Un schéma well-balanced pour le modèle de Ripa

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Outline

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\Theta/2\right) = -gh\Theta\partial_x z \\ \partial_t h\Theta + \partial_x h\Theta u = 0 \end{cases}$$

- h: water height
- u: velocity
- g: gravity constant
- $z(x)$: given smooth topography function
- Θ: temperature
- Set of physical admissible states:

$$\Omega = \left\{ w = (h, hu, h\Theta)^T \in \mathbb{R}^3, \quad h > 0, \quad \Theta > 0 \right\}.$$

The Ripa model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\Theta/2 \right) = -gh\Theta\partial_x z \\ \partial_t h\Theta + \partial_x h\Theta u = 0 \end{cases}$$

Steady states at rest

$$\begin{cases} u = 0 \\ \partial_x (h^2\Theta/2) = -h\Theta\partial_x z. \end{cases}$$

No explicit expression of all the steady states.

Lake at rest type steady states

$$\begin{cases} u = 0 \\ \Theta = \text{cst} \\ h + z = \text{cst} \end{cases} \qquad \begin{cases} u = 0 \\ z = \text{cst} \\ h^2\Theta = \text{cst} \end{cases} \qquad \begin{cases} u = 0 \\ h = \text{cst} \\ z + \frac{h}{2}\ln\Theta = \text{cst} \end{cases}$$

Main properties

- Eigenvalues: $u \pm \sqrt{gh\Theta}$ (GN), u (LD), 0 (LD)
- The Ripa model is hyperbolic outside of resonance $(u = \pm\sqrt{gh\Theta})$.
- The smooth solutions satisfy the additional conservation law

$$\partial_t(\eta(w) + g\Theta hz) + \partial_x(G(w) + g\Theta hzu) = 0,$$

where we have set

$$\eta(w) = h\frac{u^2}{2} + g\Theta\frac{h^2}{2} \quad \text{and} \quad G(w) = \left(h\frac{u^2}{2} + g\Theta h^2\right)u.$$

Problem: $w \mapsto \eta(w)$ is neither convex nor concave

Equivalent reformulation for smooth solution: $\Theta = \varphi(\theta)$

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\varphi(\theta)/2\right) = -gh\varphi(\theta)\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \end{cases}$$

Entropy inequality

The smooth solutions satisfy the additional conservation law

$$\partial_t(\tilde{\eta}(w) + g\varphi(\theta)hz) + \partial_x(\tilde{G}(w) + g\varphi(\theta)hzu) = 0,$$

where we have set

$$\tilde{\eta}(w) = h\frac{u^2}{2} + g\varphi(\theta)\frac{h^2}{2} \quad \text{and} \quad \tilde{G}(w) = \left(h\frac{u^2}{2} + g\varphi(\theta)h^2\right)u.$$

The partial entropy function $w \mapsto \tilde{\eta}$ is convex iif

$$\varphi''(\theta)\varphi(\theta) - \frac{1}{2}\varphi'(\theta)^2 > 0 \quad \text{and} \quad \varphi(\theta) - \theta\varphi'(\theta) + \frac{\theta^2}{2}\varphi''(\theta) > 0$$

Example: $\varphi(\theta) = e^\theta$

Objectives

Design a finite volume method with the following properties:

- Preservation of the set Ω
- Exact preservation of the three lake at rest type steady states
- Approximate preservation of all the steady states at rest
- Discrete entropy inequality

The relaxation method (without source term)

- Aim: derive a numerical scheme to approximate the solutions of

$$\partial_t w + \partial_x f(w) = 0.$$

- Relaxation system

$$\partial_t W + \partial_x F(W) = \frac{1}{\varepsilon} R(W),$$

$\rightarrow$ Should formally give back the original system when $\varepsilon \to 0$

- Relaxation scheme: splitting strategy
  1. Time evolution: We evolve the initial data by the Godunov scheme for the system $\partial_t W + \partial_x F(W) = 0$ (i.e. $\varepsilon = +\infty$).
     $\rightarrow$ We need the exact solution of the Riemann problem.
  2. Relaxation: We take into account the relaxation source term by solving $\partial_t W = \frac{1}{\varepsilon} R(W)$ then taking the limit for $\varepsilon \to 0$.

The Suliciu model ([Suliciu '98], [Bouchut '04]...)

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x (hu^2 + \pi) = -gh\varphi(\theta)\partial_x z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x (h\pi + a^2)u = \frac{h}{\varepsilon}(gh^2\varphi(\theta)/2 - \pi) \\ \partial_t z = 0 \end{cases}$$

Equilibrium

$$\pi = gh^2\varphi(\theta)/2$$

Eigenvalues: $u - \frac{a}{h}, \quad 0, \quad u, \quad u + \frac{a}{h}$

Advantage:

- All the fields are linearly degenerate

Difficulties:

- The order of the eigenvalues is not fixed *a priori*
- The Riemann invariants for the eigenvalue 0 are strongly nonlinear

Relaxation model with moving topography

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x (hu^2 + \pi) = -gh\varphi(\theta)\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x (h\pi + a^2) u = \frac{h}{\varepsilon}(gh^2\varphi(\theta)/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \end{cases}$$

Equilibrium

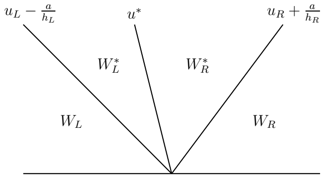
$$\pi = gh^2\varphi(\theta)/2$$

$$Z = z$$

- Eigenvalues: $u - \frac{a}{h}, \quad u, \quad u + \frac{a}{h}$
- Fixed order of the eigenvalues: $u - \frac{a}{h} < u < u + \frac{a}{h}$
- All the fields are linearly degenerate
- There is a missing Riemann invariant in order to determine a unique Riemann solution
  $\Rightarrow$ We need a closure relation

Relaxation model with moving topography

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x(hu^2 + \pi) = -gh\varphi(\theta)\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x(h\pi + a^2)u = \frac{h}{\varepsilon}(gh^2\varphi(\theta)/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \end{cases}$$



To approximate the equation

$$\partial_x(h^2\varphi(\theta)/2) = -h\varphi(\theta)\partial_x z \quad \underset{\substack{\text{equilibrium} \\ \pi = gh^2\varphi(\theta)/2 \\ Z = z}}{\Longleftrightarrow} \quad \partial_x \pi = -gh\varphi(\theta)\partial_x Z,$$

we propose the following closure relation:

$$\frac{\pi_R^* - \pi_L^*}{\Delta x} = -g\overline{h}(W_L, W_R)\varphi(\overline{\theta}(W_L, W_R))\frac{Z_R - Z_L}{\Delta x},$$

where $\overline{h}$ and $\overline{\theta}$ are suitable average (defined later)

Approximate Riemann solver

Equilibrium state:

$$W^{eq}(w) = (h, hu, h\theta, gh\varphi(\theta)/2, hz)^T$$

Theorem

With the closure equation, the Riemann problem admits a unique solution $W_{\mathcal{R}}\left(\frac{x}{t}, W_L, W_R\right)$.
Moreover,

$$w_{\mathcal{R}}\left(\frac{x}{t}, w_L, w_R\right) := W_{\mathcal{R}}^{(h,hu,h\theta)}\left(\frac{x}{t}, W^{eq}(w_L), W^{eq}(w_R)\right)$$

defines an approximate Riemann solver (in the sense of Harten, Lax and van Leer) for the Ripa model.

Complete relaxation reformulation

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x(hu^2 + \pi) = -g\overline{h}(X^-, X^+)\varphi(\overline{\theta}(X^-, X^+))\partial_x Z \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t h\pi + \partial_x(h\pi + a^2)u = \frac{h}{\varepsilon}(gh^2\varphi(\theta)/2 - \pi) \\ \partial_t hZ + \partial_x hZu = \frac{h}{\varepsilon}(z - Z) \\ \partial_t X^- + (u-\delta)\partial_x X^- = \frac{1}{\varepsilon}(W - X^-) \\ \partial_t X^+ + (u+\delta)\partial_x X^+ = \frac{1}{\varepsilon}(W - X^+) \end{cases}$$

Equilibrium

$$\pi = gh^2\varphi(\theta)/2 \qquad Z = z \qquad X^\pm = W$$

- For $\delta > 0$ small enough, the system is hyperbolic with eigenvalues

$$u - \frac{a}{h} < u - \delta < u < u + \delta < u + \frac{a}{h}.$$

- The system has a complete set of Riemann invariants.
- Leads to the same approximate Riemann solver $w_{\mathcal{R}}\left(\frac{x}{t}, w_L, w_R\right)$.

Cargo-LeRoux formulation $(z(x) = x)$

We introduce a potential $q = \int^x gh\varphi(\theta)\,dx$. Then we have

$$\partial_t q = \int^x g\partial_t(h\varphi(\theta))\,dx = -\int^x g\partial_x(h\varphi(\theta)u)\,dx = -gh\varphi(\theta)u = -u\partial_x q.$$

So q is governed by

$$\partial_t hq + \partial_x hqu = 0.$$

Equivalent reformulation of the Ripa model:

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + gh^2\varphi(\theta)/2\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = 0. \end{cases}$$

Relaxation model with the Cargo-LeRoux formulation

- Case $z(x) = x$ [Chalons et al. '10]

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \pi\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = 0 \\ \partial_t h\pi + \partial_x (h\pi + a^2)u = \frac{h}{\varepsilon}(gh^2\varphi(\theta)/2 - \pi) \end{cases}$$

Equilibrium

$$\pi = gh^2\varphi(\theta)/2$$

- Extension for general topography

  With $q = \int^x gh\varphi(\theta)\partial_x z dx$, q no longer satisfies a transport equation. We enforce the natural relaxation model

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(hu^2 + \pi\right) + \partial_x q = 0 \\ \partial_t h\theta + \partial_x h\theta u = 0 \\ \partial_t hq + \partial_x hqu = \frac{h}{\varepsilon}(\int^x gh\varphi(\theta)\partial_x z dx - q) \\ \partial_t h\pi + \partial_x (h\pi + a^2)u = \frac{h}{\varepsilon}(gh^2\varphi(\theta)/2 - \pi) \end{cases}$$

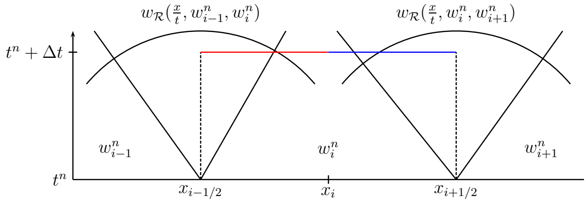
Equilibrium

$$\pi = gh^2\varphi(\theta)/2$$

$$q = \int^x gh\varphi(\theta)\partial_x z dx$$

The relaxation scheme

w_i^n: approximation of the solution on the cell $(x_{i-1/2}, x_{i+1/2})$ at time t^n



CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left| u_i^n \pm \frac{a}{h_i^n} \right| \leq \frac{1}{2}$$

The update at time $t^{n+1} = t^n + \Delta t$ is defined by

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_i} w_{\mathcal{R}} \left(\frac{x - x_{i-1/2}}{\Delta t}, w_{i-1}^n, w_i^n \right) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+1/2}} w_{\mathcal{R}} \left(\frac{x - x_{i+1/2}}{\Delta t}, w_i^n, w_{i+1}^n \right) dx$$

Properties of the relaxation scheme (1)

Theorem (Robustness)

Assume the parameter a satisfies the following inequalities:

$$u_L - \frac{a}{h_L} < u^* < u_R + \frac{a}{h_R}.$$

Then the relaxation scheme preserves the set of physical states:

$$\forall i \in \mathbb{Z},\ h_i^n > 0 \text{ and } \theta_i^n > 0 \quad \Rightarrow \quad \forall i \in \mathbb{Z},\ h_i^{n+1} > 0 \text{ and } \theta_i^{n+1} > 0.$$

Theorem (Well-balancedness)

If the initial data is an approximation of the ODE defining the steady states as follows:

$$\begin{cases} u_i^0 = 0, \\ \frac{(h_{i+1}^0)^2 \varphi(\theta_{i+1}^0)/2 - (h_i^0)^2 \varphi(\theta_i^0)/2}{\Delta x} = -\overline{h}(w_i^0, w_{i+1}^0)\varphi(\overline{\theta}(w_i^0, w_{i+1}^0))\frac{z_{i+1} - z_i}{\Delta x}. \end{cases}$$

Then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Properties of the relaxation scheme (2)

Theorem (Exact preservation of the lake at rest type steady states)

Assume the average functions $\overline{h}$ and $\overline{\theta}$ are defined by

$$\overline{h}(W_L, W_R) = \frac{1}{2}(h_L + h_R)$$

$$\overline{\theta}(W_L, W_R) = \begin{cases} \varphi^{-1}\left(\frac{\varphi(\theta_R)-\varphi(\theta_L)}{\ln(\varphi(\theta_R))-\ln(\varphi(\theta_L))}\right) & \text{if } \theta_L \neq \theta_R, \\ \theta_L & \text{if } \theta_L = \theta_R. \end{cases}$$

Then the relaxation scheme preserves exactly the lake at rest type steady states: if the initial data w_i^0 is given by

$$\begin{cases} u_i^0 = 0, \\ \theta_i^0 = cst, \\ h_i^0 + z_i = cst, \end{cases} \quad or \quad \begin{cases} u_i^0 = 0, \\ z_i = cst, \\ (h_i^0)^2 \varphi(\theta_i^0) = cst, \end{cases} \quad or \quad \begin{cases} u_i^0 = 0, \\ h_i^0 = cst, \\ z_i + h_i^0 \ln(\varphi(\theta_i^0))/2 = cst, \end{cases}$$

then $w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}.$

Properties of the relaxation scheme (3)

$$\overline{\eta}(w) = \tilde{\eta}(w) + gh\varphi(\theta)z, \qquad \overline{G}(w) = \tilde{G}(w) + gh\varphi(\theta)zu$$

Theorem (Consistency with the entropy inequality)

For a large enough, we have

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \overline{\eta}\left(w^{eq}\left(\frac{x}{\Delta t}, w_L, w_R\right)\right) dx - \frac{\overline{\eta}(w_L) + \overline{\eta}(w_R)}{2} + \frac{\Delta t}{\Delta x}(\overline{G}(w_R) - \overline{G}(w_L)) \leq O(z_R - z_L).$$

Theorem

Assume z is $1+\varepsilon$-Hölder continuous, with $\varepsilon > 0$.
Then the scheme is entropy preserving in the sense of Lax-Wendroff: if it converges to a function w, then w is an entropy solution.

Nonlinear steady state

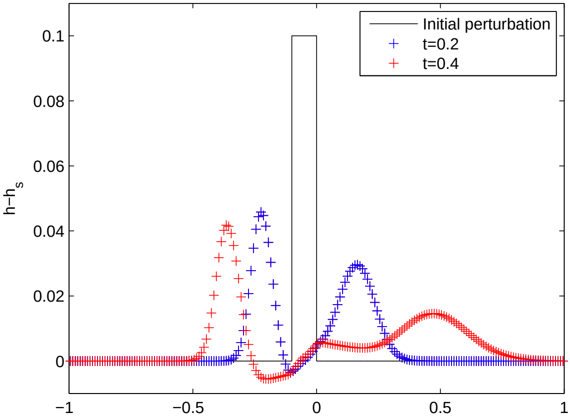
- Topography: $z(x) = -2e^x$
- Steady state solution: $(h_s, u_s, \Theta_s)(x) = (e^x, 0, e^{2x})$

  We check easily $\partial_x(h_s^2\Theta_s/2) = -h_s\Theta_s\partial_x z$
- L^1 error:

N	Water height		Velocity	
100	8.79E-05	–	2.13E-04	–
200	2.23E-05	1.98	5.41E-05	1.98
400	5.60E-06	1.99	1.36E-05	1.99
800	1.41E-06	1.99	3.42E-06	1.99
1600	3.52E-07	2.00	8.56E-07	2.00
3200	8.81E-08	2.00	2.14E-07	2.00

Perturbation of a nonlinear steady state

- Topography:
  $z(x) = -2e^x$
- Steady state solution:
  $(h_s, u_s, \Theta_s)(x) = (e^x, 0, e^{2x})$
- Initial perturbation:
  $\delta h(x,0) = 0.1\chi_{[-0.1,0]}(x)$

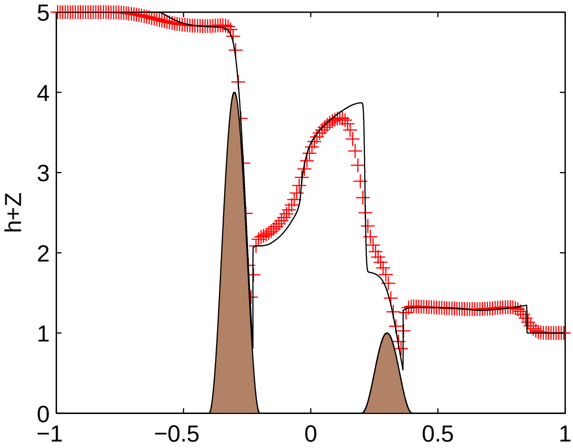


Dam break over a non-flat bottom

[Chertock, Kurganov & Liu '13]

- Initial condition:

  $(h+z,u,\theta)(x,0)=\begin{cases}(5,0,1), & x<0\\(1,0,5), & x>0\end{cases}$



Conclusions

- Relaxation scheme for the Ripa model:
  - Robust
  - Well-balanced
  - Entropy preserving
- Extension of the method to the Euler equations with gravity

Perspectives

- Extension to 2D
- High-order

Thank you for your attention!