Relaxation techniques for the Euler equations with gravity

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July 7th 2014, Würzburg

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Outline

The system of Euler equation with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p \right) = -\rho \partial_x \Phi \\ \partial_t E + \partial_x (E + p) u = -\rho u \partial_x \Phi \end{cases}$$

- ρ : density
  u : velocity
  $E = \rho e + \rho u^2/2$: total energy, with e the internal energy
  $\tau := 1/\rho$: specific volume
  $p = p(\tau, e)$: pressure given by a general law
  $\Phi(x)$: given smooth gravitational potential
- Hyperbolicity assumption:

$$c^2 := \tau^2 (p \partial_e p - \partial_\tau p) > 0$$

The system of Euler equation with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = -\rho \partial_x \Phi \\ \partial_t E + \partial_x (E + p) u = -\rho u \partial_x \Phi \end{cases}$$

- We define
  - the vector of conservative variables $w = (\rho, \rho u, E)^T$,
  - the flux function $f(w) = (\rho u, \rho u^2 + p, (E + p)u)^T$,
  - the source term $s(w) = (0, -\rho, -\rho u)^T$,

  to rewrite the system into the compact form

$$\partial_t w + \partial_x f(w) = s(w) \partial_x \Phi.$$

- The set of physical admissible states is

$$\Omega = \left\{ w \in \mathbb{R}^3, \quad \rho > 0, \quad e > 0 \right\}.$$

Steady states

The steady states at rest are described by

$$\begin{cases} u = 0 \\ \partial_x p = -\rho \partial_x \Phi \end{cases}$$

No explicit expression of all the steady states at rest

Hydrostatic atmosphere

$$\begin{cases} u(x) = 0, \\ \rho(x) = \alpha e^{-\beta \Phi(x)}, \\ p(x) = \frac{\alpha}{\beta} e^{-\beta \Phi(x)}, \end{cases} \qquad \text{with } \alpha > 0 \text{ and } \beta > 0$$

Entropy

The pressure is assumed to satisfy the second law of thermodynamics
$\Rightarrow$ existence of a specific entropy $s(\tau, e)$ which satisfies

$$-Tds = de + pd\tau, \quad \text{with } T > 0 \text{ the temperature}$$

Lemma

The smooth solutions satisfy the additional conservation laws

$$\partial_t \rho\mathcal{F}(s) + \partial_x \rho\mathcal{F}(s)u = 0,$$

for all smooth function $\mathcal{F}$.
Moreover the application $w \mapsto \rho\mathcal{F}(s)$ is strictly convex iff

$$\mathcal{F}'(s) > 0 \quad \text{and} \quad \frac{1}{c_p}\mathcal{F}'(s) + \mathcal{F}''(s) > 0.$$

Objectives

Derive a numerical scheme which has the following properties:

- Preservation of the set Ω
- Accurate approximation of all the steady states at rest
- Exact capture of the hydrostatic atmosphere
- Discrete entropy inequalities
- General gravitational potential and general pressure law

Means

- Finite volume method (approximate Riemann solver)
- Relaxation method

References

- [Chalons et al. '10] : constant gravity field ($\Phi(x) = gx$)
- [Käpelli & Mishra '13] : isentropic steady states

The relaxation method without source term

- Initial system:
$$\partial_t w + \partial_x f(w) = 0. \quad (1)$$
- Relaxation system:
$$\partial_t W + \partial_x F(W) = \frac{1}{\varepsilon} R(W), \quad (2)$$
  - (2) should formally gives back (1) when $\varepsilon \to 0$.
  - (2) should be "simpler" than (1) (e.g. only linearly degenerate fields)
- The relaxation scheme is based on a splitting strategy:

  Time evolution: We evolve the initial data by the Godunov scheme for the system $\partial_t W + \partial_x F(W) = 0$ (i.e. $\varepsilon = +\infty$).
  $\Rightarrow$ We need the exact solution of the Riemann problem

  Relaxation: We take into account the relaxation source term by solving $\partial_t W = \frac{1}{\varepsilon} R(W)$ then taking the limit for $\varepsilon \to 0$.

The Suliciu model([Suliciu '98], [Bouchut '04]...)

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho \partial_x \Phi \\ \partial_t E + \partial_x (E + \pi) u = -\rho u \partial_x \Phi \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u = \frac{\rho}{\varepsilon}(p(\tau, e) - \pi) \\ \partial_t \Phi = 0 \end{cases}$$

Equilibrium

$$\pi = p(\tau, e)$$

Eigenvalues: $u - \frac{a}{\rho}, \quad 0, \quad u, \quad u + \frac{a}{\rho}$

Advantage:

- All the fields are linearly degenerate

Difficulties:

- The order of the eigenvalues is not fixed *a priori*
- The Riemann invariants for the eigenvalue 0 are strongly nonlinear

Relaxation model with moving gravity

$$\begin{cases} \partial_t\rho + \partial_x\rho u = 0 \\ \partial_t\rho u + \partial_x(\rho u^2 + \pi) = -\rho\partial_x Z \\ \partial_t E + \partial_x(E+\pi)u = -\rho u\partial_x Z \\ \partial_t\rho\pi + \partial_x(\rho\pi + a^2)u = \frac{\rho}{\varepsilon}(p(\tau,e)-\pi) \\ \partial_t\rho Z + \partial_x\rho Z u = \frac{\rho}{\varepsilon}(\Phi - Z) \end{cases}$$

Equilibrium

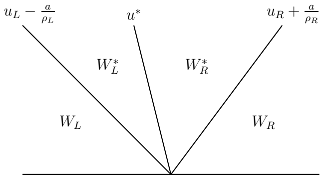
$$\pi = p(\tau, e)$$

$$Z = \Phi$$

- Eigenvalues: $u - \frac{a}{\rho}, \quad u, \quad u + \frac{a}{\rho}$
- All the fields are linearly degenerate
- Fixed order of the eigenvalues: $u - \frac{a}{\rho} < u < u + \frac{a}{\rho}$
- There is a missing Riemann invariant in order to determine a unique Riemann solution
  $\Rightarrow$ We need a closure relation

Relaxation model with moving gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho \partial_x Z \\ \partial_t E + \partial_x (E + \pi) u = -\rho u \partial_x Z \\ \partial_t \rho\pi + \partial_x(\rho\pi + a^2) u = \frac{\rho}{\varepsilon}(p(\tau, e) - \pi) \\ \partial_t \rho Z + \partial_x \rho Z u = \frac{\rho}{\varepsilon}(\Phi - Z) \end{cases}$$



To approximate the equation

$$\partial_x p = -\rho \partial_x \Phi \quad \underset{\substack{\text{equilibrium} \\ \pi = p(\tau,e) \\ Z = \Phi}}{\Longleftrightarrow} \quad \partial_x \pi = -\rho \partial_x Z,$$

we propose the following closure relation:

$$\frac{\pi_R^* - \pi_L^*}{\Delta x} = -\overline{\rho}(\rho_L, \rho_R) \frac{Z_R - Z_L}{\Delta x},$$

where $\overline{\rho}$ is a suitable ρ–average (defined later)

Approximate Riemann solver

Equilibrium state:

$$W^{eq}(w) = (\rho, \rho u, E, \rho p(\tau, e), \rho\Phi)^T$$

Theorem

With the closure equation, the Riemann problem admits a unique solution $W_{\mathcal{R}}\left(\frac{x}{t}, W_L, W_R\right)$.
Moreover,

$$w^{eq}\left(\frac{x}{t}, w_L, w_R\right) := W_{\mathcal{R}}^{(\rho,\rho u,E)}\left(\frac{x}{t}, W^{eq}(w_L), W^{eq}(w_R)\right)$$

defines an approximate Riemann solver (in the sense of Harten, Lax and van Leer) for the Euler equations with gravity.

Complete relaxation reformulation

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + \pi) = -\overline{\rho}(X^-, X^+) \partial_x Z \\ \partial_t E + \partial_x (E + \pi) u = -\overline{\rho}(X^-, X^+) u \partial_x Z \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u = \frac{\rho}{\varepsilon}(p(\tau, e) - \pi) \\ \partial_t \rho Z + \partial_x \rho Z u = \frac{\rho}{\varepsilon}(\Phi - Z) \\ \partial_t X^- + (u - \delta) \partial_x X^- = \frac{1}{\varepsilon}(\rho - X^-) \\ \partial_t X^+ + (u + \delta) \partial_x X^+ = \frac{1}{\varepsilon}(\rho - X^+) \end{cases}$$

Equilibrium

$$\pi = p(\tau, e) \qquad Z = \Phi \qquad X^{\pm} = \rho$$

- For $\delta > 0$ small enough, the system is hyperbolic with eigenvalues

$$u - \frac{a}{\rho} < u - \delta < u < u + \delta < u + \frac{a}{\rho}.$$

- The system has a complete set of Riemann invariants.
- Leads to the same approximate Riemann solver $w^{eq}\left(\frac{x}{t}, w_L, w_R\right)$.

Cargo-LeRoux formulation ($\Phi(x) = gx$)

We introduce a potential $q = \int^x g\rho dx$. Then we have

$$\partial_t q = \int^x g\partial_t \rho dx = -\int^x g\partial_x \rho u dx = -g\rho u = -u\partial_x q.$$

So q is governed by

$$\partial_t \rho q + \partial_x \rho q u = 0.$$

Equivalent reformulation of the Euler equations with gravity:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) + \partial_x q = 0 \\ \partial_t E + \partial_x (E + p) u + u \partial_x q = 0 \\ \partial_t \rho q + \partial_x \rho q u = 0. \end{cases}$$

Relaxation model with the Cargo-LeRoux formulation

- Case $\Phi(x) = gx$ [Chalons et al. '10]

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + \pi\right) + \partial_x q = 0 \\ \partial_t E + \partial_x (E + \pi) u + u \partial_x q = 0 \\ \partial_t \rho q + \partial_x \rho q u = 0 \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u = \frac{\rho}{\varepsilon}(p(\tau, e) - \pi) \end{cases}$$

Equilibrium

$$\pi = p(\tau, e)$$

- Extension for a general gravitational field

If we define the potential by $q = \int^x \rho \partial_x \Phi \, dx$, it no longer satisfies a transport equation. We enforce the natural relaxation model

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + \pi\right) + \partial_x q = 0 \\ \partial_t E + \partial_x (E + \pi) u + u \partial_x q = 0 \\ \partial_t \rho q + \partial_x \rho q u = \frac{\rho}{\varepsilon}\left(\int^x \rho \partial_x \Phi \, dx - q\right) \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u = \frac{\rho}{\varepsilon}(p(\tau, e) - \pi) \end{cases}$$

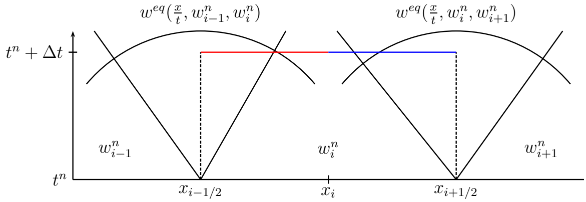
Equilibrium

$$\pi = p(\tau, e)$$

$$q = \int^x \rho \partial_x \Phi \, dx$$

The relaxation scheme

w_i^n: approximation of the solution on the cell $(x_{i-1/2}, x_{i+1/2})$ at time t^n



CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left| u_i^n \pm \frac{a}{\rho_i^n} \right| \leq \frac{1}{2}$$

The update at time $t^{n+1} = t^n + \Delta t$ is defined by

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_i} w^{eq}\left(\frac{x - x_{i-1/2}}{\Delta t}, w_{i-1}^n, w_i^n \right) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+1/2}} w^{eq}\left(\frac{x - x_{i+1/2}}{\Delta t}, w_i^n, w_{i+1}^n \right) dx$$

Properties of the relaxation scheme (2)

Theorem (Robustness)

Assume the parameter a satisfies the following inequalities:

$$u_L - \frac{a}{\rho_L} < u^* < u_R + \frac{a}{\rho_R},$$

$$e_L + \frac{{\pi_L^\star}^2 - {p_L}^2}{2a^2} > 0, \qquad e_R + \frac{{\pi_R^\star}^2 - {p_R}^2}{2a^2} > 0$$

Then the relaxation scheme preserves the set of physical states Ω.

Theorem (Well-balancedness)

The relaxation scheme preserves exactly the initial data satisfying

$$\begin{cases} u_i^0 = 0, \\ \dfrac{p_{i+1}^0 - p_i^0}{\Delta x} = -\overline{\rho}(\rho_i^0, \rho_{i+1}^0)\dfrac{\Phi_{i+1} - \Phi_i}{\Delta x}. \end{cases}$$

Then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Properties of the relaxation scheme (1)

Theorem (Exact preservation of the hydrostatic atmosphere)

Assume the average function $\overline{\rho}$ *is defined by*

$$\overline{\rho}(\rho_L, \rho_R) = \begin{cases} \dfrac{\rho_R - \rho_L}{\ln(\rho_R) - \ln(\rho_L)} & \text{if } \rho_L \neq \rho_R, \\ \rho_L & \text{if } \rho_L = \rho_R. \end{cases}$$

Then the relaxation scheme preserves exactly the hydrostatic atmosphere:
if the initial data w_i^0 *is given by*

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = \alpha e^{-\beta \Phi_i}, \\ p_i^0 = \frac{\alpha}{\beta} e^{-\beta \Phi_i}, \end{cases}$$

then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Theorem

Assume the parameter a satisfy the following Whitham conditions:

$$a^2 > p(\tau_{L,R}, e_{L,R})\partial_e p(\tau_{L,R}, e_{L,R}) - \partial_\tau p(\tau_{L,R}, e_{L,R}),$$

$$a^2 > p(\tau^*_{L,R}, e^*_{L,R})\partial_e p(\tau^*_{L,R}, e^*_{L,R}) - \partial_\tau p(\tau^*_{L,R}, e^*_{L,R}).$$

Then the relaxation scheme satisfies

$$\rho_i^{n+1}\mathcal{F}(s_i^{n+1}) \leq \rho_i^n \mathcal{F}(s_i^n) - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(s)u\}^n_{i+1/2} - \{\rho\mathcal{F}(s)u\}^n_{i-1/2}\right)$$

for all smooth function $\mathcal{F}$ such that $w \mapsto \rho\mathcal{F}(s)$ is strictly convex. The entropy numerical flux is defined by

$$\{\rho\mathcal{F}(s)u\}^n_{i+1/2} = F^\rho_{i+1/2} \times \begin{cases} \mathcal{F}(s_i^n) & \text{if } F^\rho_{i+1/2} > 0, \\ \mathcal{F}(s_{i+1}^n) & \text{if } F^\rho_{i+1/2} < 0, \end{cases}$$

with $F^\rho_{i+1/2}$ the ρ–component of the numerical flux.

Sketch of the proof (1)

1. We introduce

$$I(\pi, \tau) = \pi + a^2\tau \quad \text{and} \quad J(\pi, e) = e - \frac{\pi^2}{2a^2}.$$

   I and J are strong Riemann invariants, i.e they satisfy

$$\partial_t \Psi(I, J) + u\partial_x \Psi(I, J) = 0,$$

   for all smooth function $\Psi : \mathbb{R}^2 \to \mathbb{R}$.

2. There exists functions $\overline{\tau}(I, J)$ and $\overline{e}(I, J)$ which satisfy

$$\overline{\tau}\left(I_{|\pi=p(\tau,e)}, J_{|\pi=p(\tau,e)}\right) = \tau \quad \text{and} \quad \overline{e}\left(I_{|\pi=p(\tau,e)}, J_{|\pi=p(\tau,e)}\right) = e$$

3. We define $\overline{s}(W) = s\left(\overline{\tau}(I, J), \overline{e}(I, J)\right)$. This function reaches its minimum on the equilibrium manifold:

$$\overline{s}(W) \geq \overline{s}\left(W_{|\pi=p(\tau,e)}\right) = s\left(W_{|\pi=p(\tau,e)}\right).$$

Sketch of the proof (2)

4. $\overline{s}$ only depends on I and $J \Rightarrow \partial_t \rho\mathcal{F}(\overline{s}) + \partial_x \rho\mathcal{F}(\overline{s})u = 0$.
   We integrate this equation on $[x_{i-1/2}, x_{i+1/2}) \times [t^n, t^{n+1})$:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (\rho\mathcal{F}(\overline{s}))(W_{\Delta x}(x, t^{n+1}))dx = \int_{x_{i-1/2}}^{x_{i+1/2}} (\rho\mathcal{F}(\overline{s}))(W_{\Delta x}(x, t^{n}))dx - \int_{t^n}^{t^{n+1}} (\rho\mathcal{F}(\overline{s})u)(W_{\Delta x}(x_{i+1/2}, t))dt + \int_{t^n}^{t^{n+1}} (\rho\mathcal{F}(\overline{s})u)(W_{\Delta x}(x_{i-1/2}, t))dt.$$

   We deduce

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} (\rho\mathcal{F}(\overline{s}))(W_{\Delta x}(x, t^{n+1}))dx = \rho_i^n \mathcal{F}(s_i^n) - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(s)u\}_{i+1/2}^n - \{\rho\mathcal{F}(s)u\}_{i-1/2}^n\right).$$

Sketch of the proof (3)

5. By definition of the scheme:

$$\rho_i^{n+1}\mathcal{F}(s_i^{n+1}) = (\rho\mathcal{F}(s))\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} W_{\Delta x}(x, t^{n+1})_{|\pi=p(\tau,e)}\,dx\right).$$

Jensen inequality ($w \mapsto \rho\mathcal{F}(s)$ is convex):

$$\rho_i^{n+1}\mathcal{F}(s_i^{n+1}) \leq \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}(\rho\mathcal{F}(s))(W_{\Delta x}(x, t^{n+1})_{|\pi=p(\tau,e)})\,dx.$$

$\mathcal{F}$ is increasing + the entropy is minimal on the equilibrium manifold:

$$\rho_i^{n+1}\mathcal{F}(s_i^{n+1}) \leq \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}(\rho\mathcal{F}(\overline{s}))(W_{\Delta x}(x, t^{n+1}))\,dx.$$

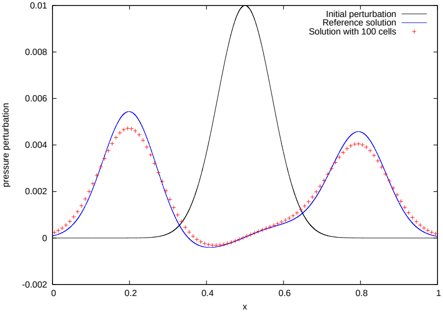
Hydrostatic atmosphere

- Computational domain: $[0,1]$ until time $T = 0.25$
- $\Phi(x) = x^2$
- Initial condition: $\begin{cases} \rho_0(x) = e^{-x^2} \\ u_0(x) = 0 \\ p_0(x) = e^{-x^2} \end{cases}$
- L^1 error:

N	Density	Velocity
100	1.15E-16	7.62E-17
200	1.77E-16	1.34E-16
400	3.01E-16	1.14E-16
800	4.37E-16	1.33E-16
1600	7.32E-16	1.91E-16
3200	1.18E-15	2.52E-16

Small perturbation of an hydrostatic atmosphere

- Computational domain: $[0,1]$ until time $T = 0.25$
- $\Phi(x) = x$
- Initial condition: $\begin{cases} \rho_0(x) = e^{-x} \\ u_0(x) = 0 \\ p_0(x) = e^{-x} + 0.01\, e^{100(x-0.5)^2} \end{cases}$
- Pressure perturbation: $\delta p = p - p_0$



Non-hydrostatic steady state

- Computational domain: $[0, 1]$ with periodic boundary conditions until time $T = 1$
- $\Phi(x) = \sin(2\pi x)$
- Initial condition: $\begin{cases} \rho_0(x) = 3 + 2\sin(2\pi x) \\ u_0(x) = 0 \\ p_0(x) = 3 + 3\sin(2\pi x) - \frac{1}{2}\cos(4\pi x) \end{cases}$

  We check easyly $\partial_x p_0 + \rho_0 \partial_x \Phi = 0$
- L^1 error:

N	Density		Velocity	
100	4.46E-05	–	2.03E-05	–
200	7.11E-06	2.65	5.29E-06	1.94
400	1.23E-06	2.53	1.34E-06	1.98
800	2.35E-07	2.39	3.37E-07	1.99
1600	5.02E-08	2.23	8.44E-08	2.00
3200	1.15E-08	2.13	2.11E-08	2.00

Perspectives

- Extension to second-order
  - MUSCL technique written as a convex combination of first-order scheme
  - Difficulty to preserve the well-balanced property
  - Possible solution: use a unusual set of variables for the reconstruction
- Extension to unstructured 2D
  - Convex combination of 1D scheme by interface