Infimum of powers in Garside groups

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Based on the following works

L-Lee '07 Translation numbers in a Garside group are rational with uniformly bounded denominators, J. Pure Appl. Algebra 211, 732–743.

L-Lee '08 Abelian subgroups of Garside groups, Comm. Algebra 36, 1121–1139.

L-Lee '08 Some power of an element in a Garside group is conjugate to a periodically geodesic element, Bull. Lond. Math. Soc. 40, 593–603.

Garside Groups

Garside groups

- introduced by Dehornoy-Paris in 1999
- a lattice-theoretic generalization of braid groups & finite-type Artin groups

Examples of Garside groups

- Braid groups, finite-type Artin groups
- Free abelian groups of finite rank, $\mathbb{Z}^n$
- Torus knot groups (the fundamental group of the complement of a (p, q)-torus knot): for coprime $p, q > 1$

$$\langle x, y \mid x^p = y^q \rangle$$

Garside groups: notaions

For a Garside group G, there are

- Garside monoid $G^+ \subset G$, Garside element $\Delta \in G^+$,
- partial order $\preceq$ (defined as $a \preceq b$ if $a^{-1}b \in G^+$).

Notations:

- For $g \in G^+$, the norm of g is
  $\|g\| = \sup\{\, \ell \mid g = g_1 \cdots g_\ell \quad \text{where } 1 \neq g_i \in G^+ \quad \forall i \,\}$.
- $\mathcal{D} = [1, \Delta] = \{\, a \in G^+ \mid a \preceq \Delta \,\}$ generates G^+.
  The elements are called simple elements.
- Lattice operations: for $a, b \in G$, $a \wedge b$ (resp. $a \vee b$) is the gcd (resp. lcm) of a and b w.r.t. $\preceq$.

A Garside group G is called finite-type if $\mathcal{D}$ is a finite set, and infinite-type otherwise.

In either case, $\|g\| < \infty$ for any $g \in G^+$, especially $\|\Delta\| < \infty$.

Normal form

Let G be a finite- or infinite-type Garside group and $g \in G$.

g admits a unique expression, called the normal form,

$$g = \Delta^u a_1 \cdots a_\ell$$

s.t. for every i, $1 \prec a_i \prec \Delta$ & $a_i = (a_i a_{i+1} \cdots a_\ell) \wedge \Delta$.

Define

- Infimum: $\inf(g) := u = \max\{r \mid \Delta^r \preceq g\}$,
- Supremum: $\sup(g) := u + \ell = \min\{r \mid g \preceq \Delta^r\}$,
- Canonical-length: $\operatorname{len}(g) := \ell = \sup(g) - \inf(g)$.

Then

- $\inf(g) \leqslant \sup(g)$,
- $\sup(g) = -\inf(g^{-1})$ $\quad (\because g \preceq \Delta^r \iff \Delta^{-r} \preceq g^{-1})$.

Conjugacy classes

Let G be a finite- or infinite-type Garside group and $g \in G$.

Definition

- Conjugacy class: $[g] := \{x^{-1}gx \mid x \in G\}$
- Summit infimum: $\inf_s(g) := \max\{\inf(h) \mid h \in [g]\}$
- Summit supremum: $\sup_s(g) := \min\{\sup(h) \mid h \in [g]\}$

Super summit sets

Let G be a finite- or infinite-type Garside group and $g \in G$.

Definition

The super summit set (SSS) of g is

$$[g]^S := \{h \in [g] \mid \inf(h) = \inf_s(g) \ \& \ \sup(h) = \sup_s(g)\}.$$

Elements of SSSs are called super summit elements.

If G is of finite-type, then $[g]^S$ is a finite set.

If G is of infinite-type, we cannot guarantee that $[g]^S$ is finite.

Cycling & Decycling

Let G be a finite- or infinite-type Garside group and $g \in G$.

Normal form: $g = \boxed{\Delta^u a_1 \cdots a_\ell} = \Delta^u a_1 \Delta^{-u} \Delta^u a_2 \cdots a_{\ell-1} a_\ell$

Definition

- Automorphism: $\tau(g) = \Delta^{-1} g \Delta$
- Cycling: $\mathbf{c}(g) = \Delta^u a_2 \cdots a_\ell \Delta^u a_1 \Delta^{-u}$

  $= (\Delta^u a_1 \Delta^{-u})^{-1} g \, (\Delta^u a_1 \Delta^{-u})$
- Decycling: $\mathbf{d}(g) = a_\ell \Delta^u a_1 \cdots a_{\ell-1} = a_\ell g a_\ell^{-1}$

  $= (\Delta^u a_1 \cdots a_{\ell-1})^{-1} g \, (\Delta^u a_1 \cdots a_{\ell-1})$

Cycling & SSS

Let G be a finite- or infinite-type Garside group and $g \in G$.

Properties of $\tau(\cdot)$, $\mathbf{c}(\cdot)$, $\mathbf{d}(\cdot)$:

- $\tau \circ \mathbf{c} = \mathbf{c} \circ \tau, \quad \tau \circ \mathbf{d} = \mathbf{d} \circ \tau, \quad \mathbf{c} \circ \mathbf{d} = \mathbf{d} \circ \mathbf{c}$;
- $\inf(g) = \inf(\tau(g)) \leqslant \sup(\tau(g)) = \sup(g)$;
- $\inf(g) \leqslant \inf(\mathbf{c}(g)) \leqslant \sup(\mathbf{c}(g)) \leqslant \sup(g)$;
- $\inf(g) \leqslant \inf(\mathbf{d}(g)) \leqslant \sup(\mathbf{d}(g)) \leqslant \sup(g)$.

So $[g]^S$ is closed under $\tau(\cdot)$, $\mathbf{c}(\cdot)$, $\mathbf{d}(\cdot)$.

Computing $h \in [g]^S$ from g

Let G be a finite- or infinite-type Garside group and $g \in G$.

Theorem (Elrifai-Morton '94, Birman-Ko-Lee '01)

- $\inf(g) < \inf_s(g) \implies \inf(g) < \inf\left(\mathbf{c}^{\|\Delta\|-1}(g)\right)$,
- $\sup(g) > \sup_s(g) \implies \sup(g) > \sup\left(\mathbf{d}^{\|\Delta\|-1}(g)\right)$.
- $(\mathbf{c}^n \circ \mathbf{d}^m)(g) \in [g]^S$ *for some* $n, m \geqslant 0$.

Therefore

- $[g]^S \neq \emptyset$,
- we can compute an element of $[g]^S$ from g in finte-time.

Computing $[g]^S$ from $h \in [g]^S$

Let G be a finite- or infinite-type Garside group and $g \in G$. From the Convexity Theorems in [Elrifai-Morton '94, Franco-González-Meneses '03],

- $C_S(g) = \left\{x \in G \mid x^{-1}gx \in [g]^S\right\}$ is closed under $\wedge$;
- for any $h, h' \in [g]^S$, there exists a finite seq of elts in $[g]^S$

$$h = h_0 \to h_1 \to \cdots \to h_m = h'$$

s.t. for each i, $h_i = x_i^{-1} h_{i-1} x_i$ for some $x_i \in [1, \Delta]$.

So if G is of finite-type, we can compute all the elements of $[g]^S$ from an element of $[g]^S$ in finite-time.

SSSs & Conjugacy problem

Let G be a group.

Conjugacy decision problem (CDP) Given $g_1, g_2 \in G$,
decide whether or not g_1 is conjugate to g_2.

Conjugacy search problem (CSP) Given conjugate $g_1, g_2 \in G$,
find $x \in G$ s.t. $g_2 = x^{-1} g_1 x$.

If G is a finite-type Garside group, from (g_1, g_2), we can compute $[g_1]^S$, $h \in [g_2]^S$, and the corresponding conjugators in finite-time. By checking whether $h \in [g_1]^S$ or not, we can solve the CDP & CSP in finite-time.

Questions about SSSs

Let G be a finite- or infinite-type Garside group and $g \in G$.

Q1. Is there $h \in [g]^S$ s.t. $h^n \in [g^n]^S$ for all $n \in \mathbb{Z}$?

Q2. Is there an explicit formula in n for the function

$$f(n) = \inf_s(g^n) \ ?$$

A way to see Q1

Let G be a finite- or infinite-type Garside group and $g \in G$.

Recall: Q1. Is there $h \in [g]^S$ s.t. $h^n \in [g^n]^S$ for all $n \in \mathbb{Z}$?

Definition

The stable super summit set (stable SSS) of g is defined as

$$[g]^{St} = \{ h \in [g]^S \mid h^n \in [g^n]^S \quad \forall n \in \mathbb{Z} \}.$$

Q1'. Is $[g]^{St}$ non-empty?

Another way to see Q1

Let G be a finite- or infinite-type Garside group and $g \in G$.

Recall: Q1. Is there $h \in [g]^S$ s.t. $h^n \in [g^n]^S$ for all $n \in \mathbb{Z}$?

$x^{-1}gx = h \iff x^{-1}g^n x = h^n$ for all $n \in \mathbb{Z}$.

Let $\langle g \rangle = \{ g^n \mid n \in \mathbb{Z} \}$.

Q1". Is there $x \in G$ s.t.

$\underbrace{x^{-1}\langle g \rangle x}_{\{ x^{-1}g^n x | n \in \mathbb{Z} \}}$ consists of super summit elements only?

$\langle g \rangle$ is an abelian subgroup of G. More generally,

Q3. For an abelian subgroup H of G, is there $x \in G$ s.t.

$x^{-1}Hx$ consists of super summit elements only?

Super summitness of commutative elements

Let G be a finite- or infinite-type Garside group and $g, h, x, y \in G$.
Normal form: $g = \boxed{\Delta^u a_1 \cdots a_\ell} = \Delta^u a_1 \Delta^{-u} \Delta^u a_2 \cdots a_{\ell-1} a_\ell$
Let $x = \Delta^u a_1 \Delta^{-u}$, $y = a_\ell^{-1}$. (Then $\mathbf{c}(g) = x^{-1}gx$, $\mathbf{d}(g) = y^{-1}gy$.)

Lemma

If $h \in [h]^S$ and $gh = hg$, then $x^{-1}hx, y^{-1}hy \in [h]^S$.

So, given mutually commutative elements $g_1, \ldots, g_n \in G$, we can compute in finite-time $x \in G$ s.t.

$$x^{-1}g_1x, \ldots, x^{-1}g_nx \quad \text{are all super summit elements.}$$

Super summitness of abelian subgroups

Recall: Q3. For an abelian subgroup H of G, is there $x \in G$ s.t.
$x^{-1}Hx$ consists of super summit elements only?

[Charney-Meier-Whittlesey '04] Every finite-type Garside group has finite virtual cohomological dimension (VCD).

Let G be a finite-type Garside group.
Then every abelian subgroup of G is finitely-generated.

Theorem

Let H be an abelian subgroup of G. Then we can compute in finite-time $x \in G$ s.t. $x^{-1}Hx$ consists of super summit elts only.

Stable super summit sets

Recall: Q1'. Is $[g]^{St}$ non-empty?

Theorem

Let G be a finite-type Garside group and $g \in G$. Then

- $[g]^{St}$ *is non-empty;*
- $C_{St}(g) := \{x \in G \mid x^{-1}gx \in [g]^{St}\}$ *is closed under* $\wedge$, $\vee$ *and multiplication by* $\Delta^{\pm 1}$ *on the right;*
- $[g]^{St}$ *is closed under* $\tau(\cdot)$, $\mathbf{c}(\cdot)$, $\mathbf{d}(\cdot)$.

In the case where G is of infinite-type, we do not know whether $[g]^{St}$ is non-empty.

Step 1 for Q2: Relation between $\inf_s(g^n)$ & $\inf_s(g)$

Recall: Q2. Is there an explicit formula in n for the function

$$f(n) = \inf_s(g^n)?$$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Theorem (Lee '07)

$$\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1 \qquad \forall n \geqslant 1.$$

$\inf_s(g) \leqslant \dfrac{\inf_s(g^n)}{n}$ is obvious.

Idea for $\dfrac{\inf_s(g^n)}{n} < \inf_s(g) + 1$

Let $G(n) = \mathbb{Z} \ltimes G^n$ where $\mathbb{Z} = \langle \delta \rangle$.

Let $\mathbb{Z}$ act on G^n by

$$(g_1, g_2, \ldots, g_n)^\delta = (g_n, g_1, g_2, \ldots, g_{n-1}).$$

Then $G(n)$ is a Garside group with Garside elt $(\delta, (\Delta, \ldots, \Delta))$, and for any $k \in \mathbb{Z}$

- $\inf\left(\delta^k, (g_1, \ldots, g_n)\right) = \min\{k, \inf(g_1), \ldots, \inf(g_n)\}$;
- if $g \in [g]^S \subset G$, then $\alpha = \left(\delta^k, (g, \ldots, g)\right) \in [\alpha]^S \subset G(n)$;
- if $k \equiv 1 \bmod n$, then $g_1 \cdots g_n \underset{\text{conj}}{\sim} h_1 \cdots h_n$ in G iff $\left(\delta^k, (g_1, \ldots, g_n)\right) \underset{\text{conj}}{\sim} \left(\delta^k, (h_1, \ldots, h_n)\right)$ in $G(n)$.

Step 2 for Q2: limit of $\dfrac{\inf(g^n)}{n}$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Define

$$t_{\inf}(g) = \limsup_{n\to\infty} \frac{\inf(g^n)}{n}, \qquad t_{\sup}(g) = \liminf_{n\to\infty} \frac{\sup(g^n)}{n}.$$

Then

- $t_{\inf}(g) = t_{\inf}(x^{-1}gx)$ for all $x \in G$.
- $t_{\inf}(g) = \displaystyle\lim_{n\to\infty} \frac{\inf(g^n)}{n}, \quad t_{\sup}(g) = \lim_{n\to\infty} \frac{\sup(g^n)}{n}.$
- $t_{\sup}(g) = -\,t_{\inf}(g^{-1})$.
- $t_{\inf}(g^n) = n \cdot t_{\inf}(g)$ for all $n \geqslant 1$.

Relation between $t_{\inf}(g)$ and $\inf_s(g)$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Recall: $\inf_s(g) \leqslant \dfrac{\inf_s(g^n)}{n} < \inf_s(g) + 1 \qquad \forall n \geqslant 1.$

Since $t_{\inf}(g)$ and $\inf_s(g)$ are conjugacy invariant, we may assume $g \in [g]^S$. Then

$$\inf_s(g) = \inf(g) \leqslant \frac{\inf(g^n)}{n} \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1.$$

Therefore

$$\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g) + 1.$$

Solving Q2

Let G be a finite- or infinite-type Garside group and $g \in G$.

From $\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g) + 1$, for any $n \geqslant 1$,

$$\inf_s(g^n) \leqslant t_{\inf}(g^n) = n \cdot t_{\inf}(g) \leqslant \inf_s(g^n) + 1.$$

Therefore

$$\inf_s(g^n) = \begin{cases} \lfloor n \cdot t_{\inf}(g) \rfloor & \text{if} \quad n \cdot t_{\inf}(g) \notin \mathbb{Z}, \\ n \cdot t_{\inf}(g) \quad \text{or} \quad n \cdot t_{\inf}(g) - 1 & \text{o.w.} \end{cases}$$

Notice: If $t_{\inf}(g)$ is irrational, then $n \cdot t_{\inf}(g) \notin \mathbb{Z}$ for all $n \geqslant 1$.

Rationality of $t_{\inf}(g)$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Lemma

$t_{\inf}(g) - \lfloor t_{\inf}(g) \rfloor \ = 0 \quad \textit{or} \quad \geqslant \|\Delta\|^{-1}.$

Theorem

$t_{\inf}(g)$ is rational with uniformly bounded denominator:
$t_{\inf}(g) = \dfrac{p}{q}$ *for some $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant \|\Delta\|$.*

How to compute $t_{\inf}(g)$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Recall:

- $t_{\inf}(g) = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant \|\Delta\|$.
- For all $n \geqslant 1$, $\inf_s(g^n) \leqslant n \cdot t_{\inf}(g) \leqslant \inf_s(g^n) + 1$.

$$\implies t_{\inf}(g) \in \left[\frac{\inf_s(g^n)}{n}, \frac{\inf_s(g^n)}{n} + \frac{1}{n} \right].$$

Let $T = \left\{ \dfrac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ 1 \leqslant q \leqslant \|\Delta\| \right\}$. For any $n \geqslant \|\Delta\|^2$,

$$T \cap \left[\frac{\inf_s(g^n)}{n}, \frac{\inf_s(g^n)}{n} + \frac{1}{n} \right] = \{ t_{\inf}(g) \}.$$

So we can compute $t_{\inf}(g)$ in finite-time.

Sharper relation between $t_{\inf}(g)$ and $\inf_s(g)$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Recall: $\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g) + 1$;

Theorem

If G is of finite-type, then $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$.

The proof applies Shur's theorem to $M = |\mathcal{D}|$.

Shur's Theorem $\forall M > 0$, $\exists L > 0$ s.t.

$\forall$ partition $\{T_1, \ldots, T_M\}$ of $\{1, \ldots, L\}$, $\exists k$ s.t.

$n, m, n + m \in T_k$ for some $n, m \in \{1, \ldots, L\}$.

Solution to Q2

Recall: Q2. Is there an explicit formula in n for the function

$$f(n) = \inf_s(g^n)?$$

Let G be a finite-type Garside group and $g \in G$.

From $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$, for any $n \geqslant 1$,

$$\inf_s(g^n) \leqslant t_{\inf}(g^n) = n \cdot t_{\inf}(g) < \inf_s(g^n) + 1.$$

Therefore

$$\inf_s(g^n) = \lfloor n \cdot t_{\inf}(g) \rfloor \quad \text{for all } n \geqslant 1.$$

More improved way to compute $t_{\inf}(g)$

Let G be a finite-type Garside group and $g \in G$.

Recall:

- $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$,
- $t_{\inf}(g) = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant \|\Delta\|$.

Then

- $t_{\inf}(g) = \dfrac{t_{\inf}(g^k)}{k} \geqslant \dfrac{\inf_s(g^k)}{k}$ for all $k \geqslant 1$.
- $\inf_s(g^q) = \lfloor t_{\inf}(g^q) \rfloor = \lfloor q\, t_{\inf}(g) \rfloor = p$. Thus

$$t_{\inf}(g) = \frac{p}{q} = \frac{\inf_s(g^q)}{q}.$$

Since $1 \leqslant q \leqslant \|\Delta\|$, we have

Theorem

$$t_{\inf}(g) = \max\left\{ \frac{\inf_s(g^k)}{k} \;\middle|\; k = 1, \ldots, \|\Delta\| \right\}$$

Application of $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$

Let G be a finite- or infinite-type Garside group and $g \in G$.

Definition

g is inf-straight if $\inf(g) = t_{\inf}(g)$.

Lemma

g is inf-straight iff $\inf(g^n) = n \cdot \inf(g) \quad \forall n \geqslant 1$.

Theorem

Let G be a finite-type Garside group and $g \in G$.

g is inf-straight up to conjugacy iff $t_{\inf}(g) \in \mathbb{Z}$.

Periodically geodesic elements

Let G be a finite- or infinite-type Garside group and $g \in G$.

Definition

g is periodically geodesic if

$$|g^n|_{\mathcal{D}} = |n| \cdot |g|_{\mathcal{D}} \qquad \forall n \in \mathbb{Z}.$$

Notice: $|g^{-1}|_{\mathcal{D}} = |g|_{\mathcal{D}} = -\inf(g),\ \sup(g)$ or $\sup(g) - \inf(g)$.

So if g is inf- & sup-straight, then g is periodically geodesic.

Periodically geodesic powers

Theorem

Let G be a finite-type Garside group and $g \in G$. Then g^k is periodically geodesic up to conjugacy for some $1 \leqslant k \leqslant \|\Delta\|^2$.

Key to the proof.

- g is inf-straight up to conjugacy iff $t_{\inf}(g) \in \mathbb{Z}$.
- $t_{\inf}(g) = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant \|\Delta\|$.
- $t_{\inf}(g^n) = n \cdot t_{\inf}(g)$ for all $n \geqslant 1$. □

Application of $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$

Recall: $[g]^{St} = \{ h \in G \mid h^n \in [g^n]^S \quad \forall n \in \mathbb{Z} \}$

Let G be a finite-type Garside group and $g \in G$.

Theorem

If $h^n \in [g^n]^S$ for $n = 1, \ldots, \|\Delta\|$,
then $h^n \in [g^n]^S$ for all $n \in \mathbb{Z}$ (i.e. $h \in [g]^{St}$).

Computing $h \in [g]^{St}$ from g

Let G be a finite- or infinite-type Garside group and $g \in G$.

Normal form: $g = \boxed{\Delta^u a_1 \cdots a_\ell} = \Delta^u a_1 \Delta^{-u} \Delta^u a_2 \cdots a_{\ell-1} a_\ell$

Let $x = \Delta^u a_1 \Delta^{-u}$, $y = a_\ell^{-1}$. (Then $\mathbf{c}(g) = x^{-1}gx$, $\mathbf{d}(g) = y^{-1}gy$.)

Recall: If $h \in [h]^S$ and $gh = hg$, then $x^{-1}hx, y^{-1}hy \in [h]^S$.

$g, g^2, g^3, \ldots$ commute with each other.

So we can compute in finite-time $x \in G$ s.t.

$$\underbrace{x^{-1}gx}_{h},\ \underbrace{x^{-1}g^2x}_{h^2}, \ldots,\ \underbrace{x^{-1}g^{\|\Delta\|}x}_{h^{\|\Delta\|}} \quad \text{are all super summit elements.}$$

Then $h \in [g]^{St}$ if G is of finite-type.

Computing $[g]^{St}$ from $h \in [g]^{St}$

Let G be a finite- or infinite-type Garside group and $g \in G$.
For any $h, h' \in [g]^{St}$, there exists a finite sequence of elts in $[g]^{St}$

$$h = h_0 \to h_1 \to \cdots \to h_m = h'$$

s.t. for each i, $h_i = x_i^{-1} h_{i-1} x_i$ for some $x_i \in [1, \Delta]$.

So we can compute $[g]^{St}$ from $h \in [g]^{St}$ in finite-time if G is of finite-type.

Application of $t_{\inf}(\cdot)$: translation numbers

Let G be an arbitrary finitely-generated group,
let X be a finite set of semigroup generators for G, and
let $|\cdot|_X$ be the shortest word-length in the alphabet $X \cup X^{-1}$.

The translation number of $g \in G$ w.r.t. X is

$$t_X(g) = \liminf_{n\to\infty} \frac{|g^n|_X}{n} = \lim_{n\to\infty} \frac{|g^n|_X}{n}.$$

The notion of translation numbers came from Riemannian geometry, and was abstracted as above.

Translation numbers: Examples

- If g is a torsion element, then $t_X(g) = 0$.
- Let $G = \langle x, y \mid x^2 = y^3 \rangle$ and $X = \{x, y\}$.
  Then $t_X(y) = \dfrac{2}{3}$ because

$$\frac{|y^{3k}|_X}{3k} = \frac{|x^{2k}|_X}{3k} = \frac{2k}{3k} = \frac{2}{3}.$$

- Let $G = \langle x, y \mid xy = yx \rangle$, $X = \{x, y\}$ and $g = x^a y^b$.
  Then $t_X(g) = |g|_X = |a| + |b|$, the ℓ_1-norm of g.
- Let G be the free group generated by X.
  Then $t_X(g)$ is the length of a cyclically reduced word representing g.

Discreteness of translation numbers

[Kapovich '97, Conner '00] A finitely-generated group G is

1. translation separable if, for some finite set X of semigroup generators for G, the translation numbers of non-torsion elements are strictly positive;
2. translation discrete if it is translation separable and, for some finite set X of semigroup generators for G, the set $t_X(G)$ has 0 as an isolated point;
3. strongly translation discrete if it is translation separable and, for some finite set X of semigroup generators for G and for any real number r, the number of conjugacy classes $[g]$ with $t_X(g) \leqslant r$ is finite.

Strongly Trans. Discrete $\Rightarrow$ Trans. Discrete $\Rightarrow$ Trans. Separable

Algebraic, geometric aspects of translation numbers

Let G be a finitely-generated group.

- [Gersten-Short '91] If G is translation separable, then every solvable subgroup of G is finitely-generated and virtually abelian.
- [Conner '00] If G is translation separable, solvable and of finite VCD, then G is metabelian-by-finite.
- [Kapovich '97] If G is translation discrete, G cannot contain subgroups isomorphic to $\mathbb{Q}$ or the group of p-adic numbers $\mathbb{Q}_p$.
- [Conner '00] If G is translation discrete, then every solvable subgroup of finite VCD is a finite extension of $\mathbb{Z}^m$.

Translation numbers in geometric or combinatorial groups

- [Gersten-Short '91] Biautomatic gps are translation separable.
- [Alibegović '00] Out(F_n) is translation discrete.
- [Kapovich '97] C(4)-T(4)-P, C(3)-T(6)-P and C(6)-P small cancelation groups are strongly translation discrete.
- [Gromov '87, Baumslag-Gersten-Shapiro-Short '91, Swenson '95] Word hyperbolic groups are strongly translation discrete. Moreover, the translation numbers are rational with uniformly bounded denominators.

Translation numbers in Garside groups

- [Bestvina '99] Finite-type Artin groups are translation discrete (tool: normal form complex).
- [Dehornoy-Paris '99] Finite-type Garside gps are biautomatic. [Gersten-Short '91] Biautomatic gps are translation separable. So finite-type Garside groups are translation separable.
- [Charney-Meier-Whittlesey '04] Finite-type Garside groups with tame Garside element are translation discrete (tool: normal form complex).
- [Lee '07] Finite-type Garside groups are strongly translation discrete (tool: Garside theory).

Questions about translation numbers in Garside groups

Recall:

- Finite-type Garside groups are strongly translation discrete.
- Word hyperbolic groups are strongly translation discrete. Moreover, the translation numbers are rational with uniformly bounded denominators.

Q. Rationality of translation numbers in finite-type Garside gps?

Q. Discreteness of translation numbers in infinite-type Garside gps?

Translation numbers in infinite-type Garside groups

- Translation numbers are defined for finitely-generated groups.
- In case of infinite-type Garside groups, we assume they are finitely-generated.
- Let a finitely-generated gp G be a Garside gp with $|\mathcal{D}| = \infty$. Then there exists a finite subset $\mathcal{D}'$ of $\mathcal{D}$ which generates G.
- For every $g \in G$, since $|g|_{\mathcal{D}} \leqslant |g|_{\mathcal{D}'}$,

$$t_{\mathcal{D}}(g) \leqslant t_{\mathcal{D}'}(g).$$

Continued

From $t_{\mathcal{D}}(g) \leqslant t_{\mathcal{D}'}(g)$ where $\mathcal{D}$ is infinite and $\mathcal{D}'$ is finite,

- if $t_{\mathcal{D}}(g) > 0$, then $t_{\mathcal{D}'}(g) > 0$, ($\rightsquigarrow$ translation separable)
- if $t_{\mathcal{D}}(G)$ has 0 as an isolated point, so does $t_{\mathcal{D}'}(G)$, ($\rightsquigarrow$ translation discrete)
- if the conjugacy classes $[g]$ with $t_{\mathcal{D}}(g) \leqslant r$ are finitely many, so are the conjugacy classes $[g]$ with $t_{\mathcal{D}'}(g) \leqslant r$. ($\rightsquigarrow$ strongly translation discrete)

Therefore it is meaningful to study translation numbers in infinite-type Garside groups, using given Garside structures.

Relation between $t_{\mathcal{D}}(g)$ and $t_{\inf}(g)$

Let G be a finite- or infinite-type Garside group and $g \in G$. Recall:

- $$|g|_{\mathcal{D}} = \begin{cases} \sup(g) & \text{if } \inf(g) \geqslant 0, \\ -\inf(g) & \text{if } \sup(g) \leqslant 0, \\ \sup(g) - \inf(g) & \text{o.w.} \end{cases}$$
- $t_{\mathcal{D}}(g) = t_{\mathcal{D}}(x^{-1}gx)$ for any $x \in G$.

Using the above, we have

$$t_{\mathcal{D}}(g) = \begin{cases} t_{\sup}(g) & \text{if } \inf_s(g) \geqslant 0, \\ -t_{\inf}(g) & \text{if } \sup_s(g) \leqslant 0, \\ t_{\sup}(g) - t_{\inf}(g) & \text{o.w.} \end{cases}$$

Rationality of translation numbers in Garside groups

Let G be a finite- or infinite-type Garside group and $g \in G$.

Recall:

- $t_{\mathcal{D}}(g) = t_{\sup}(g), \ -t_{\inf}(g)$ or $t_{\sup}(g) - t_{\inf}(g)$.
- $t_{\inf}(g)$ is rational with uniformly bounded denominator: $t_{\inf}(g) = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant \|\Delta\|$.

So $t_{\mathcal{D}}(g)$ is rational with uniformly bounded denominator:
$t_{\mathcal{D}}(g) = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant \|\Delta\|^2$.

Recall: If G is of finite-type, G is strongly translation discrete.

But if G is of infinite-type, we do not even know wheter G is translation seperable or not.

Application to the algebraic flat torus theorem

Theorem (Algebraic flat torus theorem for a group G)

Every abelian subgroup of G is quasi-isometric to $\mathbb{Z}^m$ for some m.

Definition

Let (X, d_X), (Y, d_Y) be metric spaces. Let $\lambda \geqslant 1$, $\epsilon \geqslant 0$, $\delta \geqslant 0$.

A map $f : X \to Y$ is a $(\lambda, \epsilon, \delta)$-quasi-isometry if

- Y lies in the δ-neighborhood of $f(X)$, and
- $\forall x_1, x_2 \in X$,
  $\frac{1}{\lambda} \cdot d_X(x_1, x_2) - \epsilon \leqslant d_Y(f(x_1), f(x_2)) \leqslant \lambda \cdot d_X(x_1, x_2) + \epsilon.$

Algebraic flat torus theorem for Garside groups

- Finite-type Garside groups are biautomatic [Dehornoy-Paris '99].
- Biautomatic groups are semihyperbolic [Alonso-Bridson '95, Bridson-Haefliger '99].
- The Algebraic flat torus theorem holds for semihyperbolic groups [Alonso-Bridson '95].

So the algebraic flat torus theorem holds for finite-type Garside groups.

Algebraic flat torus theorem by Garside theory

Theorem

Let H be an abelian subgroup of a finite-type Garside group G.
Then H is quasi-isometric to $\mathbb{Z}^m$ for some m.

Let $g, h \in G$.

- In G, $|\cdot|_{\mathcal{D}}$ induces a metric $d_1(g,h) := |g^{-1}h|_{\mathcal{D}}$.
- $t_{\mathcal{D}}(\cdot)$ can be used as a norm on H because
  - (i) $t_{\mathcal{D}}(g^n) = |n| \cdot t_{\mathcal{D}}(g)$ for all $n \in \mathbb{Z}$.
  - (ii) If $gh = hg$, $t_{\mathcal{D}}(gh) \leqslant t_{\mathcal{D}}(g) + t_{\mathcal{D}}(h)$.
  - (iii) If $g \neq 1$, $t_{\mathcal{D}}(g) > 0$.
    ($\because$ G is torsion-free & translation seperable.)

So $t_{\mathcal{D}}(\cdot)$ induces a metric $d_2(g,h) := t_{\mathcal{D}}(g^{-1}h)$ on H.

Proof for the algebraic flat torus theorem

Recall:

- H has finite-rank, say m,
- $\exists x \in G$ s.t. $x^{-1}Hx$ consists of super summit elements.

$(H, |\cdot|_{\mathcal{D}})$

$\simeq \longleftarrow$ $|h|_{\mathcal{D}} - 2|x|_{\mathcal{D}} \leqslant |x^{-1}hx|_{\mathcal{D}} \leqslant |h|_{\mathcal{D}} + 2|x|_{\mathcal{D}} \quad \forall h \in H.$

$(x^{-1}Hx, |\cdot|_{\mathcal{D}})$

$\simeq \longleftarrow$ [Lee '07] If $g \in [g]^S$ then $|g|_{\mathcal{D}} - 2 \leqslant t_{\mathcal{D}}(g) \leqslant |g|_{\mathcal{D}}$.

$(x^{-1}Hx, t_{\mathcal{D}}(\cdot))$

$\simeq \longleftarrow$ Finite dimensional normed spaces with the same rank are equivalent.

$(\mathbb{Z}^m, \|\cdot\|_\infty)$