Robustesse et stabilité des schémas d'ordre élevé pour approcher les systèmes de lois de conservation

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1. A high-order entropy preserving scheme with *a posteriori* limitation
   - Motivations
   - From one to all discrete entropy inequalities
   - The e-MOOD scheme for the Euler equations

2. Second-order schemes based on dual mesh gradient reconstruction
   - MUSCL scheme and CFL condition
   - The DMGR scheme
   - Numerical results

Introduction

- Hyperbolic system of conservation laws in 1D

$$\partial_t w + \partial_x f(w) = 0$$

$w : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^d$: unknown state vector
$f : \Omega \to \mathbb{R}^d$: flux function

- Ω convex set of physical states
- Entropy inequalities:

$$\partial_t \eta(w) + \partial_x \mathcal{G}(w) \leq 0,$$

where $w \mapsto \eta(w)$ is convex and $\nabla_w f \nabla_w \eta = \nabla_w \mathcal{G}$

- Objectives:
  - Study the entropy stability of high-order schemes
  - Derive a high-order numerical scheme for the Euler equations which is entropy preserving in the sense of Lax-Wendroff

Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t E + \partial_x (E + p) u = 0 \end{cases}$$

- Ideal gas law: $p = (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right), \quad \gamma \in (1, 3]$
- Set of physical states: $\Omega = \left\{ w \in \mathbb{R}^3,\ \rho > 0,\ p > 0 \right\}$
- Entropy inequalities:

$$\partial_t \rho \mathcal{F}(\ln(s)) + \partial_x \rho \mathcal{F}(\ln(s)) u \leq 0, \quad \text{with } s = \frac{p}{\rho^\gamma}$$

and $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ a smooth function such that

$$\mathcal{F}'(y) < 0 \quad \text{and} \quad \mathcal{F}'(y) < \gamma \mathcal{F}''(y), \quad \forall y \in \mathbb{R}$$

Scheme notations

- Space discretization: cells $K_i = [x_{i-1/2}, x_{i+1/2}]$ with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$
- w_i^n: approximate solution at time t^n on the cell K_i
- Update at time $t^{n+1} = t^n + \Delta t$ given by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

  where $F_{i+1/2}^n = F\left(w_{i-s+1}^n, \cdots, w_{i+s}^n\right)$ and F is a consistant numerical flux ($F(w, \cdots, w) = f(w)$)
- We introduce the piecewise constant function

$$w^\Delta(x,t) = w_i^n, \quad \text{for } (x,t) \in K_i \times [t^n, t^{n+1})$$

- The sequence $(\Delta x, \Delta t)$ is devoted to converge to $(0,0)$, the ratio $\frac{\Delta t}{\Delta x}$ being kept constant.

Lax-Wendroff Theorem

Theorem

(i) *Assume the following hypotheses:*

- *There exists a compact* $K \subset \Omega$ *such that* $w^\Delta \in K$*;*
- w^Δ *converges in* $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ *to a function* w*.*

Then w *is a weak solution.*

(ii) *Assume the additional hypothesis:*

- *For all entropy pair* $(\eta, \mathcal{G})$*, there exists an entropy numerical flux* G*, consistant with* $\mathcal{G}$ $(G(w, \cdots, w) = \mathcal{G}(w))$*, such that we have the discrete entropy inequality (DEI)*

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0,$$

with $G_{i+1/2}^n = G\left(w_{i-s+1}^n, \cdots, w_{i+s}^n\right)$*.*

Then w *is an entropic solution.*

Example: the MUSCL scheme

- We assume the first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F(w_i^n, w_{i+1}^n) - F(w_{i-1}^n, w_i^n)\right)$$

satisfies the DEI

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^n, w_{i+1}^n) - G(w_{i-1}^n, w_i^n)}{\Delta x} \leq 0.$$

- Let L be a limiter function (minmod, superbee...). We define a limited increment on each cell by

$$\mu_i^n = L\left(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n\right)$$

- Reconstructed states at interfaces : $w_i^{n,\pm} = w_i^n \pm \frac{1}{2}\mu_i^n$
- The MUSCL scheme is defined by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F(w_i^{n,+}, w_{i+1}^{n,-}) - F(w_{i-1}^{n,+}, w_i^{n,-})\right),$$

DEI satisfied by the MUSCL scheme

- The known DEI satisfied by the MUSCL scheme all write

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^{n,+}, w_{i+1}^{n,-}) - G(w_{i-1}^{n,+}, w_i^{n,-})}{\Delta x} \leq \frac{P_i^n - \eta(w_i^n)}{\Delta t}$$

where $P_i^n = P\left(w_i^n, \mu_i^n, \Delta x, \eta\right)$.

- Examples of operator P:

$$P_1(w, \mu, \Delta x, \eta) = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \eta\left(w + \frac{x}{\Delta x}\mu\right) dx \quad \text{[Bouchut \textit{et al.} '96]}$$

$$P_2(w, \mu, \Delta x, \eta) = \frac{\eta(w - \mu/2) + \eta(w + \mu/2)}{2} \quad \text{[Berthon '05]}$$

Convergence study

- The discrete entropy inequality

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^{n,+}, w_{i+1}^{n,-}) - G(w_{i-1}^{n,+}, w_i^{n,-})}{\Delta x} \leq \frac{P_i^n - \eta(w_i^n)}{\Delta t}$$

converges weakly to

$$\partial_t \eta(w) + \partial_x \mathcal{G}(w) \leq \delta,$$

where δ is a positive measure.

Conjecture (Hou-LeFloch '94)

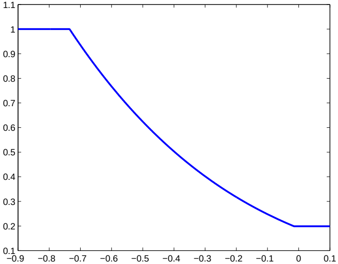
- $\delta = 0$ in the areas where w is smooth
- $\delta > 0$ on the curves of discontinuity of w

Numerical study: test cases (Euler equations)

Entropy error (total mass of the right-hand side):

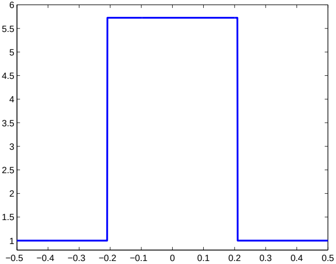
$$I^{\Delta} = \Delta x \sum_{i,n} \left(P_i^n - \eta(w_i^n) \right)$$

1–rarefaction



$I^{\Delta} \stackrel{?}{\to} 0$

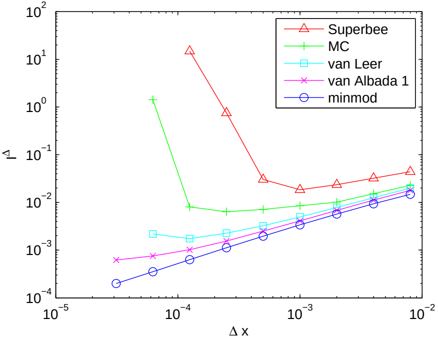
Double shock



$I^{\Delta} \stackrel{?}{\to} c > 0$

Numerical results obtained with a first-order time scheme

1-rarefaction



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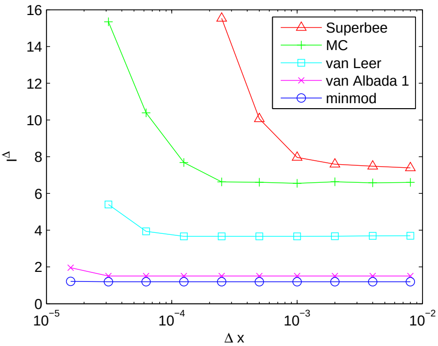

Numerical results obtained with a second-order time scheme

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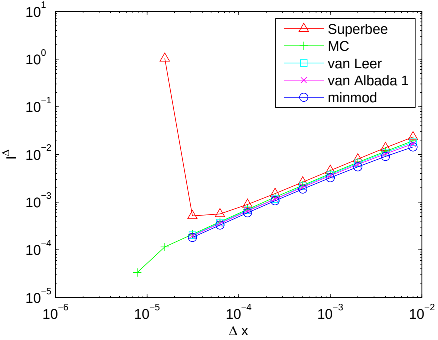

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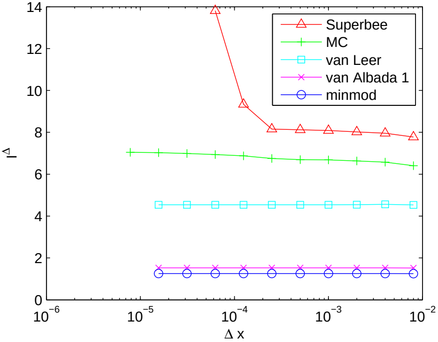


Conclusion of motivations

- Numerical results confirm the Hou-le Floch conjecture: when the scheme converges, the measure δ seems to be concentrated on the curves of discontinuity of w.
- This does not imply that the limit is not entropic, but the usual discrete entropy inequalities are not relevant to apply the Lax-Wendroff theorem.
- We have to enforce the stronger discrete entropy inequalities

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0.$$

- We suggest to extend the *a posteriori* methods (MOOD) introduced in [Clain, Diot & Loubère '11].

The family of entropies for the Euler equations

Lemma

The entropy pairs $(\eta, \mathcal{G})$ of the Euler system rewrite

$$\eta = \rho\psi(r), \quad \mathcal{G} = \rho\psi(r)u,$$

where $r = -\frac{p^{1/\gamma}}{\rho}$ and ψ is a smooth increasing convex function.

We consider the scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right),$$

where $w_i^n = (\rho_i^n, \rho_i^n u_i^n, E_i^n)^T$ and $F_{i+1/2} = (F^{\rho}_{i+1/2}, F^{\rho u}_{i+1/2}, F^{E}_{i+1/2})^T$.

We introduce $r_{i+1/2}^n = \begin{cases} r_{i+1}^n & \text{if } F_{i+1/2}^\rho < 0 \\ r_i^n & \text{if } F_{i+1/2}^\rho > 0 \end{cases}.$

Theorem

Assume the scheme preserves Ω. Assume the scheme satisfies the specific discrete entropy inequality

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^\rho r_{i+1/2}^n - F_{i-1/2}^\rho r_{i-1/2}^n\right).$$

Assume the additional CFL like condition

$$\frac{\Delta t}{\Delta x}\left(\max\left(0, F_{i+1/2}^\rho\right) - \min\left(0, F_{i-1/2}^\rho\right)\right) \leq \rho_i^n.$$

Then the scheme is entropy preserving: for all smooth increasing convex function ψ, we have

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n \psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^\rho \psi(r_{i+1/2}^n) - F_{i-1/2}^\rho \psi(r_{i-1/2}^n)\right).$$

Proof of the Theorem (1)

Using the upwind definition of $r_{i+1/2}^n$, the specific DEI writes

$$r_i^{n+1} \leq \frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n,$$

where we have set

$$\begin{aligned} a &= \frac{\Delta t}{2\Delta x}\left(F_{i-1/2}^\rho + \left|F_{i-1/2}^\rho\right|\right), \\ b &= \rho_i^n - \frac{\Delta t}{2\Delta x}\left(F_{i+1/2}^\rho + \left|F_{i+1/2}^\rho\right| - F_{i-1/2}^\rho + \left|F_{i-1/2}^\rho\right|\right), \\ c &= \frac{\Delta t}{2\Delta x}\left(\left|F_{i+1/2}^\rho\right| - F_{i+1/2}^\rho\right). \end{aligned}$$

- We have $a + b + c = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^\rho - F_{i-1/2}^\rho\right) = \rho_i^{n+1}$.
- $a \geq 0$, $c \geq 0$
- $b \geq 0$ thanks to the CFL like condition

$\Rightarrow$ r_i^{n+1} is less than a convex combination of r_{i-1}^n, r_i^n and r_{i+1}^n.

Proof of the Theorem (2)

- We consider an entropy pair $(\rho\psi(r), \rho\psi(r)u)$ with ψ a smooth increasing convex function.
- ψ is increasing:

$$\psi\left(r_i^{n+1}\right) \leq \psi\left(\frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n\right)$$

- Jensen inequality (ψ is convex):

$$\psi\left(r_i^{n+1}\right) \leq \frac{a}{\rho_i^{n+1}} \psi\left(r_{i-1}^n\right) + \frac{b}{\rho_i^{n+1}} \psi\left(r_i^n\right) + \frac{c}{\rho_i^{n+1}} \psi\left(r_{i+1}^n\right)$$

- Replacing a, b and c by their value, we get

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n \psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^{\rho}\psi_{i+1/2}^n - F_{i-1/2}^{\rho}\psi_{i-1/2}\right),$$

with $\psi_{i+1/2}^n = \begin{cases} \psi\left(r_{i+1}^n\right) & \text{if} \quad F_{i+1/2}^{\rho} < 0 \\ \psi\left(r_i^n\right) & \text{if} \quad F_{i+1/2}^{\rho} > 0 \end{cases}.$

First-order scheme

We consider a first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^n, w_{i+1}^n\right) - F\left(w_{i-1}^n, w_i^n\right)\right).$$

For a time step restricted according to the CFL condition

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left|\lambda^{\pm}\left(w_i^n, w_{i+1}^n\right)\right| \leq \frac{1}{2},$$

the first-order scheme is assumed to satisfy:

- Robustness: $\forall i \in \mathbb{Z}, \quad w_i^n \in \Omega \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \quad w_i^{n+1} \in \Omega$
- Stability:

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\Big(F^\rho\left(w_i^n, w_{i+1}^n\right) r_{i+1/2}^n - F^\rho\left(w_{i-1}^n, w_i^n\right) r_{i-1/2}^n\Big).$$

Example: the HLLC/Suliciu relaxation scheme

Reconstruction procedure

- We consider high-order reconstructed states $w_i^{n,\pm}$ on the cell K_i at the interfaces $x_{i\pm1/2}$.
- These reconstructed states can be obtained by any reconstruction procedure (MUSCL, ENO/WENO, PPM...).
- Assumptions:
  - The reconstruction is Ω-preserving: $w_i^{n,\pm} \in \Omega$;
  - The reconstruction is conservative:

$$w_i^n = \frac{1}{2}\left(w_i^{n,-} + w_i^{n,+}\right).$$

The e-MOOD algorithm

1. Reconstruction step: For all $i \in \mathbb{Z}$, we evaluate high-order reconstructed states $w_i^{n,\pm}$ located at the interfaces $x_{i\pm1/2}$.
2. Evolution step: We compute a candidate solution as follows:

$$w_i^{n+1,\star} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^{n,+}, w_{i+1}^{n,-}\right) - F\left(w_{i-1}^{n,+}, w_i^{n,-}\right)\right).$$

3. A posteriori limitation step: We have the following alternative:
   - if for all $i \in \mathbb{Z}$, we have

$$\rho^{n+1,\star} r_i^{n+1,\star} \leq \rho_i^n r(w_i^n) - \frac{\Delta t}{\Delta x}\Big(F^\rho\left(w_i^{n,+}, w_{i+1}^{n,-}\right) r_{i+1/2}^n - F^\rho\left(w_{i-1}^{n,+}, w_i^{n,-}\right) r_{i-1/2}^n\Big), \quad (1)$$

     then the solution is valid and the updated solution at time $t^n + \Delta t$ is defined by $w_i^{n+1} = w_i^{n+1,\star}$;
   - otherwise, for all $i \in \mathbb{Z}$ such that (1) is not satisfied, we set $w_i^{n,\pm} = w_i^n$ and we go back to step 2.

Theorem

Assume the time step Δt is chosen in order to satisfy the two following CFL like conditions:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left| \lambda^{\pm} \left(w_i^{n,+}, w_{i+1}^{n,-} \right) \right|, \left| \lambda^{\pm} \left(w_i^{n,-}, w_i^{n,+} \right) \right| \right) \leq \frac{1}{4},$$

$$\frac{\Delta t}{\Delta x} \left(\max \left(0, F^{\rho}_{i+1/2} \right) - \min \left(0, F^{\rho}_{i-1/2} \right) \right) \leq \rho_i^n.$$

Then the updated states w_i^{n+1}, given by the e-MOOD scheme, belong to Ω. Moreover, for all smooth increasing convex function ψ, the e-MOOD scheme satisfies

$$\frac{1}{\Delta t} \left(\rho_i^{n+1} \psi(r_i^{n+1}) - \rho_i^n \psi\left(r_i^n\right) \right) + \frac{1}{\Delta x} \left(F^{\rho} \left(w_i^{n,+}, w_{i+1}^{n,-} \right) \psi(r^n_{i+1/2}) - F^{\rho} \left(w_{i-1}^{n,+}, w_i^{n,-} \right) \psi(r^n_{i-1/2}) \right) \leq 0.$$

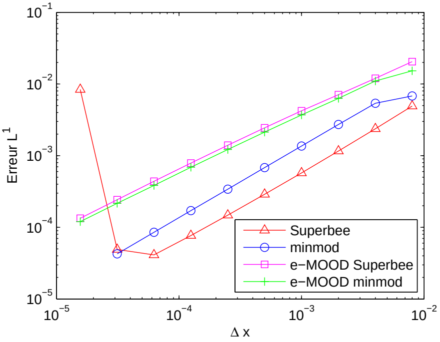
The e-MOOD scheme is thus entropy preserving.

Numerical results obtained with a second-order time scheme

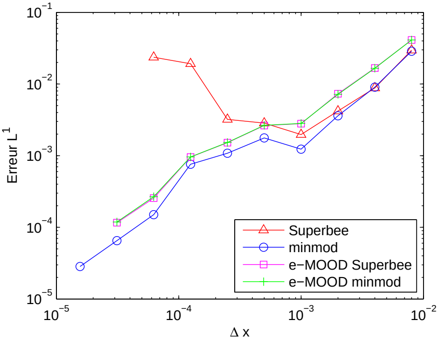
L^1 error:

$$\sum_i \left| \rho_i^N - \rho_{ex}(x_i, T) \right|$$

1-rarefaction



Double shock



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Introduction

- Hyperbolic system of conservation laws in 2D

$$\partial_t w + \partial_x f(w) + \partial_y g(w) = 0$$

  $w : \mathbb{R}^+ \times \mathbb{R}^2 \to \Omega \subset \mathbb{R}^d$: unknown state vector
  $f, g : \Omega \to \mathbb{R}^d$: flux functions
- Ω convex set of physical states
- Objective: derive a numerical scheme
  - second order accurate
  - Ω–preserving
  - works on unstructured meshes
  - with an optimized CFL condition

Motivations: CFL condition

- First-order CFL condition for a polygonal cell [Perthame & Shu '96]:

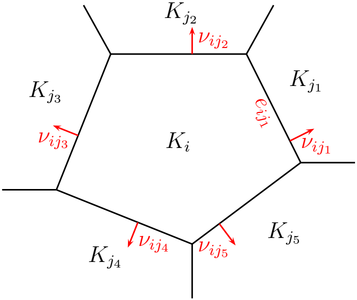
$$\Delta t \frac{\text{perimeter}}{\text{area}} \max\{\text{speed}\} \leq \frac{1}{2}$$

- First-order CFL condition on a square:

$$\frac{\Delta t}{\Delta x} \max\{\text{speed}\} \leq \frac{1}{4}$$

- $\Rightarrow$ Inconsistency. Usual first-order CFL conditions are not optimal.

Mesh notations



Geometry of the cell K_i

- polygonal cells K_i (perimeter $\mathcal{P}_i$, area $|K_i|$)
- $\gamma(i)$: index set of the cells neighbouring K_i
- e_{ij}: common edge between K_i and K_j (length $|e_{ij}|$)
- ν_{ij}: unit outward normal to e_{ij}

First-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \varphi\left(w_i^n, w_j^n, \nu_{ij}\right)$$

- 2D numerical flux: φ Godunov-type in each direction ν [Harten, Lax & van Leer '83]:

$$\varphi(w_L, w_R, \nu) = h_\nu(w_L) + \frac{\delta}{2\Delta t} w_L - \frac{1}{\Delta t} \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_\nu\left(\frac{x}{\Delta t}, w_L, w_R\right) dx$$

  with respect to the CFL condition $\dfrac{\Delta t}{\delta} \max\left|\lambda^\pm(w_L, w_R, \nu)\right| \leq \dfrac{1}{2}$

  - $h_\nu(w) = \nu_x f(w) + \nu_y g(w)$: flux in the ν–direction, with $\nu = (\nu_x, \nu_y)^T$
  - $\widetilde{w}_\nu$ approximate Riemann solver valued in Ω

- Consistency: $\varphi(w, w, \nu) = h_\nu(w)$
- Conservation: $\varphi(w_L, w_R, \nu) = -\varphi(w_R, w_L, -\nu)$

First-order scheme: CFL condition

Under the CFL condition $\dfrac{\Delta t}{\delta} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \dfrac{1}{2}$, we have

$$w_i^{n+1} = \left(1 - \frac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \right) w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| h_{\nu_{ij}}(w_i^n)$$
$$+ \frac{1}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_{\nu_{ij}} \left(\frac{x}{\Delta t}, w_i^n, w_j^n \right) dx$$

First-order scheme: CFL condition

Under the CFL condition $\frac{\Delta t}{\delta} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \frac{1}{2}$, we have

$$w_i^{n+1} = \left(1 - \frac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \right) w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| h_{\nu_{ij}}(w_i^n) + \frac{1}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_{\nu_{ij}} \left(\frac{x}{\Delta t}, w_i^n, w_j^n \right) dx$$

$$\sum_{j \in \gamma(i)} |e_{ij}| h_{\nu_{ij}}(w_i^n) = \begin{pmatrix} f \\ g \end{pmatrix} (w_i^n) \cdot \sum_{j \in \gamma(i)} |e_{ij}| \nu_{ij} = 0 \text{ by Green's formula}$$

First-order scheme: CFL condition

Under the CFL condition $\frac{\Delta t}{\delta} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \frac{1}{2}$, we have

$$w_i^{n+1} = \left(1 - \frac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \right) w_i^n - 0$$

$$+ \frac{1}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \int_{-\frac{\delta}{2}}^{0} \widetilde{w}_{\nu_{ij}} \left(\frac{x}{\Delta t}, w_i^n, w_j^n \right) dx$$

Taking $\delta = \frac{2|K_i|}{\mathcal{P}_i}$, we have $1 - \frac{\delta}{2|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| = 0$.

The CFL condition becomes

$$\frac{\Delta t}{|K_i|}\mathcal{P}_i \max_{j\in\gamma(i)} \left|\lambda^{\pm}(w_i^n, w_j^n, \nu_{ij})\right| \leq 1$$

and we have

$$\begin{aligned} w_i^{n+1} &= \frac{1}{|K_i|}\sum_{j\in\gamma(i)} |e_{ij}| \int_{-\frac{|K_i|}{\mathcal{P}_i}}^{0} \widetilde{w}_{\nu_{ij}}\left(\frac{x}{\Delta t}, w_i^n, w_j^n\right) dx \\ &= \frac{1}{\mathcal{P}_i}\sum_{j\in\gamma(i)} |e_{ij}|\widehat{w}_{ij} \end{aligned}$$

with $\widehat{w}_{ij} = \dfrac{\mathcal{P}_i}{|K_i|}\displaystyle\int_{-\frac{|K_i|}{\mathcal{P}_i}}^{0} \widetilde{w}_{\nu_{ij}}\left(\frac{x}{\Delta t}, w_i^n, w_j^n\right) dx$

$\widehat{w}_{ij} \in \Omega$ as the mean value of a function valued in the convex Ω

$w_i^{n+1} \in \Omega$ as a convex combination of the $\widehat{w}_{ij}$

Theorem (Robustness of the first-order scheme)

Assume the following CFL condition is satisfied:

$$\Delta t \frac{\mathcal{P}_i}{|K_i|} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq 1, \quad \forall i \in \mathbb{Z}.$$

Then the first-order scheme preserves Ω*:*

$$\forall i \in \mathbb{Z}, \quad w_i^n \in \Omega \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \quad w_i^{n+1} \in \Omega.$$

Remark : this CFL can be written

$$\Delta t \frac{|e_i|}{|K_i|} \max_{j \in \gamma(i)} \left| \lambda^{\pm}(w_i^n, w_j^n, \nu_{ij}) \right| \leq \frac{1}{n_i}$$

n_i number of edges of the cell K_i
$|e_i| = \frac{1}{n_i} \mathcal{P}_i$ mean length of the edges
$\Rightarrow$ Consistency with the CFL condition for a square

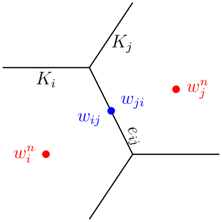
MUSCL scheme ([van Leer '79], [Perthame & Shu '96]...)

First-order scheme on the cell K_i

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \varphi \left(w_i^n, w_j^n, \nu_{ij} \right)$$

Second-order MUSCL scheme on the cell K_i

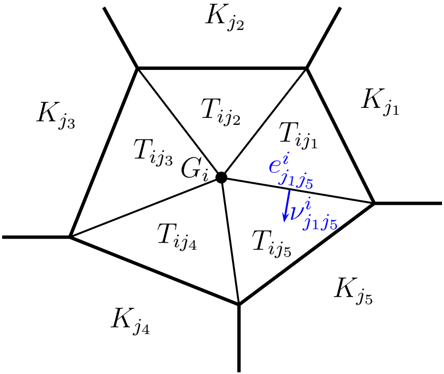
$$w_i^{n+1} = w_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \gamma(i)} |e_{ij}| \varphi \left(w_{ij}, w_{ji}, \nu_{ij} \right)$$



w_{ij} and w_{ji} are second-order approximations at the interface between K_i and K_j

→ How to compute w_{ij} ?

Subcells decomposition



Subcells decomposition of the cell K_i

- T_{ij}: triangle formed by the mass center G_i and the edge e_{ij} (perimeter $\mathcal{P}_{ij}$, area $|T_{ij}|$)
- $\gamma(i,j)$: index set of the two subcells neighbouring T_{ij} in K_i
- e^i_{jk}: common edge between T_{ij} and T_{ik} (length $|e^i_{jk}|$)
- ν^i_{jk}: unit outward normal to e^i_{jk}

Theorem (Robustness of the MUSCL scheme)

Assume the following hypotheses:

(i) *The reconstruction preserves* Ω*:*

$$\forall i \in \mathbb{Z}, \quad w_i^n \in \Omega \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \quad \forall j \in \gamma(i), \quad w_{ij} \in \Omega.$$

(ii) *The reconstruction satisfies the conservation property*

$$\sum_{j \in \gamma(i)} \frac{|T_{ij}|}{|K_i|} w_{ij} = w_i^n.$$

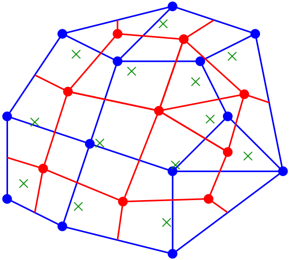
(iii) *The following CFL condition is satisfied for all* $i \in \mathbb{Z}$*:*

$$\Delta t \max_{j \in \gamma(i)} \frac{\mathcal{P}_{ij}}{|T_{ij}|} \max_{k \in \gamma(i,j)} \left| \lambda^{\pm}(w_{ij}, w_{ji}, \nu_{ij}), \lambda^{\pm}(w_{ij}, w_{ik}, \nu_{jk}^{i}) \right| \leq 1$$

Then the MUSCL scheme preserves Ω*:*

$$\forall i \in \mathbb{Z}, \quad w_i^n \in \Omega \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \quad w_i^{n+1} \in \Omega.$$

The DMGR scheme

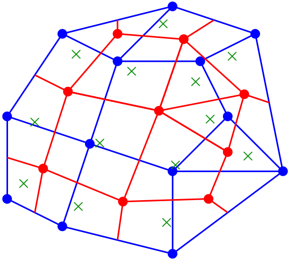


- ● Vertex of the primal mesh (unknown state)
- ● Center of a primal cell = Vertex of the dual mesh (known state)
- × Center of a dual cell (known state)

Figure: Primal mesh (blue) and dual mesh (red)

1. We write a MUSCL scheme on both primal and dual meshes
$\Rightarrow$ At time t^n, we know a state at the center of each primal and dual cell

The DMGR scheme



- Vertex of the primal mesh (unknown state)
- Center of a primal cell = Vertex of the dual mesh (known state)
- × Center of a dual cell (known state)

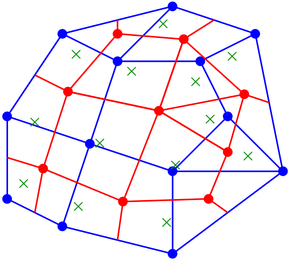
Figure: Primal mesh (blue) and dual mesh (red)

Assume we have a reconstruction procedure on a generic cell K:

$$\begin{cases} \text{state at the center} \\ \text{states at the vertices} \end{cases} \mapsto \quad \text{linear reconstruction } \widetilde{w}$$

This procedure will be detailed after.

The DMGR scheme

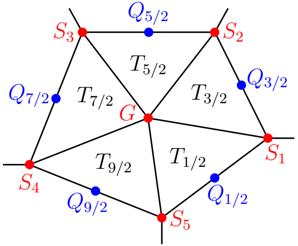


- Vertex of the primal mesh (unknown state)
- Center of a primal cell = Vertex of the dual mesh (known state)
- × Center of a dual cell (known state)

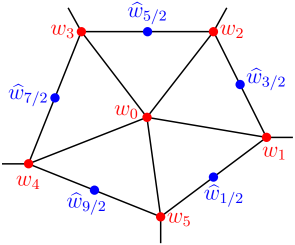
Figure: Primal mesh (blue) and dual mesh (red)

2. We can apply the reconstruction procedure on the dual cells
   $\Rightarrow$ We get a linear function $\widetilde{w}_i^d$ on each dual cell
3. To get the state at the primal vertex S_i^p, we take $\widetilde{w}_i^d(S_i^p)$
   $\Rightarrow$ We can apply the reconstruction procedure on the primal cells to get a linear function $\widetilde{w}_i^p$

Reconstruction procedure



Geometry of the cell K



Known states and reconstructed states

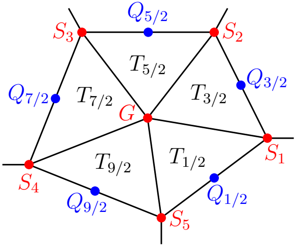
The states $\widehat{w}_{j-1/2}$ have to satisfy:

- $\widehat{w}_{j-1/2} \in \Omega$
- $\displaystyle\sum_j \frac{|T_{j-1/2}|}{|K|} \widehat{w}_{j-1/2} = w_0$

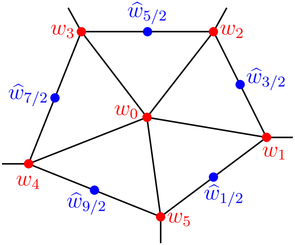
If we take $\widehat{w}_{j-1/2} = \widetilde{w}(Q_{j-1/2})$ with $\widetilde{w}$ a linear function on K, we have

$$\sum_j \frac{|T_{j-1/2}|}{|K|} \widehat{w}_{j-1/2} = w_0 \quad \Leftrightarrow \quad \widetilde{w}(G) = w_0$$

Reconstruction procedure



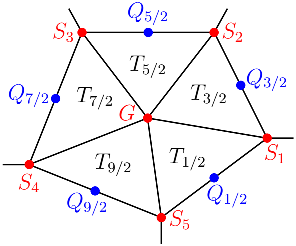
Geometry of the cell K



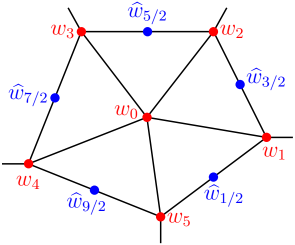
Known states and reconstructed states

1. Gradient reconstruction
   We define a continuous function $\overline{w} : K \to \mathbb{R}^d$ piecewise linear on each triangle $T_{j-1/2}$ and such that $\overline{w}(S_j) = w_j$ and $\overline{w}(G) = w_0$.

Reconstruction procedure



Geometry of the cell K

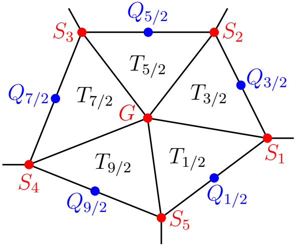


Known states and reconstructed states

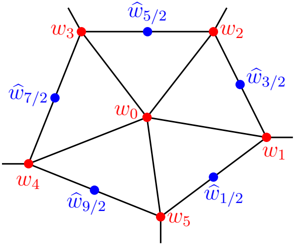
2. Projection
   For a slope $\alpha \in M_{d,2}(\mathbb{R})$, we define $\widetilde{w}_\alpha(X) = w_0 + \alpha \cdot (X - G)$, the linear function whose gradient is α.
   Let μ be the slope resulting from the L^2–projection of $\overline{w}$:

$$\int_K \|\overline{w}(X) - \widetilde{w}_\mu(X)\|^2 \, dX = \min_{\alpha \in M_{d,2}(\mathbb{R})} \int_K \|\overline{w}(X) - \widetilde{w}_\alpha(X)\|^2 \, dX.$$

Reconstruction procedure



Geometry of the cell K



Known states and reconstructed states

3. Limitation of the slope μ

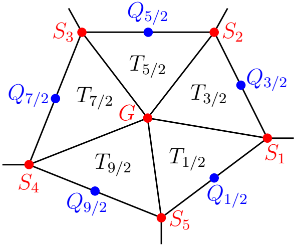
   We define the optimal slope limiter by:

$$\alpha_{j-1/2} = \sup\left\{\theta \in [0,1], \widetilde{w}_{s\mu}(Q_{j-1/2}) \in \Omega, \forall s \in [0,\theta]\right\},$$

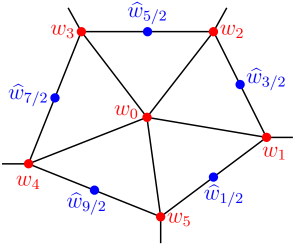
$$\beta = \min_j \alpha_{j-1/2} - \epsilon,$$

   where $\epsilon > 0$ is a small paramater s.t. $\widetilde{w}_{\beta\mu}(Q_{j-1/2}) \in \Omega, \quad \forall j.$

Reconstruction procedure



Geometry of the cell K



Known states and reconstructed states

4. Finally, the reconstructed states are given by $\widehat{w}_{j-1/2} = \widetilde{w}_{\beta\mu}(Q_{j-1/2})$.

$$\text{Limitation procedure} \quad \Rightarrow \quad \widehat{w}_{j-1/2} \in \Omega$$

$$\widetilde{w}(G) = w_0 \quad \Rightarrow \quad \sum_j \frac{|T_{j-1/2}|}{|K|} \widehat{w}_{j-1/2} = w_0$$

$\Rightarrow$ **The DMGR scheme preserves Ω**

Numerical results: 2D Euler equations

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix} + \partial_y \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} = 0$$

- ρ: density
- (u, v): velocity
- E: total energy
- p: pressure given by the ideal gas law

$$p = (\gamma - 1)\left(E - \frac{\rho}{2}\left(u^2 + v^2\right)\right), \quad \text{with } \gamma \in (1, 3]$$

- Set of physical states

$$\Omega = \left\{(\rho, \rho u, \rho v, E) \in \mathbb{R}^4; \quad \rho > 0, \quad E - \frac{\rho}{2}\left(u^2 + v^2\right) > 0\right\}$$

2D Riemann problems

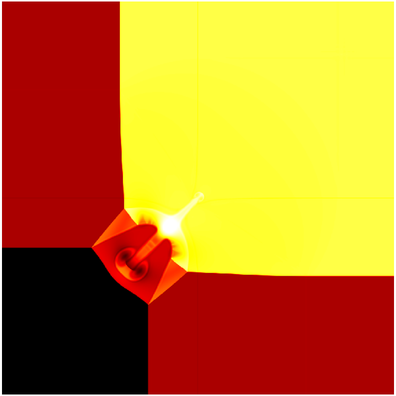


Figure: Four shocks 2D Riemann problem on a Cartesian mesh with 1.5×10^6 DOF

Figure: Four contact discontinuities 2D Riemann problem on a Cartesian mesh with 1.5×10^6 DOF

Double Mach reflection on a ramp

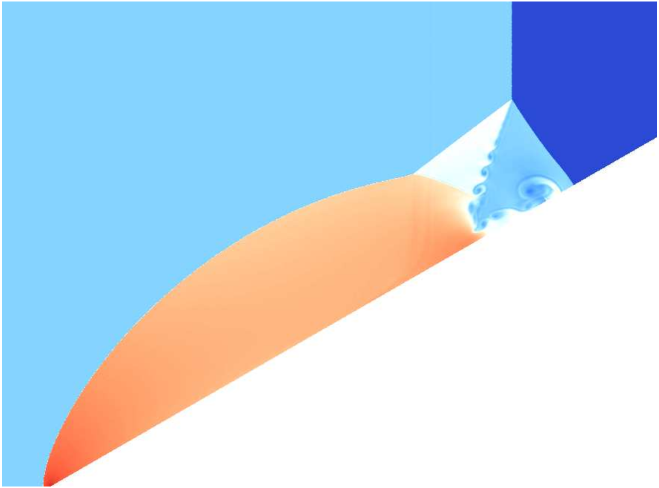


Figure: Double Mach reflection on a ramp on an unstructured mesh with 3×10^6 DOF

Mach 3 wind tunnel with a step

Figure: Mach 3 tunnel with a step on an unstructured mesh with 1.5×10^6 DOF

Conclusions and perspectives

Conclusions

- **e-MOOD scheme**: high-order entropy preserving scheme for the Euler equations in 1D
- **DMGR scheme**: second-order robust scheme for system of conservation laws on 2D unstructured meshes

Perspectives

- Extension of the DMGR scheme to higher-order
- Combination of the DMGR and e-MOOD methods to get high-order entropy preserving schemes in 2D
- Extension of the e-MOOD scheme to other systems

Thank you for your attention!