Nonsymmetric generalisations of Artin monoids

An Artin monoid.

$$A = \langle a, b \mid aba = bab \rangle$$

A~~n~~ Coxeter gp:

$$W = \langle a, b \mid aba = bab, \begin{matrix} a^2 = 1 \\ b^2 = 1 \end{matrix} \rangle$$

a Coxeter monoid:

$$W' = \left\langle a, b \;\middle|\; aba = bab, \begin{array}{l} a^2 = a \\ b^2 = b \end{array} \right\rangle$$

an AI monoid:

$$A'' = \left\langle a, b \;\middle|\; aba = baba \right\rangle$$

a CI monoid.

$$W'' = \left\langle a, b \,\middle|\, \begin{array}{l} aba = baba = abab \\ a^2 = a, \quad b^2 = b \end{array} \right\rangle \checkmark$$

<u>Def</u> a CI-matrix is a map $m: S \times S \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ st

(1) $m(a,b) = 1 \iff a = b$

(2) $m(a,b) = \infty \iff m(b,a) = \infty$

(3) $|m(a,b) - m(b,a)| \leq 1$

$\forall a, b,\ m(a,b) \neq \infty$

Associated is a CI monoid gen'd by S subject to the rels

(a) $a^2 = a \quad \forall a \in S$

(b) $[a, b; m(a,b)] =$

$= [b, a; m(b,a)]$

$\forall a, b \in S$ where

$[x, y; 2n] = (xy)^n$

$[x, y; 2n+1] = (xy)^n x$

"braid relation"

$(m(a,b) \neq \infty)$

(c) $[a, b; m(a,b)] = [a, b; m(a,b)+1]$

$\forall\, a, b \in S$

$(m(a,b) \neq \infty)$

The AI-monoid is

$\langle S \mid (b) \rangle$

Motivation

If $x, y \in M$ = monoid

write $x \leq y \overset{\text{def}}{\iff} y \in xM$

<u>Thm</u> Let M be a monoid gen'd by 2 elements a, b, $\#\{1, a, b\} \neq 2$

a, b incomparable

a, b are idempotents.

Then $(M, \leq)$ is a lattice $\iff$ $(M, \{a, b\})$ is CI-system

The Coxeter graph
has vertex set S
has an edge between
a, b labelled $m(a,b) + m(b,a)$ and direction
a to b if $m(a,b) < m(b,a)$

Type $A_3 = \circ \underset{6}{\text{———}} \circ \underset{6}{\text{———}} \circ$

$m(a,b) = m(b,a) = 2$:

(edges with label 4 are suppressed)

Thm Let (M, S) be a CI-system. Let R be the associative ring presented by gen's x_{ab} ($a, b \in S$ distinct) and rel's (1) $x_{ab} x_{cd} = 0$ unless $b = c$

(2) $[x_{ab}, x_{ba}; m(a,b) - 1] = 0$

Let V be the free R-module with basis $(e_a \mid a \in S)$.
Then $\exists$ M-action on V given by

$$e_a a = 0, \quad e_b a = e_b + x_{ba} e_a$$

whenever $a \neq b$. $\square$

Open questions

(a) Is the refl. rep faithful ?

(b) Solve the Word Problems in W, A

(c) Classify the CI-monoids W with a sink ie ie a $0 \in W$ s.t.

$W \circ W = \{0\}$

(d) Is $(A, \leq)$ a lattice?

$(W, \leq)$?

From now we look at one example.

$$M_n = \overset{a_1}{\circ} \xrightarrow[7]{} \overset{a_2}{\circ} \xrightarrow[7]{} \overset{a_3}{\circ} \quad \cdots \quad \circ \xrightarrow[7]{} \overset{a_n}{\circ}$$

(CI)

$$M_n = \langle a_1, \cdots, a_n \mid$$

(1) $a_i a_j = a_j a_i \quad |i-j| > 1$

(2) $a_i^2 = a_i$

(3) $a_i a_j a_i = a_i a_j a_i a_j$

$= a_j a_i a_j a_i \rangle$

$(j = i+1)$

Lemma ($i = a_i$, $\emptyset \in M_n$ trivial)

$$121323 = 12132$$

Pf $\underbrace{121}323 = 121\underbrace{\hat{2}323} =$

$$= 121\hat{2}32 = 12132 \qquad \square$$

More generally:

$$p(i,j) = a_i \, a_{i+1} \cdots a_j$$

$$q(i,j) = a_i \, a_{i-1} \cdots a_j$$

Then $p(i,j)^2 = p(i,j)\,p(i,j-1)$

 $q(i,j)^2 = q(i-1,j)\,q(i,j)$

<u>Def</u> (1) $F = \langle a_1, \cdots, a_n \rangle$

free monoid

(2) $\approx$ = smallest

congruence on F s.t.

$a_i a_j \approx a_j a_i \quad (|i-j| > 1)$

Thm $\forall x \in M_n$ is represented by a unique commutation class C s.t. no member is of the form abc, b = standard word ($p(i,j)^2$ or $q(i,j)^2$), $a, c \in M$

<u>Corollary</u> $\exists$ fast algorithm to compute the normal form.

$\exists$ fast sol to the WP in M_n.

Corollary $\# M_3 = \infty$

Pf $(1232)^k$

is a normal form $\forall k \geq 0$ $\square$

But

Prop M_n has a sink.

Expected soon:

(1) $(M_n, \leq)$ is a lattice

(2) (A, Δ) is a Garside monoid.