Germ-monoids *vs* Garside monoids for complex braid groups

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Definition

The reflection groups of type (e, e, r) are defined in terms of positive integral parameters e, r :

$$\mathbf{G}(e,e,r) = \left\{ \begin{array}{c} r \times r \text{ monomial matrices} \\ (x_{ij}) \text{ over } \{0\} \cup \mu_e \end{array} \;\middle|\; \prod_{x_{ij} \neq 0} x_{ij} = 1 \right\}.$$

Short history

'54 a classification : $\mathbf{G}(de, e, r) + \mathbf{G}_4 + \cdots + \mathbf{G}_{37}$ [Shephard–Todd, Canad. J. Math.]

'98 a monoid presentation for $\mathbf{B}(e, e, r)$ [Broué–Malle–Rouquier, J. reine angew. Math.]

'00 a proof of the embedding failure

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Definition

A Garside monoid is a monoid which

- *is cancellative* $abc = adc \implies b = d$
- *admits lcm's* $a \vee b = a \cdot (a \backslash b) = b \cdot (b \backslash a)$
- *admits a Garside element*

an element whose right and left divisors
- *coincide*
- *are finite in number*
- *generate the monoid*

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A Garside group is the group of fractions of a Garside monoid.

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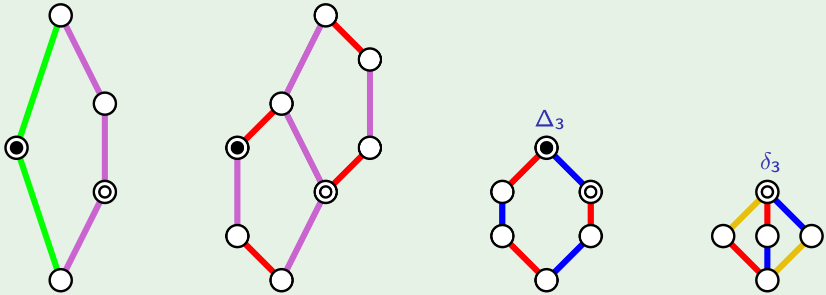
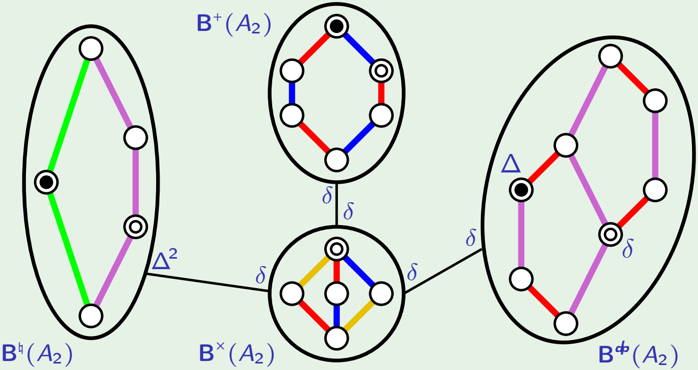
Some Garside monoids for the 3-strand braid group $\mathbf{B}(A_2)$



$\mathbf{B}^{\natural}(A_2)$ $\mathbf{B}^{\Phi}(A_2)$ $\mathbf{B}^{+}(A_2)$ $\mathbf{B}^{\times}(A_2)$

A tree product of Garside monoids for $\mathbf{B}(A_2)$



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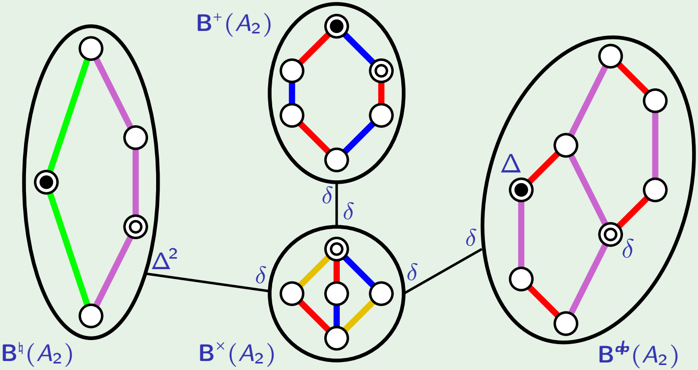
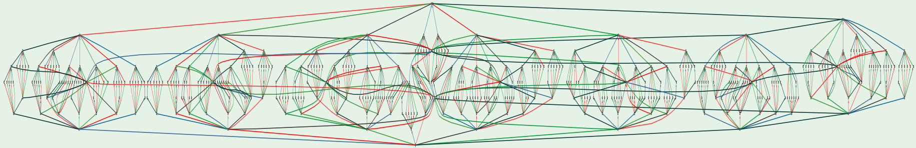
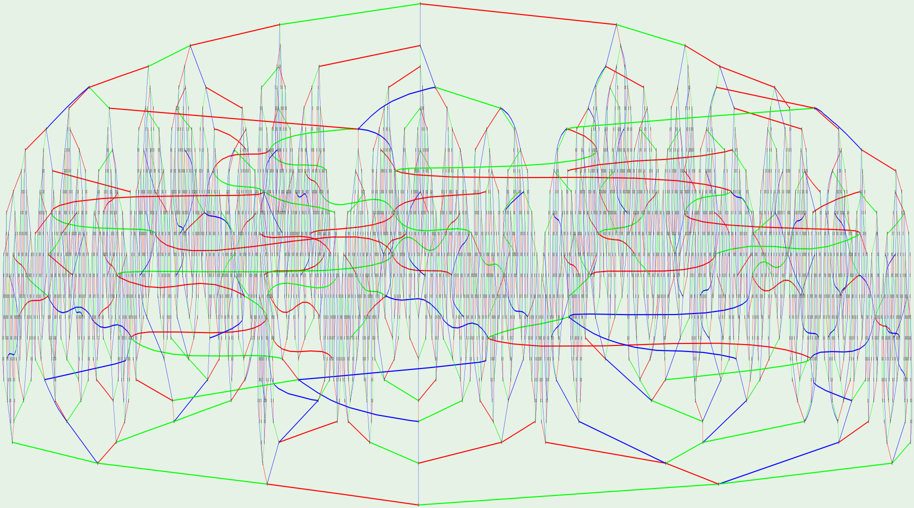
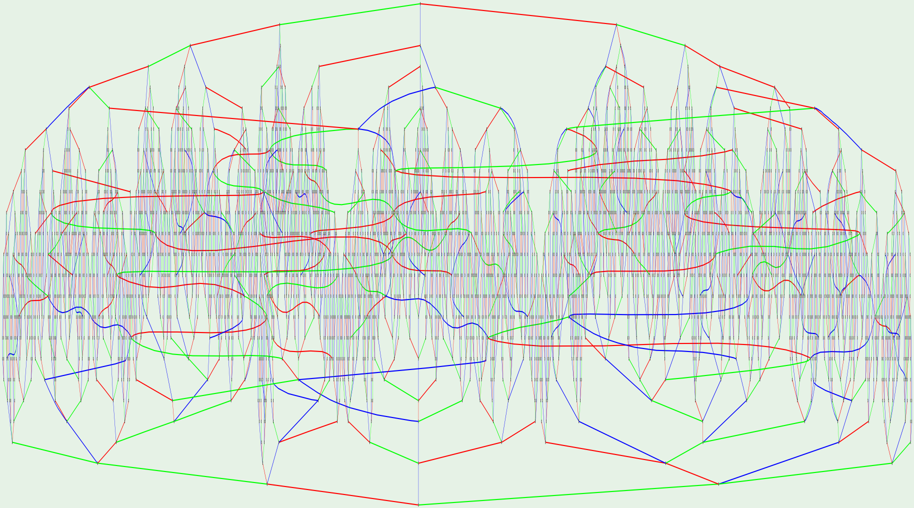




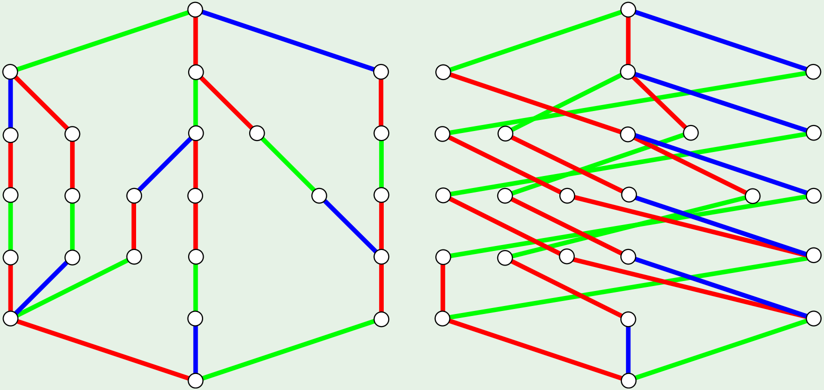
The 4004 simples of $\langle\ a, x : x^3ax^{-4}a^{-1}x^3ax^{-4}a^{-1}x^6ax^{-4}a^{-1}\rangle$



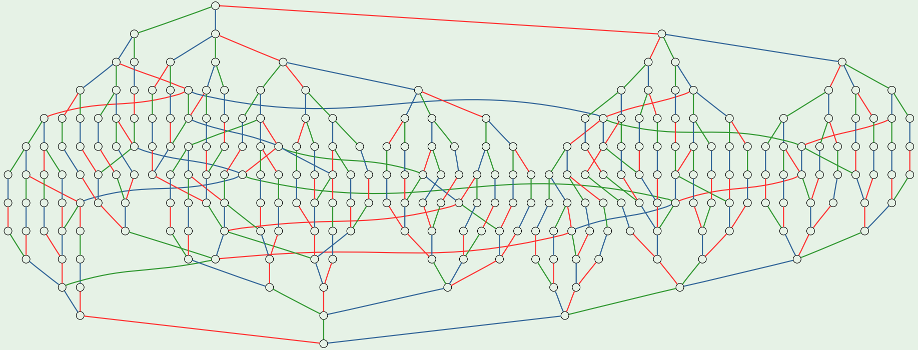
The 4004 simples of $\langle a, x_1, x_2, x_3 : ax_1 = x_3 a, x_1^3 = x_2^3, x_2^4 = x_3^4 \rangle$



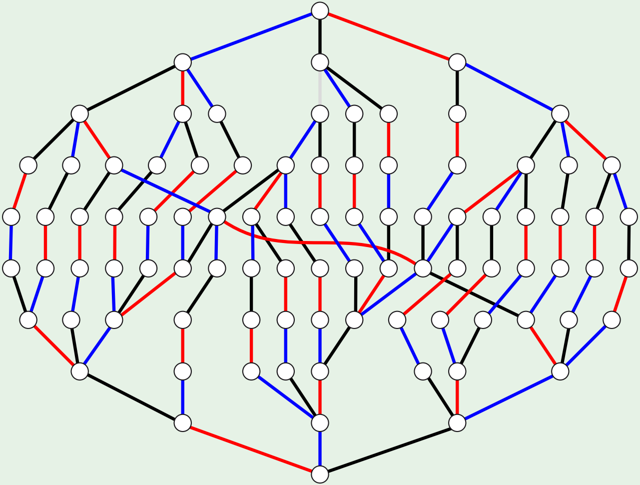
The Garside monoid M_χ for the 3-strand pure braid group $\mathbf{P}(A_2)$



The lattice of simples in $\mathbf{U}(A_3) = \langle\ \sigma_1, \sigma_2, \sigma_3 : \begin{smallmatrix}(\sigma_1\sigma_2\sigma_3)\sigma_1 = \sigma_2(\sigma_1\sigma_2\sigma_3) \\ (\sigma_1\sigma_2\sigma_3)\sigma_2 = \sigma_3(\sigma_1\sigma_2\sigma_3)\end{smallmatrix}\ \rangle$



The lattice of simples in $\mathbf{B}^+(G_{13})$



Length function, associated order and generated monoid

Let G be a group generated by X as a monoid.
A partial order on G is defined as follows :

- an X-decomposition of $g \in G$ is a sequence $(g_1, \ldots, g_n) \in X^n$ satisfying $g = g_1 \cdots g_n$.
- we denote by $\ell_X(g)$ the common length of the minimal X-decompositions of g.
- we write $g <_X h$ for $\ell_X(g) + \ell_X(g^{-1}h) = \ell_X(h)$.

To any subset S of G containing 1, the *germ-monoid* $\mathcal{C}(S)$ is defined by the following presentation :

$$\langle\ S : ab = c \text{ for } ab \in S \text{ and } a <_X ab\ \rangle.$$

Braid germ-monoids *vs* Garside germ-monoids

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P(G) & \longrightarrow & B(G) & \longrightarrow & G & \longrightarrow & 1 \\ & & & & \vdots & & \cup\!\!\mid & & \\ & & & & \mathcal{C}(S) & \longrightarrow & S & & \end{array}$$

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$$1 \longrightarrow \mathbf{P}(G) \longrightarrow \mathbf{B}(G) \longrightarrow G \longrightarrow 1$$

$$\mathcal{C}(S) \longleftarrow S \subseteq G$$

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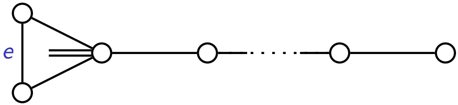
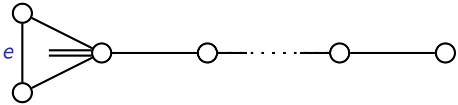


the *post-classical* braid monoid $\mathbf{B}^{\oplus}(e,e,r)$

- atoms : $e+r-2$
- simples : $\frac{(2(r-1)+e)!!}{e!!}$
- delta : $r(r-1)$
- Poincaré : $\prod_{k=1}^{r-1}(1+q+\cdots+q^{k-1}$
  $+eq^k+q^{k+1}+\cdots+q^{2k})$

the dual braid monoid $\mathbf{B}^{\times}(e,e,r)$

- atoms : $(e+r-2)(r-1)$
- simples : $\frac{er+r-e}{r}\binom{2r-2}{r-1}$
- delta : r
- Zeta : $\frac{r+e(r-1)(q-1)}{r}\prod_{k=1}^{r-1}\frac{ek+e(r-1)(q-1)}{ek}$



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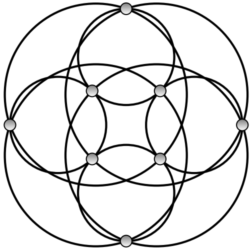

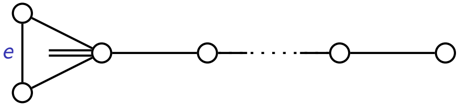

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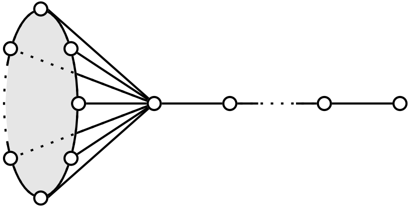

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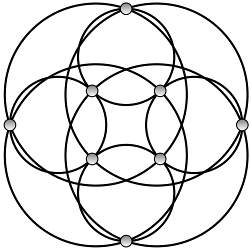

Some features of the lattice of simples in classical vs dual braid monoids

	classical	dual
lattice isomorphism	weak order on the reflection group	refinement order for non-crossing partitions
top element		
atoms		
nice factorization		

The number of simple elements in classical vs dual braid monoids

	A_n	B_n	D_n	H_3	F_4	H_4	E_6	E_7	E_8	$I_2(m)$
classical	$(n+1)!$	$2^n n!$	$2^{n-1} n!$	120	1152	14400	51840	2903040	696729600	$2m$
dual	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\binom{2n}{n}-\binom{2n-2}{n-1}$	32	105	280	833	4160	25080	$m+2$

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	classical	dual
lattice isomorphism	weak order on the reflection group	refinement order for non-crossing partitions
top element	longest element	
atoms	few reflections	
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Some features of the lattice of simples in classical vs dual braid monoids

	classical	dual
lattice isomorphism	weak order on the reflection group	refinement order for non-crossing partitions
top element	longest element	regular element
atoms	few reflections	many reflections
nice factorization		

The number of simple elements in classical vs dual braid monoids

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classical	$(n+1)!$	$2^n n!$	$2^{n-1} n!$	120	1152	14400	51840	2903040	696729600	$2m$
dual	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\binom{2n}{n}-\binom{2n-2}{n-1}$	32	105	280	833	4160	25080	$m+2$

Some features of the lattice of simples in classical vs dual braid monoids

	classical	dual
lattice isomorphism	weak order on the reflection group	refinement order for non-crossing partitions
top element	longest element	regular element
atoms	few reflections	many reflections
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Some numerical facts justifiying the duality terminology [Bessis '03]

	$\mathbf{B}^{+}(W)$	$\mathbf{B}^{\times}(W)$
Product of the atoms	$\mathbf{c}$	$\mathbf{w}_0$
Δ	$\mathbf{w}_0$	$\mathbf{c}$
Number of atoms	n	N
Length of Δ	N	n
Order of $a \mapsto a^{\Delta}$	2	h
Regular degree	h	2

A different kind of duality

	$\mathbf{B}^{\oplus}(e,e,r)$	$\mathbf{B}^{\times}(e,e,r)$
Number of atoms	$e+r-2$	$(e+r-2)(\boldsymbol{r}-\mathbf{1})$
Length of Δ	$r(\boldsymbol{r}-\mathbf{1})$	r
Order of $a \mapsto a^{\Delta}$	$\dfrac{e}{e \wedge r}$	$\dfrac{e(\boldsymbol{r}-\mathbf{1})}{e \wedge r}$

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Poincaré polynomial for $\mathbf{B}^+(W)$ with W a real reflection group

$$P_W^+(q) = \prod_{k=1}^{r}(1 + q + \cdots + q^{d_k - 1})$$

where the numbers d_k denote the reflection degrees.

$$\begin{aligned} P_{A_n}^+(q) &= \prod_{k=1}^{n}(1 + q + \cdots + q^k), \\ P_{B_n}^+(q) &= \prod_{k=1}^{n}(1 + q + \cdots + q^{2k-1}), \\ P_{D_n}^+(q) &= (1 + q + \cdots + q^{n-1})\prod_{k=1}^{n-1}(1 + q + \cdots + q^{2k-1}). \end{aligned}$$

Poincaré polynomial for $\mathbf{B}^\oplus(e, e, r)$

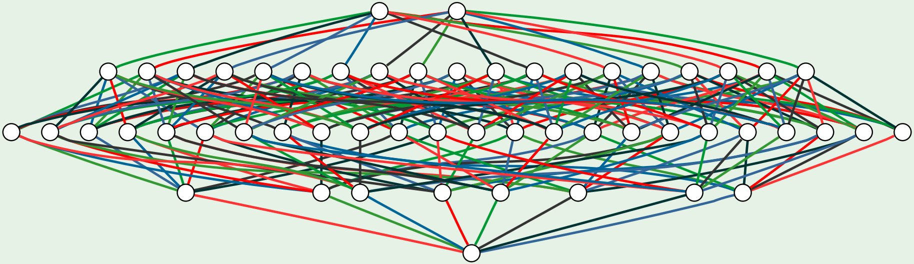
$$P_{(e,e,r)}^\oplus(q) = \prod_{k=1}^{r}(1 + q + \cdots + q^{k-2} + eq^{k-1} + q^k + \cdots + q^{2k-2}).$$

Length function, associated order and generated monoid

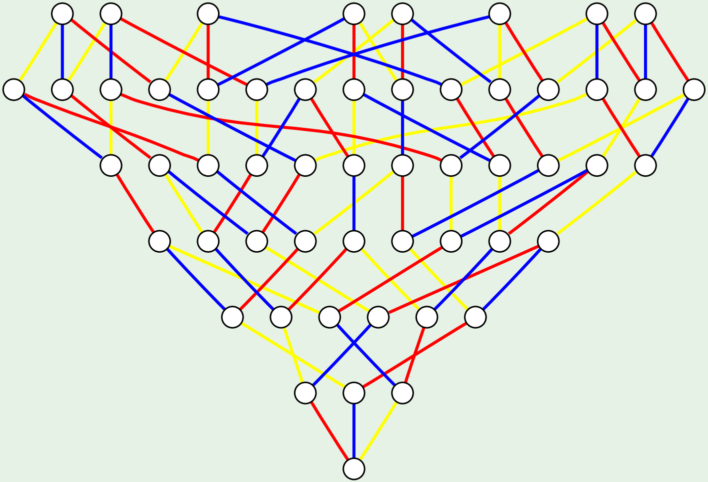
Let G be a group generated by X as a monoid.
A partial order on G is defined as follows :

- an X-decomposition of $g \in G$ is a sequence $(g_1, \ldots, g_n) \in X^n$ satisfying $g = g_1 \cdots g_n$.
- we denote by $\ell_X(g)$ the common length of the minimal X-decompositions of g.
- we write $g <_X h$ for $\ell_X(g) + \ell_X(g^{-1}h) = \ell_X(h)$.

The $<_X$-poset for $\mathbf{G}(3,3,3)$ with $X = \{a_{01}, a_{10}, a_0^{(0)}, a_0^{(1)}, a_0^{(2)}, a_1^{(0)}, a_1^{(1)}, a_1^{(2)}\}$



The $<_X$-poset for $\mathbf{G}(3,3,3)$ with $X = \{t_0, t_1, s_3\}$



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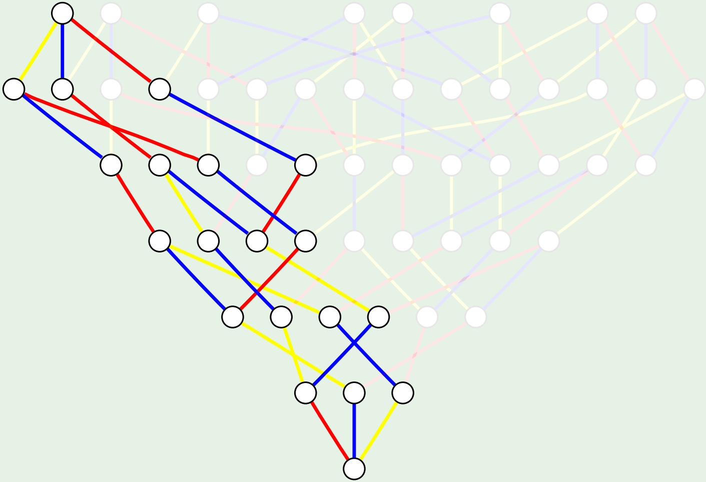

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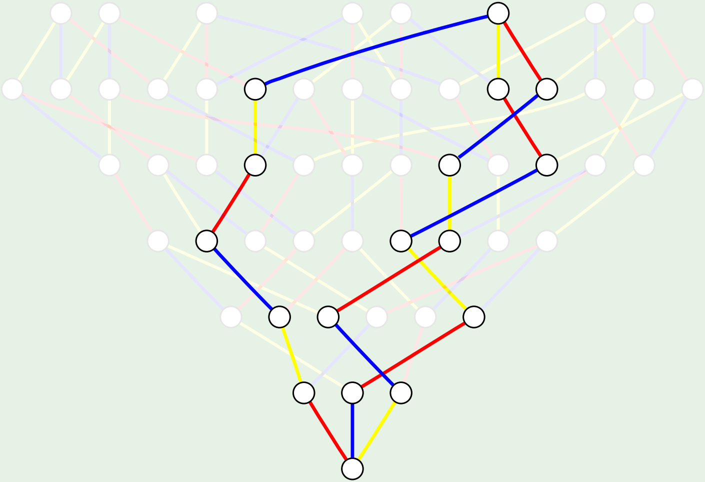

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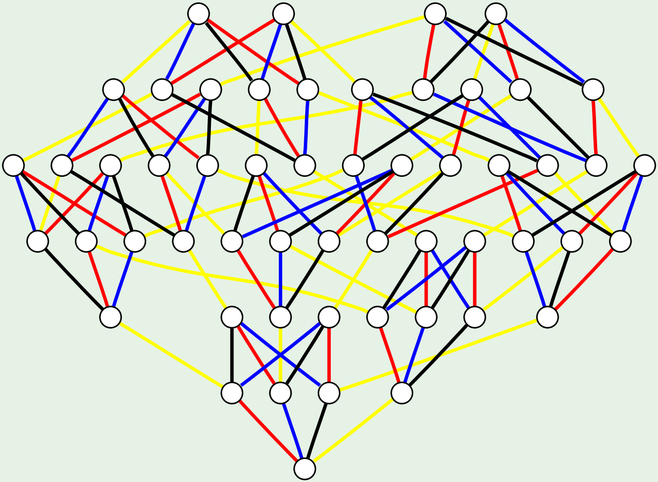

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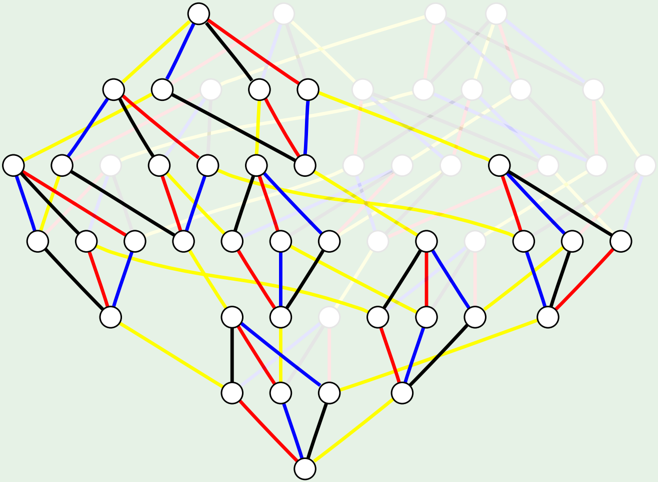

The classical braid monoid $\mathbf{B}^+(2,1,r)$ admits the presentation

$$\begin{aligned}\langle\ \tau_1,\sigma_1,\ldots,\sigma_{r-1} : \ & \sigma_1\tau_1\sigma_1\tau_1 = \tau_1\sigma_1\tau_1\sigma_1 \\ & \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{for } |i-j| = 1, \\ & \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1\ \rangle.\end{aligned}$$

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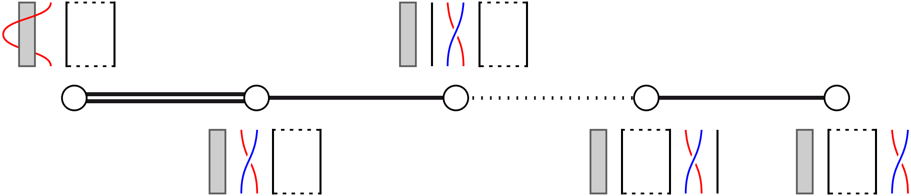

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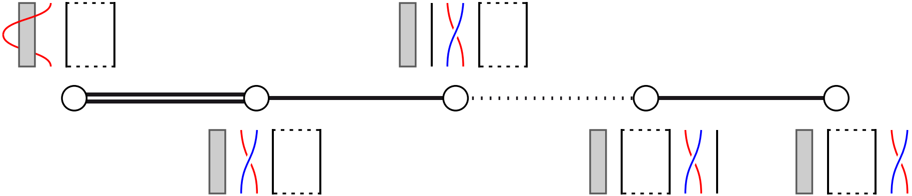

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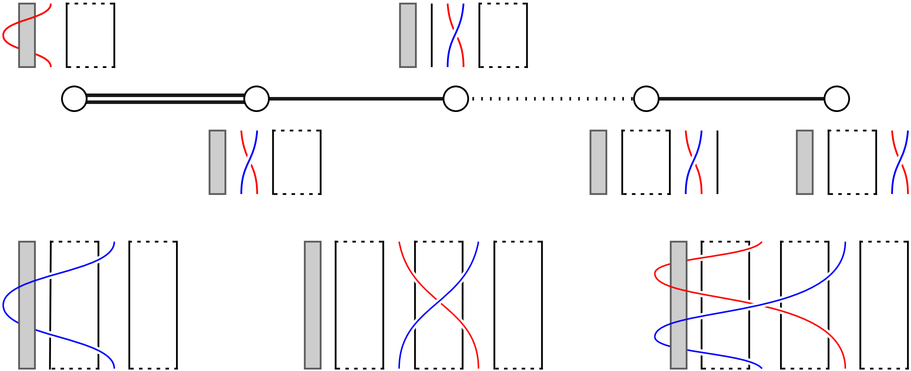
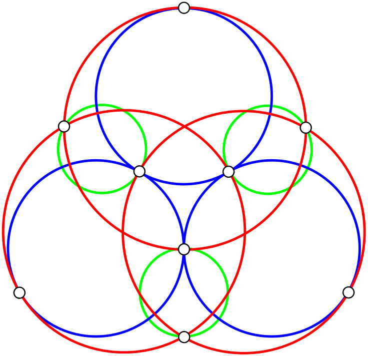
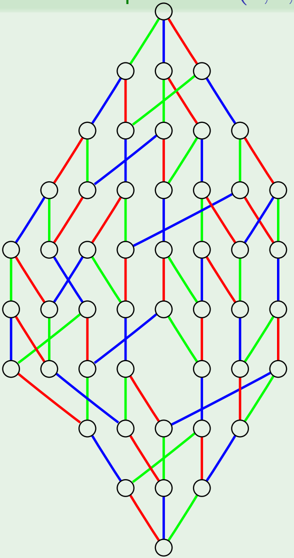


The dual braid monoid $\mathbf{B}^{\times}(2,1,r)$ admits the presentation

$$\langle\ \tau_t, \alpha_{ts}, \beta_{ts} : [\alpha_{ts}, \tau_s, \beta_{ts}, \tau_t] \text{ for } t > s,$$
$$[\alpha_{ts}, \alpha_{sr}, \alpha_{tr}],\ [\beta_{ts}, \alpha_{sr}, \beta_{tr}],\ [\alpha_{ts}, \beta_{sr}, \beta_{tr}] \text{ for } t > s > r,$$
$$[\alpha_{ts}, \tau_r],\ [\tau_t, \alpha_{sr}],\ [\beta_{tr}, \tau_s] \text{ for } t > s > r,$$
$$[\alpha_{ts}, \alpha_{rq}],\ [\alpha_{ts}, \beta_{rq}],\ [\beta_{ts}, \alpha_{rq}],$$
$$[\alpha_{tq}, \alpha_{sr}],\ [\beta_{tq}, \alpha_{sr}],\ [\beta_{tq}, \beta_{sr}] \text{ for } t > s > r > q\ \rangle.$$



The lattice of simples in $\mathbf{B}^+(2,1,3)$



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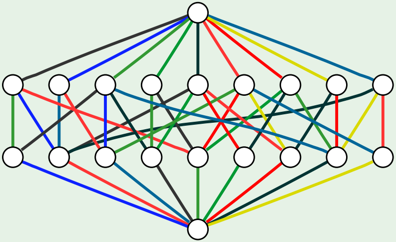

Program

- Start from a finite group G.
- Choose a subset X generating G as a monoid.
- Compute the poset $(G, \leq_X)$.
- Consider some maximal ideal $\mathcal{J}$ satisfying $G = \mathcal{J}/torsion(\mathcal{J} \cap X)$.
- Extract a (monoid) presentation M
- Check garsideness

Extra Program

Whenever G admits a braid group $B(G)$,

- Check whether the group of fractions of G is $B(G)$ as well (and not only a quotient).

2	3	**4**										
8	8	6										
	3	4	5	6	7	**9**						
	48	48	48	40	28	20						
		4	5		7		10	13		**16**		
		384	384		384		224	100		70		
			5	6		9			15		21	**25**
			3840	3840		3840			1344		364	252

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			△									

the classical braid monoid $\mathbf{B}^+(2,1,r)$

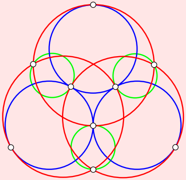
- atoms : r
- simples : $2^r r!$
- delta : r^2
- Poincaré : $\prod_{k=1}^{r}(1+q+\cdots+q^{2r-1})$

2	3	**4**										
8	8	6										
	3	4	5	6	7	**9**						
	48	48	48	40	28	20						
		4	5		7		10	13		**16**		
		384	384		384		224	100		70		
			5	6		9			15		21	**25**
			3840	3840		3840			1344		364	252

△

the dual braid monoid $\mathbf{B}^{\times}(2,1,r)$

- atoms : r^2
- simples : $\binom{2r}{r}$
- delta : r
- Zeta : $\prod_{k=1}^{r} \frac{k+r(q-1)}{k}$



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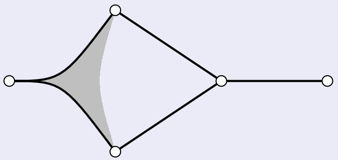
			5	6		9			15		21	**25**
			3840	3840		3840			1344		364	252
				△								

the classical++ monoid

- atoms : $r + 1$
- simples : $2^r r!$
- delta : ???
- Poincaré *vs* Zeta : –



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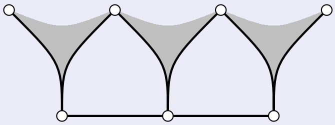
			5	6		9			15		21	**25**
			3840	3840		3840			1344		364	252
						△						

the champaign monoid

- atoms : $2r-1$
- simples : $2^r r!$
- delta : ???
- Poincaré *vs* Zeta : –



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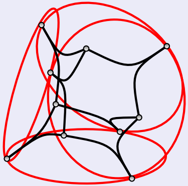
△

the triangular monoid

- atoms : $\frac{r(r+1)}{2}$
- simples : $\frac{2^r}{r+1}\binom{2r}{r}$
- delta : ???
- Poincaré *vs* Zeta : –



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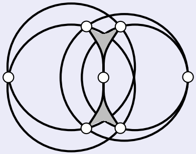
											△	

the biOHAzard monoid

- atoms : $r^2 - r + 1$
- simples : $\frac{3r-2}{2r-1}\binom{2r}{r}$
- delta : ???
- Poincaré *vs* Zeta : –



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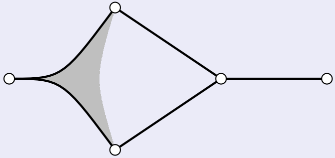

the champaign monoid

- atoms : $2r-1$
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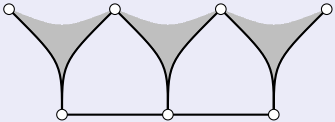

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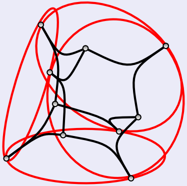

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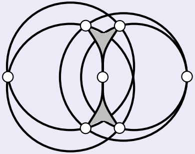

the dual braid monoid $\mathbf{B}^{\times}(2,1,r)$

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- simples : $\binom{2r}{r}$
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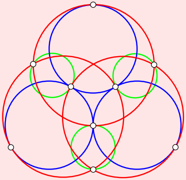

- These new Garside germ-monoids fails to be braid monoids for $\mathbf{B}(d,1,r)$.
- The corresponding Garside groups are proper quotients of $\mathbf{B}(d,1,r)$.
- The associated diagrams provide new diagrams for $\mathbf{G}(d,1,r)$.

The fake braid diagram

Simples in $\mathbb{N} \ltimes \mathbb{N}^2$

Main obstruction

$$\left\{ \begin{array}{c} 12 = 23 \\ 32 = 21 \\ 13 = 31 \end{array} \right\}$$

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in a Garside germ-monoid
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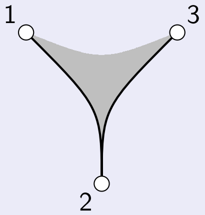

Simples in $\mathbb{N} \rtimes \mathbb{N}^2$

Main obstruction

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$$\Downarrow$$

$$121\ 2 = 21\ 21$$

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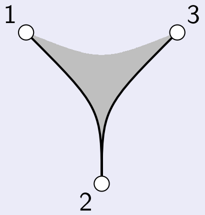

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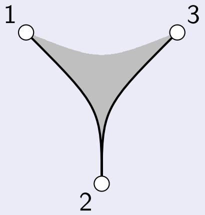

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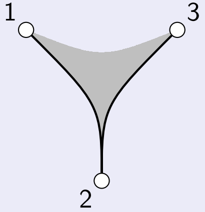
The fake braid diagram



Simples in $\mathbb{N} \rtimes \mathbb{N}^2$

Main obstruction

$$\left\{ \begin{array}{c} 12 = 23 \\ 32 = 21 \\ 13 = 31 \end{array} \right\}$$

$$\Downarrow$$

$$121^k 2 = 21^k 21 \ (k \in \mathbb{Z})$$

Any occurrence of the fake braid relation
in a Garside germ-monoid
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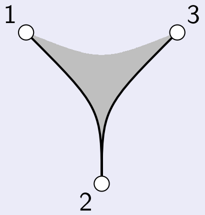

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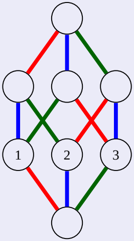

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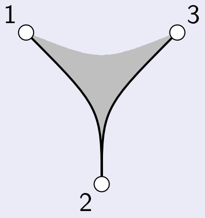

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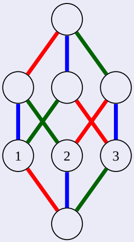

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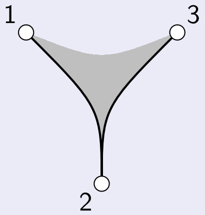

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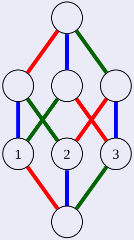

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Any occurrence of the fake braid relation
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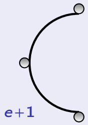
the classical braid monoid $\mathbf{B}^+(2e, e, 2)$

- atoms : 3
- simples : $\begin{cases} 4e & \text{for } e \text{ even} \\ 8e^2 + 4e + 1 & \text{for } e \text{ odd} \end{cases}$
- delta : ???
- Poincaré : –



$$\langle\ s, t, u : stu = tus \quad \underbrace{ustutut\cdots}_{e+1 \text{ factors}} = \underbrace{stututu\cdots}_{e+1 \text{ factors}}\ \rangle$$

Simples in $\mathbf{B}^+(6, 3, 2)$

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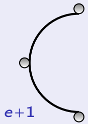

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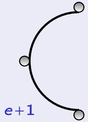

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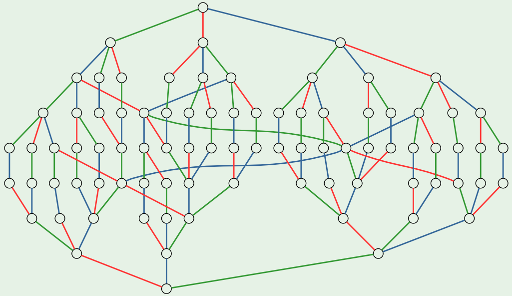

the spintop monoid

- atoms : $2e + 1$
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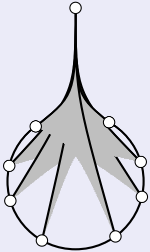

Simples in the spintop monoid associated with $\mathbf{G}(4,2,3)$

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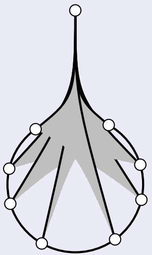

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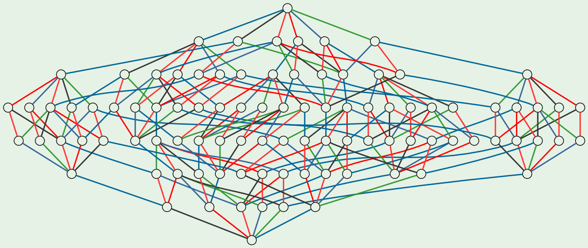

Braid monoids from germ-monoids

One can try to reconstruct (non germ Garside) braid monoids from (non braid Garside) germ-monoids.

the shell monoid for $\mathbf{B}(de, e, 2)$

- atoms : $e + 1$
- simples : ???
- delta : ???
- Poincaré *vs* Zeta : –

the rocket monoid for $\mathbf{B}(de, e, 2)$

- atoms : $e + 2$
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- Zeta : ???

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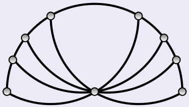

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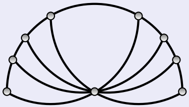

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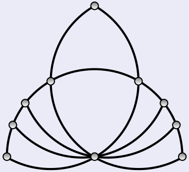

Germ-monoids from ad-hoc group

One can try to view a (non germ Garside) braid monoid as a (Garside) braid germ-monoid for a bigger group.

Consider the (Garside) braid monoid $\mathbf{B}^{\circ}(2,1,r)$ with presentation

$$\begin{aligned}\langle\ \sigma_1,\ldots,\sigma_{r-1},\tau_1,\ldots,\tau_r:\ & \sigma_j\sigma_k=\sigma_k\sigma_j \quad \text{for} \quad k>j+1\ ,\\ & \sigma_j\sigma_{j+1}\sigma_j=\sigma_{j+1}\sigma_j\sigma_{j+1}\ ,\\ & \sigma_j\tau_j\tau_{j+1}=\tau_j\tau_{j+1}\sigma_j\ ,\\ & \sigma_j\tau_j=\tau_{j+1}\sigma_j\ ,\\ & \sigma_j\tau_k=\tau_k\sigma_j \ \text{for} \quad k\neq j,j+1\ \rangle.\end{aligned}$$

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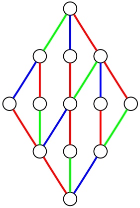

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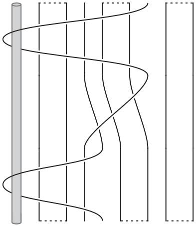
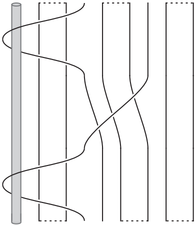

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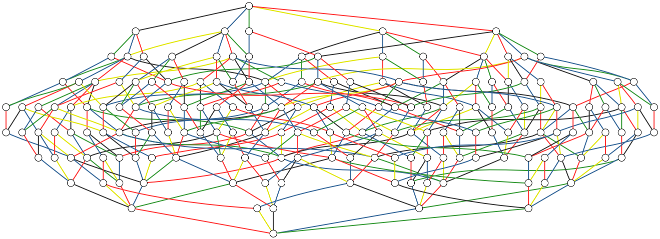

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r	0	1	2	3	4	5	6	7	8
$\lvert\{\text{simples in } \mathbf{B}^{+}(2,1,r)\}\rvert$	1	2	8	48	384	3 840	46 080	645 120	10 321 920
$\lvert\{\text{simples in } \mathbf{B}^{\circ}(2,1,r)\}\rvert$	1	2	13	184	4 607	180 106	10 138 107	776 735 268	77 727 056 619
$\lvert\{\text{simples in } \mathbf{B}^{\times}(2,1,r)\}\rvert$	1	2	6	20	70	252	924	3 432	12 870

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It is not a germ-monoid for $\mathbf{G}(2,1,r)$.

Now, $\mathbf{B}^\circ(2,1,3)$ happens to be a germ-monoid for the group $\mathbf{G}(26)$ of order 1296.