Deligne-Lusztig varieties and conjugacy of braids

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Motivation

The work of Bessis, Broué, Digne, Dudas, Malle, Michel, Rouquier in the last 20 years has connected modular character theory of finite reductive groups (in particular the Broué derived equivalence conjectures) with questions about conjugacy and centralizers in braid groups and ribbon categories.

This goes through the cohomology of Deligne-Lusztig varieties, and the action of some braid monoids on them.

My goal is to explain this picture in the case of $\mathrm{GL}_n(\mathbb{F}_q)$, but in a way which can be extended to other finite reductive groups.

Finite linear groups

The linear group over a finite field $\mathbb{F}_q$ with $q = p^r$ elements is best seen as the $\mathbb{F}_q$-rational points of the algebraic group $\mathbf{G} = \mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_q$.

If F is the *Frobenius endomorphism* given by $(a_{i,j}) \mapsto (a_{i,j}^q)$, we have $\mathrm{GL}_n(\mathbb{F}_q) = \mathbf{G}^F$ (since the elements of $\mathbb{F}_q$ are the solutions in $\mathbb{F}$ of the equation $x^q = x$).

Each F-stable subgroup of $\mathbf{G}$ gives rise to a subgroup of $\mathbf{G}^F$.

(Lang's theorem)

If $\mathbf{H}$ is a connected algebraic group with a Frobenius endomorphism F, then the map $x \mapsto x^{-1}F(x)$ is surjective.

Important subgroups

- A *maximal torus* of $\mathbf{G}$ is a maximal connected algebraic subgroup formed of diagonalizable elements.
- A *Borel subgroup* is a maximal solvable algebraic subgroup.

All maximal tori are conjugate to the subgroup $\mathbf{T}_1$ of diagonal matrices.
All Borel subgroups are conjugate to the subgroup $\mathbf{B}_1$ of upper triangular matrices.

$$\mathbf{T}_1 = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \qquad \mathbf{B}_1 = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$$

The Borel subgroups are also the stabilizers of complete flags $V_0 \subset V_1 \subset \ldots \subset V_n = V$ where $V_i \simeq \mathbb{F}^i$.
$N_{\mathbf{G}}(\mathbf{T}_1)$ is the group of monomial matrices.
$W := N_{\mathbf{G}}(\mathbf{T}_1)/\mathbf{T}_1 \simeq \mathfrak{S}_n$ is the *Weyl group*.

Parabolic subgroups and Levi subgroups

The subgroups containing a Borel subgroup are called *parabolic subgroups*. They are conjugate to some group of block-upper triangular matrices.
A parabolic subgroup $\mathbf{P}$ has a Levi decomposition $\mathbf{LV}$, where $\mathbf{L}$ is conjugate to a group of block-diagonal matrices.

$$\mathbf{P} = \begin{pmatrix} * & \cdots\cdots\cdots\cdots\cdots\cdots & & & & * \\ \vdots & & & & & \vdots \\ * & \dots & * & & & \\ & & & * & & \\ & & & * & * & \\ & & & & & * \\ & 0 & & & & * & \vdots \\ & & & & & * & * & * \end{pmatrix} \quad \mathbf{L} = \begin{pmatrix} * & \dots & * & & & & & \\ \vdots & & \vdots & & & & 0 & \\ * & \dots & * & & & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ & & & & & * & * & * \\ & 0 & & & & * & * & * \\ & & & & & * & * & * \end{pmatrix}$$

We have $N_{\mathbf{G}}(\mathbf{P}) = \mathbf{P}$ for any parabolic subgroup.

$\mathbf{G}^F$-conjugacy

Lang's theorem allows to study $\mathbf{G}^F$-conjugacy.
We call Borel subgroup of $\mathbf{G}^F$ the group $\mathbf{B}^F$ where $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}$.

All Borel subgroups of $\mathbf{G}^F$ are $\mathbf{G}^F$-conjugate.

Proof.

Any Borel subgroup is $g\mathbf{B}_1 g^{-1}$ for $g \in \mathbf{G}$.
It is F-stable if $F(g\mathbf{B}_1 g^{-1}) = g\mathbf{B}_1 g^{-1} \Leftrightarrow g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{B}_1) = \mathbf{B}_1$.
Since $\mathbf{B}_1$ is connected we may apply Lang's theorem to write $g^{-1}F(g) = b^{-1}F(b)$ with $b \in \mathbf{B}_1$.
Then $gb^{-1} = F(gb^{-1})$ and $g\mathbf{B}_1 g^{-1} = gb^{-1}\mathbf{B}_1(gb^{-1})^{-1}$ thus $g\mathbf{B}_1 g^{-1}$ *is conjugate to* $\mathbf{B}_1$ *by the F-stable element* gb^{-1}. □

$\mathbf{G}^F$-conjugacy of tori

Things are different with tori. Here $\mathbf{T}_1$ is the connected component of $N_{\mathbf{G}}(\mathbf{T}_1)$ and the group of components is W.

The $\mathbf{G}^F$-class of the F-stable torus $g\mathbf{T}_1 g^{-1}$ is parameterized by the conjugacy class of the image of $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_1)$ in W.

We have thus tori $\mathbf{T}_w$ (depending on the class of w) such that $(\mathbf{T}_w, F) \sim_{\mathbf{G}} (\mathbf{T}_1, wF)$.
Any degree n polynomial $c_0 + \ldots c_{n-1}T^{n-1} + T^n \in \mathbb{F}_q[T]$ is the characteristic polynomial of $\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \ldots & 0 & 1 \\ -c_0 & \ldots & \ldots & -c_{n-1} \end{pmatrix} \in \mathbf{G}^F$.
F does the permutation w on the roots of the characteristic polynomial of an element of $\mathbf{T}_w$.
If w is a product of cycles of lengths $n_1, \ldots n_k$ we have $\mathbf{T}_w^F \simeq \prod_i \mathbb{F}_{q^{n_i}}^\times$ thus $|\mathbf{T}_w^F| = |\mathbf{T}_1^{wF}| = \prod_i (q^{n_i} - 1)$.

Complex representations of $\mathbf{G}^F$

A first idea to construct representations of $\mathbf{G}^F$ is to induce from "similar" subgroups, like a Levi $\mathbf{L}$ isomorphic to $\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_k}$.
For $\chi \in \mathrm{Irr}(\mathbf{L}^F)$, The induced representation $\mathrm{Ind}_{\mathbf{L}^F}^{\mathbf{G}^F}(\chi)$ is "too complicated" — has too many irreducible components—, but a good construction is

(Harish-Chandra induction)

$R_{\mathbf{L}}^{\mathbf{G}}(\chi) := \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F}(\tilde{\chi})$ *where* $\tilde{\chi}$ *is the natural extension to* $\mathbf{P}^F$ *through the quotient* $1 \to \mathbf{V}^F \to \mathbf{P}^F \to \mathbf{L}^F \to 1$.

When χ is *cuspidal*, which means "cannot be obtained by induction", that is $\langle \chi, R_{\mathbf{M}}^{\mathbf{L}}(\psi) \rangle_{\mathbf{G}^F} = 0$ for all $\psi \in \mathrm{Irr}(\mathbf{M}^F)$, then the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$ is simple — indexed by the characters of the group $N_{\mathbf{G}^F}(\mathbf{L}, \chi)/\mathbf{L}^F$.

Cuspidals

Thus Harish-Chandra induction reduces the problem of parameterizing $\mathrm{Irr}(\mathbf{G}^F)$ to that of parameterizing the cuspidals.
Green (1955) had a complicated inductive construction for GL_n.
Macdonald(1970) predicted the cuspidals should be associated to twisted tori.

Deligne and Lusztig (1976) constructed a variety associated to twisted tori such that they occur in its ℓ-adic cohomology.

$\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ are abstractly isomorphic, but an explicit isomorphism depends on the axiom of choice.
However, for algebraic numbers like entries in representing matrices there is no difference between ℓ-adic and complex representations.

Relative positions

We say the complete flag $(V_i)_i$ is in relative position $w \in W$ with $(V'_i)_i$ if there is a basis $(e_i)_i$ such that e_i spans V_i/V_{i-1} and $e_{w(i)}$ spans V'_i/V'_{i-1}.

If the Borel subgroups $\mathbf{B}$ and $\mathbf{B}'$ are the respective stabilizers we denote this $\mathbf{B} \xrightarrow{w} \mathbf{B}'$. Equivalently $(\mathbf{B}, \mathbf{B}') \sim_{\mathbf{G}} (\mathbf{B}_1, {}^w\mathbf{B}_1)$.
Any two Borel subgroups have a relative position; equivalently we have the *Bruhat decomposition* $\mathbf{G} = \coprod_w \mathbf{B}_1 w \mathbf{B}_1$.
Two Borel subgroups are in relative position $1 \in W$ if and only if they coincide.

$W = \mathfrak{S}_n$ is a Coxeter group with Coxeter generators $(i, i+1)$. The length with respect to these generators is the number of inversions. The dimension of $\mathbf{B}_1 w \mathbf{B}_1$ is $\dim \mathbf{B}_1 + l(w)$.
The Borel $\mathbf{B}_1$ of upper triangular matrices is in relative position the longest element of W (the permutation $(1,n)(2,n-1)\ldots$) with the Borel subgroup formed of the lower triangular matrices.

Deligne-Lusztig variety

The Deligne-Lusztig variety *associated to* $w \in W$ *is*
$\mathbf{X}_w = \{\mathbf{B} \mid \mathbf{B} \xrightarrow{w} F(\mathbf{B})\}$

For $\mathbf{B} \in \mathbf{X}_w$ there exists g such that ${}^g(\mathbf{B}, F(\mathbf{B})) = (\mathbf{B}_1, {}^w\mathbf{B}_1)$.
g can be chosen so its conjugates a torus of $\mathbf{B} \cap F(\mathbf{B})$ to $\mathbf{T}_1$ then $(\mathbf{B}, F) \sim_{\mathbf{G}} (\mathbf{B}_1, wF)$.
$\mathbf{B}_1 \cap {}^w\mathbf{B}_1$ contains the wF-stable torus $\mathbf{T}_1$ so $\mathbf{B}$ and $F(\mathbf{B})$ contain an F-stable torus of type w.

When $w = 1$ then $\mathbf{X}_1 = \{\mathbf{B} \mid (\mathbf{B}, F(\mathbf{B})) \sim_{\mathbf{G}} (\mathbf{B}_1, \mathbf{B}_1)\}$ is the discrete variety of F-stable Borel subgroups.
In general $\dim \mathbf{X}_w = l(w)$.

$\mathbf{G}^F$ acts on $\mathbf{X}_w$.

Deligne-Lusztig unipotent characters

For a prime $\ell \neq p$ a variety $\mathbf{X}$ over $\mathbb{F}$ has ℓ-adic cohomology groups with compact support $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ which are $\overline{\mathbb{Q}}_\ell$-vector spaces which can be $\neq 0$ only for $i \in \{0, \ldots, 2\dim \mathbf{X}\}$.

The Deligne-Lusztig unipotent *characters are*
$R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id})(g) := \sum_{i=0}^{2l(w)} (-1)^i \,\mathrm{Trace}(g \mid H^i_c(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell))$.

It can be shown that they depend only on the $\mathbf{G}^F$-class of $\mathbf{T}_w$, thus on the class of w in W.
$H^{2\dim \mathbf{X}}_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ is a vector space of dimension the number of irreducible components of maximal dimension of $\mathbf{X}$.
When $w = 1$ we have $H^i_c(\mathbf{X}_1, \overline{\mathbb{Q}}_\ell) = 0$ for $i > 0$ and $H^0_c(\mathbf{X}_1, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}_1^F]$.
The definition agrees with the previous notation $R^{\mathbf{G}}_{\mathbf{T}_1}(\mathrm{Id}) = \mathrm{Ind}_{\mathbf{B}_1^F}^{\mathbf{G}^F} \mathrm{Id}$.

Deligne-Lusztig characters

To any $\theta \in \mathrm{Irr}(\mathbf{T}_w^F)$ one can associate a natural sheaf $\mathcal{F}_\theta$ on $\mathbf{X}_w$.

The Deligne-Lusztig character associated to θ is
$R_{\mathbf{T}_w}^{\mathbf{G}}(\theta) := \sum_i (-1)^i H_c^i(\mathbf{X}_w, \mathcal{F}_\theta)$.

- $\langle R_{\mathbf{T}_w}^{\mathbf{G}}(\theta), R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F} = |N_{\mathbf{G}^F}(\mathbf{T}_w, \theta)/\mathbf{T}_w^F|$.
- *Every irreducible representation occurs in some $R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)$.*

For GL_n the decomposition of $R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)$ can be described simply.
For example,

$$R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) = \sum_{\chi \in \mathrm{Irr}(W)} \chi(w) U_\chi$$

where U_χ are irreducible characters.
It is more complicated for other groups.

Modular representation theory

The algebra $\mathbb{C}\mathbf{G}^F$ is semi-simple so its blocks (indecomposable factors) are the irreducible representations.
When ℓ is a prime dividing $|\mathbf{G}^F|$, the algebra $\overline{\mathbb{F}}_\ell\mathbf{G}^F$ is not semi-simple, and the first problem of *modular representation theory* is to understand its blocks.
They are the same as the blocks of $\overline{\mathbb{Z}}_\ell\mathbf{G}^F$ which are slightly easier to understand since $\overline{\mathbb{Z}}_\ell\mathbf{G}^F$ can be compared with $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$ which is semisimple.
The *principal* block is the one containing the identity representation.

(A particular case of Broué's conjecture)

If S is an abelian ℓ-Sylow subgroup, there is an equivalence of derived categories between the principal blocks of $\overline{\mathbb{Z}}_\ell\mathbf{G}^F$ and $\overline{\mathbb{Z}}_\ell N_{\mathbf{G}^F}(S)$.

Geometric Sylow subgroups

We have $|\mathbf{G}^F| = \prod_{i=0}^{n-1}(q^n - q^i) = q^{n(n-1)/2} \prod_{d \le n} \Phi_d(q)^{\lfloor \frac{n}{d} \rfloor}$.

If $\ell \neq p$ is a prime, the ℓ-Sylow subgroup of $\mathbf{G}^F$ is abelian if and only if $\ell > n$. Then ℓ divides a single $\Phi_d(q)$.

This implies that a ℓ-Sylow subgroup S is a subgroup of $\mathbf{T}_w^F$ where w is a product of $\lfloor \frac{n}{d} \rfloor$ d-cycles, since then $|\mathbf{T}_w^F| = (q^d - 1)^{\lfloor \frac{n}{d} \rfloor}(q-1)^{n \bmod d}$.
The minimal connected algebraic subgroup $\mathbf{S}$ containing S is a subtorus such that $|\mathbf{S}^F| = \Phi_d(q)^{\lfloor \frac{n}{d} \rfloor}$.
We have $C_{\mathbf{G}}(\mathbf{S}) = C_{\mathbf{G}}(S)$ and $N_{\mathbf{G}}(\mathbf{S}) = N_{\mathbf{G}}(S)$. We call $\mathbf{S}$ a *geometric Sylow subgroup*.
As the centralizer of a torus is a Levi subgroup, we have $C_{\mathbf{G}}(\mathbf{S}) = \mathbf{L}$ where $\mathbf{L}$ is a Levi subgroup containing $\mathbf{T}_w$. Similarly $N_{\mathbf{G}}(\mathbf{S}) = N_{\mathbf{G}}(\mathbf{L})$.

Geometric version of Broué's conjectures

We first look at the case $d|n$ or $d|n-1$.
In this case $C_{\mathbf{G}}(S) = \mathbf{T}_w$ and the derived equivalence predicted by Broué's conjecture should be realized "through the cohomology complex" of $\mathbf{X}_w$ for an appropriate choice of w in its conjugacy class.
In particular, for the "right" choice of w one should have

- *$\langle H_c^i(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell), H_c^j(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)\rangle_{\mathbf{G}^F} = 0$ for $i \neq j$.*
- *There is an action of $N_{\mathbf{G}^F}(S)/C_{\mathbf{G}^F}(S)$ on $H_c^*(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)$ commuting to that of $\mathbf{G}^F$.*

Here $N_{\mathbf{G}^F}(S)/C_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{T}_w)/\mathbf{T}_w \simeq C_W(w)$.

Action of $C_W(w)$

The action of $C_W(w)$ as $\mathbf{G}^F$-endomorphisms of $H_c^*(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)$ is difficult to construct.

- w is a regular element in the sense of Springer — it has an eigenvector outside the reflecting hyperplanes of W viewed as a complex reflection group.
- $C_w(w)$ is a complex reflection group — $G(d, 1, \lfloor \frac{n}{d} \rfloor) \simeq \mathbb{Z}/d \wr \mathfrak{S}_{\lfloor \frac{n}{d} \rfloor}$.
- There is an action of a braid monoid for the braid group of $G(d, 1, \lfloor \frac{n}{d} \rfloor)$ on $\mathbf{X}_\mathbf{w}$.
- This action, on the cohomology, factors through a cyclotomic Hecke algebra for $C_W(w)$.
- We conclude using Tits' theorem which gives an isomorphism of the cyclotomic Hecke algebra with the group algebra.

Composing relative positions

Double cosets $\mathbf{B}_1 w \mathbf{B}_1$ multiply as basis elements of the Hecke algebra for W.
In particular

If $l(w') + l(w'') = l(w'w'')$ *then* $\mathbf{B}_1 w' \mathbf{B}_1 w'' \mathbf{B}_1 = \mathbf{B}_1 w' w'' \mathbf{B}_1$.

Otherwise $\mathbf{B}_1 w' \mathbf{B}_1 w'' \mathbf{B}_1$ is a union of double cosets $\mathbf{B}_1 w \mathbf{B}_1$ where w runs over some elements smaller than $w'w''$ for the Bruhat order.
It follows that

If $l(w') + l(w'') = l(w'w'')$ *then*

- *if* $\mathbf{B} \xrightarrow{w'} \mathbf{B}'$ *and* $\mathbf{B}' \xrightarrow{w''} \mathbf{B}''$ *then* $\mathbf{B} \xrightarrow{w'w''} \mathbf{B}'$.
- *Conversely if* $\mathbf{B} \xrightarrow{w'w''} \mathbf{B}'$ *there is a unique* $\mathbf{B}'$ *such that* $\mathbf{B} \xrightarrow{w'} \mathbf{B}'$ *and* $\mathbf{B}' \xrightarrow{w''} \mathbf{B}''$.

$\mathbf{G}^F$-endomorphisms of $\mathbf{X}_w$

The idea for constructing $\mathbf{G}^F$-endomorphisms of $\mathbf{X}_w$ is:

- If $w = xy$ with $l(w) = l(x) + l(y)$,

given $\mathbf{B} \xrightarrow{w} F(\mathbf{B})$ there is a unique $\mathbf{B}'$ such that $\mathbf{B} \xrightarrow{x} \mathbf{B}' \xrightarrow{y} F(\mathbf{B})$.

- If we have also $l(yx) = l(y) + l(x)$,

then since $\mathbf{B}' \xrightarrow{y} F(\mathbf{B}) \xrightarrow{x} F(\mathbf{B}')$, we have $\mathbf{B}' \in \mathbf{X}_{yx}$,
thus $\mathbf{B} \mapsto \mathbf{B}'$ defines a map $\mathbf{X}_w \xrightarrow{D_x} \mathbf{X}_{yx}$ which is $\mathbf{G}^F$-equivariant.

- If in addition x commutes to w

D_x is an endomorphism.
There are too many conditions so this do not construct enough endomorphisms.

The braid monoid

If

$$W = \langle S \mid s_i^2 = 1 \text{ for } s_i \in S \text{ and the braid relations}\rangle$$

The braid monoid is

$$B^+ = \langle \mathbf{S} \mid \text{ the braid relations}\rangle.$$

The generators of Coxeter system (W, S) lift to the set of generators $\mathbf{S}$ (here $s_i = (i, i+1)$ lift to $\mathbf{s}_i$).

The whole of W lifts to the subset $\mathbf{W} \in B^+$ formed of reduced braids (also called permutation braids), that is braids $\mathbf{w}$ whose image $w \in W$ has same length as $\mathbf{w}$ (in terms of $\mathbf{s}_i$).

Germs

A *germ* for a monoid M is a subset $G \subset M$ such that

$$\langle G \mid gg' = g'' \text{ when } g, g', gg' = g'' \in G \rangle$$

is a presentation of M.
An example is a Garside family G for a monoid M.

$\mathbf{W} = \{$*the set of reduced braids*$\}$ *is a Garside family for the ordinary braid monoid and thus a germ for* B^+.

For $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$ with images $w, w' \in W$ we have $\mathbf{ww}' \in \mathbf{W}$ if and only if $l(w) + l(w') = l(ww')$, and then $\mathbf{ww}'$ lifts ww'.
Δ is the lift of the longest element of W (here the permutation $(1, n)(2, n-1)\ldots$).
Every positive braid is a product $\mathbf{w}_1 \ldots \mathbf{w}_r$ with $\mathbf{w}_i \in \mathbf{W}$.

Varieties associated to braids

We associate a variety
$\mathbf{X}(w_1, \ldots, w_r) = \{\mathbf{B}_0, \ldots, \mathbf{B}_r \mid \mathbf{B}_{i-1} \xrightarrow{w_i} \mathbf{B}_i \text{ and } \mathbf{B}_r = F(\mathbf{B}_0)\}$ to any sequence w_i of elements of W.
The fact that

- Relative positions compose exactly when the partial multiplication is defined in $\mathbf{W}$.
- $\mathbf{W}$ is a germ for B^+.

shows that $\mathbf{X}(w_1, \ldots, w_r)$ depends up to isomorphism only on the product $\mathbf{b} = \mathbf{w}_1 \ldots \mathbf{w}_r \in B$.
We would like to know that we have a well-defined variety associated to $\mathbf{b}$, that is that there is a *unique isomorphism* between two models associated to different decompositions of the same braid $\mathbf{b}$.
This is true thanks to

(Deligne)

A representation of $\mathbf{W}$ *into a monoidal category extends uniquely to a representation of* B^+ *into that monoidal category.*

Deligne's theorem

We apply Deligne's theorem to the varieties

$$\mathcal{O}(w_1, \ldots, w_r) = \{\mathbf{B}_0, \ldots, \mathbf{B}_r \mid \mathbf{B}_{i-1} \xrightarrow{w_i} \mathbf{B}_i\}.$$

There is a unique morphism

$$\mathcal{O}(w_1, \ldots, w_r) \times_{\mathbf{G}/\mathbf{B}_1} \mathcal{O}(w'_1, \ldots, w'_{r'}) \to \mathcal{O}(w_1, \ldots, w_r, w'_1, \ldots, w'_{r'})$$

defined on the fibered product, that is on pairs of sequences where the last term of the first agrees with the first term of the second, defining a structure of monoidal category.
The restriction to 1-term sequences is a representation of $\mathbf{W}$ into a monoidal category. By Deligne's theorem it extends to B^+, giving a well-defined variety $\mathcal{O}(\mathbf{b})$ attached to any braid, which for any decomposition $\mathbf{b} = \mathbf{w}_1 \ldots \mathbf{w}_r$ as product of elements of $\mathbf{W}$, is isomorphic to $\mathcal{O}(w_1, \ldots, w_r)$.
We define $\mathbf{X}_\mathbf{b}$ by taking the intersection with the graph of the Frobenius.

Questions about braids

Now, whenever we have a divisor $\mathbf{x}$ of $\mathbf{b}$ in B^+ there is a well-defined morphism $\mathbf{X}_\mathbf{b} \xrightarrow{D_\mathbf{x}} \mathbf{X}_{\mathbf{x}^{-1}\mathbf{b}\mathbf{x}}$ commuting to the action of $\mathbf{G}^F$ on both varieties.
Thus, we have a translation as questions for braids of the questions about $\mathbf{G}^F$-endomorphisms of Deligne-Lusztig varieties.

- *A* conjugation *in a monoid M is an equality $wx = xw'$ which conjugates w to w'. We denote this $w \xrightarrow{x} w'$.*
- *The conjugation is* cyclic *if $w = xy$ and $w' = yx$.*

The *conjugacy category* is the category with objects $w \in M$ and morphisms generated by conjugating elements.
The *cyclic conjugacy category* of B^+ maps naturally to the category of Deligne-Lusztig varieties with $\mathbf{G}^F$-morphisms $D_\mathbf{x}$. An object $\mathbf{b}$ maps to $\mathbf{X}_\mathbf{b}$, and a conjugation: $\mathbf{xy} \xrightarrow{\mathbf{x}} \mathbf{yx}$ maps to the morphism $D_\mathbf{x}$.
The endomorphisms we built come from the "centralizer in the cyclic conjugacy category" which is *a priori* smaller than in the full category.

Regular elements and braids

Recall that a regular element of W is an element with an eigenvector outside of the reflecting hyperplanes.
For $\mathfrak{S}_n$ it is a product of $\lfloor \frac{n}{d} \rfloor$ d-cycles; the eigenvector is for the eigenvalue for the eigenvalue $e^{2i\pi/d}$. This implies they are of order d.
For the conjectures one must find a particular element in the conjugacy class such that the endomorphisms of the variety $\mathbf{X}_w$ are nice (in particular there are many of them).
From the results of Springer [1974] and Broué-M. [1995] we have

- *In the class of regular elements of order d, there exists elements such that $\mathbf{w}^d = \Delta^2$.*
- *The only d such that Δ^2 has d-th roots are the divisors of n and $n-1$.*
- *For any $\mathbf{b} \in B^+$ such that $\mathbf{b}^d = \Delta^2$, the image of $\mathbf{b}$ in W is a regular element of order d.*

The second item is due to Kerekjarto and Eilenberg who classified periodic braids.

Conjugacy of roots

(Digne-M.)

If $\mathbf{b} \in B^+$ has some power divisible by Δ, then any $\mathbf{b}'$ conjugate to $\mathbf{b}$ is conjugate by cyclic conjugacy.

This is true in any Garside category.

(Gonzalez-Meneses)

Roots of a given order of a given element are conjugate in B_n.

Is this true in other spherical Artin monoids?
For roots of periodic elements in complex braid groups it results in many cases from the results of Bessis.
The morphisms D_x are "equivalences of étale sites" thus by the above two results there is basically only one variety associated to a d-th roots of Δ^2.

Minimal length elements

By Bessis-Digne-M., the centralizer of a periodic braid is the braid group of $G(d, 1, \lfloor \frac{n}{d} \rfloor)$. Again by Bessis' work this extends to many other cases.

(Lusztig, He-Nie)

If w is of minimal length in its conjugacy class and not in a proper parabolic subgroup, then

- *Some power of* $\mathbf{w}$ *is divisible by* Δ.
- $\mathbf{w}$ *is conjugate in* B^+ *to any* $\mathbf{w}'$ *lifting another element of minimal length in the same* W*-conjugacy class.*
- *The morphism* $C_{B^+}(\mathbf{w}) \to C_W(w)$ *is surjective.*

These results are true for all finite Coxeter groups.
For the last statement can the assumption be replaced by "any $\mathbf{w}$ such that some power is divisible by Δ"?

The general case

In the general case d does not divide n or $n-1$ then $C_{\mathbf{G}}(\mathbf{S})$ is a Levi subgroup, and there is a similar conjecture but using Deligne-Lusztig varieties associated to parabolic subgroups.

A parabolic subgroup containing $\mathbf{B}_1$ is determined by a subset $I \subset S$ such that W_I, the subgroup generated by I, is the Weyl group of the Levi. We denote by $\mathbf{P}_I$ this parabolic subgroup.

A pair of parabolic subgroups $(\mathbf{P}', \mathbf{P}'')$ with a common Levi subgroup is $\mathbf{G}$-conjugate to the pair $(\mathbf{P}_I, w\mathbf{P}_J w^{-1})$ for some $I, J \subset S$ and $w \in W$ such that $I = wJw^{-1}$. Thus

Let $\mathcal{I}$ be the set of $\{J \subset S\}$ which are W-conjugate to I. Let $\mathcal{C}(\mathcal{I})$ be the category with objects $\mathcal{I}$ and morphisms the conjugations $I = wJw^{-1} \xrightarrow{w} J$.

The "relative position" of $\mathbf{P}'$ and $\mathbf{P}''$ is determined by a morphism in $\mathcal{C}(\mathcal{I})$.

Parabolic Deligne-Lusztig varieties

We write this $\mathbf{P}'\ {}_I\xrightarrow{w}{}_J\ \mathbf{P}''$. Since only the coset $W_I w W_J$ counts, we may assume that w is I-reduced, or equivalently $\mathbf{w}$ is divisible on the left by no element of $\mathbf{I}$.
We define the *parabolic Deligne-Lusztig variety*
$\mathbf{X}(I \xrightarrow{w} I) = \{\mathbf{P} \mid\ {}_I\xrightarrow{w}{}_I\ F(\mathbf{P})\}$.

Let $\mathcal{I}$ be the set of conjugates in $\mathbf{S}$ of $\mathbf{I} \subset \mathbf{S}$.

The ribbon category $B(\mathcal{I})$ *has objects the elements of* $\mathcal{I}$*, and morphisms the* $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ *where* $\mathbf{b} \in B^+$ *is such that*

- $\mathbf{I} = \mathbf{b}\mathbf{J}\mathbf{b}^{-1}$.
- *No element of* $\mathbf{I}$ *divides* $\mathbf{b}$ *on the left (we say* $\mathbf{b}$ *is* $\mathbf{I}$*-reduced).*

Varieties associated to ribbons

Example of ribbon

In B_4 with $\mathbf{S} = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ a $\{\mathbf{s}_2\}$-ribbon.

The category $B(\mathcal{I})$ has a Garside family formed of the morphisms where $\mathbf{b} \in \mathbf{W}$.

We associate the variety $\mathbf{X}(I \xrightarrow{w} I)$ to the morphism $\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}$.
To a morphism $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{I} = \mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_1 \ldots \mathbf{I}_{n-1} \xrightarrow{\mathbf{w}_n} \mathbf{I}$ with $\mathbf{w}_i \in \mathbf{W}$ we associate the variety $\{\mathbf{P}, \mathbf{P}_1, \ldots \mathbf{P}_n \mid \mathbf{P} \xrightarrow{I,w,I_1} \mathbf{P}_1 \ldots \mathbf{P}_n \xrightarrow{I_{n-1},w_n,I} F(\mathbf{P})\}$.

Varieties associated to ribbons

By extending Deligne's theorem to representations of a Garside category into a bicategory, one can show that there is a canonical isomorphism between the varieties attached to two decompositions of $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$. This allows to attach "parabolic Deligne-Lusztig varieties" $\mathbf{X}(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{I})$ to morphisms in $B(\mathcal{I})$.
The cyclic conjugacy of the ribbon category occurs here: whenever we have a divisor $\mathbf{x}$ of $\mathbf{b}$ in $B(\mathcal{I})$ there is a well-defined morphism $\mathbf{X}(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{I}) \xrightarrow{D_{\mathbf{x}}} \mathbf{X}(\mathbf{x}^{-1}\mathbf{I}\mathbf{x} \xrightarrow{\mathbf{x}^{-1}\mathbf{b}\mathbf{x}} \mathbf{x}^{-1}\mathbf{I}\mathbf{x})$.

The category with objects the parabolic varieties $\mathbf{X}(\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{I})$ and morphisms the compositions of the $D_{\mathbf{x}}$ identifies to the cyclic conjugacy category $\operatorname{cyc} B^+(\mathcal{I})$ of $B(\mathcal{I})$.

Eigenspaces and roots in the braid group

For any finite Coxeter group W,

Proposition (Digne-M., He-Nie)

Let $\zeta = e^{2ik\pi/d}$, where $2k \le d$ (k prime to d). Let $V_\zeta \subset V$ be a subspace on which $w \in W$ acts by ζ. Then, up to W-conjugacy we have

- *$C_W(V_\zeta) = W_I$ for some $I \subset S$ (thus $wIw^{-1} = I$).*
- *For the I-reduced element w the lift $\mathbf{w}$ to the braid monoid satisfies $\mathbf{w}^d = (\Delta^2/\Delta_I^2)^k$*

Roots

We have a kind of converse

Let $\mathbf{w} \in B^+$ and d such that $\mathbf{w}^d = \Delta^2/\Delta^2_{\mathbf{I}}$ for some $\mathbf{I} \subset \mathbf{S}$. Then

- $\mathbf{w}\mathbf{I}\mathbf{w}^{-1} = \mathbf{I}$, *thus* $\mathbf{w}$ *defines a morphism* $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}) \in B(\mathcal{I})$.
- *Let V_d be the $\zeta_d = e^{2i\pi/d}$-eigenspace of w. Then $C_W(V_d) \subset W_I$*

Further, the following conditions are equivalent

- $\mathbf{w}$ *is "not extendible", that is there does not exist $\mathbf{J} \subset \mathbf{I}$ and $\mathbf{v} \in B^+_{\mathbf{I}}$ such that $(\mathbf{vw})^d = \Delta^2/\Delta^2_{\mathbf{J}}$.*
- $C_W(V) = W_I$, *and V_d is a maximal ζ_d-eigenspace of W.*

Varieties for roots

The same questions arise in this generalized context, with less answers.

Assume that $(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}) \in B(\mathcal{I})$ is such that some power of $\mathbf{w}$ is divisible by $\Delta_{\mathbf{I}}^{-1}\Delta$. Then $\mathrm{End}_{\mathrm{cyc}\, B^+(\mathcal{I})}(\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I})$ coincide with the endomorphisms in the conjugacy category $\{\mathbf{x} \mid (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I}) \xrightarrow{\mathbf{I} \xrightarrow{\mathbf{x}} \mathbf{I}} (\mathbf{I} \xrightarrow{\mathbf{w}} \mathbf{I})\}$.

In particular $\mathbf{x} \in C_{B^+}(\mathbf{w})$. Coming back to the case of roots of Δ^2/Δ^2_I, the group $C_W(wW_I)/W_I$ is a complex reflection group.

Conjecture

$\{\mathbf{x} \in C_{B^+}(\mathbf{w}) \mid \mathbf{x}\mathbf{I}\mathbf{x}^{-1} = \mathbf{I}$ and $\mathbf{x}$ is $\mathbf{I}$-reduced$\}$ is a monoid for the braid group of that complex reflection group.

The really general case

We made a simplifying assumption all along: that F acts trivially on W.
This is not always the case; for instance the unitary group is defined taking the same algebraic group $\mathrm{GL}_n(\mathbb{F})$, but composing the Frobenius of $\mathrm{GL}_n(\mathbb{F}_q)$ with the transpose and inverse maps.
Then this new F acts on W by conjugation by the longest element.
Now we have to replace conjugacy by F-conjugacy: $w \xrightarrow{v} v^{-1}wF(v)$.
The $\mathbf{G}^F$-classes of tori are parameterized by F-conjugacy classes.
We get morphisms of Deligne-Lusztig varieties in the "cyclic F-congugacy category".
We have to look at F-centralizers — here already the results of Bessis are not (yet?) applicable.