An Entropic MOOD Scheme for the Euler Equations

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Introduction

- Hyperbolic system of conservation laws

$$\begin{cases} \partial_t w + \partial_x f(w) = 0 \\ w(x,0) = w_0(x) \end{cases}$$

$w : \mathbb{R}^+ \times \mathbb{R} \to \Omega$: unknown state vector
$f : \Omega \to \mathbb{R}^d$: continuous flux function
$w_0 \in L^1_{\text{loc}}(\mathbb{R};\Omega)$: initial condition

- $\Omega \subset \mathbb{R}^d$ convex set of physical states
- Objective: study the stability of high-order space-time schemes

Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t E + \partial_x (E + p) u = 0 \end{cases}$$

- ρ: density
- u: velocity
- E: total energy
- p: pressure given by the perfect gas law

$$p = (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right), \quad \gamma \in (1, 3]$$

- Set of physical states:

$$\Omega = \left\{ w \in \mathbb{R}^3,\ \rho > 0,\ p > 0 \right\}$$

Weak solutions and entropy solutions

- A function $w \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ is a weak solution if $\forall \phi \in C^1_c(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^d)$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} (w \cdot \partial_t \phi + f(w) \cdot \partial_x \phi)\, dt dx + \int_{\mathbb{R}} w(x,0) \cdot \phi(x,0) dx = 0.$$

- A convex function $S \in C^2(\Omega; \mathbb{R})$ is an entropy for the system if there exists an entropy flux $g \in C^2(\Omega; \mathbb{R})$ such that $\nabla f(w) \nabla S(w) = \nabla g(w)$, $\forall w \in \Omega$.
- A weak solution w is an entropy solution if for any entropy pair (S, g) of the system, and $\forall \phi \in C^1_c(\mathbb{R} \times \mathbb{R}^+; \mathbb{R})$, $\phi \geq 0$, w satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} (S(w) \partial_t \phi + g(w) \partial_x \phi)\, dt dx + \int_{\mathbb{R}} S(w(x,0)) \phi(x,0) dx \geq 0.$$

Space and time discretizations

- Space discretization: cells $[x_{i-1/2}, x_{i+1/2}]$ with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$
- Time discretization: $t^n = n\Delta t$
- Rectangular cells in the (x, t)-plane:

$$R_i^n = [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^{n+1})$$

- The sequence $(\Delta x, \Delta t)$ of discretization steps is devoted to converge to $(0, 0)$, the ratio $\frac{\Delta t}{\Delta x}$ being kept constant.

A general high-order space-time scheme

Initial condition

$$w_i^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w_0(x)\, dx$$

Runge-Kutta time discretization (Gottlieb-Shu)

$$w_i^{n,(\ell)} = \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \left(w_i^{n,(j)} - \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(F_{i+1/2}^{n,(j)} - F_{i-1/2}^{n,(j)} \right) \right), \quad \ell = 1, \cdots, m$$

$$w_i^{n,(0)} = w_i^n, \quad w_i^{n+1} = w_i^{n,(m)}$$

Assumptions: $\alpha_{\ell,j} \geq 0, \quad \beta_{\ell,j} \geq 0, \quad \sum_{j=0}^{l-1} \alpha_{\ell,j} = 1$

Space discretization

$$F_{i+1/2}^{n,(j)} = F\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)} \right)$$

Assumptions: F continuous and consistent $(F(w, \cdots, w) = f(w))$

Lax-Wendroff Theorem

We introduce the piecewise constant functions

$$w^{\Delta}(x,t) = w_i^n, \quad \text{for } (x,t) \in R_i^n,$$

$$w^{\Delta,(\ell)}(x,t) = w_i^{n,(\ell)}, \quad \text{for } (x,t) \in R_i^n.$$

Theorem

We assume the following hypotheses:

(i) *There exists a compact $K \subset \Omega$ such that $w^{\Delta,(\ell)} \in K$, for all $\ell = 0, \cdots, m$;*

(ii) *w^{Δ} converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ to a function w.*

Then w is a weak solution.

Entropic version of the Lax-Wendroff Theorem

(S, g) being an entropy pair of the system, an entropy numerical flux is a continuous function $G : (\Omega)^{2s} \to \mathbb{R}$ which is consistent with g, i.e. $G(w, \cdots, w) = g(w)$.

Theorem

Under the hypotheses of Lax-Wendroff Theorem, we assume that for all entropy pair (S, g) there exists an entropy numerical flux G such that we have the discrete entropy inequality (DEI)

$$S\left(w_i^{n,(l)}\right) \leq \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \left(S\left(w_i^{n,(j)}\right) - \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{\ell,j} \right) \right),$$

with $G_{i+1/2}^{n,(j)} = G\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)}\right)$.
Then w is an entropic solution.

Problem: No such DEI has ever been proven as soon as the scheme is second-order in space.

Example: the second-order MUSCL scheme

- We consider $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ a slope limiter and we define the limited slope $\mu_i^{n,(j)} = L\left(w_i^{n,(j)} - w_{i-1}^{n,(j)}, w_{i+1}^{n,(j)} - w_i^{n,(j)}\right)$.
- The MUSCL flux is defined by

$$F_{i+1/2}^{n,(j)} = F\left(w_i^{n,(j)} + \frac{1}{2}\mu_i^{n,j}, w_{i+1}^{n,(j)} - \frac{1}{2}\mu_{i+1}^{n,(j)}\right),$$

where F is a first-order numerical flux.
- DEI satisfied by the MUSCL scheme:

$$S\left(w_i^{n,(l)}\right) \leq \sum_{j=0}^{\ell-1} \alpha_{\ell,j}\left(S\left(w_i^{n,(j)}\right) - \beta_{\ell,j}\frac{\Delta t}{\Delta x}\left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{\ell,j}\right)\right) + \sum_{j=0}^{\ell-1} \alpha_{\ell,j}\left(P_i^{n,(j)} - S\left(w_i^{n,(j)}\right)\right)$$

where $P_i^{n,(j)} = P\left(w_i^{n,(j)}, \mu_i^{n,(j)}, \Delta x, S\right)$

Operators P

- Examples:

$$P_1(w, \mu, \Delta x, S) = \frac{S(w - \mu/2) + S(w + \mu/2)}{2} \quad \text{(Berthon)}$$

$$P_2(w, \mu, \Delta x, S) = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} S\left(w + \frac{x}{\Delta x}\mu\right) dx \quad \text{(Bouchut)}$$

- In general, the operators P satisfy: $\exists C > 0$ such that

$$0 \leq P(w, \mu, \Delta x, S) - S(w) \leq C \|\nabla^2 S(w)\| \|\mu\|^2$$

- We define the piecewise constant function

$$D^\Delta(x, t) = \sum_{j=0}^{m-1} \alpha_{m,j} \frac{P_i^{n,(j)} - S\left(w_i^{n,(j)}\right)}{\Delta t}, \quad \text{for } (x, t) \in R_i^n$$

Convergence of D^Δ: theoretical study

- Let μ be the weak-star limit of the sequence D^Δ. Let β be the entropy dissipation measure defined as the weak-star limit of the sequence

$$b^\Delta(x,t) = \sum_{j=0}^{m-1} \alpha_{m,j} \frac{\left\| w_i^{n,(j)} - w_{i-1}^{n,(j)} \right\|^2}{\Delta x}, \quad \text{for } (x,t) \in R_i^n.$$

- μ is absolutely continuous with respect to β.

Conjecture (Hou-le Floch)

The entropy dissipation measure β is concentrated on the curves of discontinuity of w.

Numerical study: test cases

1–rarefaction

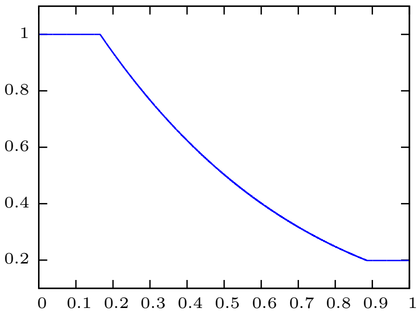

Figure: Exact solution in density

shock–shock

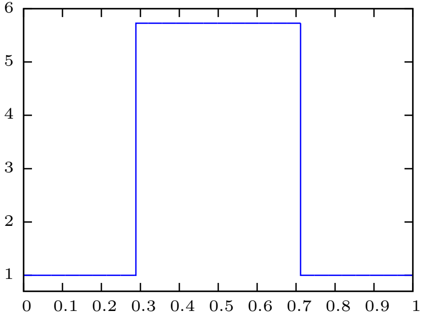

Figure: Exact solution in density

- L^1 error for the convergence: $E^\Delta = \sum_i \left|\rho_i^N - \rho_{ex}(x_i, T)\right|$
- Convergence of D^Δ: $I^\Delta = \int_{[0,1]\times[0,T]} D^\Delta(x,t)\,dxdt$

1–rarefaction: convergence of first-order time schemes

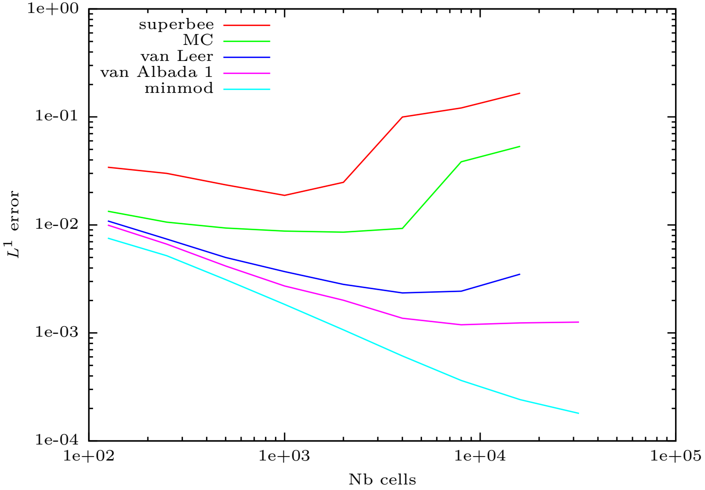

Figure: Convergence of second-order space / first-order time schemes

1–rarefaction: what superbee does (1)

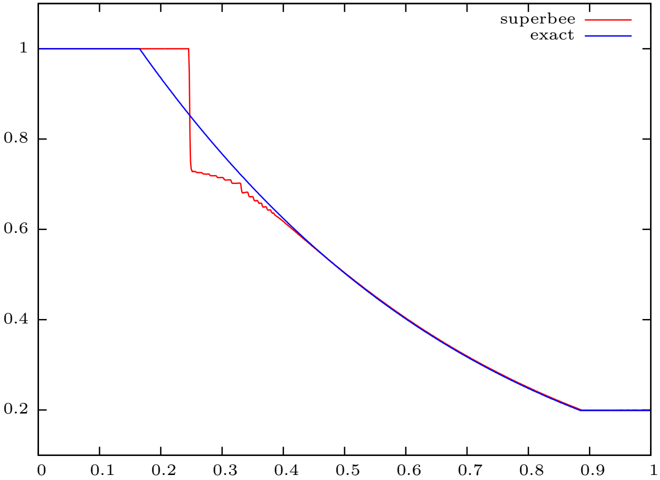

Figure: Solution given by the superbee limiter with 1000 cells

1–rarefaction: what superbee does (2)

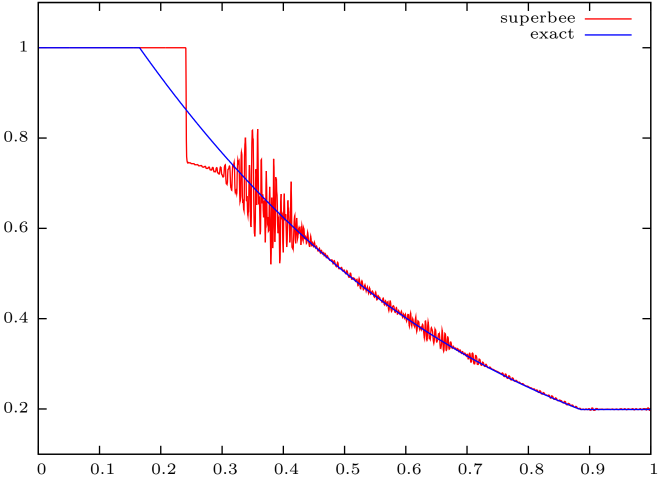

Figure: Solution given by the superbee limiter with 2000 cells

1–rarefaction: convergence of I^Δ for first-order time schemes

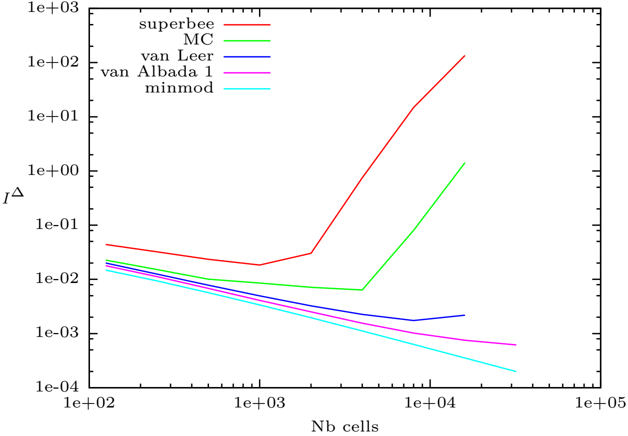

Figure: Convergence of I^Δ for second-order space / first-order time schemes

1–rarefaction: convergence of second-order time schemes

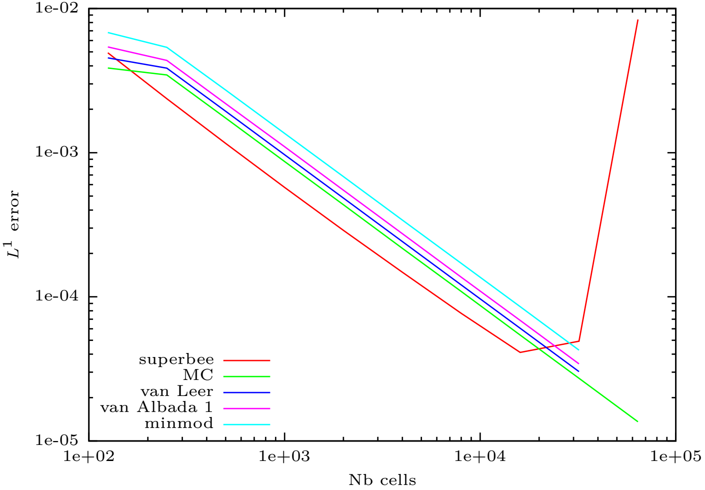

Figure: Convergence of second-order space-time schemes

1–rarefaction: convergence of I^Δ for second-order time schemes

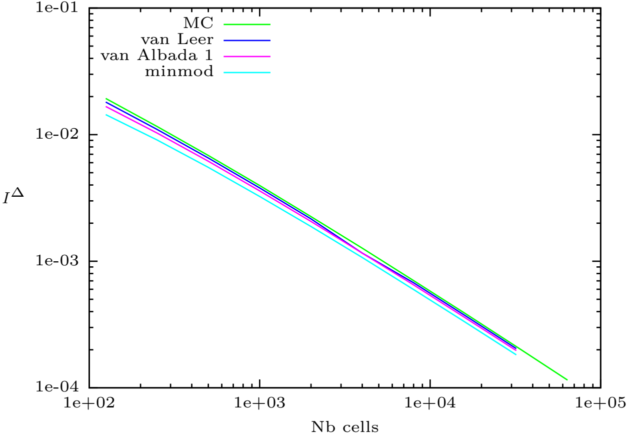

Figure: Convergence of I^Δ for second-order space-time schemes

Shock-Shock: convergence of first-order time schemes

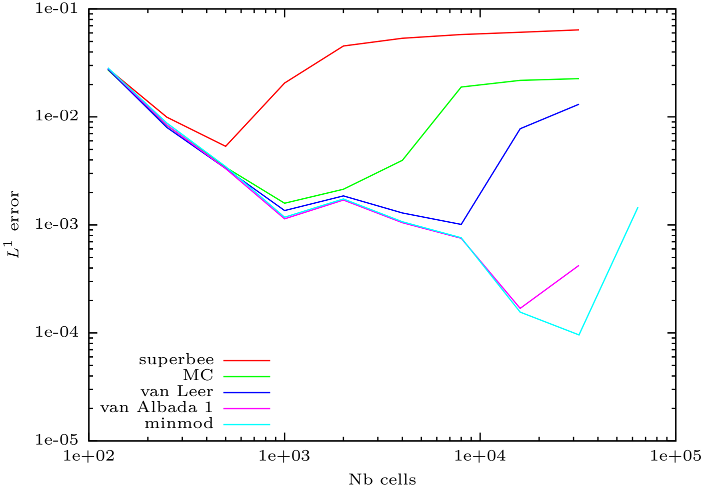

Figure: Convergence of second-order space / first-order time schemes

Shock-Shock: convergence of I^{Δ} for the minmod limiter

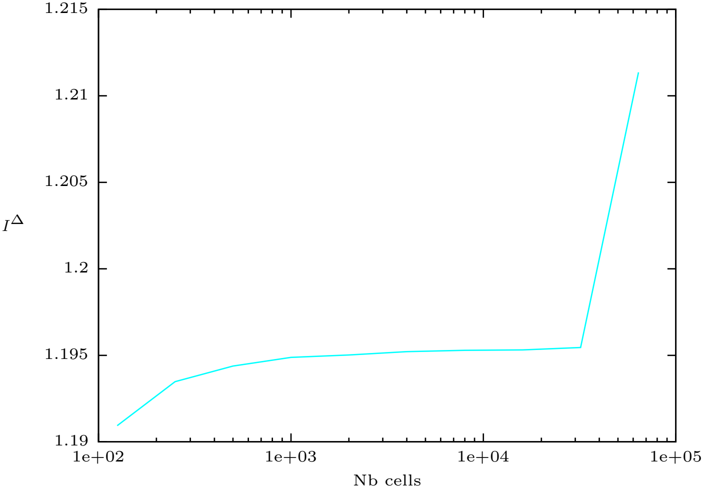

Figure: Convergence of I^{Δ} for the minmod limiter / first-order time scheme

Shock-Shock: convergence of I^{Δ} for first-order time schemes

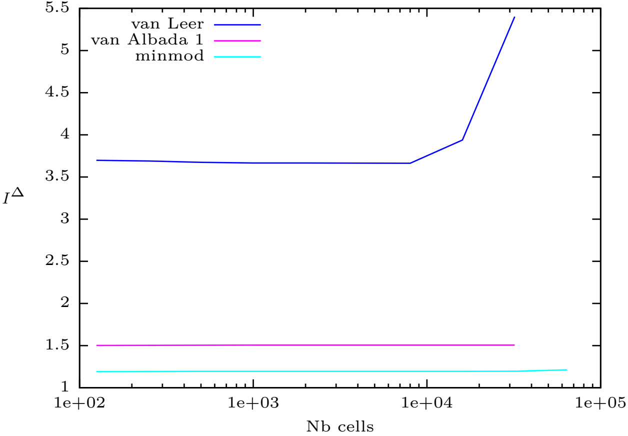

Figure: Convergence of I^{Δ} for second-order space / first-order time schemes

Shock-Shock: convergence of second-order time schemes

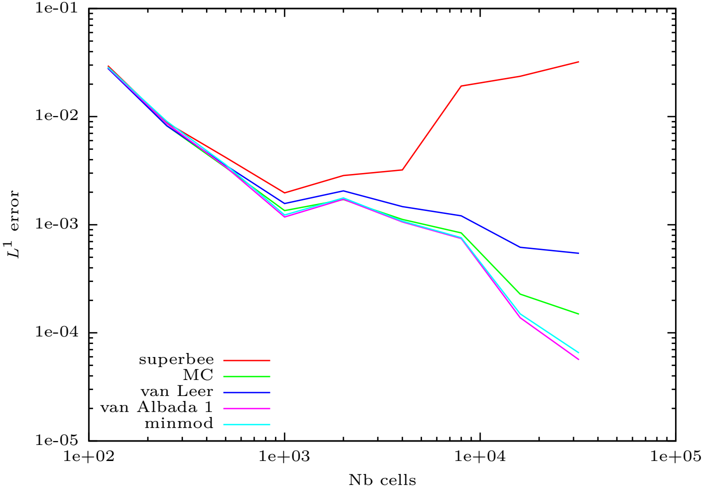

Figure: Convergence of second-order space-time schemes

Shock-Shock: convergence of I^{Δ} for second-order time schemes

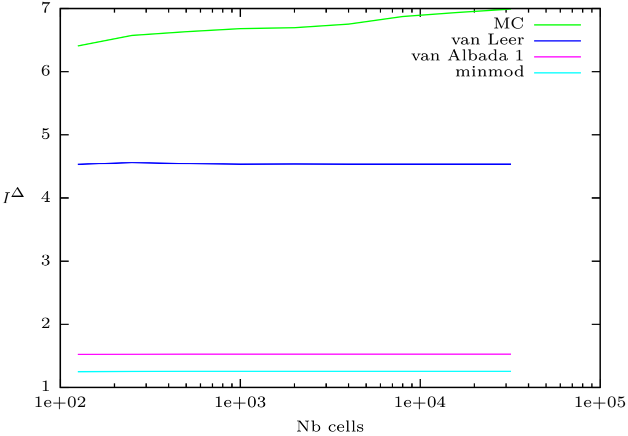

Figure: Convergence of I^{Δ} for second-order space-time scheme

Conclusion

- Numerical results confirm the Hou-le Floch conjecture: when the scheme converges, the weak-star limit μ of D^{Δ} seems to be concentrated on the curves of discontinuity of w.
- This does not imply that the limit is not entropic, but only that the usual DEI are not the suitable tool to prove a Lax-Wendroff theorem.
- We have to focus on the stronger DEI

$$S\left(w_i^{n,(l)}\right) \leq \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \left(S\left(w_i^{n,(j)}\right) - \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{\ell,j} \right) \right).$$

- Most of the limiters seem to be unstable to small perturbations, though with a very low explosion rate.

The family of entropies for the Euler equations

The Euler system possesses a family of entropy pairs (S, g) written

$$S = -\rho h(s), \quad g = -\rho u h(s),$$

where $s = \ln\left(\frac{p}{\rho^\gamma}\right)$ is the specific entropy and h is a smooth function satisfying

$$h'(s) > 0, \quad h'(s) - \gamma h''(s) > 0.$$

Lemma (reformulation)

The entropy pairs of the Euler system write

$$S(r) = \rho\psi(r), \quad g(r) = \rho u \psi(r),$$

where $r = \frac{\rho^{1/\gamma}}{p}$ and ψ is a smooth decreasing convex function.

We consider the scheme $w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right)$, where $w_i^n = (\rho_i^n, \rho_i^n u_i^n, E_i^n)^T$ and $F_{i+1/2} = (F^{\rho}_{i+1/2}, F^{\rho u}_{i+1/2}, F^{E}_{i+1/2})^T$.

Theorem

Assume the scheme is Ω-preserving. Assume the DEI

$$-\rho_i^{n+1} r_i^{n+1} \leq -\rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(-F^{\rho}_{i+1/2} r^n_{i+1/2} + F^{\rho}_{i-1/2} r^n_{i-1/2}\right)$$

with $r^n_{i+1/2} = \begin{cases} r^n_{i+1} & \text{if} \quad F^{\rho}_{i+1/2} < 0 \\ r^n_i & \text{if} \quad F^{\rho}_{i+1/2} > 0 \end{cases}$. *Assume the additional CFL like condition (Larrouturou)*

$$\frac{\Delta t}{\Delta x}\left(\max(0, F^{\rho}_{i+1/2}) - \min(0, F^{\rho}_{i-1/2})\right) \leq \rho_i^n.$$

Then the scheme satisfies all the discrete entropy inequalities.

Example : the HLLC/Suliciu relaxation scheme

Proof of the Theorem (1)

The numerical flux can be written

$$F^{\rho}_{i+1/2} r_{i+1/2} = F^{\rho}_{i+1/2} \frac{r_i^n + r_{i+1}^n}{2} - \left| F^{\rho}_{i+1/2} \right| \frac{r_{i+1}^n - r_i^n}{2}.$$

The DEI then writes

$$r_i^{n+1} \geq \frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n,$$

where we have set

$$a = \frac{\Delta t}{2\Delta x} \left(F^{\rho}_{i-1/2} + \left| F^{\rho}_{i-1/2} \right| \right),$$

$$b = \rho_i^n - \frac{\Delta t}{2\Delta x} \left(F^{\rho}_{i+1/2} + \left| F^{\rho}_{i+1/2} \right| - F^{\rho}_{i-1/2} + \left| F^{\rho}_{i-1/2} \right| \right),$$

$$c = \frac{\Delta t}{2\Delta x} \left(\left| F^{\rho}_{i+1/2} \right| - F^{\rho}_{i+1/2} \right).$$

Proof of the Theorem (2)

- - We have $a + b + c = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^{\rho}_{i+1/2} - F^{\rho}_{i-1/2}\right) = \rho_i^{n+1}$.
  - $a \geq 0,\ c \geq 0$
  - $b \geq 0$ thanks to the CFL like condition

  $\Rightarrow r_i^{n+1}$ is greater than a convex combination of r_{i-1}^n, r_i^n and r_{i+1}^n.
- We consider an entropy pair which can writes $(S, g) = (\rho\psi(r), \rho u\psi(r))$ with ψ a smooth decreasing convex function thanks to the Lemma.
- ψ is decreasing:

$$\psi\left(r_i^{n+1}\right) \leq \psi\left(\frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n\right)$$

- Jensen inequality (ψ is convex):

$$\psi\left(r_i^{n+1}\right) \leq \frac{a}{\rho_i^{n+1}}\psi\left(r_{i-1}^n\right) + \frac{b}{\rho_i^{n+1}}\psi\left(r_i^n\right) + \frac{c}{\rho_i^{n+1}}\psi\left(r_{i+1}^n\right)$$

Proof of the Theorem (3)

- We replace a, b and c by their value to obtain

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n\psi(r_i^n) - \frac{\Delta t}{2\Delta x}\Big(F^\rho_{i+1/2}(\psi(r_i^n)+\psi(r_{i+1}^n)) \\ -|F^\rho_{i+1/2}|(\psi(r_{i+1}^n)-\psi(r_i^n)) - F^\rho_{i-1/2}(\psi(r_{i-1}^n)+\psi(r_i^n)) \\ +|F^\rho_{i-1/2}|(\psi(r_i^n)-\psi(r_{i-1}^n))\Big).$$

- We define $\psi_{i+1/2}^n = \begin{cases} \psi\left(r_{i+1}^n\right) & \text{if} \quad F^\rho_{i+1/2} < 0 \\ \psi\left(r_i^n\right) & \text{if} \quad F^\rho_{i+1/2} > 0 \end{cases}$.
- We have shown the DEI (for the entropy pair $(\rho\psi(r), \rho u\psi(r))$)

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n\psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2}\psi^n_{i+1/2} - F^\rho_{i-1/2}\psi_{i-1/2}\right).$$

□

First-order scheme

We consider a first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^n, w_{i+1}^n\right) - F\left(w_{i-1}^n, w_i^n\right)\right).$$

For a time step restricted according to the CFL condition

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left|\lambda^{\pm}\left(w_i^n, w_{i+1}^n\right)\right| \leq \frac{1}{2},$$

the first-order scheme is assumed to satisfy:

(i) $w_i^n \in \Omega, \quad \forall i \in \mathbb{Z} \quad \Rightarrow \quad w_i^{n+1} \in \Omega, \quad \forall i \in \mathbb{Z}$

(ii) $\forall i \in \mathbb{Z}$, the following DEI is satisfied:

$$\begin{aligned} -\rho_i^{n+1} r_i^{n+1} \leq -\rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\Big(&-F^{\rho}\left(w_i^n, w_{i+1}^n\right) r_{i+1/2}^n \\ &+F^{\rho}\left(w_{i-1}^n, w_i^n\right) r_{i-1/2}^n\Big). \end{aligned}$$

Example: HLLC scheme

High-order reconstruction

- A reconstruction function is a continuous function $\mathcal{R} : \Omega^{2s+1} \to \Omega$ such that $\mathcal{R}(w, \cdots, w) = w$, for all $w \in \Omega$.
- A high-order reconstruction function is usually a reconstruction function based on high degree polynomial reconstruction.
- Here, we consider two reconstruction functions $\mathcal{R}_-$ and $\mathcal{R}_+$. The associated MUSCL scheme is then given by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(F\left(\mathcal{W}_{i,+}, \mathcal{W}_{i+1,-}\right) - F\left(\mathcal{W}_{i-1,+}, \mathcal{W}_{i,-}\right) \right),$$

with $\mathcal{W}_{i,\pm} = \mathcal{R}_\pm \left(w_{i-s}^n, \cdots, w_{i+1}^n \right)$

- Example: second-order MUSCL scheme:

$$\mathcal{R}_\pm \left(w_{i-1}^n, w_i^n, w_{i+1}^n \right) = w_i^n \pm \frac{1}{2} L\left(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n \right),$$

where L is a slope limiter.

The E-MOOD algorithm

1. **Evaluation of the reconstructed states.** The reconstructed states are given by $\mathcal{W}_{i,\pm} = \mathcal{R}_{\pm}\left(w_{i-s}^n, \cdots, w_{i+s}^n\right)$
2. **Computation of the candidate solution w_i^*.** We compute a candidate solution w_i^* using the MUSCL scheme

$$w_i^* = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\mathcal{W}_{i,+}, \mathcal{W}_{i+1,-}\right) - F\left(\mathcal{W}_{i-1,+}, \mathcal{W}_{i,-}\right)\right).$$

3. **DEI test.** If w_i^* does not satisfy the DEI test

$$-\rho_i^* r_i^* \leq -\rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(-F_{i+1/2}^{\rho} r_{i+1/2}^n + F_{i-1/2}^{\rho} r_{i-1/2}^n\right),$$

with $F_{i+1/2}^{\rho} = F^{\rho}\left(\mathcal{W}_{i,+}, \mathcal{W}_{i+1,-}\right)$, then we set $\mathcal{W}_{i,\pm} = w_i^n$

4. **Stopping criterion.**
    - If the DEI test is satisfied on all the cells, the candidate solution is valid and we set $w_i^{n+1} = w_i^*$
    - else the solution is recomputed from step 2

Stability and robustness of the E-MOOD scheme

Theorem

Assume the time step Δt is chosen in order to satisfy the two following CFL like conditions:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left| \lambda^{\pm} \left(w_{i,+}, w_{i+1,-} \right) \right|, \left| \lambda^{\pm} \left(w_{i,-}, w_{i,+} \right) \right| \right) \leq \frac{1}{4}$$

$$\frac{\Delta t}{\Delta x} \left(\max(0, F^{\rho}_{i+1/2}) - \min(0, F^{\rho}_{i-1/2}) \right) \leq \rho_i^n.$$

Then the E-MOOD method provides an updated solution w_i^{n+1} after a finite number of iterations. It is physically admissible, and it satisfies all the entropy inequalities.

1–rarefaction: first-order time schemes

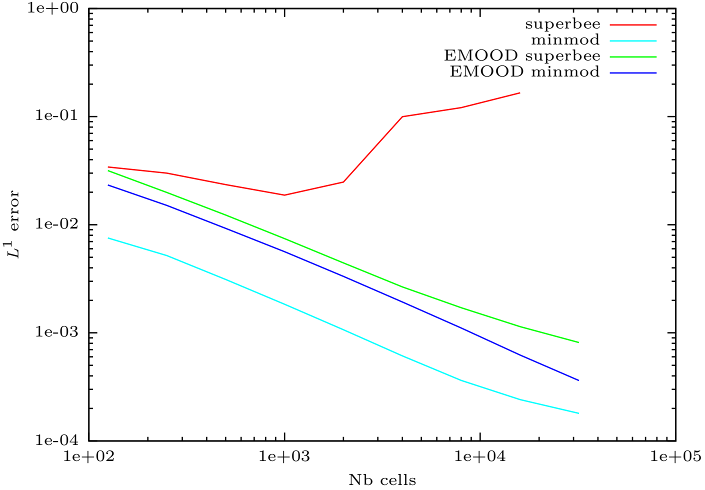

Figure: Convergence of first-order time schemes: E-MOOD vs MUSCL

1–rarefaction: second-order time schemes

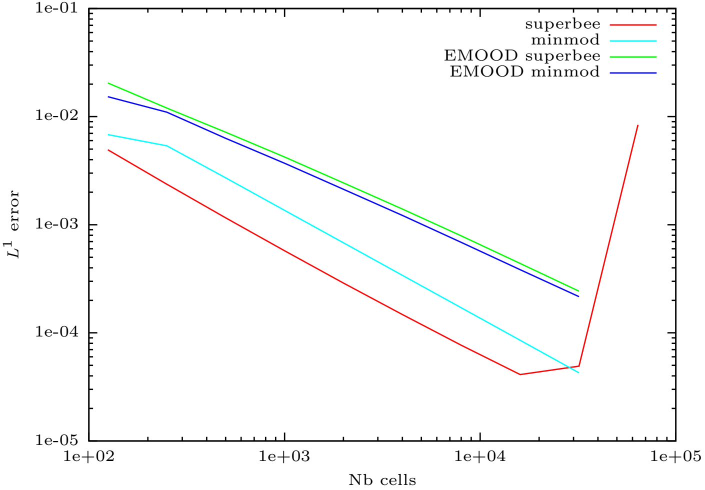

Figure: Convergence of second-order time schemes: E-MOOD vs MUSCL

Shock-Shock: first-order time schemes

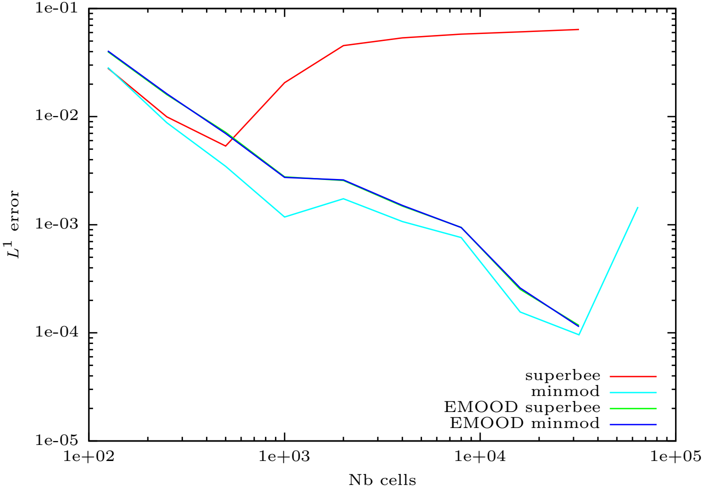

Figure: Convergence of first-order time schemes: E-MOOD vs MUSCL

Shock-Shock: second-order time schemes

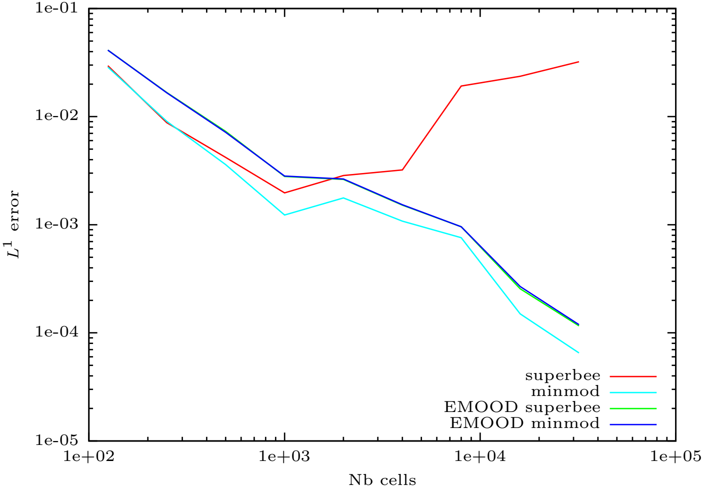

Figure: Convergence of second-order time schemes: E-MOOD vs MUSCL

Smooth problem

- $\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.2 \text{ or } x > 0.8 \\ 1 + \exp\left(\frac{(x-0.5)^2}{(x-0.2)(x-0.8)}\right) & \text{if } 0.2 \leq x \leq 0.8 \end{cases}$
  $u_0(x) = 1,\ p_0(x) = 1$
- Periodic boundary conditions

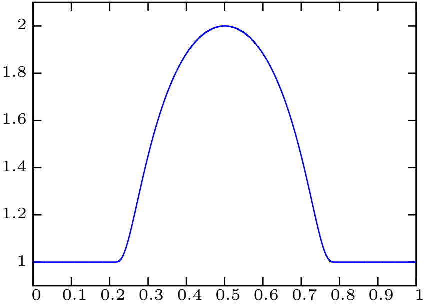

Figure: Initial and final solution in density for the smooth problem

Smooth problem: convergence

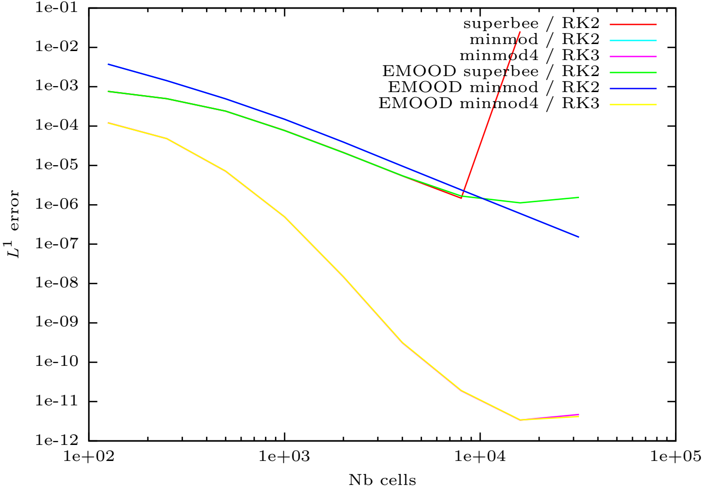

Figure: Convergence: E-MOOD vs MUSCL

Thank you for your attention!!