Vrije Universiteit Brussel

Some concrete classes of finitely presented monoids

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Overview

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3. Proving this theorem (on semigroup level) for a bigger class of monoids (that also contains the class of left divisibility monoids defined by $\binom{n}{2}$ relations).
4. Work in progress: Is the semigroup algebra of such a monoid Noetherian?

Definition: monoid of left I-type

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Definition

FaM_n free abelian monoid with generators $u_1, \ldots, u_n$.
Monoid $S = \langle x_1, \ldots, x_n \rangle$ is of **left I-type** if there exists a bijection (left I-structure) $v : FaM_n \to S$ such that

$$v(1) = 1 \quad \text{en} \quad \{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\}$$

for all $a \in FaM_n$.

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for all $a \in FaM_n$.

Proposition (E. Jespers, J. Okniński)

A monoid of I-type has a presentation of the form
$\langle x_1, \ldots, x_n \mid x_{f_{u_i}(j)} x_{f_1(i)} = x_{f_{u_j}(i)} x_{f_1(j)},\ 1 \leqslant i < j \leqslant n \rangle$.

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   $S \cong T = \{(c, \sigma_c) \mid c \in FaM_2\} \subseteq G$ where

$$\sigma_c = \begin{cases} \sigma & \text{if } c \text{ odd length} \\ 1 & \text{if } c \text{ even length} \end{cases}$$

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   $v : FaM_2 \to T : c \mapsto (c, \sigma_c)$ right I-structure

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Remark: if S is a left divisibility monoid, then $(S, \leqslant_l)$ is a poset.

Proposition (D. Kuske)

$(S, \cdot, 1)$ *left divisibility monoid and* X *set of its atoms. Let* $\sim$ *denote the least congruence on the free monoid* X^* *containing* $E := \{(ab, cd) \mid a, b, c, d \in X \text{ and } a \cdot b = c \cdot d\}$. *Then* $\sim$ *is the kernel of the natural epimorphism* $[.] : X^* \to S$. *In particular,* $S \cong X^*/\sim$.

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Theorem (D. Kuske)

T finite set, E set of equations of the form $ab = cd$, with $a, b, c, d \in T$. Let $\sim$ be the least congruence on T^* containing E. $S := T^*/\sim$ is a left divisibility monoid $\Leftrightarrow$ $\forall\, a, b, c, a', b', c' \in T$:

(i) $(\downarrow (a \cdot b \cdot c), \leqslant_l)$ *is a distributive lattice;*

(ii) $a \cdot b \cdot c = a \cdot b' \cdot c' \Rightarrow b \cdot c = b' \cdot c'$ *and* $b \cdot c \cdot a = b' \cdot c' \cdot a \Rightarrow b \cdot c = b' \cdot c'$;

(iii) $a \cdot b = a' \cdot b'$, $a \cdot c = a' \cdot c'$ *and* $a \neq a'$ *imply* $b = c$.

Corollary

Let $S = \langle x_1, \ldots, x_n \rangle$ be a left divisibility monoid. Then there are at most $\binom{n}{2}$ defining relations for S.

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$$\{ab = ba\}$$
$$\{aa = bb\}$$

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3. (Christian Pech) There are (up to isomorphism) precisely 15 left divisibility monoids on three generators a, b and c, namely the free monoid and those defined by the following sets of equations

$$\begin{array}{lll}
\{ab = ba\} & \{ab = ba, bc = cb\} & \{ab = ba, bc = cb, ac = ca\} \\
\{aa = bb\} & \{ab = bc, ac = ca\} & \{aa = bb, bc = cb, ac = ca\} \\
\{ab = bc\} & \{ab = bc, ba = cb\} & \{ab = bc, ba = cb, ac = ca\} \\
\{aa = bc\} & \{ab = bc, ac = cb\} & \{aa = bb, ac = cb, ca = bc\} \\
 & \{aa = bc, cc = ba\} & \{aa = bc, bb = ca, cc = ab\}
\end{array}$$

Furthermore there are 219 left divisibility with four generators and 8371 with five generators.

Definition and examples: Garside monoids

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Definition

A monoid S is a **Garside monoid** if

1. S is generated by its atoms
2. $\|x\| < \infty$
3. S is cancellative
4. $x \vee y$, $x \wedge y$, $x \tilde{\vee} y$, $x \tilde{\wedge} y$ exist
5. $\exists$ Garside element Δ (i.e. $\downarrow \Delta = \Delta \downarrow$ is finite and generates S)

Examples:

1. $FaM_2 = \langle a, b \mid ab = ba \rangle$, Garside element: ab
2. (P. Dehornoy) $S = \langle a, b \mid aba = b^2 \rangle$, Garside element: b^3

I-type vs left divisibility

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Theorem (E. Jespers, J. Okniński, MVC)

1. *If S is a monoid of I-type, then S is a left divisibility monoid.*

I-type vs left divisibility

Theorem (E. Jespers, J. Okniński, MVC)

1. *If S is a monoid of I-type, then S is a left divisibility monoid.*
2. *Conversely, if $S = \langle x_1, \ldots, x_n \rangle$ is a left divisibility monoid defined by $\binom{n}{2}$ relations, then S is a monoid of I-type.*

Garside vs left divisibility

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Theorem (M. Picantin)

1. *Let $S = \langle x_1, \ldots, x_n \rangle$ be a left divisibility monoid. Then S is Garside monoid*

   $\Leftrightarrow$ x_i and x_j have a commun upperboud for the left divisor relation $\forall i, j$ (i.e. x_i and x_j admit right common multiples).

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   $\Leftrightarrow$ *x_i and x_j have a commun upperboud for the left divisor relation $\forall i, j$ (i.e. x_i and x_j admit right common multiples).*
   $\Leftrightarrow$ *S is defined by $\binom{n}{2}$ relations.*
2. *Let S be a Garside monoid. Then S is a left divisibility monoid*
   $\Leftrightarrow$ *the lattice $(S, \vee, \wedge)$ of its simple elements is a hypercube.*

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2. *If $S = \langle x_1, \ldots, x_n \rangle$ is Garside monoid defined by $\binom{n}{2}$ relations and $x_i x_j$ appears at most once in the defining relations for any, then S monoid of I-type.*

Conclusion

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Theorem

A monoid is of I-type if and only if it is both a left divisibility and a Garside monoid.

Definitions

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Definition

Monoid of **skew type**: $S = \langle x_1, \ldots, x_n \mid R \rangle$.
R finite set of $\binom{n}{2}$ relations of the form $x_i x_j = x_k x_l$ with $i \neq j$, $k \neq l$, $(i,j) \neq (k,l)$ and every word $x_i x_j$ (with $i \neq j$) appears in exactly one defining relation.

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Monoid of **quadratic type**: $S = \langle x_1, \ldots, x_n \mid R \rangle$.
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Definitions

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$S = \langle x_1, \dots, x_n \rangle$, $n \geqslant 2$ monoid defined by quadratic relations such that for every $x, y \in X = \{x_1, \dots, x_n\}$ the word xy appears in at most one of the defining relations.
Associated bijective map $r : X \times X \to X \times X$ defined by

$$r(x_i, x_j) = (x_k, x_l)$$

if $x_i x_j = x_k x_l$ is a defining relation for S, and $r(x_i, x_j) = (x_i, x_j)$ otherwise.

Definition

S is **right non-degenerate**, if for all $x, x' \in X$, there exist unique elements $y, y' \in X$ such that $r(x, y) = (x', y')$.

Lemma

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Lemma (E. Jespers, J. Okniński, MVC)

$S = \langle x_1, \ldots, x_n \mid R \rangle$ monoid of quadratic type. Are equivalent:

1. *S is right non-degenerate;*
2. (i) *there are no defining relations of the form $xy = xy'$, for $y \neq y'$;*
   (ii) *if $xy = x'y'$ and $xz = x'z'$, with $x, x', y, y', z, z' \in X$ and $x \neq x'$ it follows that $y = z$.*

In particular, if S is right non-degenerate, then for any $x \in X$, there exists a unique $y \in X$ so that xy does not appear in any defining relation (we also say xy is not rewritable).

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Note: left divisibility monoids with $\binom{n}{2}$ relations and right non-degenerate monoids of skew type satisfy these conditions.

Property

Proposition (E. Jespers, J. Okniński, MVC)

$S = \langle X; R \rangle$ right non-degenate monoid of quadratic type. For every $w, a \in S$, there exist $k \geqslant 1$ and $b \in S$ such that $w^k a = ab$ (S satisfies the over-jumping property).

Lemmas

S monoid with generating set $X = \{x_1, \ldots, x_n\}$ and $Y \subseteq X$.

$$S_Y = \bigcap_{y \in Y} yS \quad \text{and} \quad S_j = \bigcup_{Y \subseteq X, |Y| = j} S_Y.$$

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2. *$S = \langle x_1, \ldots, x_n \rangle$, $n \geqslant 2$ right non-degenate monoid of quadratic type. Then $S_1 \backslash S_2$ is the set of non-rewritable elements of S.*

Proposition

Proposition (E. Jespers, J. Okniński, MVC)

Let $S = \langle x_1, \ldots, x_n \mid R \rangle$ be a right non-degenerate monoid of quadratic type. Then

$$S \backslash \{1\} = \{w_1 \cdots w_q \mid 1 \leqslant q \leqslant n, w_i \in A_k \text{ for some } 1 \leqslant k \leqslant n\},$$

where $S_1 \backslash S_2 = A_1 \cup \cdots \cup A_n$ is the set of all non-rewritable elements in S and each A_j consists of subwords of an infinite periodic word of period $\leqslant n$. In particular: $GKdim(S) \leqslant n$.

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Lemma (E. Jespers, J. Okniński, MVC)

$S = \langle x_1, \ldots, x_n \rangle$, $n \geqslant 2$ right non-degenate monoid of quadratic type, $Y \subseteq X$, $|Y| = i - 1$ and $Z \subseteq Y$. If $b \in D_Z$ and $|b| = k$, then

$$(S_{i-1})^k \cap D_Y \subseteq bS.$$

Theorem

Theorem (E. Jespers, J. Okniński, MVC)

A right non-degenate monoid S of quadratic type satisfies the ascending chain condition on right ideals.

Proof

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- By contradiction: assume $s_1, s_2, \ldots \in S \setminus S_{i+1}$ such that strictly ac of right ideals:

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- By going to a subsequence $(a_j)_j$ of $(s_j)_j$: there exists a minimal non-negative integer r for wich there is a strictly ac of right ideals:

$$a_1 S \subset a_1 S \cup a_2 S \subset a_1 S \cup a_2 S \cup a_3 S \subset \cdots,$$

  such that $a_j \in S_i^r \backslash S_i^{r+1}$

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  such that $a_j \in S_i^r \backslash S_i^{r+1}$
- By going to a subsequence again: $a_j \in D_Z$ for some $Z \subseteq X$

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- To prove: S/S_i acc $\Rightarrow S/S_{i+1}$ acc
- By contradiction: assume $s_1, s_2, \ldots \in S \setminus S_{i+1}$ such that strictly ac of right ideals:

$$s_1 S \subset s_1 S \cup s_2 S \subset s_1 S \cup s_2 S \cup s_3 S \subset \cdots$$

- By going to a subsequence $(a_j)_j$ of $(s_j)_j$: there exists a minimal non-negative integer r for wich there is a strictly ac of right ideals:

$$a_1 S \subset a_1 S \cup a_2 S \subset a_1 S \cup a_2 S \cup a_3 S \subset \cdots ,$$

  such that $a_j \in S_i^r \backslash S_i^{r+1}$
- By going to a subsequence again: $a_j \in D_Z$ for some $Z \subseteq X$
- $|Z| = i$ (using the IH)

Proof

- Fix $x \in Z$ and write $a_j = v_j t_j$, where v_j is the non-rewritable word of maximal length, starting with x

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- There are infinitely many distinct $v_j = (x_{i_1} \cdots x_{i_q})(x_{i_{q+1}} \cdots x_{i_p})^{\alpha_j}(x_{i_{q+1}} \cdots x_{i_{s_j}})$,

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- By going to a subsequence: $v_j = uw^{\alpha_j}z$, hence $a_j = uw^{\alpha_j}zt_j = ub_j$
- $b_1 S \subset b_1 S \cup b_2 S \subset b_1 S \cup b_2 S \cup b_3 S \subset \cdots$ strictly ac and $b_j \in S_i^r \backslash S_i^{r+1}$

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- Using the over-jumping property: $b_j = vc_j$ with $c_j \in S_i^{r'} \setminus S_i^{r'+1}$ and $r' < r$
- By the minimality of r, the chain $c_1 S \subset c_1 S \cup c_2 S \subset c_1 S \cup c_2 S \cup c_3 S \subset \cdots$ is not strictly ascending

Corollary

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Corollary

Let $S = \langle x_1, \ldots, x_n \rangle$, $n \geqslant 2$, be a right non-degenate monoid of quadratic type and assume additionally that S is cancellative. Then the group of quotients of S is abelian-by-finite.

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