# Second-order MUSCL schemes based on Dual Mesh Gradient Reconstruction (DMGR)

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# Introduction

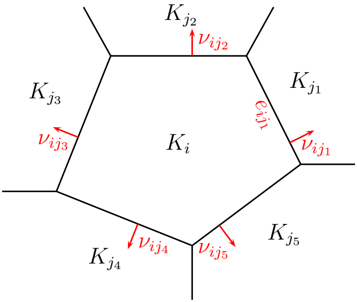
- Hyperbolic system of conservation laws in 2D

$$\partial_t w + \partial_x f(w) + \partial_y g(w) = 0$$

  $w : \mathbb{R}^+ \times \mathbb{R}^2 \to \Omega \subset \mathbb{R}^d$ : unknown state vector
  $f, g : \Omega \to \mathbb{R}^d$ : flux functions

- $\Omega$ convex set of physical states
- Objective : derive a numerical scheme
  - Second order accurate
  - $\Omega$–preserving
  - Unstructured meshes
  - CFL condition

# Mesh notations




Geometry of the cell $K_i$

- polygonal cells $K_i$ (perimeter $\mathcal{P}_i$, area $|K_i|$)
- $\gamma(i)$: index set of the cells neighbouring $K_i$
- $e_{ij}$: common edge between $K_i$ and $K_j$ (length $|e_{ij}|$)
- $\nu_{ij}$: unit outward normal to $e_{ij}$

# First-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \varphi\left(w_i^n, w_j^n, \nu_{ij}\right)$$

- $\varphi$ 2D Godunov-type flux (Harten, Lax, van Leer):

$$\varphi(w_L, w_R, \nu) = h_\nu(w_L) + \frac{\delta}{2\Delta t} w_L - \frac{1}{\Delta t} \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_\nu \left(\frac{x}{\Delta t}, w_L, w_R\right) dx$$

  with respect to the CFL condition $\dfrac{\Delta t}{\delta} \max \left|\lambda^{\pm}(w_L, w_R, \nu)\right| \leq \dfrac{1}{2}$
  - $h_\nu(w) = \nu_x f(w) + \nu_y g(w)$: flux in the $\nu$–direction, with $\nu = (\nu_x, \nu_y)^T$
  - $\widetilde{w}_\nu$ approximate Riemann solver valued in $\Omega$
- Consistency : $\varphi(w, w, \nu) = h_\nu(w)$
- Conservation : $\varphi(w_L, w_R, \nu) = -\varphi(w_R, w_L, -\nu)$

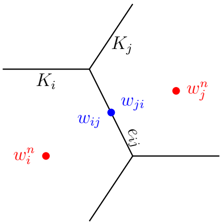
# MUSCL scheme (Van Leer, Perthame-Shu...)

First-order scheme on the cell $K_i$

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \varphi \left( w_i^n, w_j^n, \nu_{ij} \right)$$

Second-order MUSCL scheme on the cell $K_i$

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \varphi \left( w_{ij}, w_{ji}, \nu_{ij} \right)$$




$w_{ij}$ and $w_{ji}$ are second-order approximations at the interface between $K_i$ and $K_j$

$\rightarrow$ **How to compute $w_{ij}$ ?**

# Motivations

- First-order CFL condition for a polygonal cell (Perthame-Shu):

$$\Delta t \frac{\text{perimeter}}{\text{area}} \max\{\text{speed}\} \leq \frac{1}{2}$$

- First-order CFL condition on a square:

$$\frac{\Delta t}{\Delta x} \max\{\text{speed}\} \leq \frac{1}{4}$$

- $\Rightarrow$ Inconsistency. Usual first-order CFL conditions are not optimal.

# First-order scheme: CFL condition

Under the CFL condition $\dfrac{\Delta t}{\delta} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \dfrac{1}{2}$, we have

$$w_i^{n+1} = \left( 1 - \frac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \right) w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| h_{\nu_{ij}}(w_i^n) + \frac{1}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_{\nu_{ij}} \left( \frac{x}{\Delta t}, w_i^n, w_j^n \right) dx$$

# First-order scheme: CFL condition

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$$\sum_{j \in \gamma(i)} |e_{ij}| h_{\nu_{ij}}(w_i^n) = \begin{pmatrix} f \\ g \end{pmatrix} (w_i^n) \cdot \sum_{j \in \gamma(i)} |e_{ij}| \nu_{ij} = 0 \text{ by Green's formula}$$

# First-order scheme: CFL condition

Under the CFL condition $\dfrac{\Delta t}{\delta} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \dfrac{1}{2}$, we have

$$w_i^{n+1} = \left( 1 - \frac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \right) w_i^n - 0 + \frac{1}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_{\nu_{ij}} \left( \frac{x}{\Delta t}, w_i^n, w_j^n \right) dx$$

Taking $\delta = \dfrac{2|K_i|}{\mathcal{P}_i}$, we have $1 - \dfrac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| = 0$.

The CFL condition becomes

$$\frac{\Delta t}{|K_i|}\mathcal{P}_i \max_{j\in\gamma(i)} \left|\lambda^{\pm}(w_i^n, w_j^n, \nu_{ij})\right| \leq 1$$

and we have

$$\begin{aligned} w_i^{n+1} &= \frac{1}{|K_i|}\sum_{j\in\gamma(i)} |e_{ij}| \int_{-\frac{|K_i|}{\mathcal{P}_i}}^{0} \widetilde{w}_{\nu_{ij}}\left(\frac{x}{\Delta t}, w_i^n, w_j^n\right) dx \\ &= \frac{1}{\mathcal{P}_i}\sum_{j\in\gamma(i)} |e_{ij}|\widehat{w}_{ij} \end{aligned}$$

with $\widehat{w}_{ij} = \dfrac{\mathcal{P}_i}{|K_i|}\displaystyle\int_{-\frac{|K_i|}{\mathcal{P}_i}}^{0} \widetilde{w}_{\nu_{ij}}\left(\frac{x}{\Delta t}, w_i^n, w_j^n\right) dx$

$\widehat{w}_{ij} \in \Omega$ as the mean value of a function valued in the convex $\Omega$

$w_i^{n+1} \in \Omega$ as a convex combination of the $\widehat{w}_{ij}$

## Theorem : Robustness of the first-order scheme

If the following hypothesis are satisfied

(i) $w_i^n \in \Omega, \forall i \in \mathbb{Z}$

(ii) We have the CFL condition

$$\Delta t \frac{\mathcal{P}_i}{|K_i|} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq 1, \forall i \in \mathbb{Z}$$

Then the states $w_i^{n+1}$ remain in $\Omega$.

Remark : this CFL can be written

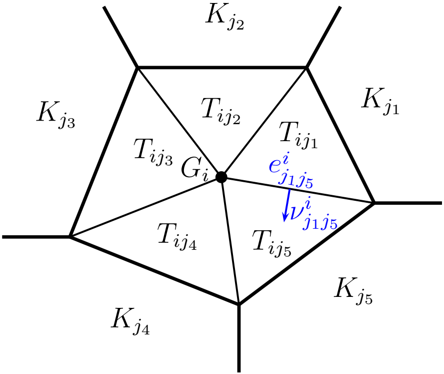
$$\Delta t \frac{|e_i|}{|K_i|} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \frac{1}{n_i}$$

$n_i$ number of edges of the cell $K_i$

$|e_i| = \frac{1}{n_i}\mathcal{P}_i$ mean length of the edges

$\Rightarrow$ Consistency with the CFL condition for a square

# Mesh notations




Subcells decomposition of the cell $K_i$

- $T_{ij}$: triangle formed by the mass center $G_i$ and the edge $e_{ij}$ (perimeter $\mathcal{P}_{ij}$, area $|T_{ij}|$)
- $\gamma(i,j)$: index set of the two subcells neighbouring $T_{ij}$ in $K_i$
- $e^i_{jk}$: common edge between $T_{ij}$ and $T_{ik}$ (length $|e^i_{jk}|$)
- $\nu^i_{jk}$: unit outward normal to $e^i_{jk}$

## Theorem : Robustness of the MUSCL scheme

If the following hypothesis are satisfied

(i) The initial states $w_i^n$ and the reconstructed states $w_{ij}$ are in $\Omega$

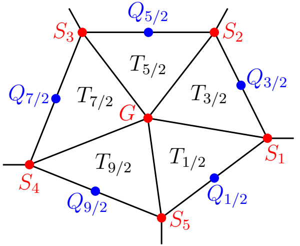
(ii) The reconstruction satisfies the conservation property
$$\sum_{j\in\gamma(i)} \frac{|T_{ij}|}{|K_i|} w_{ij} = w_i^n$$

(iii) We have the CFL condition $\forall i \in \mathbb{Z}$

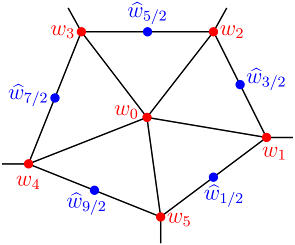
$$\Delta t \max_{j\in\gamma(i)} \frac{\mathcal{P}_{ij}}{|T_{ij}|} \max_{k\in\gamma(i,j)} \left|\lambda^{\pm}(w_{ij}, w_{ji}, \nu_{ij}), \lambda^{\pm}(w_{ij}, w_{ik}, \nu_{jk}^{i})\right| \leq 1$$

Then the states $w_i^{n+1}$ remain in $\Omega$.

# Reconstruction process




Geometry of the cell $K$




Known states and reconstructed states

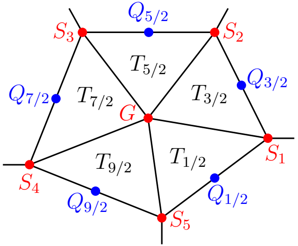
The states $\widehat{w}_{j-1/2}$ have to satisfy:

- $\widehat{w}_{j-1/2} \in \Omega$
- $\displaystyle\sum_j \frac{|T_{j-1/2}|}{|K|} \widehat{w}_{j-1/2} = w_0$

If we take $\widehat{w}_{j-1/2} = \widetilde{w}(Q_{j-1/2})$ with $\widetilde{w}$ a linear function on $K$, we have

$$\sum_j \frac{|T_{j-1/2}|}{|K|} \widehat{w}_{j-1/2} = w_0 \Longleftrightarrow \widetilde{w}(G) = w_0$$

# Reconstruction process

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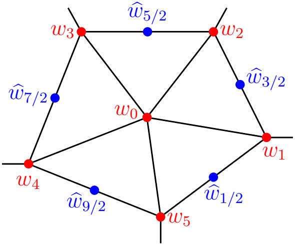
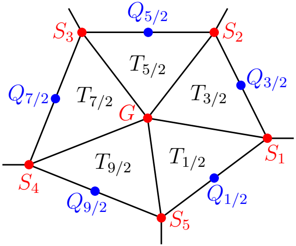



Known states and reconstructed states

1. Gradient reconstruction
   We define a continuous function $\overline{w} : K \to \mathbb{R}^d$ piecewise linear on each triangle $T_{j-1/2}$ and such that $\overline{w}(S_j) = w_j$ and $\overline{w}(G) = w_0$.

# Reconstruction process

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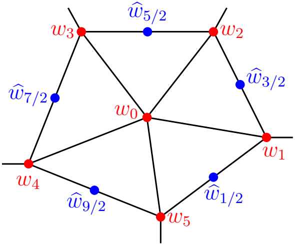



Known states and reconstructed states

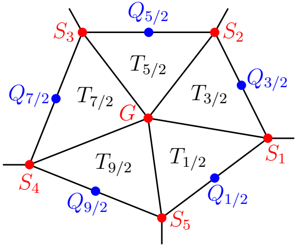
2. Projection

   For a matrix $\alpha \in \mathbb{R}^d \times \mathbb{R}^d$, we define $\widetilde{w}_\alpha(X) = w_0 + \alpha \cdot (X - G)$, the linear function whose gradient is $\alpha$.

   Let $\mu$ be the gradient resulting from the $L^2$–projection of $\overline{w}$:

$$\int_K \left\|\overline{w}(X) - \widetilde{w}_\mu(X)\right\|^2 dX = \min_{\alpha \in \mathbb{R}^d \times \mathbb{R}^d} \int_K \left\|\overline{w}(X) - \widetilde{w}_\alpha(X)\right\|^2 dX.$$

# Reconstruction process

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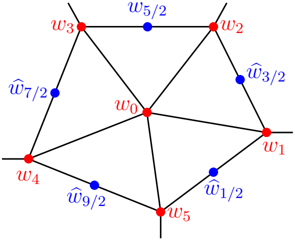


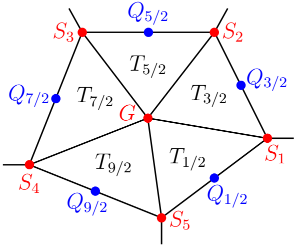

Known states and reconstructed states

3. Limitation of the slope $\mu$
   We consider the sets of admissible slope limiters:

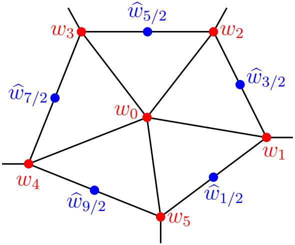
$$F_{j-1/2} = \left\{ \theta \in [0,1], \widetilde{w}_{s\mu}\left(Q_{j-1/2}\right) \in \Omega, \forall s \in [0,\theta] \right\}.$$

We define the optimal slope limiter $\beta = \min_j \sup(F_{j-1/2}) - \epsilon$, where $\epsilon > 0$ is a small paramater such that $\beta \in \bigcup_j F_{j-1/2}$.

# Reconstruction process




Geometry of the cell $K$



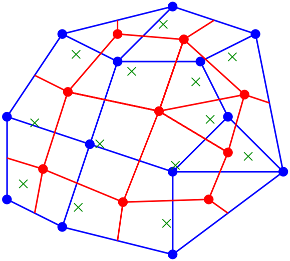

Known states and reconstructed states

4. Finally, the reconstructed states are given by $\widehat{w}_{j-1/2} = \widetilde{w}_{\beta\mu}(Q_{j-1/2})$.

$$\begin{array}{lcl} \text{Limitation procedure} & \Rightarrow & \widehat{w}_{j-1/2} \in \Omega \\ \widetilde{w}(G) = w_0 & \Rightarrow & \displaystyle\sum_j \frac{|T_{j-1/2}|}{|K|} \widehat{w}_{j-1/2} = w_0 \end{array}$$

$\Rightarrow$ **The DMGR scheme is robust**

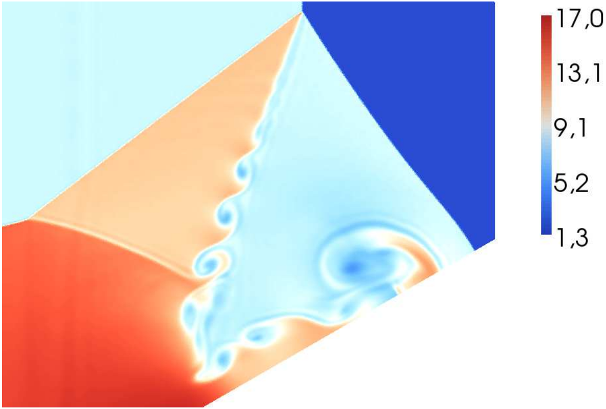
# Primal and dual mesh



- Vertex of the primal mesh (unknown state)
- Center of a primal cell = Vertex of the dual mesh (known state)
- × Center of a dual cell (known state)

- We write a MUSCL scheme on both primal and dual meshes
- These schemes give a state at the center of each primal and dual cell
- We can apply the reconstruction procedure on the dual cells $\rightarrow$ we get a linear function $\widetilde{w}_i^d$ on each dual cell
- To get the state at a primal vertex $S_i^p$, we take $\widetilde{w}_i^d(S_i^p)$.

# Double Mach reflection on a ramp




Density solution
$2.10^6$ cells, $3.10^6$ DOF

# Mach 3 wind tunnel with a step


Density solution
$1.10^6$ cells, $1.5 \times 10^6$ DOF

Thank you for your attention!!