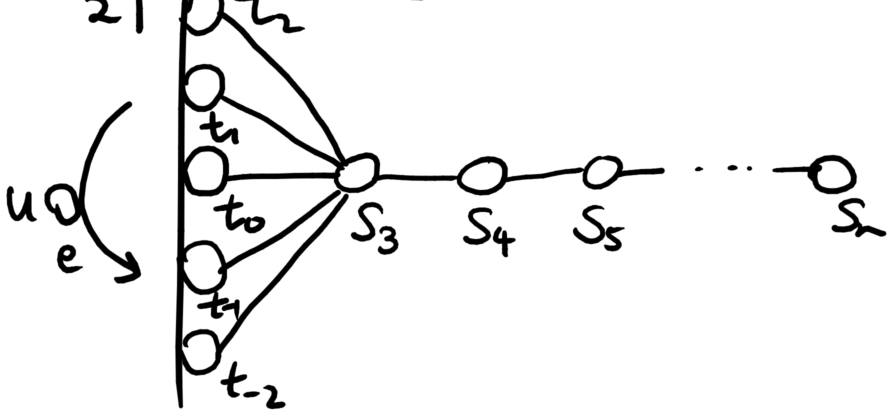




$B(e,e,r)$



$B(de,e,r)$

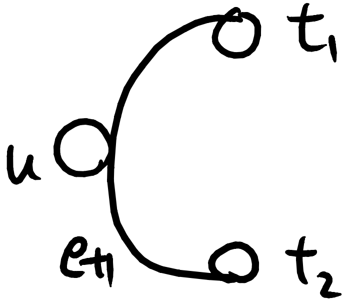


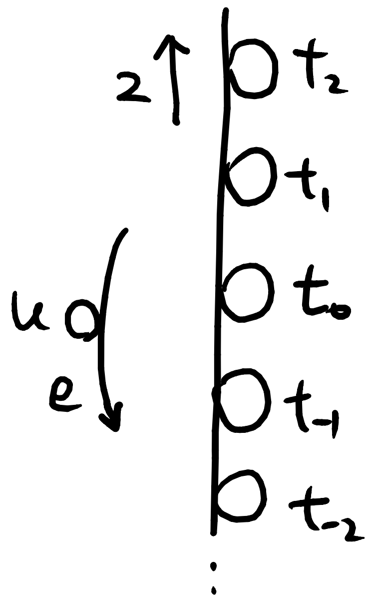
Thm The new presentation is correct

proof) ① add new generators $\{t_3, t_4, \dots\} \cup \{t_0, t_{-1}, t_{-2}, \dots\}$

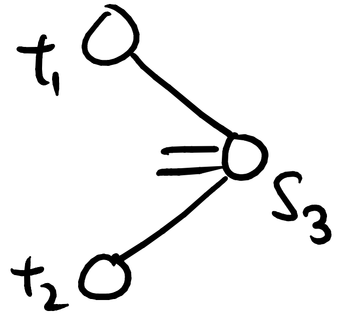
with new relations $\quad \cdots = t_{-1}t_0 = t_0 t_1 = t_1 t_2 = t_2 t_3 = \cdots$.

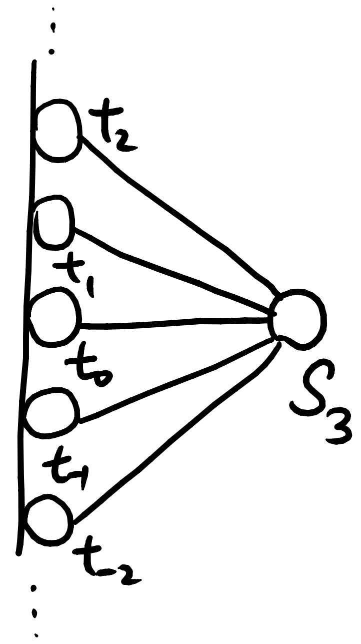
② $$\left.\begin{array}{l} u t_1 t_2 = t_1 t_2 u \\ \underbrace{u t_1 t_2 t_1 \cdots}_{e_{+1}} = t_2 \underbrace{u t_1 t_2 \cdots}_{e_{+1}} \end{array}\right\} \iff u t_i = t_{i+e} u \quad \forall i \in \mathbb{Z}$$





③ $\left.\begin{array}{l} s_3 t_i s_3 = t_i s_3 t_i \quad i=1,2 \\ s_3 t_1 t_2 s_3 t_1 t_2 = t_1 t_2 s_3 t_1 t_2 s_3 \end{array}\right\} \Longleftrightarrow s_3 t_i s_3 = t_i s_3 t_i, \ \forall i \in \mathbb{Z}$





④ $t_i s_j = s_j t_i, \quad j \geq 4, \ i=1,2 \Longleftrightarrow t_i s_j = s_j t_i, \ j \geq 4, \ \forall i \in \mathbb{Z}$

///

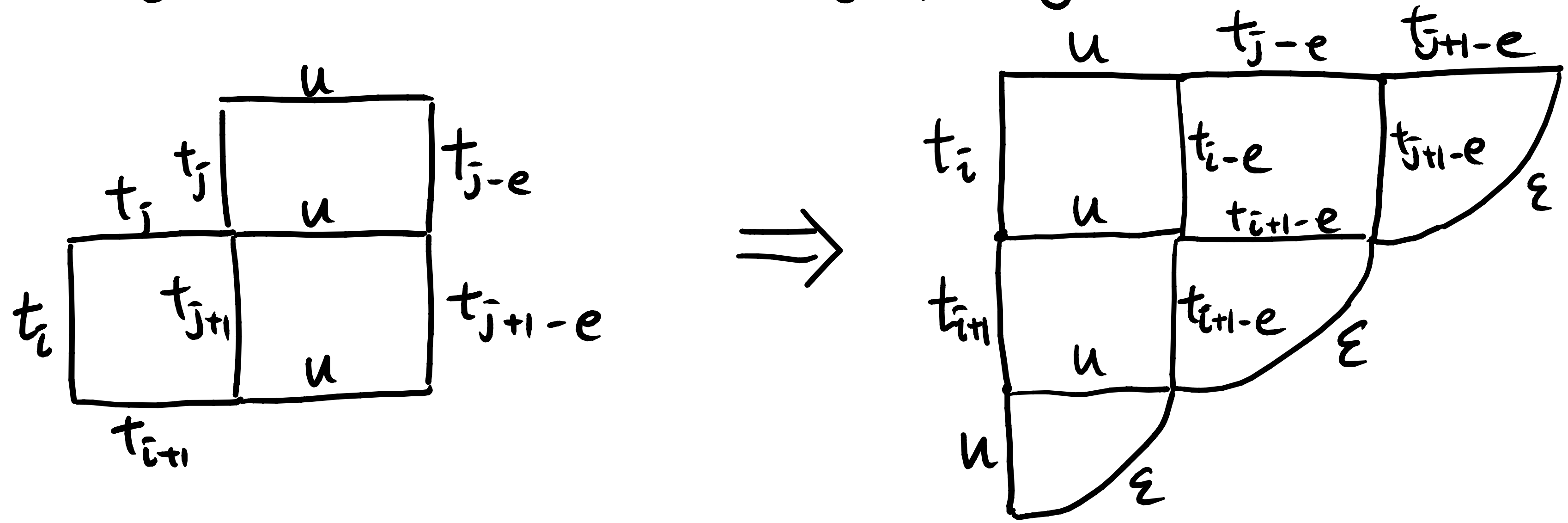
Thm The monoid defined by the new presentation is a Garside monoid of infinite type.

proof) ① The presentation is complemented.

② The presentation is complete.

- The cube condition holds for every triple (x, y, z).

e.g.) $(x, y, z) = (t_i, u, t_j)$, $i \neq j$



Let $\Delta = u^{\frac{r}{e \wedge r}} (s_r s_{r-1} \cdots s_3 t_1 t_2)^{\frac{e(r-1)}{e \wedge r}}$.

[BMR] Δ is the generator of the center of $B(de,e,r)$

③ Δ is a Garside element.

- $\mathrm{Div}(\Delta)$ contains all the generators.
- $x \Delta \equiv_+ \Delta x$ for any generator x.

$\therefore$ The monoid satisfies the Ore's condition, hence it embeds in its group of fractions.

- Because Δ is central, $\Delta = ab \Rightarrow ba = b(ab)b^{-1} = b\Delta b^{-1} = \Delta$.

$\therefore \{\text{left-divisors of } \Delta\} = \{\text{right-divisors of } \Delta\}$.

◎ Periodic elements

Let $S = s_r s_{r-1} \cdots s_3$. Then $\Delta = (S t_1 t_2)^{\frac{e(r-1)}{e \wedge r}} u^{\frac{r}{e \wedge r}}$

Let $\varepsilon = (S t_1 S t_2 \cdots S t_e) u$. Then $\varepsilon^{\frac{r}{e \wedge r}} = \Delta$.

<u>Q</u> Is every periodic element a power of ε?

Proof of $\quad \varepsilon^{\frac{r}{e\wedge r}} = \Delta$.)

$$\varepsilon^k = (St_1 St_2 \cdots St_e) u \; (St_1 St_2 \cdots St_e) u \; \cdots$$
$$= (St_1 St_2 \cdots St_e)(St_{e+1} St_{e+2} \cdots St_{2e}) \cdots \; u^k$$
$$= (St_1 St_2 \cdots St_{ke}) u^k$$

$$St_{\ell+1} St_{\ell+2} \cdots St_{\ell+r} = (St_1t_2)^{r-1}. \quad \forall \ell \in \mathbb{Z}$$

$$\therefore \varepsilon^{\frac{r}{e\wedge r}} = (St_1 St_2 \cdots St_{\frac{r}{e\wedge r}e}) \; u^{\frac{r}{e\wedge r}}$$
$$= (St_1 St_2 \cdots St_r)^{\frac{e}{e\wedge r}} \; u^{\frac{r}{e\wedge r}}$$
$$= \left\{(St_1t_2)^{r-1}\right\}^{\frac{e}{e\wedge r}} \; u^{\frac{r}{e\wedge r}}$$
$$= (St_1t_2)^{\frac{e(r-1)}{e\wedge r}} \; u^{\frac{r}{e\wedge r}}$$

///

Questions

① Does $B(de,e,r)$ admit a finite type Garside structure?

② What can we do with this quasi-Garside structure of $B(de,e,r)$?

Some additional tools:

- $1 \longrightarrow \langle\langle u \rangle\rangle \longrightarrow B(de,e,r) \longrightarrow B(e,e,r) \longrightarrow 1$
- $1 \longrightarrow B(\tilde{A}_{r-1}) \longrightarrow B(de,e,r) \rightleftarrows \mathbb{Z} = \langle u \rangle \longrightarrow 1$
- Garside group of fake type $F(de,e,r)$

....

- Translation separable?

  (· translation number $t_X(g) = \lim_{n\to\infty} \frac{\|g^n\|_X}{n}$, where X is a "finite" generating set

  · G is "translation separable" if $t_X(g)=0 \iff g=1$.)

- Uniqueness of roots up to conjugacy of periodic elts?

  $\underset{k\neq 0}{g^k = h^k} \in \mathrm{Center}(B(de,e,r)) \overset{?}{\Longrightarrow} g$ conjugate to h

- Solvable conjugacy problem?
- Automatic?

...