Artin groups of Euclidean type

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Coxeter groups

The central motivating examples of Coxeter groups are the discrete groups generated by reflections acting geometrically on spheres and euclidean spaces. These arise in the study of regular polytopes and in Lie theory.

General Coxeter groups were introduced by Jacques Tits in the early 1960s. They are defined by simple presentations that can be encoded in diagrams. And in the paper that introduced these groups, Tits proved that every general Coxeter group has a faithful linear representation preserving a symmetric bilinear form and thus has a solvable word problem.

Coarse classification

Coxeter groups can be coarsely classified by the signature of the symmetric bilinear forms they preserve.

- If there are no negative eigenvalues it is spherical.
- If there is one negative eigenvalue it is hyperbolic.
- If there is more than one negative eigenvalue it is higher-rank.
- If there are zero eigenvalues, we add the adjective weakly.

But weakly spherical Coxeter groups are better known as euclidean or affine Coxeter groups since they act geometrically on euclidean space.

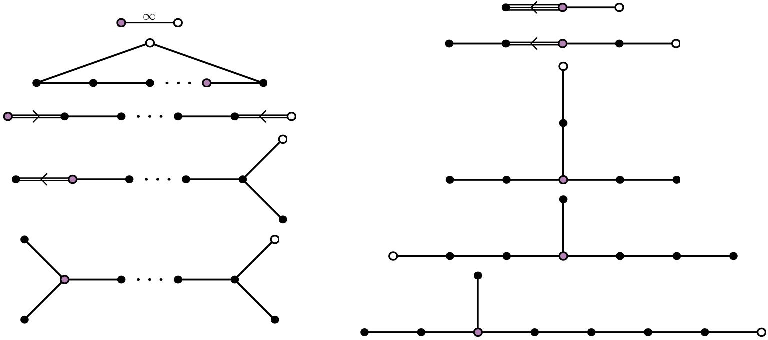
Spherical and euclidean

The classification of *spherical* and *euclidean* Coxeter groups is classical. Their presentations are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams, respectively. They use simplified conventions that are sufficient for these groups, but not for general Coxeter groups.

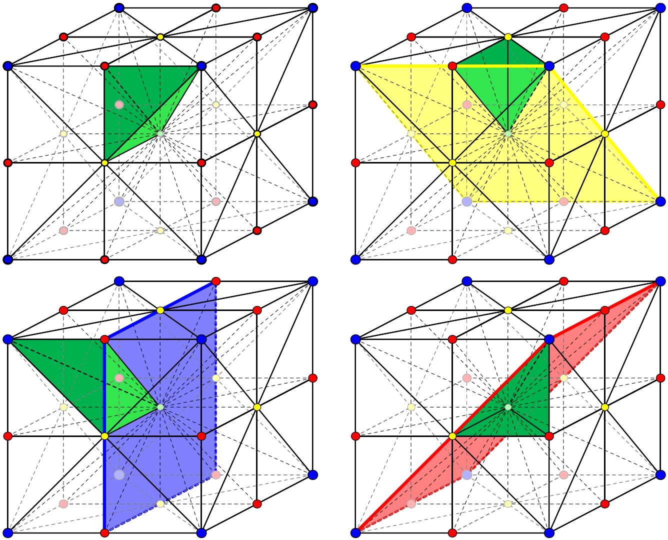
The extended Dynkin diagrams are shown on the next slide.

- Four infinite families: $\widetilde{A}_n$, $\widetilde{C}_n$, $\widetilde{B}_n$, and $\widetilde{D}_n$.
- Five exceptions: $\widetilde{G}_2$, $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$.

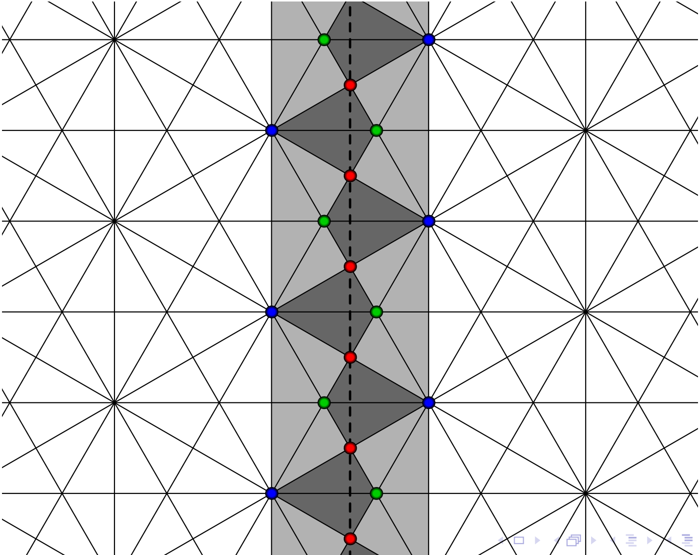
Extended Dynkin diagrams



The spherical Coxeter group $\mathrm{Cox}(B_3)$



The euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



Crystallographic groups

The class of *crystallographic groups* are those groups which act geometrically on euclidean space but without the restriction that they be generated by reflections. The class contains the irreducible euclidean Coxeter groups, but it is larger.

• Free abelian groups are crystallographic groups but not Coxeter groups.

• The index 2 subgroup of orientation-preserving isometries of an irreducible euclidean Coxeter group is crystallographic but not a Coxeter group.

Artin groups

Artin groups first appear in print in 1972 (Brieskorn and Saito, Deligne). General Artin groups are defined by simple presentations that can be encoded in the same diagrams as Coxeter groups and then coarsely classified in the same way.

Those early papers connected spherical Artin groups to the fundamental groups of spaces derived from complexified hyperplane complements and successfully analyzed their structure.

Given the centrality of euclidean Coxeter groups and the elegance of their structure, one might have expected euclidean Artin groups to be well understood shortly thereafter. It is now 40 years later and these groups are still revealing their secrets.

Basic Questions

In a recent article Eddy Godelle and Luis Paris highlight four basic conjectures about Artin groups:

Conjectures

A) *Every Artin group is torsion-free*
B) *Every non-spherical Artin group has trivial center*
C) *Every Artin group has a solvable word problem*
D) *Artin groups satisfy the* $K(\pi, 1)$ *conjecture*

They also remark:

> *"A challenging question in the domain is to prove Conjectures A, B, C, and D for the so-called Artin-Tits groups of affine type, that is, those Artin-Tits groups for which the associated Coxeter group is affine."*

Euclidean Artin Groups

Rob Sulway and I, based on work with Noel Brady and John Crisp, provide positive solutions to Conjectures *A*, *B* and *C* for all euclidean Artin groups and we also make progress on Conjecture *D*.

In particular, we prove the following:

Theorem (M-Sulway)

Every irreducible euclidean Artin group $\textsc{Art}(\widetilde{X}_n)$ *is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

Known results

The easiest way to introduce our approach to these groups is to review some history.

Three types of known results about euclidean Artin groups:

- Folklore results about the groups $\textsc{Art}(\widetilde{A}_n)$.
- Craig Squier on $\textsc{Art}(\widetilde{A}_2)$, $\textsc{Art}(\widetilde{C}_2)$ and $\textsc{Art}(\widetilde{G}_2)$.
- François Digne on $\textsc{Art}(\widetilde{A}_n)$ and $\textsc{Art}(\widetilde{C}_n)$.

To my knowledge, these are the only euclidean Artin groups that were previously fully understood.

Middle groups

The euclidean Artin group $\text{ART}(\widetilde{A}_{n-1})$ has long been well understood because it can be viewed as a subgroup of the spherical Artin group $\text{ART}(B_n)$.

We call the symmetries of $\mathbb{Z}^n$ generated by coordinate permutations and integral translations the annular symmetric group or the middle group $\text{MID}(B_n)$, for lack of a better name.

The group $\text{MID}(B_n)$ is generated by $\mathcal{S} = \{s_{ij}\}$ and let $\mathcal{T} = \{t_i\}$ where s_{ij} is the reflection that switches coordinates i and j and t_i is the translation that adds 1 to the i-th coordinate.

The group $\text{MID}(B_n)$ is a semidirect product $\mathbb{Z}^n \rtimes \text{SYM}_n$ with $\mathcal{T}$ generating the normal free abelian subgroup.

The middle group and its presentation

- $\textsc{Mid}(B_n)$ is generated by $\{t_1\} \cup \{s_{12}, s_{23}, \ldots, s_{n-1n}\}$.
- t_1 and s_{12} satisfy the identity $t_1 s_{12} t_1 s_{12} = s_{12} t_1 s_{12} t_1$.
- The element t_1 commutes with the other generators.
- Thus $\textsc{Mid}(B_n)$ is a quotient of $\textsc{Art}(B_n)$ but in this group t_1 has infinite order and the s_{ij} have order 2.
- Adding the relation $t_1^2 = 1$ gives a quotient onto $\textsc{Cox}(B_n)$.
- Factorizations of $t_1 s_{12} s_{23} \cdots s_{n-1n}$ form a noncrossing partition lattice of type B.

The middle group and its relatives

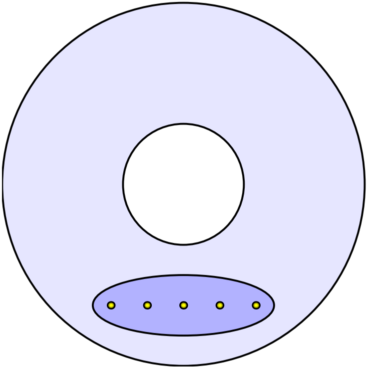
The middle group is closely related to several Coxeter groups and Artin groups, hence its name.

$$\begin{array}{ccccc} \textsc{Art}(\widetilde{A}_{n-1}) & \hookrightarrow & \textsc{Art}(B_n) & \twoheadrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \textsc{Cox}(\widetilde{A}_{n-1}) & \hookrightarrow & \textsc{Mid}(B_n) & \twoheadrightarrow & \mathbb{Z} \\ & & \downarrow & & \\ & & \textsc{Cox}(B_n) & & \end{array}$$

We call these groups the euclidean braid group, the annular braid group, the euclidean symmetric group, the annular symmetric group, and the signed symmetric group.

Explaining the names

We avoid the name affine braid group because both $\mathrm{ART}(\widetilde{A}_{n-1})$ and $\mathrm{ART}(B_n)$ have been called this in the literature.



Recognizing a middle group

The following configuration of euclidean isometries generates a group isomorphic to the middle group.

Theorem

Let S be a simple system for a symmetric group SYM_n with root system Φ and let $t = t_\lambda$ be a translation. If λ is not contained in the span of Φ and t does not commute with exactly one of the elements of S, then the group generated by $S \cup \{t\}$ is isomorphic to $\mathrm{MID}(B_n)$.

Planar Artin groups

In 1987 Craig Squier successfully analyzed the structure of the three irreducible euclidean Artin groups $\mathrm{ART}(\widetilde{A}_2)$, $\mathrm{ART}(\widetilde{C}_2)$ and $\mathrm{ART}(\widetilde{G}_2)$ that correspond to the three irreducible euclidean Coxeter groups acting on the euclidean plane.

He worked directly with the presentations and analyzed them using Bass-Serre decompositions, that is, as amalgamated products and HNN extensions of well-known groups.

His techniques do not appear to generalize to higher dimensions.

Dual presentations

The final results about euclidean Artin groups are two papers by François Digne in the early 2000s.

Theorem (Digne)

The groups $\mathrm{ART}(\widetilde{A}_n)$ *and* $\mathrm{ART}(\widetilde{C}_n)$ *have Garside structures.*

In both cases he shows that the dual presentations - obtained by considering the poset of all factorizations of a carefully chosen Coxeter element over the generating set of all reflections - is Garside (= quasi-Garside).

Our first attempt was to follow his approach.

Artin groups as Garside groups?

There were other indications that all Artin groups might have dual presentations that are Garside.

- The free group.
- Artin groups with 3 generators.
- Artin groups in which all $m_{ij} \geq 6$.

The last two are unpublished results with John Crisp.

Conjecture

All Artin groups have dual presentations that are Garside.

Dual presentations and Garside structures

This conjecture is too optimistic and false.

Theorem (Crisp-M)

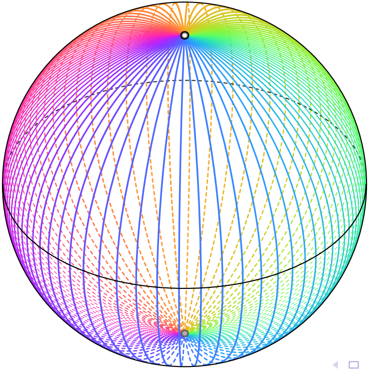
The unique dual presentation of $\mathrm{ART}(\widetilde{X}_n)$ *is a Garside structure when* X *is C or G and it is not a Garside structure when* X *is B, D, E or* F*. When the group has type A there are distinct dual presentations and the one investigated by Digne is the only one that is a Garside structure.*

This is proved by computing the dual presentations explicitly and geometrically and then showing that they fail to have the lattice property.

Intervals

Definition

z is between x and y when $d(x,z) + d(z,y) = d(x,y)$. All points between x and y form the interval $[x,y]$. Intervals are posets with $z \leq w$ iff $d(x,z) + d(z,w) + d(w,y) = d(x,y)$.



Intervals from groups

Cayley graphs are metric spaces and thus they have intervals.

Definition

Since posets can be recovered from their Hasse diagrams, let $[g, h]$ denote the edge-labeled directed graph inside the Cayley graph that is the union of shortest paths from v_g to v_h. It is also the Hasse diagram of the poset order on $[g, h]$.

Remark

Cayley graphs are homogeneous so the interval $[g, h]$ is isomorphic (as an edge-labeled directed graph) to the interval $[1, g^{-1}h]$. Thus we can restrict to intervals of the form $[1, g]$.

Interval groups and dual presentations

Groups lead to intervals which lead to new groups.

Definition

Let $[1, g]$ be an interval in a marked group G. The interval group G_g is the group generated by the labels of edges in the interval subject to the relations that are visible in the interval.

Definition

Let w be a Coxter element in an irreducible euclidean Coxeter group $W = \text{Cox}(\widetilde{X}_n)$ generated by the set of all reflections. The dual Artin group $\text{Art}^*(\widetilde{X}_n, w)$ is the interval group W_w.

Coxeter intervals vs. Artin intervals

It is not known in general whether Artin groups and dual Artin groups are isomorphic. This is because the duals are defined as an interval group over the corresponding Coxeter group.

• Computing intervals in $\text{ART}(\widetilde{X}_n)$ is not really doable in general but the interval group would definitely be the Artin group.

• Computing intervals in $\text{COX}(\widetilde{X}_n)$ is doable but it is less clear whether we are presenting the correct group.

Theorem (M-Sulway)

For every choice of Coxeter element in an irreducible euclidean type $\widetilde{X}_n$, the dual Artin group $\text{ART}^*(\widetilde{X}_n, w)$ is naturally isomorphic to $\text{ART}(\widetilde{X}_n)$.

Elliptic and Hyperbolic isometries

Euclidean isometries are called elliptic when they fix points and hyperbolic when they do not.

Elliptic isometries have fixed sets.

Hyperbolic isometries have minsets.

Definition

Let α be a euclidean isometry and let *d* be the infimum of the distances that points are moved. The set of points that are moved exactly distance *d* form a nonempty affine subspace called $\textsc{Min}(\alpha)$, the minset of α,

Coxeter elements

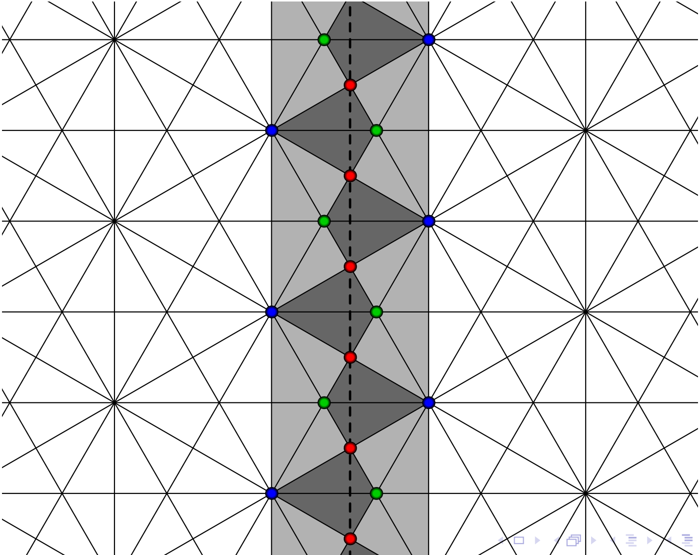
Definition

If w is a Coxeter element of an irreducible euclidean Coxeter group then $\textsc{Min}(w)$ is a line called its axis. The top-dimensional simplices whose interior nontrivially intersects the axis are called axial simplices and the vertices of these simplices are axial vertices.

Theorem

In irreducible euclidean Coxeter groups whose diagrams are trees (i.e. every type except $\widetilde{A}_n$), the axial simplices of Coxeter element w are exactly those chambers whose bipartite Coxeter elements produce w and its inverse.

The euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



Reflection generators

Let w be a Coxeter element of an irreducible euclidean Coxeter group $\textsc{Cox}(\widetilde{X}_n)$.

Theorem

A reflection labels an edge in the interval $[1, w]$ iff its fixed hyperplane contains an axial vertex.

The set of reflections labeling edges in $[1, w]$ consists of every reflection whose hyperplane crosses the Coxeter axis and those reflections which move points horizontally and bound the convex hull of the axial simplices. We call these the vertical and horizontal reflections below w.

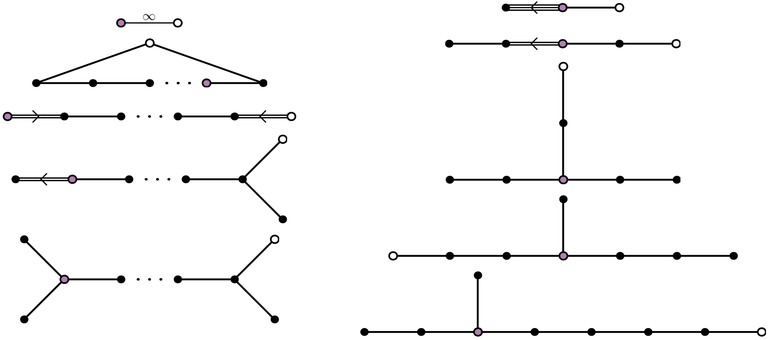
Horizontal Roots

Definition

The horizontal reflections below w have roots that are orthogonal to the Coxeter axis of w. The horizontal roots form a root system inside the full root system that we call the horizontal root system.

In the diagrams on the next page the horizontal root system corresponds to the subdiagram obtained by removing both non-black vertices. For type $\widetilde{A}_n$ there is more than one possibility.

Extended Dynkin diagrams



Horizontal Root Systems

Type	Horizontal root system
A_n	$\Phi_{A_{p-1}} \cup \Phi_{A_{q-1}}$
C_n	$\Phi_{A_{n-1}}$
B_n	$\Phi_{A_1} \cup \Phi_{A_{n-2}}$
D_n	$\Phi_{A_1} \cup \Phi_{A_1} \cup \Phi_{A_{n-3}}$
G_2	Φ_{A_1}
F_4	$\Phi_{A_2} \cup \Phi_{A_1}$
E_6	$\Phi_{A_2} \cup \Phi_{A_1} \cup \Phi_{A_2}$
E_7	$\Phi_{A_2} \cup \Phi_{A_1} \cup \Phi_{A_3}$
E_8	$\Phi_{A_2} \cup \Phi_{A_1} \cup \Phi_{A_4}$

Theorem (Crisp-M)

The dual presentation is a Garside structure iff the horizontal root system is irreducible.

Artin groups as subgroups

The proof that euclidean Artin groups are understandable does **not** prove that they have Garside structures.

We show instead that they are **subgroups** of Garside groups.

Theorem (M-Sulway)

For every choice of a Coxeter element w in an irreducible euclidean Coxeter group $\text{Cox}(\widetilde{X}_n)$, the Artin group $\text{Art}(\widetilde{X}_n)$ is isomorphic to a subgroup of a Garside group we call $\text{Gar}(\widetilde{X}_n, w)$.

These Garside groups are completely new examples.

Six groups

$$\begin{array}{ccccc}
\textsc{Art}(\widetilde{X}_n) & = & \textsc{Art}^*(\widetilde{X}_n, w) & \hookrightarrow & \textsc{Gar}(\widetilde{X}_n, w) \\
\downarrow & & \downarrow & & \downarrow \\
\textsc{Cox}(\widetilde{X}_n) & = & \textsc{Cox}^*(\widetilde{X}_n, w) & \hookrightarrow & \textsc{Cryst}(\widetilde{X}_n, w)
\end{array}$$

For each Coxeter element w in an irreducible euclidean Coxeter group of type $\widetilde{X}_n$ we define several groups. The notations in the middle column refer to the Coxeter group and the Artin group as defined by their dual presentations and it is these presentations that facilitate the connection between the Coxeter group $\textsc{Cox}(\widetilde{X}_n)$ and the crystallographic group $\textsc{Cryst}(\widetilde{X}_n, w)$ and between the Artin group $\textsc{Art}(\widetilde{X}_n)$ and the crystallographic Garside group $\textsc{Gar}(\widetilde{X}_n, w)$.

Crystallographic Garside Groups

The new Garside groups are based on intervals in crystallographic groups closely related to the irreducible euclidean Coxeter groups.

Theorem (M-Sulway)

Let $W = \textsc{Cox}(\widetilde{X}_n)$ be an irreducible euclidean Coxeter group and let R be its set of reflections. For each Coxeter element $w \in W$ there exists a crystallographic group $\textsc{Cryst}(\widetilde{X}_n, w)$ containing W with a generating set $R \cup T$ so that the weighted factorizations of w over $R \cup T$ form a balanced lattice. As a consequence, this interval defines a group $\textsc{Gar}(\widetilde{X}_n, w)$ with a Garside structure.

The grid

The interval $[1, w]$ can be given a coarse structure as follows.

$$
\begin{tikzcd}
T & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r] & \bullet & w & (h, e) \\
R_V & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r, dotted, no head] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & & (e, e) \\
1 & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r, dotted, no head] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & & (e, h) \\
 & & R_H & & & & &
\end{tikzcd}
$$

Multiplying by horizontal reflections move left/right.
Multiplying by vertical reflections move up/down (or possibly left/right inside the middle row).

Products of middle groups

The failures of the lattice property always involve two elements in the top row which are minimum common multiples for two elements in the bottom row.

If we focus on the subgroup generated by R_H and T we see a finite induced poset consisting of just the top and bottom rows.

This finite poset is almost a poset product of noncrossing partitions lattices of type B, i.e. the subgroup they generate is nearly a direct product of middle groups and the corresponding interval group is nearly a direct product of type B Artin groups.

The problem is that the translations in T are diagonal in the sense that they have effects in each irreducible factor.

Factored translations

In order to complete the near product of groups, we factor the pure translations below w so that they only travel in one irreducible component of the horizontal root system and the direction of the Coxeter axis.

We call the set of factored translations T_F and we define various groups based on various sets of generators.

Name	Symbol	Generating set
Horizontal	C_H	R_H
Diagonal	C_D	$R_H \cup T_D$
Coxeter	C_C	$R_H \cup R_V$ $(\cup\ T_D)$
Factorable	C_F	$R_H \cup T_F$ $(\cup\ T_D)$
Crystallographic	C	$R_H \cup R_V \cup T_F$ $(\cup\ T_D)$

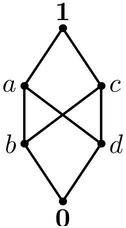
Ten groups

Ten groups defined for each choice of a Coxeter element in an irreducible euclidean Coxeter group and some of the maps between them.

$$
\begin{tikzcd}
 & & G_F \arrow[rr] \arrow[dd, two heads] & & G \arrow[dd, two heads] \\
G_H \arrow[r] \arrow[dd, two heads] & G_D \arrow[ur] \arrow[rr] \arrow[dd, two heads] & & G_A \arrow[ur] \arrow[dd, two heads] & \\
 & & C_F \arrow[rr, hook] & & C \\
C_H \arrow[r, hook] & C_D \arrow[ur, hook] \arrow[rr, hook] & & C_C \arrow[ur, hook] &
\end{tikzcd}
$$

Lattices, bowties and GAP

As in the spherical case, the most difficult part is to find a uniform proof that these crystallographic intervals are lattices.



I am still searching but in the meantime I wrote GAP routines to test low dimensional examples looking for bowties.

The computer verifies that they are lattices for $n \leq 8$. This covers all of the exceptional cases and only leaves types *B* and *D* (for $n \geq 9$) which are done individually.

Where to next?

These results raise several questions.

• Are we sure that Artin groups as presently defined are the groups we really care about? Or are these crystallographic Garside groups the first instance of a natural geometric completion process?

• Now that we understand the word problem for the euclidean types, can we devise an Artin group intrinsic solution that avoids the crystallographic Garside groups?

• What about hyperbolic Artin groups? Do similar procedures work there?

References

This talk are based on three papers. The first two are available from my preprints page (they are not yet on the arXiv).

• Noel Brady and Jon McCammond, “Factoring euclidean isometries”.

• (John Crisp and) Jon McCammond, “Dual euclidean Artin groups and the failure of the lattice property”.

• Jon McCammond and Robert Sulway, “Euclidean Artin groups”

The conclusion of the trilogy is mostly written, but not yet complete and not yet available.