From Broué's conjectures to questions in braid group theory

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Background

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Notation

- $\mathbf{G}$ *is a reductive connected algebraic group with rational structure given by the Frobenius endomorphism* F
- $\mathbf{T} \subset \mathbf{B}$ *rational maximal torus and Borel subgroup.*
- $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ *Weyl group;* (W, S) *is a Coxeter system.*
- *For* $I \subset S$*, define* $\mathbf{P}_I$ *standard parabolic subgroup,* $\mathbf{L}_I$ *standard Levi subgroup and* W_I *parabolic subroup of* W*.*

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Definition

For $I \subset S$, $w \in N_W(W_I)$ let $X(I, w) = \{\mathbf{P} \mid (\mathbf{P}, F(\mathbf{P})) \sim_{\mathbf{G}} ((\mathbf{P}_I)^w, \mathbf{P}_I)\}$.

Note that only the coset $W_I w$ matters so we can choose for w the shortest coset representative. For such a w we have $I^w = I$ and $\dim(X(w, I)) = l(w)$.

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Notation

For subsets I and J of S and $w \in W$ we write $\mathbf{P} \xrightarrow{I,w,J} \mathbf{Q}$ to indicate that the pair of parabolic subgroups $(\mathbf{P}, \mathbf{Q})$ is conjugate to $((\mathbf{P}_I)^w, \mathbf{P}_J)$.

In particular a Deligne-Lusztig variety is $\{\mathbf{P} \mid \mathbf{P} \xrightarrow{I,w,I} F(\mathbf{P})\}$ with $I^w = I$.

Lehrer-Springer elements

The pairs (I, w) involved in Broué's conjectures are such that w has a maximal $\exp(2i\pi/d)$-eigenspace V_ζ for some integer d and $C_W(V_\zeta) = W_I$. (We have put $\zeta = \exp(2i\pi/d)$)

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The associated braid group is denoted by $B(w)$. We want to make $B(w)$ act on $X(I, w)$, this action inducing an action on the cohomology of the cyclotomic Hecke algebra of W with predicted parameters.

Choice of w

Which w to choose in the W-conjugacy class of the coset $W_I w$ so that (in particular) $X(I, w)$ has the right dimension $\frac{|\Phi - \Phi_I|}{d}$?

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If we introduce the braid group B associated to W and the canonical lifting $w \mapsto \mathbf{w}$ from W to B, this can be expressed as
$\frac{|\Phi-\Phi_I|}{d} == \frac{l(\boldsymbol{\pi}\boldsymbol{\pi}_I^{-1})}{d}$ where $\boldsymbol{\pi}_I = \mathbf{w}_I^2$ and $\boldsymbol{\pi} = \mathbf{w}_S^2$.

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Theorem

In the conjugacy class of $W_I w$ as above, there exists always a $W_J v$ such that $\mathbf{v}^d = \boldsymbol{\pi}\boldsymbol{\pi}_J^{-1}$.

In particular $l(v) = \frac{l(\boldsymbol{\pi}\boldsymbol{\pi}_J^{-1})}{d}$

Conversely

Theorem

If w is an I-reduced element such that $I^w = I$ and $\mathbf{w}^d = \boldsymbol{\pi}\boldsymbol{\pi}_I^{-1}$, and if there is no pair $(J \subset I, \mathbf{w}' = \mathbf{wv})$ which the same properties as $(I, \mathbf{w})$ then w is a Lehrer-Springer element (ie, the ζ-eigenspace of w is maximal).

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(I-reduced means shortest coset representative).
So we want to define an action of $B(w)$ on $X(I, w)$ when $\mathbf{w}$ is as in the above theorem.

Recall that $\mathbf{P} \xrightarrow{I,w,J} \mathbf{Q}$ means that $(\mathbf{P}, \mathbf{Q}) \sim_{\mathbf{G}} ((\mathbf{P}_I)^w, \mathbf{P}_J)$.

In particular a Deligne-Lusztig variety is $\{\mathbf{P} \mid \mathbf{P} \xrightarrow{I,w,I} F(\mathbf{P})\}$ with $I^w = I$.

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To define operators on these varieties we use

Proposition

Let $I, J, K \subset S$ and $w_1, w_2 \in W$ be such that $I^{w_1} = J$, $J^{w_2} = K$, with w_1 and w_2 shortest coset representatives and $l(w_1) + l(w_2) = l(w_1 w_2)$; then

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Do we get a morphism $D_{w_1} : \mathbf{P} \mapsto \mathbf{P}_1$ from $\mathbf{X}(I, w_1 w_2)$ to $\mathbf{X}(J, w_2 w_1)$?

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Do we get a morphism $D_{w_1} : \mathbf{P} \mapsto \mathbf{P}_1$ from $\mathbf{X}(I, w_1 w_2)$ to $\mathbf{X}(J, w_2 w_1)$? Only if $l(w_2 w_1) = l(w_2) + l(w_1)$.

More generally we can consider varieties associated to sequences of elements in W For a sequence $(w_1, \ldots, w_r)$ of elements of W and a sequence $(I_1, \ldots, I_r)$ of subsets of S such that $I_j^{w_j} = I_{j+1}$ and $I_r^{w_r} = I_1$ we define

$$X(I, (w_1, \ldots, w_r)) = \{(\mathbf{P}_1, \ldots, \mathbf{P}_r) \mid \mathbf{P}_j \xrightarrow{I_j, w_j, I_{j+1}} \mathbf{P}_{j+1}\}$$

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So we can define a morphism of varieties sending

$$\mathbf{P}_1 \xrightarrow{I_1, w_1, I_2} \mathbf{P}_2 \xrightarrow{I_2, w_2, I_3} \mathbf{P}_3 \ldots \mathbf{P}_r \xrightarrow{I_r, w_r, I_{r+1}} F(\mathbf{P}_1)$$

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Theorem (After Deligne)

The variety $X(I, (w_1, \ldots, w_r))$ *depends only on the element* $b = \mathbf{w}_1 \ldots \mathbf{w}_r \in B^+$.

We write $X(\mathbf{I}, b)$ for $b \in B^+$ such that $\mathbf{I}^b = \mathbf{I}$ and b is $\mathbf{I}$-reduced in the sense that it has no divisor in the parabolic submonoid $B_{\mathbf{I}}^+$ ($\mathbf{I}$ is the canonical lifting of I in B).

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For b and $\mathbf{I}$ as above, if $b_1 \in B^+$ left divides b and $\mathbf{I}^{b_1} = \mathbf{J} \subset \mathbf{S}$ we get a morphism $D_{b_1} : X(\mathbf{I}, b) \to X(\mathbf{J}, b_1^{-1} b b_1)$.

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Notation

We write $B(\mathbf{w},\mathbf{I})$ for the subgroup of B generated by $\mathbf{I}$-reduced elements in $C_B(\mathbf{w})\cap N_B(\mathbf{I})$.

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Let w be a Lehrer-Springer element such that $\mathbf{w}^d = \boldsymbol{\pi}\boldsymbol{\pi}_I^{-1}$; what is the relation between $B(w)$ and $B(\mathbf{w}, \mathbf{I})$?

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Question (2)

Describe $B(\mathbf{w}, \mathbf{I})$ for such a $\mathbf{w}$.

This leads to the following questions:

Question (1)

Let w be a Lehrer-Springer element such that $\mathbf{w}^d = \boldsymbol{\pi}\boldsymbol{\pi}_I^{-1}$; what is the relation between $B(w)$ and $B(\mathbf{w}, \mathbf{I})$?

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Question (2)

Describe $B(\mathbf{w}, \mathbf{I})$ for such a $\mathbf{w}$.

More precisely

Question (2′)

Is the group $B(\mathbf{w}, \mathbf{I})$ generated by the above conjugations?

Introducing categories

We have a category whose objects are pairs $(\mathbf{J}, b)$, with $\mathbf{J} \subset \mathbf{S}$, $b \in B^+$, b being $\mathbf{J}$-reduced and satisfying $\mathbf{J}^b = \mathbf{J}$. Elementary morphisms from $(\mathbf{J}, b)$ to $(\mathbf{K}, b')$ are elements $b_1 \in B^+$ such that b_1 divides b in B^+, $\mathbf{K} = \mathbf{J}^{b_1}$ and $b' = b_1^{-1} b b_1$.

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The endomorphisms of $(\mathbf{I}, \mathbf{w})$ in this category are elements of $B(\mathbf{w}, \mathbf{I})$.
Using the properties of the above category we prove that question $2'$ has a positive answer.
The main property is that the category is a "Garside" category (ie has similar property of divisibility as the braid monoid).
The concept of Garside category comes mainly from Krammer and Bessis.
Bessis has used these categories to answer question (1) (for I empty) giving $B(w) \simeq C_B(\mathbf{w})$ for a regular w.

epilogue

Besides answering question 1 much more is left to be done:

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(2) Prove that this gives the full endomorphism algebra of the cohomology viewed as a representation of $\mathbf{G}^F$.

Not much is known except when $I = \emptyset$. In this case the main results are positive answers

- to (1) for Coxeter elements (Lusztig 1976), for $w = w_S$ (D.-Michel-Rouquier 2006), for type A_n and B_n (D.-Michel 2006).
- to (2) for Coxeter elements (Lusztig) and for n-th roots of π in type A_n (D.-M.).