Schémas well-balanced permettant de capturer des états d'équilibre non-explicites

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Outline

The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2\right) = -gh\partial_x z \end{cases}$$

- h: water height
- u: velocity
- g: gravity constant
- $z(x)$: given smooth topography function
- Set of physical admissible states:

$$\Omega = \left\{ w = (h, hu)^T \in \mathbb{R}^2, \quad h > 0 \right\}.$$

The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2 \right) = -gh\partial_x z \end{cases}$$

Steady states

The steady states at rest are described by

$$\begin{cases} u = 0 \\ \partial_x(h^2/2) = -h\partial_x z \end{cases} \quad \Leftrightarrow \quad \begin{cases} u = 0 \\ h + z = \text{ cst.} \end{cases}$$

There is only one steady state at rest (up to a constant): the lake at rest.

The shallow-water model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2/2 \right) = -gh\partial_x z \end{cases}$$

Well-balanced scheme

- w_i^n: approximation of the solution on the cell $K_i = (x_{i-1/2}, x_{i+1/2})$ at time t^n
- z_i: approximation of the topgraphy $z(x)$ on the cell K_i
- A numerical scheme is well-balanced if

$$\forall i \in \mathbb{Z}, \quad h_i^0 + z_i = H \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \forall n \in \mathbb{N}, \quad w_i^{n+1} = w_i^n.$$

- There exists numerous well-balanced schemes for the shallow-water model: [Gosse '00], [Gallouët, Hérard & Seguin '03], [Audusse et al. '04]...

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2 \theta/2 \right) = -gh\theta \partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

- θ: temperature
- Set of physical admissible states:

$$\Omega = \left\{ w = (h, hu, h\theta)^T \in \mathbb{R}^3, \quad h > 0, \quad \theta > 0 \right\}.$$

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

Steady states

The steady states at rest are governed by the ODE

$$\begin{cases} u = 0 \\ \partial_x(h^2\theta/2) = -h\theta\partial_x z. \end{cases}$$

We cannot obtain an explicit expression of all the steady states.

Particular steady states

$$\begin{cases} u = 0 \\ \theta = \text{cst} \\ h + z = \text{cst} \end{cases} \qquad \begin{cases} u = 0 \\ z = \text{cst} \\ h^2\theta = \text{cst} \end{cases} \qquad \begin{cases} u = 0 \\ h = \text{cst} \\ z + \frac{h}{2}\ln\theta = \text{cst} \end{cases}$$

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

Objectives

- Robust finite volume method: preservation of the set Ω
- Exact capture of the three particular steady states
- Exact/Approximated preservation of all the steady states at rest

The relaxation framework

Initial system

$$\partial_t w + \partial_x f(w) = 0$$

$$w \in \Omega \subset \mathbb{R}^d$$

The relaxation framework

Initial system	Relaxation system
$\partial_t w + \partial_x f(w) = 0$	$\partial_t W + \partial_x F(w) = \frac{1}{\varepsilon} R(W)$
$w \in \Omega \subset \mathbb{R}^d$	$W \in \mathcal{O} \subset \mathbb{R}^N, \quad N > d$

The relaxation framework

$\mathcal{Q}$

Initial system	Relaxation system
$\partial_t w + \partial_x f(w) = 0$	$\partial_t W + \partial_x F(w) = \frac{1}{\varepsilon} R(W)$
$w \in \Omega \subset \mathbb{R}^d$	$W \in \mathcal{O} \subset \mathbb{R}^N, \quad N > d$

- Matrix $\mathcal{Q} \in M_{d,N}(\mathbb{R})$ s.t. $\mathcal{Q}\mathcal{O} = \Omega$ and $\mathcal{Q}R(W) = 0, \quad \forall W \in \mathcal{O}$

The relaxation framework

$\mathcal{Q}$

Initial system	Relaxation system
$\partial_t w + \partial_x f(w) = 0$	$\partial_t W + \partial_x F(w) = \frac{1}{\varepsilon} R(W)$
$w \in \Omega \subset \mathbb{R}^d$	$W \in \mathcal{O} \subset \mathbb{R}^N, \quad N > d$

$\mathcal{E}$

- Matrix $\mathcal{Q} \in M_{d,N}(\mathbb{R})$ s.t. $\mathcal{Q}\mathcal{O} = \Omega$ and $\mathcal{Q}R(W) = 0, \quad \forall W \in \mathcal{O}$
- For all $w \in \Omega$, there exists a unique equilibrium $\mathcal{E}(w)$ such that
  - $\mathcal{Q}\mathcal{E}(w) = w$
  - $R(\mathcal{E}(w)) = 0$

The relaxation framework

$\mathcal{Q}$

Initial system	Relaxation system
$\partial_t w + \partial_x f(w) = 0$	$\partial_t W + \partial_x F(w) = \frac{1}{\varepsilon} R(W)$
$w \in \Omega \subset \mathbb{R}^d$	$W \in \mathcal{O} \subset \mathbb{R}^N, \quad N > d$

$\mathcal{E}$

- Matrix $\mathcal{Q} \in M_{d,N}(\mathbb{R})$ s.t. $\mathcal{Q}\mathcal{O} = \Omega$ and $\mathcal{Q}R(W) = 0, \quad \forall W \in \mathcal{O}$
- For all $w \in \Omega$, there exists a unique equilibrium $\mathcal{E}(w)$ such that
  - $\mathcal{Q}\mathcal{E}(w) = w$
  - $R(\mathcal{E}(w)) = 0$
- Equilibrium manifold: $\mathcal{M} \subset \mathcal{O} := \{\mathcal{E}(w),\ w \in \Omega\}$

$$W \in \mathcal{M} \quad \Leftrightarrow \quad R(W) = 0 \quad \Leftrightarrow \quad \mathcal{E}(\mathcal{Q}\mathcal{W}) = W$$

- Compatibility of the flux functions: $\mathcal{Q}F(\mathcal{E}(w)) = f(w)$

Relaxation scheme

Assume we know a piecewise constant approximation at time t^n given by w_i^n on the cell K_i.

The relaxation scheme is based on a splitting strategy:

1. Time evolution: We use the Godunov scheme for the system $\partial_t W + \partial_x F(W) = 0$ (i.e. $\varepsilon = +\infty$), with initial data given by

$$W^n_{\Delta x}(x) = \mathcal{E}(w_i^n), \quad \text{for } x \in K_i.$$

   This gives us an update solution $W_i^{n+1,-}$ on cell K_i.
   $\rightarrow$ We need to know the exact solution of the Riemann problem.

2. Relaxation: We take into account the relaxation source term by solving $\partial_t W = \frac{1}{\varepsilon} R(W)$, with $W_i^{n+1,-}$ as initial data, then taking the limit for $\varepsilon \to 0$.
   Let W_i^{n+1} be the solution of this ODE, the updated state w_i^{n+1} is then given by $w_i^{n+1} = \mathcal{Q} W_i^{n+1}$.
   Remark : multiplying the ODE by $\mathcal{Q}$, we get $\partial_t \mathcal{Q} W = 0$, so the update solution satisfies $w_i^{n+1} = \mathcal{Q} W_i^{n+1,-}$.

The Suliciu model ([Suliciu '98], [Bouchut '04]...)

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x(hu^2 + \pi) = -gh\theta\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x(u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \\ \partial_t z = 0 \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2$$

Eigenvalues	Riemann invariants
$u \pm \frac{a}{h}$	$u \pm \frac{a}{h}, \quad \pi \mp au, \quad \theta, \quad z$
u $(\times 2)$	$u, \quad \pi, \quad z$
0	$hu, \quad \pi + \frac{a^2}{h}, \quad \theta, \quad g\theta z + \frac{u^2}{2} - \frac{a^2}{2h^2}$

Difficulties to compute the solution of the Riemann problem:

- The order of the eigenvalues is not determined *a priori*.
- There are strong nonlinearities in the Riemann invariants for the eigenvalue 0.

Relaxation model with moving topography

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x (hu^2 + \pi) = -gh\theta\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x (u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \end{cases}$$

Equilibrium

$\pi = gh^2\theta/2$

$Z = z$

Eigenvalues	Riemann invariants
$u \pm \frac{a}{h}$	$u \pm \frac{a}{h}$, $\pi \mp au$, θ, Z
u $(\times 3)$	u

- The order of the eigenvalues is fixed: $u - \frac{a}{h} < u < u + \frac{a}{h}$
- There is a missing invariant for the eigenvalue u
  $\Rightarrow$ we need a closure equation

The closure equation

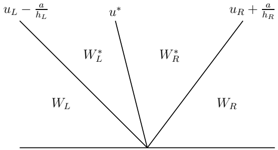


Figure: Structure of the Riemann problem

To mimic the ODE defining the steady states

$$\partial_x(gh^2\theta/2) = -gh\theta\partial_x z \quad \underset{\text{equilibrium}}{\iff} \quad \partial_x\pi = -gh\theta\partial_x Z,$$

we propose the closure equation

$$\frac{\pi_R^* - \pi_L^*}{\Delta x} = -g\overline{h}(W_L, W_R)\overline{\theta}(W_L, W_R)\frac{Z_R - Z_L}{\Delta x},$$

where $\overline{h}$ and $\overline{\theta}$ are suitable averages (defined later).

Approximate Riemann solver

Equilibrium state:

$$W^{eq}(w) = (h, hu, h\theta, gh\theta/2, hz)^T$$

Theorem

With the closure equation, the Riemann problem admits a unique solution $W_{\mathcal{R}}\left(\frac{x}{t}, W_L, W_R\right)$.
Moreover,

$$w_{\mathcal{R}}\left(\frac{x}{t}, w_L, w_R\right) := W_{\mathcal{R}}^{(h,hu,h\theta)}\left(\frac{x}{t}, W^{eq}(w_L), W^{eq}(w_R)\right)$$

defines an approximate Riemann solver (in the sense of Harten, Lax and van Leer) for the Ripa model.

Complete relaxation reformulation

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x(hu^2 + \pi) = -g\overline{h}(X^-, X^+)\overline{\theta}(X^-, X^+)\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x(u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \\ \partial_t X^- + (u - \delta)\partial_x X^- = \frac{1}{\varepsilon}(W - X^-) \\ \partial_t X^+ + (u + \delta)\partial_x X^+ = \frac{1}{\varepsilon}(W - X^+) \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2 \qquad Z = z \qquad X^\pm = W$$

- For $\delta > 0$ small enough, the system is hyperbolic with eigenvalues

$$u - \frac{a}{h} < u - \delta < u < u + \delta < u + \frac{a}{h}.$$

- The system has a complete set of Riemann invariants.
- Leads to the same approximate Riemann solver $w_{\mathcal{R}}\left(\frac{x}{t}, w_L, w_R\right)$.

Cargo-LeRoux formulation $(z(x) = x)$

We introduce a potential $q = \int^x gh\theta\, dx$. Then we have

$$\partial_t q = \int^x g\partial_t(h\theta)\, dx = -\int^x g\partial_x(h\theta u)\, dx = -gh\theta u = -u\partial_x q.$$

So q is governed by

$$\partial_t hq + \partial_x hqu = 0.$$

Equivalent reformulation of the Ripa model:

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\theta/2\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = 0. \end{cases}$$

Relaxation model with the Cargo-LeRoux formulation

- Case $z(x) = x$ [Chalons et al. '10]

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \pi\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = 0 \\ \partial_t h\pi + \partial_x(u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \end{cases}$$

Equilibrium

$$\pi = gh^2\theta/2$$

- Extension for general topography

  If we define the potential by $q = \int^x gh\theta\partial_x z dx$, it no longer satisfies a transport equation. We enforce the natural relaxation model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \pi\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = \frac{h}{\varepsilon}(\int^x gh\theta\partial_x z dx - q) \\ \partial_t h\pi + \partial_x(u(h\pi + a^2)) = \frac{h}{\varepsilon}(gh^2\theta/2 - \pi) \end{cases}$$

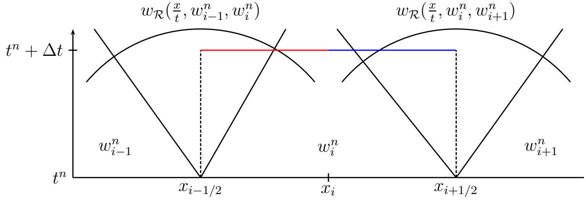
Equilibrium

$$\pi = gh^2\theta/2$$

$$q = \int^x gh\theta\partial_x z dx$$

The relaxation scheme

w_i^n: approximation of the solution on the cell $(x_{i-1/2}, x_{i+1/2})$ at time t^n



CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{i\in\mathbb{Z}} \left| u_i^n \pm \frac{a}{h_i^n} \right| \leq \frac{1}{2}$$

The update at time $t^{n+1} = t^n + \Delta t$ is defined by

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_i} w_{\mathcal{R}} \left(\frac{x - x_{i-1/2}}{\Delta t}, w_{i-1}^n, w_i^n \right) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+1/2}} w_{\mathcal{R}} \left(\frac{x - x_{i+1/2}}{\Delta t}, w_i^n, w_{i+1}^n \right) dx$$

Properties of the relaxation scheme (1)

Theorem (Exact preservation of the particular steady states)

Assume the average functions $\overline{h}$ and $\overline{\theta}$ are defined by

$$\overline{h}(W_L, W_R) = \frac{1}{2}(h_L + h_R), \quad \overline{\theta}(W_L, W_R) = \begin{cases} \frac{\theta_R - \theta_L}{\ln(\theta_R) - \ln(\theta_L)} & \text{if } \theta_L \neq \theta_R, \\ \theta_L & \text{if } \theta_L = \theta_R. \end{cases}$$

Then the relaxation scheme preserves exactly the particular steady states: if the initial data w_i^0 is given by

$$\begin{cases} u_i^0 = 0, \\ \theta_i^0 = \theta, \\ h_i^0 + z_i = H, \end{cases} \quad \text{or} \quad \begin{cases} u_i^0 = 0, \\ z_i = Z, \\ \left(h_i^0\right)^2 \theta_i^0 = P, \end{cases} \quad \text{or} \quad \begin{cases} u_i^0 = 0, \\ h_i^0 = H, \\ z_i + h_i^0 \ln(\theta_i^0)/2 = P, \end{cases}$$

then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Properties of the relaxation scheme (2)

Theorem (Well-balancedness)

If the initial data is an approximation of the ODE defining the steady states as follows:

$$\begin{cases} u_i^0 = 0, \\ \frac{(h_{i+1}^0)^2\theta_{i+1}^0/2-(h_i^0)^2\theta_i^0/2}{\Delta x} = -\bar{h}(w_i^0, w_{i+1}^0)\bar{\theta}(w_i^0, w_{i+1}^0)\frac{z_{i+1}-z_i}{\Delta x}. \end{cases}$$

Then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Theorem (Robustness)

Assume the parameter a satisfies the following inequalities:

$$u_L - \frac{a}{h_L} < u^* < u_R + \frac{a}{h_R}.$$

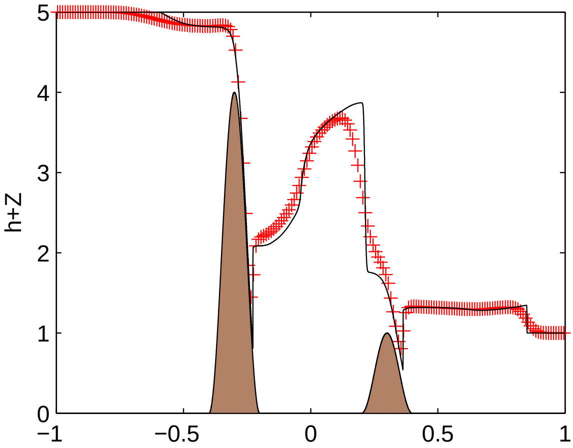
Then the relaxation scheme preserves the set of physical states:

$$\forall i \in \mathbb{Z},\ h_i^n > 0 \text{ and } \theta_i^n > 0 \quad \Rightarrow \quad \forall i \in \mathbb{Z},\ h_i^{n+1} > 0 \text{ and } \theta_i^{n+1} > 0.$$

Dam break over a non-flat bottom

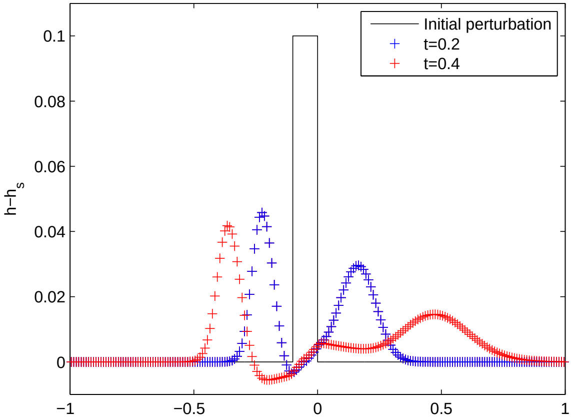
[Chertock, Kurganov & Liu '13]

- Initial condition:
  $(h{+}z,u,\theta)(x,0){=}$
  $$\begin{cases}(5,0,1), & x<0\\ (1,0,5), & x>0\end{cases}$$



Perturbation of a nonlinear steady state

- Topography:
  $z(x) = -2e^x$
- Steady state solution:
  $(h_s, u_s, \theta_s)(x) = (e^x, 0, e^{2x})$
- Initial perturbation:
  $\delta h(x, 0) = 0.1\chi_{[-0.1, 0]}(x)$



The system of Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p \right) = -\rho \partial_x \phi \\ \partial_t E + \partial_x (u(E + p)) = -\rho u \partial_x \phi \end{cases}$$

- ρ: density
  u: velocity
  $E = \rho e + \rho u^2/2$: total energy, with e the internal energy
  $p = p(\rho, e)$: pressure given by a general law
  $\phi(x)$: gravitational potential (example: $\phi(x) = gx$)
- Set of physical admissible states:

$$\Omega = \left\{ w = (\rho, \rho u, E)^T \in \mathbb{R}^3, \quad \rho > 0, \quad e > 0 \right\}.$$

The system of Euler equations with gravity

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = -\rho \partial_x \phi \\ \partial_t E + \partial_x (u(E+p)) = -\rho u \partial_x \phi \end{cases}$$

Steady states

The steady states at rest are governed by the ODE

$$\begin{cases} u \equiv 0, \\ \partial_x p = -\rho \partial_x \phi. \end{cases}$$

We cannot obtain an explicit expression of all the steady states.

Particular steady state: hydrostatic atmosphere

$$\begin{cases} u(x) = 0 \\ \rho(x) = \alpha e^{-\beta \phi(x)} \qquad \alpha > 0, \quad \beta > 0 \\ p(x) = \frac{\alpha}{\beta} e^{-\beta \phi(x)} \end{cases}$$

Relaxation scheme for the Euler equations with gravity

Using a similar method than for Ripa, we get a relaxation scheme which satisfies the following properties:

- Preservation of the set Ω

$$\rho_i^n > 0, e_i^n > 0 \quad \Rightarrow \quad \rho_i^{n+1} > 0, e_i^{n+1} > 0$$

- Exact preservation of the hydrostatic atmosphere

$$\begin{cases} u_i^0 = 0 \\ \rho_i^0 = \alpha e^{-\beta \phi_i} \\ p_i^0 = \frac{\alpha}{\beta} e^{-\beta \phi_i} \end{cases} \quad \Rightarrow \quad w_i^{n+1} = w_i^n$$

- Exact preservation of approximations of all the steady states

$$\begin{cases} u_i^0 = 0 \\ \frac{p_{i+1} - p_i}{\Delta x} + \overline{\rho}_{i+1/2} \frac{\phi_{i+1} - \phi_i}{\Delta x} = 0 \end{cases} \quad \Rightarrow \quad w_i^{n+1} = w_i^n$$

Conclusions

- Robust well-balanced scheme for the Ripa model
- Extension of the method to the Euler equations with gravity

Perspectives

- Entropy property for the relaxation schemes
- Extension to 2D (or 3D)
- Development of high-order well-balanced schemes for systems with source terms.

Thank you for your attention!