Schémas well-balanced permettant de capturer des états d'équilibre non-explicites

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Séminaire Analyse numérique - Équations aux dérivées partielles,
Lille, 30 janvier 2014

Outline

1. Well-balanced schemes to capture non-explicit steady states
2. Entropy stability of high-order schemes

Joint work with:

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1. Well-balanced schemes to capture non-explicit steady states
   - Introduction: From shallow-water to Ripa model
   - Relaxation models
   - Relaxation scheme and main properties
   - Numerical results

2. Entropy stability of high-order schemes
   - Motivations
   - From one to all discrete entropy inequalities
   - The e-MOOD scheme for the Euler equations

The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2 \right) = -gh\partial_x z \end{cases}$$

- h: water height
- u: velocity
- g: gravity constant
- $z(x)$: smooth topography function
- Set of physical admissible states:

$$\Omega = \left\{ w = (h, hu)^T \in \mathbb{R}^2, \quad h > 0 \right\}.$$

The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2\right) = -gh\partial_x z \end{cases}$$

Steady states

The steady states at rest are described by

$$\begin{cases} u = 0 \\ \partial_x (h^2/2) = -h\partial_x z \end{cases} \Leftrightarrow \begin{cases} u = 0 \\ h + z = \text{ cst.} \end{cases}$$

There is only one steady state at rest (up to a constant): the lake at rest.

The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2 \right) = -gh\partial_x z \end{cases}$$

Well-balanced scheme

- w_i^n: approximation of the solution on the cell $(x_{i-1/2}, x_{i+1/2})$ at time t^n
- z_i: approximation of the topgraphy $z(x)$ on the cell $(x_{i-1/2}, x_{i+1/2})$
- A numerical scheme is well-balanced if

$$\forall i \in \mathbb{Z}, \quad h_i^0 + z_i = H \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \forall n \in \mathbb{N}, \quad w_i^{n+1} = w_i^n.$$

- There exists numerous well-balanced schemes for the shallow-water model: [Gosse '00], [Gallouët, Hérard & Seguin '03], [Audusse et al. '04]...

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

- θ: temperature
- Set of physical admissible states:

$$\Omega = \left\{ w = (h, hu, h\theta)^T \in \mathbb{R}^3, \quad h > 0, \quad \theta > 0 \right\}.$$

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

Steady states

The steady states at rest are governed by the ODE

$$\begin{cases} u = 0 \\ \partial_x(h^2\theta/2) = -h\theta\partial_x z. \end{cases}$$

We cannot obtain an explicit expression of all the steady states.

Particular steady states

$$\begin{cases} u = 0 \\ \theta = \text{cst} \\ h + z = \text{cst} \end{cases} \qquad \begin{cases} u = 0 \\ z = \text{cst} \\ h^2\theta = \text{cst} \end{cases} \qquad \begin{cases} u = 0 \\ h = \text{cst} \\ z + \frac{h}{2}\ln\theta = \text{cst} \end{cases}$$

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

Objectives

- Robust finite volume method: preservation of the set Ω
- Exact capture of the three particular steady states
- Exact/Approximated preservation of all the steady states at rest

The relaxation method without source term

- Initial system:

$$\partial_t w + \partial_x f(w) = 0. \tag{1}$$

- Relaxation system:

$$\partial_t W + \partial_x F(W) = \frac{1}{\varepsilon} R(W), \tag{2}$$

  - (2) should formally gives back (1) when $\varepsilon \to 0$.
  - (2) should be "simpler" than (1) (e.g. only linearly degenerate fields)

- The relaxation scheme is based on a splitting strategy:

  Time evolution: We evolve the initial data by the Godunov scheme for the system $\partial_t W + \partial_x F(W) = 0$ (i.e. $\varepsilon = +\infty$).

  Relaxation: We take into account the relaxation source term by solving $\partial_t W = \frac{1}{\varepsilon} R(W)$ then taking the limit for $\varepsilon \to 0$.

The Suliciu model ([Suliciu '98], [Bouchut '04]...)

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x (hu^2 + \pi) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x (u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \\ \partial_t z = 0 \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2$$

Eigenvalues	Riemann invariants
$u \pm \frac{a}{h}$	$u \pm \frac{a}{h}, \quad \pi \mp au, \quad \theta, \quad z$
$u\ (\times 2)$	$u, \quad \pi, \quad z$
0	$hu, \quad \pi + \frac{a^2}{h}, \quad \theta, \quad g\theta z + \frac{u^2}{2} - \frac{a^2}{2h^2}$

Difficulties to compute the solution of the Riemann problem:

- The order of the eigenvalues is not determined *a priori*.
- There are strong nonlinearities in the Riemann invariants for the eigenvalue 0.

Relaxation model with moving topography

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x(hu^2 + \pi) = -gh\theta\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x(u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2$$

$$Z = z$$

Eigenvalues	Riemann invariants
$u \pm \frac{a}{h}$	$u \pm \frac{a}{h}$, $\pi \mp au$, θ, Z
u $(\times 3)$	u

- The order of the eigenvalues is fixed: $u - \frac{a}{h} < u < u + \frac{a}{h}$
- There is a missing invariant for the eigenvalue u
  $\Rightarrow$ we need a closure equation

The closure equation

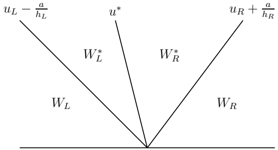


Figure: Structure of the Riemann problem

To mimic the ODE defining the steady states

$$\partial_x(gh^2\theta/2) = -gh\theta\partial_x z \quad \underset{\text{equilibrium}}{\Longleftrightarrow} \quad \partial_x \pi = -gh\theta\partial_x Z,$$

we propose the closure equation

$$\pi_R^* - \pi_L^* = -g\overline{h}(W_L, W_R)\overline{\theta}(W_L, W_R)(Z_R - Z_L),$$

where $\overline{h}$ and $\overline{\theta}$ are suitable averages (defined later).

Approximate Riemann solver

Equilibrium state:

$$W^{eq}(w) = (h, hu, h\theta, gh\theta/2, hz)^T$$

Theorem

With the closure equation, the Riemann problem admits a unique solution $W_{\mathcal{R}}\left(\frac{x}{t}, W_L, W_R\right)$.
Moreover,

$$w_{\mathcal{R}}\left(\frac{x}{t}, w_L, w_R\right) := W_{\mathcal{R}}^{(h,hu,h\theta)}\left(\frac{x}{t}, W^{eq}(w_L), W^{eq}(w_R)\right)$$

defines an approximate Riemann solver (in the sense of Harten, Lax and van Leer) for the Ripa model.

Complete relaxation reformulation

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x (hu^2 + \pi) = -g\overline{h}(X^-, X^+)\overline{\theta}(X^-, X^+)\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x (u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \\ \partial_t X^- + (u-\delta)\partial_x X^- = \frac{1}{\varepsilon}(W - X^-) \\ \partial_t X^+ + (u+\delta)\partial_x X^+ = \frac{1}{\varepsilon}(W - X^+) \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2 \qquad Z = z \qquad X^{\pm} = W$$

- For $\delta > 0$ small enough, the system is hyperbolic with eigenvalues

$$u - \frac{a}{h} < u - \delta < u < u + \delta < u + \frac{a}{h}.$$

- The system has a complete set of Riemann invariants.
- Leads to the same approximate Riemann solver $w_{\mathcal{R}}\left(\frac{x}{t}, w_L, w_R\right)$.

Cargo-LeRoux formulation $(z(x) = x)$

We introduce a potential $q = \int^x gh\theta\,dx$. Then we have

$$\partial_t q = \int^x g\partial_t(h\theta)\,dx = -\int^x g\partial_x(h\theta u)\,dx = -gh\theta u = -u\partial_x q.$$

So q is governed by

$$\partial_t hq + \partial_x hqu = 0.$$

Equivalent reformulation of the Ripa model:

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = 0. \end{cases}$$

Relaxation model with the Cargo-LeRoux formulation

- Case $z(x) = x$ [Chalons et al. '10]

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \pi\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = 0 \\ \partial_t h\pi + \partial_x (u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2$$

- Extension for general topography

  If we define the potential by $q = \int^x gh\theta\partial_x z dx$, it no longer satisfies a transport equation. We enforce the natural relaxation model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \pi\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = \frac{h}{\varepsilon}(\int^x gh\theta\partial_x z dx - q) \\ \partial_t h\pi + \partial_x (u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \end{cases}$$

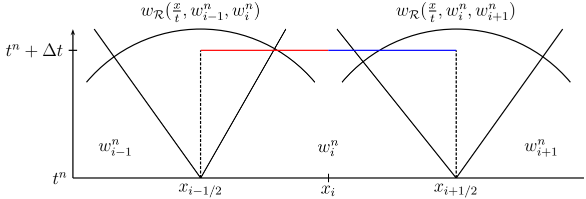
Equilibrium

$$\pi = gh^2\theta/2$$

$$q = \int^x gh\theta\partial_x z dx$$

The relaxation scheme

w_i^n: approximation of the solution on the cell $(x_{i-1/2}, x_{i+1/2})$ at time t^n



CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left| u_i^n \pm \frac{a}{h_i^n} \right| \leq \frac{1}{2}$$

The update at time $t^{n+1} = t^n + \Delta t$ is defined by

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_i} w_{\mathcal{R}} \left(\frac{x - x_{i-1/2}}{\Delta t}, w_{i-1}^n, w_i^n \right) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+1/2}} w_{\mathcal{R}} \left(\frac{x - x_{i+1/2}}{\Delta t}, w_i^n, w_{i+1}^n \right) dx$$

Properties of the relaxation scheme (1)

Theorem (Exact preservation of the particular steady states)

Assume the average functions $\overline{h}$ and $\overline{\theta}$ are defined by

$$\overline{h}(W_L, W_R) = \frac{1}{2}(h_L + h_R), \quad \overline{\theta}(W_L, W_R) = \begin{cases} \frac{\theta_R - \theta_L}{\ln(\theta_R) - \ln(\theta_L)} & \text{if } \theta_L \neq \theta_R, \\ \theta_L & \text{if } \theta_L = \theta_R. \end{cases}$$

Then the relaxation scheme preserves exactly the particular steady states: if the initial data w_i^0 is given by

$$\begin{cases} u_i^0 = 0, \\ \theta_i^0 = \theta, \\ h_i^0 + z_i = H, \end{cases} \quad \text{or} \quad \begin{cases} u_i^0 = 0, \\ z_i = Z, \\ \left(h_i^0\right)^2 \theta_i^0 = P, \end{cases} \quad \text{or} \quad \begin{cases} u_i^0 = 0, \\ h_i^0 = H, \\ z_i + h_i^0 \ln(\theta_i^0)/2 = P, \end{cases}$$

then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Properties of the relaxation scheme (2)

Theorem (Well-balancedness)

If the initial data is an approximation of the ODE defining the steady states as follows:

$$\begin{cases} u_i^0 = 0, \\ \frac{(h_{i+1}^0)^2\theta_{i+1}^0/2-(h_i^0)^2\theta_i^0/2}{\Delta x} = -\bar{h}(w_i^0, w_{i+1}^0)\bar{\theta}(w_i^0, w_{i+1}^0)\frac{z_{i+1}-z_i}{\Delta x}. \end{cases}$$

Then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Theorem (Robustness)

Assume the parameter a *satisfies the following inequalities:*

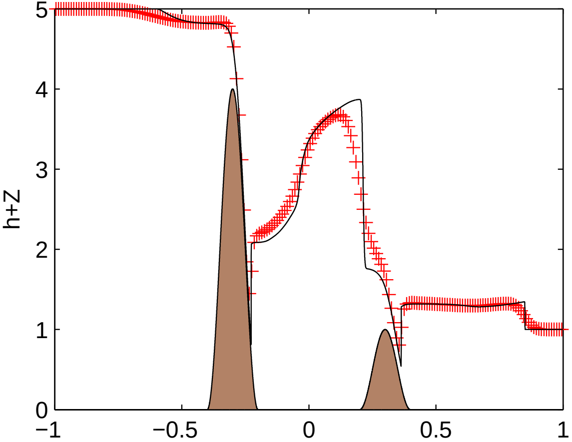
$$u_L - \frac{a}{h_L} < u^* < u_R + \frac{a}{h_R}.$$

Then the relaxation scheme preserves the set of physical states:

$$\forall i \in \mathbb{Z},\ h_i^n > 0 \text{ and } \theta_i^n > 0 \quad \Rightarrow \quad \forall i \in \mathbb{Z},\ h_i^{n+1} > 0 \text{ and } \theta_i^{n+1} > 0.$$

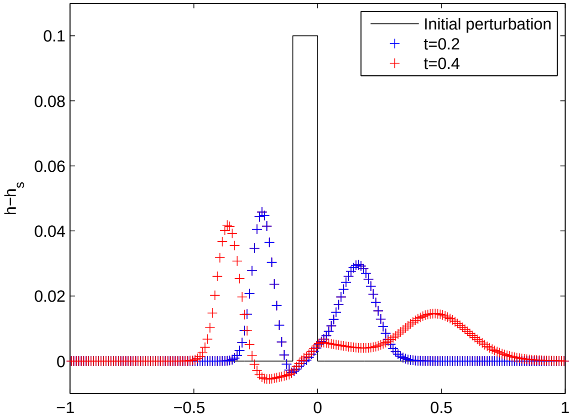
Dam break over a non-flat bottom

[Chertock, Kurganov & Liu '13]



Perturbation of a nonlinear steady state

- Topography:
  $z(x) = -2e^x$
- Steady state solution:
  $(h_s, u_s, \theta_s)(x) = (e^x, 0, e^{2x})$
- Initial perturbation:
  $\delta h(x,0) = 0.1\chi_{[-0.1,0]}(x)$



Introduction

- Hyperbolic system of conservation laws in 1D

$$\partial_t w + \partial_x f(w) = 0$$

$w : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^d$: unknown state vector
$f : \Omega \to \mathbb{R}^d$: flux function

- Ω convex set of physical states
- Objective: Study the entropy stability of high-order schemes

Weak solutions and entropy solutions

- A function $w \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ is a weak solution if $\forall \phi \in C^1_c(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^d)$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} \left(w \cdot \partial_t \phi + f(w) \cdot \partial_x \phi \right) dt dx + \int_{\mathbb{R}} w(x,0) \cdot \phi(x,0) dx = 0.$$

- A convex function $\eta \in C^2(\Omega; \mathbb{R})$ is an entropy for the system if there exists an entropy flux $\mathcal{G} \in C^2(\Omega; \mathbb{R})$ such that $\nabla_w f(w) \nabla_w \eta(w) = \nabla_w \mathcal{G}(w)$, $\forall w \in \Omega$.
- A weak solution w is an entropy solution if for any entropy pair $(\eta, \mathcal{G})$ of the system, we have

$$\partial_t \eta(w) + \partial_x \mathcal{G}(w) \leq 0$$

in the sense of distributions.

Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t E + \partial_x (E + p) u = 0 \end{cases}$$

- Ideal gas law: $p = (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right), \quad \gamma \in (1, 3]$
- Set of physical states: $\Omega = \left\{ w = (\rho, \rho u, E)^T \in \mathbb{R}^3,\ \rho > 0,\ p > 0 \right\}$
- Entropy inequalities:

$$\partial_t \rho \mathcal{F}(\ln(s)) + \partial_x \rho \mathcal{F}(\ln(s)) u \leq 0, \quad \text{with } s = \frac{p}{\rho^\gamma}$$

and $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ a smooth function such that

$$\mathcal{F}'(y) < 0 \quad \text{and} \quad \mathcal{F}'(y) < \gamma \mathcal{F}''(y), \quad \forall y \in \mathbb{R}$$

Scheme notations

- Space discretization: cells $K_i = (x_{i-1/2}, x_{i+1/2})$ with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$
- w_i^n: approximate solution at time t^n on the cell K_i
- Update at time $t^{n+1} = t^n + \Delta t$ given by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

  where $F_{i+1/2}^n = F\left(w_{i-s+1}^n, \cdots, w_{i+s}^n\right)$ and F is a consistant numerical flux ($F(w, \cdots, w) = f(w)$)
- We introduce the piecewise constant function

$$w^{\Delta}(x,t) = w_i^n, \quad \text{for } (x,t) \in K_i \times [t^n, t^{n+1})$$

- The sequence $(\Delta x, \Delta t)$ is devoted to converge to $(0,0)$, the ratio $\frac{\Delta t}{\Delta x}$ being kept constant.

Lax-Wendroff Theorem

Theorem

(i) *Assume the following hypotheses:*

- *There exists a compact* $K \subset \Omega$ *such that* $w^\Delta \in K$*;*
- w^Δ *converges in* $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ *to a function* w*.*

Then w *is a weak solution.*

(ii) *Assume the additional hypothesis:*

- *For all entropy pair* $(\eta, \mathcal{G})$*, there exists an entropy numerical flux* G*, consistant with* $\mathcal{G}$ $(G(w, \cdots, w) = \mathcal{G}(w))$*, such that we have the discrete entropy inequality (DEI)*

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G^n_{i+1/2} - G^n_{i-1/2}}{\Delta x} \leq 0,$$

with $G^n_{i+1/2} = G\left(w^n_{i-s+1}, \cdots, w^n_{i+s}\right)$*.*

Then w *is an entropic solution.*

Example: the MUSCL scheme

- We assume the first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F(w_i^n, w_{i+1}^n) - F(w_{i-1}^n, w_i^n)\right)$$

satisfies the DEI

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^n, w_{i+1}^n) - G(w_{i-1}^n, w_i^n)}{\Delta x} \leq 0.$$

- Let L be a limiter function (minmod, superbee...). We define a limited increment on each cell by

$$\mu_i^n = L\left(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n\right)$$

- Reconstructed states at interfaces : $w_i^{n,\pm} = w_i^n \pm \frac{1}{2}\mu_i^n$
- The MUSCL scheme is defined by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F(w_i^{n,+}, w_{i+1}^{n,-}) - F(w_{i-1}^{n,+}, w_i^{n,-})\right),$$

DEI satisfied by the MUSCL scheme

- The known DEI satisfied by the MUSCL scheme all write

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^{n,+}, w_{i+1}^{n,-}) - G(w_{i-1}^{n,+}, w_i^{n,-})}{\Delta x} \leq \frac{P_i^n - \eta(w_i^n)}{\Delta t}$$

where $P_i^n = P_\eta\left(w_i^n, \mu_i^n, \Delta x\right)$.

- Examples of operator P_η:

$$P_\eta^1(w, \mu, \Delta x) = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \eta\left(w + \frac{x}{\Delta x}\mu\right) dx \quad \text{[Bouchut \textit{et al.} '96]}$$

$$P_\eta^2(w, \mu, \Delta x) = \frac{\eta(w - \mu/2) + \eta(w + \mu/2)}{2} \quad \text{[Berthon '05]}$$

- The operator P_η satisfies: $\exists C > 0$ such that

$$0 \leq P_\eta(w, \mu, \Delta x) - \eta(w) \leq C\|\nabla^2\eta(w)\|\|\mu\|^2$$

Convergence study

- The discrete entropy inequality

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^{n,+}, w_{i+1}^{n,-}) - G(w_{i-1}^{n,+}, w_i^{n,-})}{\Delta x} \leq \frac{P_i^n - \eta(w_i^n)}{\Delta t}$$

converges weakly to

$$\partial_t \eta(w) + \partial_x \mathcal{G}(w) \leq \delta,$$

where δ is a positive measure.

Conjecture (Hou-LeFloch '94)

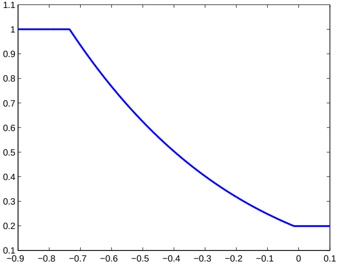
- $\delta = 0$ in the areas where w is smooth
- $\delta > 0$ on the curves of discontinuity of w

Numerical study: test cases (Euler equations)

Total mass of the right-hand side:

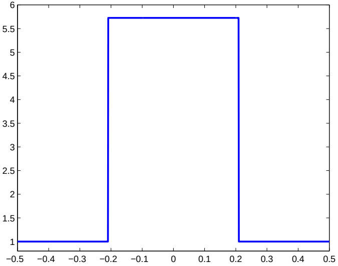
$$I^{\Delta} = \Delta x \sum_{i,n} \left(P_i^n - \eta(w_i^n) \right)$$

1–rarefaction



$I^{\Delta} \overset{?}{\to} 0$

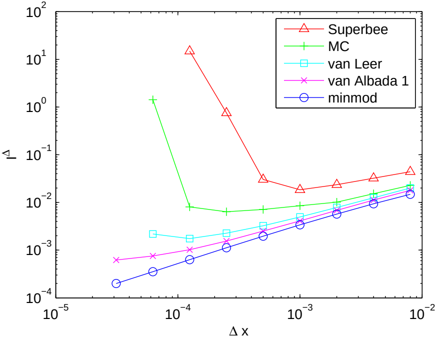
Double shock



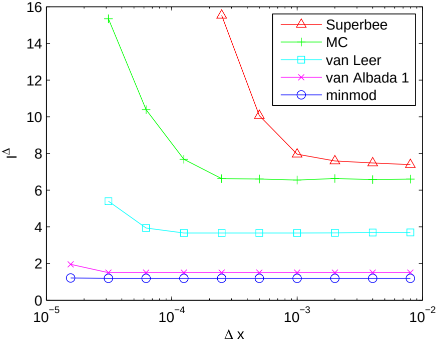
$I^{\Delta} \overset{?}{\to} c > 0$

Numerical results obtained with a first-order time scheme

1-rarefaction

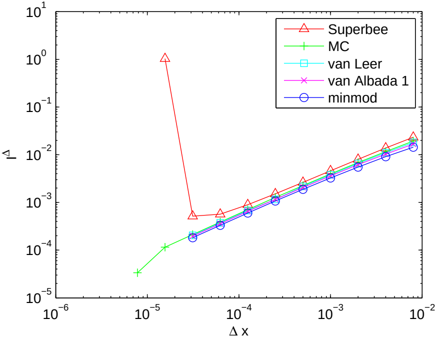


Double shock

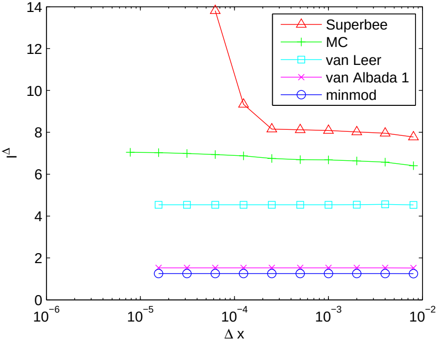


Numerical results obtained with a second-order time scheme

1-rarefaction



Double shock



Conclusion of motivations

- Numerical results confirm the Hou-le Floch conjecture: when the scheme converges, the measure δ seems to be concentrated on the curves of discontinuity of w.
- This does not imply that the limit is not entropic, but the usual discrete entropy inequalities are not relevant to apply the Lax-Wendroff theorem.
- We have to enforce the stronger discrete entropy inequalities

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0.$$

- We suggest to extend the *a posteriori* methods (MOOD) introduced in [Clain, Diot & Loubère '11].

The family of entropies for the Euler equations

Lemma

The entropy pairs $(\eta, \mathcal{G})$ of the Euler system rewrite

$$\eta = \rho\psi(r), \quad \mathcal{G} = \rho\psi(r)u,$$

where $r = -\frac{p^{1/\gamma}}{\rho}$ and ψ is a smooth increasing convex function.

We consider the scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right),$$

where $w_i^n = (\rho_i^n, \rho_i^n u_i^n, E_i^n)^T$ and $F_{i+1/2} = (F_{i+1/2}^{\rho}, F_{i+1/2}^{\rho u}, F_{i+1/2}^{E})^T$.

We introduce $r_{i+1/2}^n = \begin{cases} r_{i+1}^n & \text{if } F_{i+1/2}^\rho < 0 \\ r_i^n & \text{if } F_{i+1/2}^\rho > 0 \end{cases}.$

Theorem

Assume the scheme preserves Ω. Assume the scheme satisfies the specific discrete entropy inequality

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^\rho r_{i+1/2}^n - F_{i-1/2}^\rho r_{i-1/2}^n\right).$$

Assume the additional CFL like condition

$$\frac{\Delta t}{\Delta x}\left(\max\left(0, F_{i+1/2}^\rho\right) - \min\left(0, F_{i-1/2}^\rho\right)\right) \leq \rho_i^n.$$

Then the scheme is entropy preserving: for all smooth increasing convex function ψ, we have

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n \psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^\rho \psi(r_{i+1/2}^n) - F_{i-1/2}^\rho \psi(r_{i-1/2}^n)\right).$$

Proof of the Theorem (1)

Using the upwind definition of $r^n_{i+1/2}$, the specific DEI writes

$$r_i^{n+1} \leq \frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n,$$

where we have set

$$\begin{aligned} a &= \frac{\Delta t}{2\Delta x}\left(F^\rho_{i-1/2} + \left|F^\rho_{i-1/2}\right|\right), \\ b &= \rho_i^n - \frac{\Delta t}{2\Delta x}\left(F^\rho_{i+1/2} + \left|F^\rho_{i+1/2}\right| - F^\rho_{i-1/2} + \left|F^\rho_{i-1/2}\right|\right), \\ c &= \frac{\Delta t}{2\Delta x}\left(\left|F^\rho_{i+1/2}\right| - F^\rho_{i+1/2}\right). \end{aligned}$$

- We have $a + b + c = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2} - F^\rho_{i-1/2}\right) = \rho_i^{n+1}$.
- $a \geq 0$, $c \geq 0$
- $b \geq 0$ thanks to the CFL like condition

$\Rightarrow$ r_i^{n+1} is less than a convex combination of r_{i-1}^n, r_i^n and r_{i+1}^n.

Proof of the Theorem (2)

- We consider an entropy pair $(\rho\psi(r), \rho\psi(r)u)$ with ψ a smooth increasing convex function.
- ψ is increasing:

$$\psi\left(r_i^{n+1}\right) \leq \psi\left(\frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n\right)$$

- Jensen inequality (ψ is convex):

$$\psi\left(r_i^{n+1}\right) \leq \frac{a}{\rho_i^{n+1}} \psi\left(r_{i-1}^n\right) + \frac{b}{\rho_i^{n+1}} \psi\left(r_i^n\right) + \frac{c}{\rho_i^{n+1}} \psi\left(r_{i+1}^n\right)$$

- Replacing a, b and c by their value, we get

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n \psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2}\psi^n_{i+1/2} - F^\rho_{i-1/2}\psi_{i-1/2}\right),$$

with $\psi^n_{i+1/2} = \begin{cases} \psi\left(r_{i+1}^n\right) & \text{if} \quad F^\rho_{i+1/2} < 0 \\ \psi\left(r_i^n\right) & \text{if} \quad F^\rho_{i+1/2} > 0 \end{cases}.$

First-order scheme

We consider a first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^n, w_{i+1}^n\right) - F\left(w_{i-1}^n, w_i^n\right)\right).$$

For a time step restricted according to the CFL condition

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left|\lambda^{\pm}\left(w_i^n, w_{i+1}^n\right)\right| \leq \frac{1}{2},$$

the first-order scheme is assumed to satisfy:

- Robustness: $\forall i \in \mathbb{Z}, \quad w_i^n \in \Omega \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \quad w_i^{n+1} \in \Omega$
- Stability:

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\Big(F^{\rho}\left(w_i^n, w_{i+1}^n\right) r_{i+1/2}^n - F^{\rho}\left(w_{i-1}^n, w_i^n\right) r_{i-1/2}^n\Big).$$

Example: the HLLC/Suliciu relaxation scheme

Reconstruction procedure

- We consider high-order reconstructed states $w_i^{n,\pm}$ on the cell K_i at the interfaces $x_{i\pm 1/2}$.
- These reconstructed states can be obtained by any reconstruction procedure (MUSCL, ENO/WENO, PPM...).
- Assumptions:
  - The reconstruction is Ω-preserving: $w_i^{n,\pm} \in \Omega$;
  - The reconstruction is conservative:

$$w_i^n = \frac{1}{2}\left(w_i^{n,-} + w_i^{n,+}\right).$$

The e-MOOD algorithm

1. Reconstruction step: For all $i \in \mathbb{Z}$, we evaluate high-order reconstructed states $w_i^{n,\pm}$ located at the interfaces $x_{i\pm1/2}$.
2. Evolution step: We compute a candidate solution as follows:

$$w_i^{n+1,\star} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^{n,+}, w_{i+1}^{n,-}\right) - F\left(w_{i-1}^{n,+}, w_i^{n,-}\right)\right).$$

3. A posteriori limitation step: We have the following alternative:
   - if for all $i \in \mathbb{Z}$, we have

$$\rho^{n+1,\star} r_i^{n+1,\star} \leq \rho_i^n r(w_i^n) - \frac{\Delta t}{\Delta x}\Big(F^\rho\left(w_i^{n,+}, w_{i+1}^{n,-}\right) r_{i+1/2}^n - F^\rho\left(w_{i-1}^{n,+}, w_i^{n,-}\right) r_{i-1/2}^n\Big), \quad (3)$$

     then the solution is valid and the updated solution at time $t^n + \Delta t$ is defined by $w_i^{n+1} = w_i^{n+1,\star}$;
   - otherwise, for all $i \in \mathbb{Z}$ such that (3) is not satisfied, we set $w_i^{n,\pm} = w_i^n$ and we go back to step 2.

Theorem

Assume the time step Δt is chosen in order to satisfy the two following CFL like conditions:

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(\left|\lambda^{\pm}\left(w_i^{n,+},w_{i+1}^{n,-}\right)\right|,\left|\lambda^{\pm}\left(w_i^{n,-},w_i^{n,+}\right)\right|\right)\leq\frac{1}{4},$$

$$\frac{\Delta t}{\Delta x}\left(\max\left(0,F^{\rho}_{i+1/2}\right)-\min\left(0,F^{\rho}_{i-1/2}\right)\right)\leq\rho_i^n.$$

Then the updated states w_i^{n+1}, given by the e-MOOD scheme, belong to Ω. Moreover, for all smooth increasing convex function ψ, the e-MOOD scheme satisfies

$$\frac{1}{\Delta t}\left(\rho_i^{n+1}\psi(r_i^{n+1})-\rho_i^n\psi\left(r_i^n\right)\right)+\frac{1}{\Delta x}\Big(F^{\rho}\left(w_i^{n,+},w_{i+1}^{n,-}\right)\psi(r_{i+1/2}^n)$$
$$-F^{\rho}\left(w_{i-1}^{n,+},w_i^{n,-}\right)\psi(r_{i-1/2}^n)\Big)\leq 0.$$

The e-MOOD scheme is thus entropy preserving.

Numerical results: smooth solution

Computational domain $[0,1]$ with periodic boundary conditions

Initial data $u_0(x) = 1$, $p_0(x) = 1$, $\rho_0(x) = 1 + \chi_{[0.2,0.8]}(x) \exp\left(\frac{(x-0.5)^2}{(x-0.2)(x-0.8)}\right)$

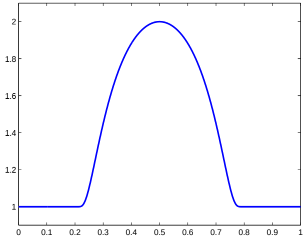
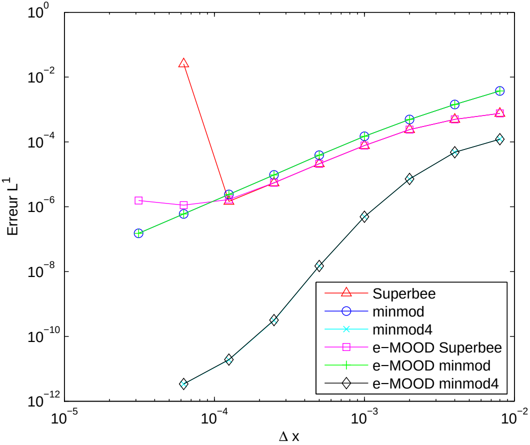


Figure: Initial density

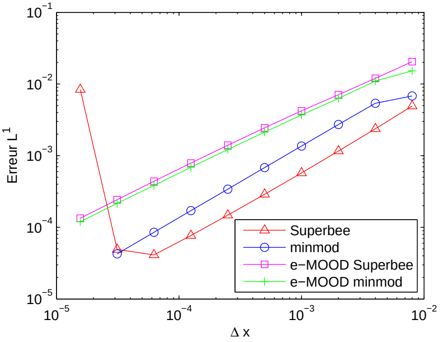


Numerical results obtained with a second-order time scheme

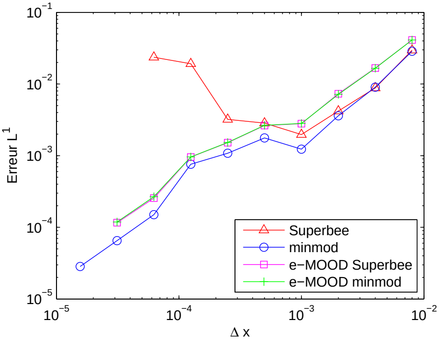
L^1 error:

$$\sum_i \left| \rho_i^N - \rho_{ex}(x_i, T) \right|$$

1-rarefaction



Double shock



Conclusions

- **Relaxation schemes for systems with source term**:
  - Robust well-balanced scheme for the Ripa model
  - Extension of the method to the Euler equations with gravity
- **e-MOOD scheme**: high-order entropy preserving scheme for the Euler equations in 1D

Perspectives

- Entropy property for the relaxation schemes
- Development of high-order well-balanced schemes for systems with source terms.
- Extension of the e-MOOD method to 2D
- Extension of the e-MOOD scheme to other systems (Euler with general pressure law, Shallow-water...)

Thank you for your attention!