Braid groups of imprimitive reflection groups

Ruth Corran

Cap Hornu
May 30, 2012

Finite complex reflection groups

V a vector space over $\mathbb{C}$ with dim(V) = r.

A **complex reflection** *s* is a non-trivial linear map which fixes a complex hyperplane H_s pointwise:

$$s:V \to V \text{ for which } \dim(H_s) = r-1$$

H_s = ker(s-1) is the **reflection hyperplane** for the reflection s

A (finite) **complex reflection group W** is a (finite) group generated by complex reflections.

Write $\mathcal{A}$ for the collection of reflection hyperplanes of a fcrg W.

Braid groups of fcrg's

The **pure braid group** of a fcrg W is the fundamental group of its hyperplane complement:

$$P(W) = \Pi_1(V-\mathcal{A},*)$$

A fcrg W acts on its hyperplane complement.

The **braid group** is the fundamental group of the corresponding quotient space:

$$B(W) = \Pi_1(V-\mathcal{A}/W,*)$$

These groups are related by the short exact sequence

$$1 \to P(W) \to B(W) \to W \to 1$$

Finite Coxeter groups & Artin groups

Reflection groups which may be represented as reflections of a real vector space have been extensively studied – they are Coxeter groups.

The finite Coxeter groups have been classified into 4 infinite families (types A_r ,B_r , D_r and the dihedral groups $I_2(e)$) and 7 exceptional groups (E_6, E_7, E_8, F_4, G_2, H_3 and H_4)

A huge amount of information about these groups is encoded in their **Coxeter diagram,** including:

- a presentation of W via reflection generators
- by throwing away finite order of ref gens, a presentation of B(W)
- a *Garside structure* arising from this presentation permitting effective calculation in the braid group.

Coxeter diagrams for finite real reflection groups

Presentation for the Coxeter group with Coxeter diagram Γ = (I,E):

Generators: s_i for all i $\in$ I

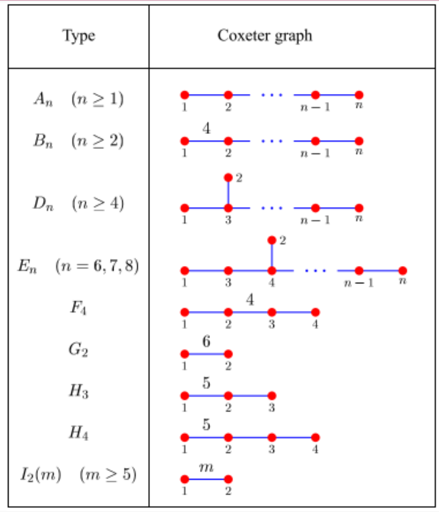
Relations: (1) s^2 = 1 for all i

(2) if an edge between nodes i and j is labelled m, there is a relation:

$$\langle s_i s_j \rangle^m = \langle s_j s_i \rangle^m$$

(3) if no edge between i , j:

$$s_i s_j = s_j s_i$$



Example:
Reflection group of type A_n = the symmetric group

A_n $(n \geq 1)$

Generators: S = {s_1, s_2, ... , s_n}

Relations: $s_i^{\,2}$ = 1 for all i=1, ... , n

$s_i\, s_j\, s_i = s_j\, s_i\, s_j$ |i-j|=1

$s_i\ s_j = s_j\, s_i$ |i-j| $\geq$ 2

The presentation is *positive* ~~and *homogeneous*~~.

The generator s_i corresponds to interchanging i and i+1.

(N.B. An edge whose label is not explicitly shown is understood to be an edge with label 3.)

THE AMERICAN UNIVERSITY OF PARIS

Example:
Braid group of type A_n = group of "geometric braids"

$A_n \quad (n \geq 1)$

Generators: $S = \{s_1, s_2, \ldots, s_n\}$

Relations: ~~$s_i^2 = 1$ for all i=1, … , n~~

$s_i s_j s_i = s_j s_i s_j \qquad |i-j|=1$

$s_i s_j = s_j s_i \qquad |i-j| \geq 2$

Braid relations

The presentation is *positive* and *homogeneous*.

(N.B. An edge whose label is not explicitly shown is understood to be an edge with label 3.)

Artin groups: braid groups of real reflection gps

Presentation for the Artin group with Coxeter diagram Γ = (I,E):

Generators: s_i for all i $\in$ I

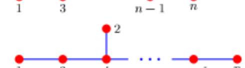
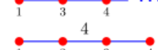
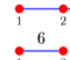
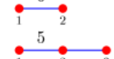
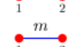
Relations: ~~**(1) $s^2 = 1$ for all i**~~

(2) if an edge between nodes i and j is labelled m, there is a relation:

$$\langle s_i s_j \rangle^m = \langle s_j s_i \rangle^m$$

(3) if no edge between i , j:

$$s_i s_j = s_j s_i$$

Type	Coxeter graph
$A_n \quad (n \geq 1)$	
$B_n \quad (n \geq 2)$	
$D_n \quad (n \geq 4)$	
$E_n \quad (n = 6,7,8)$	
F_4	
G_2	
H_3	
H_4	
$I_2(m) \quad (m \geq 5)$	

Garside structure & related notions

Work of Garside [1969], Deligne [1972], Brieskorn & Saito [1972]:

- Solutions to word and conjugacy problems in braid groups for finite real reflection groups using the Artin presentations
- Normal form based on special element Δ, which corresponds to the *longest element* in the reflection group.

Generalisation of these properties in the notion of Garside structure (Dehornoy, Paris, Digne, Michel,...), in the noughties

The particular case of a *generated group* Garside structure (Michel 1999) where the Garside structure is determined according to a length function in a related group – in this case, the reflection group.

The classification of finite complex reflection groups

Shephard-Todd (1954)

- 34 "exceptional" groups of ranks 2 to 8
- ... And an infinite family G(de,e,r), the imprimitive reflection groups:

μ_{de} = de^{th} roots of unity

M = monomial matrices over over $\mu_{de} \cup \{0\}$

$$G(de,e,r) = \{(x_{ij}) \in M \mid \Pi_{x_{ij}\neq 0} x_{ij}^{d} = 1\}$$

This infinite family contains type A: G(1,1,r), type B: G(2, 1, r), type D (G(2,2,r) and type I(e): G(e,e,2).

Broue-Malle-Rouquier [1998]

- An investigation of the geometry, algebra and combinatorics of the finite complex reflection groups, their Hecke algebras and braid groups
- Proposed diagrams for the non-real reflection groups to be "like" Coxeter diagrams
- These diagrams were a kind of generalization of Coxeter diagrams
- In general the diagrams **did** encode presentations for the braid groups as well
- The diagrams did **not** give rise to presentations which had the sort of Garside-like properties desired.

Aim: To describe diagrams for B(e,e,r) and B(de,e,r) which work "like Coxeter diagrams"

- Such that one can "read off" a presentation for B(W) (and W)
  - A positive, homogeneous presentation in particular
  - For which adding finite order to the generators of B(W) gives a presentation by reflections for W
- For which: the braid group is the group of fractions of the positive monoid M of the presentation: $B(W) = M M^{-1}$
- M is a lattice with respect to division

  $\rightsquigarrow$ solutions to the word and conjugacy problems in B(W)

  Garside structure
- Diagram automorphisms $\leftrightsquigarrow$ auts of B(W) (and W)
- (Suitable) subgraphs $\leftrightsquigarrow$ parabolic subgroups

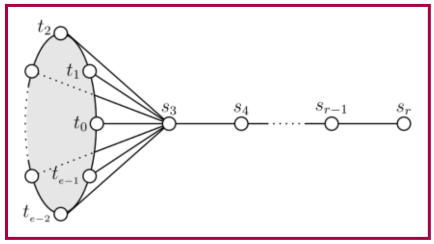
THE AMERICAN UNIVERSITY OF PARIS

1. Type (e,e,r): [C-Picantin, 2004-2011]

- Generators

  $T_e := \{t_i \mid i \in \mathbb{Z}/e\}$

  $S := \{s_j \mid 3 \leq j \leq r\}$



- Relations
  - Braid relations on S;
  - $t_i\, s_3\, t_i = s_3\, t_i\, s_3$ for all $i \in \mathbb{Z}/e$;
  - $t_i\, s_j = s_j\, t_i$ for all $i \in \mathbb{Z}/e$ and $4 \leq j \leq r$;
  - $t_i\, t_{i-1} = t_j\, t_{j-1}$ for all $i, j \in \mathbb{Z}/e$.

1. Type (e,e,r): [C-Picantin, 2004-2011]

Theorem 1. [C-P, 2011]

a) $\langle T_e \cup S \mid R \rangle$ is a group presentation for B(e,e,r)

b) The monoid M = mon$\langle T_e \cup S \mid R \rangle$ embeds in B(e,e,r) and gives rise to a Garside structure

c) $\langle T_e \cup S \mid R \cup I \rangle$ is a presentation for G(e,e,r), where the set I consists of the relations $a^2 = 1$ for all $a \in T_e \cup S$; and all the generators are reflections: for e^{th} root of unity ζ_e :

$$\overline{t_i} = \left(\begin{array}{c|c} \begin{matrix} 0 & \zeta_e^{-i} \\ \zeta_e^{i} & 0 \end{matrix} & 0 \\ \hline 0 & I_{r-2} \end{array}\right)$$

s_j corresponds to permuting the $(j-1)^{th}$ and j^{th} basis vectors

1. Type (e,e,r): [C-Picantin, 2004-2011]

Remarks 1.

1. This is not the first Garside structure for B(e,e,r): cf the dual presentation for B(e,e,r) [Bessis-Corran 2006]

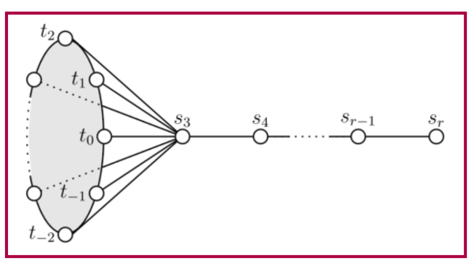
2. Conjecturally a *generated group,* using the length function defined on the fcrg G(e,e,r) via the generating set $T_e \cup S$.

2. Type (∞, ∞ ,r) : affine type A_{r-1}

- Shi (2002): B(∞, ∞ ,r) $\cong B(\widetilde{A}_{r-1})$
- Generators

  T_∞ := $\{t_i \mid i \in \mathbb{Z}\}$

  S := $\{s_j \mid 3 \leq j \leq r\}$



- Relations
  - Braid relations on S;
  - $t_i s_3 t_i = s_3 t_i s_3$ for all $i \in \mathbb{Z}$;
  - $t_i s_j = s_j t_i$ for all $i \in \mathbb{Z}$ and $4 \leq j \leq r$;
  - $t_i t_{i-1} = t_j t_{j-1}$ for all $i, j \in \mathbb{Z}$.

2. Type (∞, ∞ ,r) : affine type A_{r-1}

Theorem 2. [C]

a) $\langle T_\infty \cup S \mid R \rangle$ is a group presentation for B(∞,∞,r) ≅$B(\widetilde{A}_{r-1})$

b) The monoid M = mon$\langle T_\infty \cup S \mid R \rangle$ embeds in B(∞,∞,r) and gives rise to a (quasi-)Garside structure

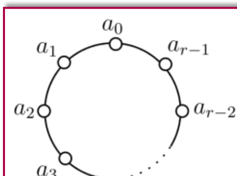
c) $\langle T_\infty \cup S \mid R \cup I \rangle$ is a presentation for W($\widetilde{A}_{r-1}$) where the set I consists of the relations $a^2 = 1$ for all $a \in T_\infty \cup S$; and all the generators are reflections: for x an indeterminate:

$$\overline{t_i} = \left(\begin{array}{cc|c} 0 & x^{-i} & \\ x^{i} & 0 & 0 \\ \hline & 0 & I_{r-2} \end{array} \right)$$

s_j corresponds to permuting the $(j-1)^{th}$ and j^{th} basis vectors

2. Type (∞, ∞ ,r) : affine type A_{r-1}

What is the isomorphism?



Define χ: (∞, ∞, r) $\rightarrow \widetilde{A}_{r-1}$

by: $\chi(s_j) = a_{j-1}$ for all j, $3 \leq j \leq r$

and $\chi(t_i) = \varphi^i(a_1)$ for all $i \in \mathbb{Z}$

...where $\varphi = \psi \circ \rho \in$ Aut($\widetilde{A}_{r-1}$)

with $\rho(a_i) = a_{i-1}$ = "rotation" diagram automorphism, and

$\psi(g) = pgp^{-1}$ = inner automorphism by $p = a_1\, a_2 \cdots a_{r-1}$

In particular:

$\varphi(a_i) = a_i$ for 2 ≤ i ≤ r-1

$\varphi(a_1) = a_0{}^h$ for some h

2. Type (∞, ∞ ,r) : affine type A_{r-1}

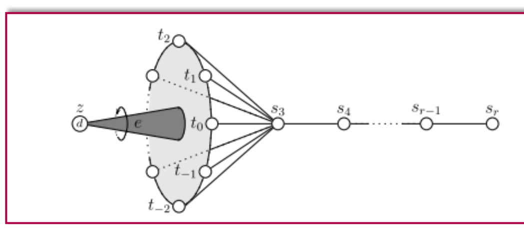
Remarks 2.

1. The isomorphism is originally due to Shi (2002)
2. Digne (2003) described a quasi-Garside structure for B($\widetilde{A}_{r-1}$ which can be considered as a"dual" Garside structure.

3. Type (δe,e,r) (δ>1)

If d, d' > 1, B(de,e,r) $\cong$ B(d'e,e,r). Write B(δe,e,r) for δ "any integer > 1".

- Generators: **{z}**
  T_∞ := {t_i | i $\in \mathbb{Z}$}
  S := {s_j | 3 $\leq$ j $\leq$ r}



- Relations
  - Braid relations on S;
  - $t_i\, s_3\, t_i = s_3\, t_i\, s_3$ for all i $\in \mathbb{Z}$;
  - $t_i\, s_j = s_j\, t_i$ for all i $\in \mathbb{Z}$ and 4 $\leq$ j $\leq$ r;
  - $t_i\, t_{i-1} = t_j\, t_{j-1}$ for all i, j $\in \mathbb{Z}$;
  - **$z\, s_j = s_j\, z$** for all j with 3 $\leq$ j $\leq$ r;
  - **$z\, t_i = t_{i-e}\, z$** for all i $\in \mathbb{Z}$.

"Applying the handle" turns the disk by e nodes

3. Type $(\delta e,e,r)$ $(\delta>1)$

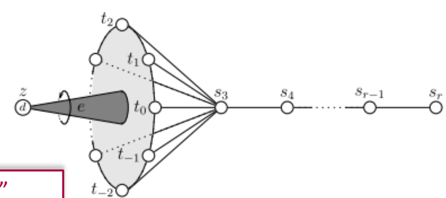
Theorem 3.1. $B(\delta e,e,r) \cong \mathbb{Z} \ltimes_e B(\widetilde{A}_{r-1})$

Here: $\mathbb{Z} = \langle z \rangle$

and the action is given by:

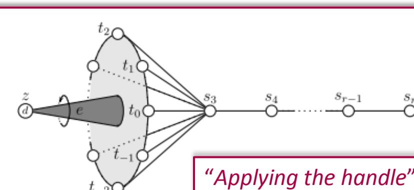
$z\, t_i\, z^{-1} = t_{i-e}$

$z\, s_j\, z^{-1} = s_j$



"Applying the handle" turns the disk by e nodes

3. Type $(\delta e,e,r)$ $(\delta>1)$



"Applying the handle" turns the disk by e nodes

Theorem 3.2.

a) $\langle \{z\} \cup T_\infty \cup S \mid R \rangle$ is a group presentation for $B(\delta e,e,r)$

b) The monoid $M = \mathrm{mon}\langle \{z\} \cup T_\infty \cup S \mid R \rangle$ embeds in $B(\delta e,e,r)$ and gives rise to a (quasi-)Garside structure

c) $\langle \{z\} \cup T_\infty \cup S \mid R \cup I \cup J_d \rangle$ is a presentation for $G(de,e,r)$ where:

- I consists of the relations $a^2 = 1$ for all $a \in T_\infty \cup S$; and
- J_d is the relation: $z^d = 1$,

and for which all the generators are reflections: the images of t_i and s_j are as before, and $z \leftrightsquigarrow \mathrm{Diag}(\zeta_d,1, \ldots, 1)$

3. Type $(\delta e, e, r)$ $(\delta > 1)$

Remarks 3.

1. The semidirect product decomposition:

   $$G(de,e,r) \cong \mathbb{Z}/d \ltimes_e G(de,de,r)$$

   is well-known and direct from the definition of G(de,e,r).

2. [Ram, Ramagge (2003)] describe the analogous decomposition for the corresponding Hecke algebras.

3. Using Digne's dual (quasi-)Garside structure for $\widetilde{A}_{r-1}$ gives rise to an alternative dual structure of type $(\delta e, e, r)$ $(\delta > 1)$.

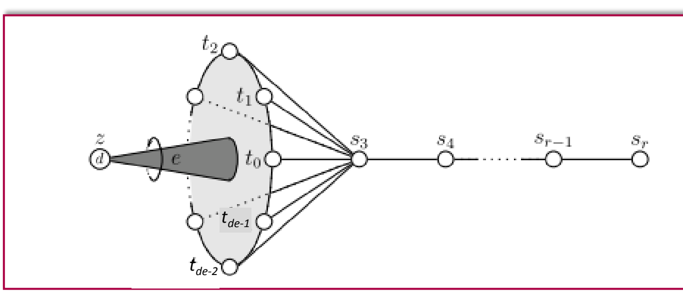
4. This is not a generated group construction.

4. Fake type $F(de,e,r) \cong \mathbb{Z} \ltimes_e B(de,de,r)$

If $z^d = 1$, the relation $z\, t_i = t_{i-e}\, z$ forces:

$$t_i = t_{i-de}$$

This corresponds to a kind of "folding" of the ∞-disk onto a disk with only *de* nodes – which corresponds to type (de,de,r).



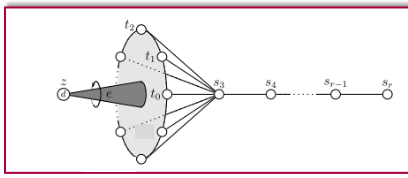
4. Fake type *F(de,e,r)* $\cong \mathbb{Z} \ltimes_e$ *B(de,de,r)*

This leads to a finite presentation for a group we denote ***F(de,e,r)*** and call the ***fake braid group*** of type (de,e,r).

- Generators: {z}

  $T_{de} := \{t_i \mid i \in \mathbb{Z}/de\}$

  $S := \{s_j \mid 3 \leq j \leq r\}$



- Relations
  - Braid relations on S;
  - $t_i\, s_3\, t_i = s_3\, t_i\, s_3$ for all $i \in \mathbb{Z}/de$;
  - $t_i\, s_j = s_j\, t_i$ for all $i \in \mathbb{Z}/de$ and $4 \leq j \leq r$;
  - $t_i\, t_{i-1} = t_j\, t_{j-1}$ for all $i, j \in \mathbb{Z}/de$;
  - $\mathbf{z\, s_j = s_j\, z}$ for all j with $3 \leq j \leq r$;
  - $\mathbf{z\, t_i = t_{i-e}\, z}$ for all $i \in \mathbb{Z}/de$.

4. Fake type *F(de,e,r)* $\cong \mathbb{Z} \ltimes_e$ *B(de,de,r)*

Remarks 4.

1. Conjecturally this is a generated group construction using the length function defined on the fcrg G(de,e,r) via the generating set $T_{de} \cup S$: When d=2 it coincides with Picantin's *Spintop monoid* [Amiens, 2011], a generated group construction from G(2e,e,2).
2. This is a **finite** Garside structure (not quasi-), which is specific to the choice of *d;* but only for a quotient of the (true) braid group B(δe,e,r). We expect in general for there to be no finite Garside structure for the true braid groups.
3. The fake braid group is intermediate between the braid group B(δe,e,r) and its reflection group: the natural surjection onto G(de,e,r) factors through F(de,e,r).

THE AMERICAN UNIVERSITY OF PARIS

5. Maps

A. The diagram automorphism "rotate the disk"

$$\uparrow: \begin{cases} t_i \to t_{i+1} \\ s_j \to s_j \end{cases}$$

This is an automorphism of type (#,*,r)
(including $\widetilde{A}_{r-1}$ and (e,e,r))

5B. Maps between B(δe,e,r) : interpretation as braids

Consider $e_1, e_2, \ldots \in \mathbb{N}_{>1}$ for which e_q divides e_{q+1} for each q.

We have: $e_{q+1}\mathbb{Z} \hookrightarrow e_q\mathbb{Z}$ by $z \mapsto z^{e_{q+1}/e_q}$.

Hence:

$$B\left(\tilde{A}_{r-1}\right) \hookrightarrow \cdots \hookrightarrow B\left(\delta e_{i+1}, e_{i+1}, r\right) \hookrightarrow B\left(\delta e_i, e_i, r\right) \hookrightarrow \cdots \hookrightarrow B(\delta, 1, r)$$

The group B(δ,1,r) is isomorphic to the braid group of type B_r which can be visualized as the group of (geometric) braids on a cylinder.

In this way, **elements of B(δe,e,r)** may be visualized as geometric braids on a cylinder with **winding number 0 mod e**.

5C. Maps between B(e,e,r): Wrapping up the disk

C. Between B(e,e,r): Wrapping up the disk

Consider $e_1, e_2, \ldots \in \mathbb{N}_{>1}$ for which e_q divides e_{q+1} for each q.

We have: $\mathbb{Z}/e_{q+1} \twoheadrightarrow \mathbb{Z}/e_q$ by $t_i \mapsto t_{i \bmod e\_q}$.

Hence:

$$B(\tilde{A}_{r+1} \twoheadrightarrow \cdots \twoheadrightarrow B(e_{q+1}, e_{q+1}, r) \twoheadrightarrow B(e_q, e_q, r) \twoheadrightarrow \cdots \twoheadrightarrow B(1,1,r)$$

THE AMERICAN UNIVERSITY OF PARIS

5D. F(de,e,r): pushout

Consider $e_1, e_2, \ldots \in \mathbb{N}_{>1}$ for which e_q divides e_{q+1} for each q.

For p<q the diagram below commutes, and the fake braid group $F(e_q, e_p, r)$ is the universal object with this property.

$$\begin{tikzcd} B(\widetilde{A}_{r-1}) \cong B(\infty,\infty,r) \arrow[r, hook] \arrow[d, two heads] & \mathbb{Z} \ltimes_{e_p} B(\infty,\infty,r) \cong B(\delta e_p, e_p, r) \arrow[d, two heads] \\ B(e_q, e_q, r) \arrow[r, hook] & \mathbb{Z} \ltimes_{e_p} B(e_q, e_q, r) \cong F(e_q, e_p, r) \end{tikzcd}$$

THE AMERICAN UNIVERSITY OF PARIS

5E. Suitable subgraphs and parabolics

Suitable sub-graphs:

1. Truncate the tail: for $4 \leq r_1 < r_2$:
   B(#,*,r_1) embeds in B(#,*,r_2).
2. The disk on e nodes alone is the dual structure for the dihedral group I_2(e):
   I_2(e) embeds in B(e,e,r).
3. The subgraphs consisting of exactly one node on the disk give e structures, all isomorphic to the classical type A_{r-1} :
   A_{r-1} embeds in B(e,e,r).

5F. Graph foldings and group embeddings

A *diagram folding* of diagram Γ_1 onto the diagram Γ_2 gives rise to a group embedding B(Γ_2) $\hookrightarrow$ B(Γ_1).

The canonical example: the folding of the diagram of type A_{2r-1} onto the diagram of type B_r gives rise to the group embedding:

$$B(B_r) \hookrightarrow B(A_{2r-1})$$

In the case of B(*, *,r): the folding of the disk onto the first node of the type B_{r-1} diagram gives rise to the group embedding:

$$B(B_{r-1}) \hookrightarrow B(e,e,r)$$

by: $a_1 \mapsto \tau = t_i t_{i-1}$