A well-balanced relaxation scheme for the Euler equations with gravity

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NumHyp 2013 conference, Aachen, September 26, 2013

The system of Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = -\rho \partial_x \phi \\ \partial_t E + \partial_x (u(E+p)) = -\rho u \partial_x \phi \end{cases}$$

- ρ: density
  u: velocity
  $E = \rho e + \rho u^2/2$: total energy, with e the internal energy
  $p = p(\rho, e)$: pressure given by a general law
  $\phi(x)$: gravitational potential (example: $\phi(x) = gx$)
- Hyperbolicity assumption:

$$c^2 := \partial_\rho p + \frac{p}{\rho^2} \partial_e p > 0$$

The system of Euler equations with gravity

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- We define
  - the vector of conservative variables $w = (\rho, \rho u, E)^T$,
  - the flux function $f(w) = (\rho u, \rho u^2 + p, u(E+p))^T$,
  - the source term $s(w) = (0, -\rho, -\rho u)^T$,

  to rewrite the system into the compact form

$$\partial_t w + \partial_x f(w) = s(w) \partial_x \phi.$$

- The set of physical admissible states is

$$\Omega = \left\{ w \in \mathbb{R}^3, \quad \rho > 0, \quad E - \rho u^2/2 > 0 \right\}.$$

The system of Euler equations with gravity

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Steady states

At the continuous level, the steady states at rest are governed by the ODE

$$\begin{cases} u \equiv 0, \\ \partial_x p = -\rho \partial_x \phi. \end{cases}$$

We cannot obtain an explicit expression of all the steady states.
→ We have to define the steady states at the discrete level.

Discrete steady states and well-balanced scheme

- Space discretization: cells $[x_{i-1/2}, x_{i+1/2})$ with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$
- w_i^n: approximation of the solution of the system at time t^n on the cell $[x_{i-1/2}, x_{i+1/2})$
- Discretization of the potential ϕ: $\phi_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x) dx$

Definition (Discrete steady states)

An approximation $(w_i^n)_{i \in \mathbb{Z}}$ is a discrete steady state, if for all $i \in \mathbb{Z}$, we have

$$u_i^n = 0, \quad \text{and} \quad p_{i+1}^n - p_i^n = -\frac{\rho_i^n + \rho_{i+1}^n}{2}(\phi_{i+1} - \phi_i).$$

Definition (Well-balanced scheme)

A numerical scheme is well-balanced if for all discrete steady state $(w_i^n)_{i \in \mathbb{Z}}$, the scheme satisfies $w_i^{n+1} = w_i^n$, for all $i \in \mathbb{Z}$.

The relaxation method without source term

- Aim: derive a numerical scheme to approximate the solutions of

$$\partial_t w + \partial_x f(w) = 0.$$

- We introduce a relaxation system

$$\partial_t W + \partial_x F(W) = \frac{1}{\varepsilon} R(W),$$

which should formally give back the original system when $\varepsilon \to 0$. Moreover the relaxation system should be "simpler" than the original system (e.g. only linearly degenerate fields)

- The relaxation scheme is based on a splitting strategy:

Time evolution: We evolve the initial data by the Godunov scheme for the system $\partial_t W + \partial_x F(W) = 0$ (i.e. $\varepsilon = +\infty$).

Relaxation: We take into account the relaxation source term by solving $\partial_t W = \frac{1}{\varepsilon} R(W)$ then taking the limit for $\varepsilon \to 0$.

The Suliciu model for the Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho \partial_x \phi \\ \partial_t E + \partial_x(u(E+\pi)) = -\rho u \partial_x \phi \\ \partial_t \rho\pi + \partial_x(u(\rho\pi + \nu^2)) = \frac{\rho}{\varepsilon}(p(\rho, e) - \pi) \end{cases}$$

The relaxation parameter $\nu > 0$ must satisfy the Whitham condition:

$$\nu^2 > \rho^2 c^2.$$

Eigenvalues	Riemann invariants
$u \pm \frac{\nu}{\rho}$	$u \pm \frac{\nu}{\rho}, \quad \pi \mp \nu u, \quad \nu^2 e - \frac{\pi^2}{2}, \quad \phi$
u $(\times 2)$	$u, \quad \pi, \quad \phi$
0	$\rho u, \quad \pi + \frac{\nu^2}{\rho}, \quad \nu^2 e - \frac{\pi^2}{2}, \quad \phi + \frac{u^2}{2} - \frac{\nu^2}{2\rho^2}$

Difficulties to compute the solution of the Riemann problem:

- the order of the eigenvalues is not determined *a priori*
- there are strong nonlinearities in the Riemann invariants for the eigenvalue 0

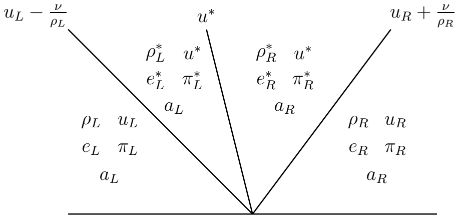
Relaxation model with moving gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho \partial_x a \\ \partial_t E + \partial_x(u(E+\pi)) = -\rho u \partial_x a \\ \partial_t \rho\pi + \partial_x(u(\rho\pi + \nu^2)) = \frac{\rho}{\varepsilon}(p(\rho, e) - \pi) \\ \partial_t a + u\partial_x a = \frac{1}{\varepsilon}(\phi - a) \end{cases}$$

Eigenvalues	Riemann invariants
$u \pm \frac{\nu}{\rho}$	$u \pm \frac{\nu}{\rho}$, $\pi \mp \nu u$, $\nu^2 e - \frac{\pi^2}{2}$, a
u $(\times 3)$	u

- The order of the eigenvalues is fixed: $u - \frac{\nu}{\rho} < u < u + \frac{\nu}{\rho}$
- There is a missing invariant for the eigenvalue u
  $\Rightarrow$ we need a closure equation

The Riemann problem



- 7 unknowns: $u^*,\ \rho_{L,R}^*,\ e_{L,R}^*,\ \pi_{L,R}^*$
- 6 equations given by the Riemann invariants $u \pm \frac{\nu}{\rho},\ \pi \mp \nu u,\ \nu^2 e - \frac{\pi^2}{2}$
- Closure equation: $\pi_R^* - \pi_L^* = -\frac{\rho_L+\rho_R}{2}(a_R - a_L)$

Solution of the Riemann problem

$$u^* = \frac{u_L + u_R}{2} - \frac{\pi_R - \pi_L}{2\nu} - \frac{\rho_L + \rho_R}{2}\frac{a_R - a_L}{2\nu}$$

$$\pi_L^* = \pi_L + \nu(u_L - u^*) \qquad \pi_R^* = \pi_R + \nu(u^* - u_R)$$

$$\frac{1}{\rho_L^*} = \frac{1}{\rho_L} + \frac{u^* - u_L}{\nu} \qquad \frac{1}{\rho_R^*} = \frac{1}{\rho_R} + \frac{u_R - u^*}{\nu}$$

$$e_L^* = e_L + \frac{{\pi_L^*}^2 - {\pi_L}^2}{2\nu^2} \qquad e_R^* = e_R + \frac{{\pi_R^*}^2 - {\pi_R}^2}{2\nu^2}$$

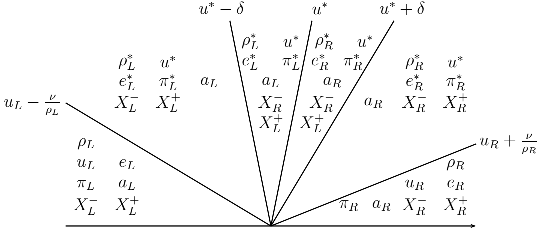
Reformulation into a fully determined model

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\frac{X^- + X^+}{2} \partial_x a \\ \partial_t E + \partial_x(u(E+\pi)) = -\frac{X^- + X^+}{2} u \partial_x a \\ \partial_t \rho\pi + \partial_x(u(\rho\pi + \nu^2)) = \frac{\rho}{\varepsilon}(p(\rho, e) - \pi) \\ \partial_t a + u \partial_x a = \frac{1}{\varepsilon}(\phi - a) \\ \partial_t X^- + (u - \delta)\partial_x X^- = \frac{1}{\varepsilon}(\rho - X^-) \\ \partial_t X^+ + (u + \delta)\partial_x X^+ = \frac{1}{\varepsilon}(\rho - X^+) \end{cases}$$

Eigenvalues	Riemann invariants
$u \pm \frac{\nu}{\rho}$	$u \pm \frac{\nu}{\rho}$, $\pi \mp \nu u$, $\nu^2 e - \frac{\pi^2}{2}$, a, X^-, X^+
u $(\times 3)$	u, $\pi + \frac{X^- + X^+}{2} a$, X^-, X^+
$u - \delta$	ρ, u, e, π, a, X^+
$u + \delta$	ρ, u, e, π, a, X^-

- For δ small enough, the order of the eigenvalues is fixed: $u - \frac{\nu}{\rho} < u - \delta < u < u + \delta < u + \frac{\nu}{\rho}$
- There is a full set of Riemann invariants

The Riemann problem for the reformulated model



- 7 unknowns: $u^*,\ \rho_{L,R}^*,\ e_{L,R}^*,\ \pi_{L,R}^*$
- 7 equations given by the Riemann invariants $u \pm \frac{\nu}{\rho},\ \pi \mp \nu u,\ \nu^2 e - \frac{\pi^2}{2},$ $\pi + \frac{X^- + X^+}{2} a$

- The equations coming from the Riemann invariants $u \pm \frac{\nu}{\rho}$, $\pi \mp \nu u$ and $\nu^2 e - \frac{\pi^2}{2}$ are the same as in the previous model.
- The last equation is $\pi_R^* - \pi_L^* = -\frac{X_R^- + X_L^+}{2}(a_R - a_L)$. For an initial data at the relaxation equilibrium (i.e. $\pi = p(\rho, e)$, $a = \phi$, $X^\pm = \rho$), we recover the closure equation of the previous model.

The two models have the "same" solution of the Riemann problem for an initial data at the relaxation equilibrium.
$\Rightarrow$ both models lead to the same numerical scheme.

The relaxation scheme

The relaxation scheme associated with both previous models writes

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F(w_i^n, w_{i+1}^n) - F(w_{i-1}^n, w_i^n)\right)$$
$$+ \frac{\Delta t}{2}\left(s^+(w_{i-1}^n, w_i^n)\frac{\phi_i - \phi_{i-1}}{\Delta x} + s^-(w_i^n, w_{i+1}^n)\frac{\phi_{i+1} - \phi_i}{\Delta x}\right),$$

where the numerical flux is defined by

$$f(w_L,w_R)=\begin{cases} \left(\rho_L u_L, \quad \rho_L u_L^2+p_L, \quad u_L(E_L+p_L)\right)^T & \text{if } u_L-\frac{\nu}{\rho_L}>0, \\ \left(\rho_L^* u^*, \quad \rho_L^*(u^*)^2+\pi_L^*, \quad u^*(E_L^*+\pi_L^*)\right)^T & \text{if } u_L-\frac{\nu}{\rho_L}<0<u^*, \\ \left(\rho_R^* u^*, \quad \rho_R^*(u^*)^2+\pi_R^*, \quad u^*(E_R^*+\pi_R^*)\right)^T & \text{if } u^*<0<u_R+\frac{\nu}{\rho_R}, \\ \left(\rho_R u_R, \quad \rho_R u_R^2+p_R, \quad u_R(E_R+p_R)\right)^T & \text{if } u_R+\frac{\nu}{\rho_R}<0, \end{cases}$$

and the numerical source terms are defined by

$$s^+(w_L,w_R)=-(\text{sgn}(u^*)+1)\left(0, \frac{\rho_L+\rho_R}{2}, \frac{\rho_L+\rho_R}{2}u^*\right)^T,$$

$$s^-(w_L,w_R)=(\text{sgn}(u^*)-1)\left(0, \frac{\rho_L+\rho_R}{2}, \frac{\rho_L+\rho_R}{2}u^*\right)^T.$$

Properties of the relaxation scheme

Theorem (Well-balancedness)

The relaxation scheme preserves the steady states at rest:

$$\forall i \in \mathbb{Z}, \begin{cases} u_i^n = 0 \\ p_{i+1}^n - p_i^n = -\frac{\rho_i^n + \rho_{i+1}^n}{2}(\phi_{i+1} - \phi_i) \end{cases} \Rightarrow \forall i \in \mathbb{Z},\ w_i^{n+1} = w_i^n$$

Theorem (Robustness)

Assume the parameter ν satisfies the following inequalities:

$$u_L - \frac{\nu}{\rho_L} < u^* < u_R + \frac{\nu}{\rho_R}, \quad e_L + \frac{{\pi_L^*}^2 - {p_L}^2}{2\nu^2} > 0, \quad e_R + \frac{{\pi_R^*}^2 - {p_R}^2}{2\nu^2} > 0.$$

Assume the following CFL condition is satisfied:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} |u_i^n \pm \nu / \rho_i^n| \leq \frac{1}{2}$$

Then the relaxation scheme preserves the set of physical states:

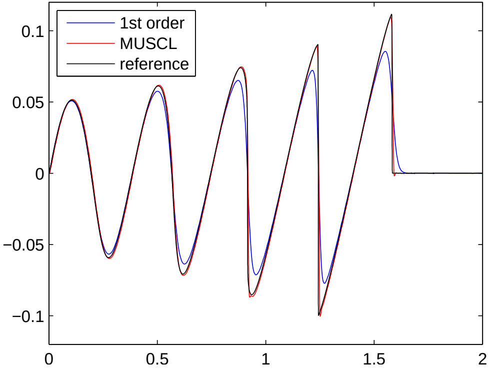
$$\forall i \in \mathbb{Z},\ \rho_i^n > 0 \text{ and } e_i^n > 0 \quad \Rightarrow \quad \forall i \in \mathbb{Z},\ \rho_i^{n+1} > 0 \text{ and } e_i^n > 0.$$

Numerical test: perturbation of an hydrostatic atmosphere

- Perfect gas law:
  $p = (\gamma - 1)\left(E - \rho u^2/2\right)$
- Constant gravitational field: $\phi(x) = gx$
- Steady state $w_s(x)$: hydrostatic atmosphere

$$\begin{cases} \rho_s(x) = \left(1 - \frac{\gamma-1}{\gamma} gx\right)^{\frac{1}{\gamma-1}} \\ u_s(x) = 0 \\ p_s(x) = \rho_s(x)^\gamma \end{cases}$$

- Boundary condition:
  $u(0,t) = 0.1 \sin(6\pi t)$
- Perturbation:
  $\delta w(x,t) = w(x,t) - w_s(x)$



Final time perturbation in velocity $\delta u(x,T)$ computed with 1.024 cells.
The reference solution is computed with 32.768 cells with the first-order relaxation scheme.