Méthodes d'ordre élevé entropiques pour les équations d'Euler

Vivien Desveaux

en collaboration avec Christophe Berthon (Nantes)

21 Mars 2017

GdT ANGE Inria

System of conservation laws

- Hyperbolic system of conservation laws in 1D

$$\partial_t w + \partial_x f(w) = 0$$

  $w : \mathbb{R}^+ \times \mathbb{R} \to \Omega \subset \mathbb{R}^d$: unknown state vector
  $f : \Omega \to \mathbb{R}^d$: flux function

- Ω set of physical states, assumed convex
- Appearance of discontinuities → Weak solutions (in the sense of distributions)
- Non-physical weak solutions → Entropy criterion

Entropy solutions

Definition

(η, g) is an entropy pair if $\eta : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are two smooth function such that

- $w \mapsto \eta(w)$ is convex ;
- $\eta'(w)f'(w) = g'(w)$, $\forall w \in \Omega$.

η is an entropy and g is a entropy flux.

Definition

An entropy solution is a weak solution w such that for all entropy pair (η, g), we have

$$\partial_t \eta(w) + \partial_x g(w) \leq 0,$$

in the sense of distributions.

Question : Does numerical schemes “converge” to an entropy solution ?

Euler equations

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t E + \partial_x (E + p) u = 0 \end{cases}$$

- ρ : density
- u : velocity
- E : total energy
- p : pressure

- Ideal gas law : $\quad p = (\gamma - 1)\left(E - \dfrac{\rho u^2}{2}\right), \quad \gamma \in (1, 3]$
- Set of physical states :
  $\Omega = \left\{ w = (\rho, \rho u, E)^T \in \mathbb{R}^3,\ \rho > 0,\ p > 0 \right\}$
- Entropy inequalities :

$$\partial_t \rho \mathcal{F}(\ln(s)) + \partial_x \rho \mathcal{F}(\ln(s)) u \leq 0, \quad \text{with } s = \frac{p}{\rho^\gamma}$$

and $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ a smooth function such that

$$\mathcal{F}'(y) < 0 \quad \text{and} \quad \mathcal{F}'(y) < \gamma \mathcal{F}''(y), \quad \forall y \in \mathbb{R}$$

Space and time discretization

- Space discretization : cells $K_i = (x_{i-1/2}, x_{i+1/2})$ with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$
- Time discretization : $t^{n+1} = t^n + \Delta t$
- w_i^n : approximate solution at time t^n on the cell K_i
- Rectangular cells in the (x, t)–plane :

$$R_i^n = K_i \times [t^n, t^{n+1})$$

- The sequence $(\Delta x, \Delta t)$ is devoted to converge to $(0, 0)$, the ratio $\frac{\Delta t}{\Delta x}$ being kept constant.

First-order time scheme

- Discretization of the initial condition :

$$w_i^0 = \frac{1}{\Delta x} \int_{K_i} w^0(x)\,dx$$

- Update at time $t^{n+1} = t^n + \Delta t$:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(F_{i+1/2} - F_{i-1/2} \right)$$

  where $F_{i+1/2} = F\left(w_{i-s+1}^n, \cdots, w_{i+s}^n\right)$ and F is a consistent numerical flux ($F(w, \cdots, w) = f(w)$)

- We introduce the piecewise function

$$w^\Delta(x,t) = w_i^n, \quad \text{for } (x,t) \in R_i^n$$

Theorem (Lax-Wendroff, first-order in time)

(i) *Assume the following hypotheses :*

- *There exists a compact* $K \subset \Omega$ *such that* $w^{\Delta} \in K$ *;*
- w^{Δ} *converges in* $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ *to a function* w*.*

Then w *is a weak solution.*

(ii) *Assume the additional hypothesis :*

- *For all entropy pair* (η, g)*, there exists an entropy numerical flux* G*, consistent with* g $(G(w, \cdots, w) = g(w))$*, such that we have the discrete entropy inequality (DEI)*

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G^n_{i+1/2} - G^n_{i-1/2}}{\Delta x} \leq 0,$$

with $G^n_{i+1/2} = G\left(w^n_{i-s+1}, \cdots, w^n_{i+s}\right)$*.*

Then w *is an entropy solution.*

Motivations

Definition

A numerical scheme is entropy preserving if it converges in the sense of Lax-Wendroff to an entropy solution.

- There exists several entropy preserving first-order space/time schemes : Godunov, HLL, Suliciu relaxation, ...
- Starting from an entropy preserving first-order space/time scheme, can we derive high-order in space and/or time schemes which still preserve the entropy ?
- In the following of the talk, we assume the first-order scheme satisfies the DEI

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0.$$

Runge-Kutta time discretization

$$w_i^{n,(\ell)} = w_i^n - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} c_{\ell,j} \left(F_{i+1/2}^{n,(j)} - F_{i-1/2}^{n,(j)} \right), \quad \ell = 1, \cdots, m$$

$$w_i^{n,(0)} = w_i^n, \quad w_i^{n+1} = w_i^{n,(m)}$$

- Numerical flux : $F_{i+1/2}^{n,(j)} = F\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)} \right)$
- Hypotheses : $c_{\ell,j} \geq 0, \quad \sum_{j=0}^{m-1} c_{m,j} = 1$
- We introduce the piecewise constant functions

$$w^{\Delta,(\ell)}(x,t) = w_i^{n,(\ell)}, \quad \text{for } (x,t) \in R_i^n.$$

Theorem (Lax-Wendroff, high-order in time)

(i) *Assume the following hypotheses :*

- *There exists a compact* $K \subset \Omega$ *s.t.* $w^{\Delta,(\ell)} \in K$, $\forall \ell = 0, \cdots, m$;
- w^{Δ} *converges in* $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ *to a function* w.

Then w *is a weak solution.*

(ii) *Assume the additional hypothesis :*

- *For all entropy pair* (η, g), *there exists an entropy numerical flux* G, *consistent with* g $(G(w, \cdots, w) = g(w))$, *such that we have the discrete entropy inequality*

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \sum_{j=0}^{m-1} c_{m,j} \frac{G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}}{\Delta x} \leq 0,$$

with $G_{i+1/2}^{n,(j)} = G\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)}\right)$.

Then w *is an entropy solution.*

Shu-Osher reformulation

The Runge-Kutta scheme can be rewritten as

$$w_i^{n,(\ell)} = \sum_{j=0}^{\ell-1} \left(\alpha_{\ell,j} w_i^{n,(j)} - \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(F_{i+1/2}^{n,(j)} - F_{i-1/2}^{n,(j)} \right) \right),$$

$$\text{with } \alpha_{\ell,j} > 0, \quad \textstyle\sum_{j=0}^{\ell-1} \alpha_{\ell,j} = 1, \quad \beta_{\ell,j} > 0.$$

The intermediate states are convex combinations of first-order time schemes with respective time steps $\frac{\beta_{\ell,j}}{\alpha_{\ell,j}} \Delta t$.

Theorem

If the first-order time scheme is entropy preserving, then the Runge-Kutta scheme is entropy preserving.

High-order space schemes

- No DEI like

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0$$

was ever proven as soon as the scheme is at least second-order in space.

- Example : second-order MUSCL scheme
  - Let L be a limiter function (minmod, superbee...). We define a limited increment on each cell by $\mu_i^n = L\left(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n\right)$.
  - Reconstructed states at interfaces : $w_i^{n,\pm} = w_i^n \pm \frac{1}{2}\mu_i^n$
  - The MUSCL scheme is defined by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^{n,+}, w_{i+1}^{n,-}\right) - F\left(w_{i-1}^{n,+}, w_i^{n,-}\right)\right),$$

where F is a first-order numerical flux.

DEI satisfied by the MUSCL scheme

- The known DEI satisfied by the MUSCL scheme all write

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^{n,+}, w_{i+1}^{n,-}) - G(w_{i-1}^{n,+}, w_i^{n,-})}{\Delta x} \leq \frac{P_i^n - \eta(w_i^n)}{\Delta t}$$

where $P_i^n = P_\eta\,(w_i^n, \mu_i^n, \Delta x)$.

- Examples of operator P_η :

$$P_\eta^1(w, \mu, \Delta x) = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \eta\left(w + \frac{x}{\Delta x}\mu\right) dx \quad \text{[Bouchut \textit{et al.} '96]}$$

$$P_\eta^2(w, \mu, \Delta x) = \frac{\eta(w - \mu/2) + \eta(w + \mu/2)}{2} \quad \text{[Berthon '05]}$$

- The operator P_η satisfies : $\exists C > 0$ such that

$$0 \leq P_\eta(w, \mu, \Delta x) - \eta(w) \leq C\|\nabla^2\eta(w)\|\|\mu\|^2$$

Convergence study

- The discrete entropy inequality

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G(w_i^{n,+}, w_{i+1}^{n,-}) - G(w_{i-1}^{n,+}, w_i^{n,-})}{\Delta x} \leq \frac{P_i^n - \eta(w_i^n)}{\Delta t}$$

converges weakly to

$$\partial_t \eta(w) + \partial_x \mathcal{G}(w) \leq \delta,$$

where δ is a positive measure.

Conjecture (Hou-LeFloch '94)

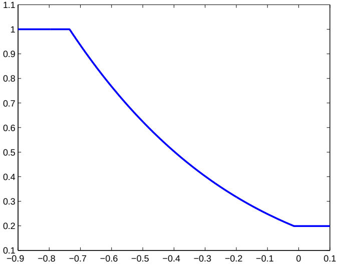
- $\delta = 0$ in the areas where w is smooth
- $\delta > 0$ on the curves of discontinuity of w

Numerical study : test cases (Euler equations)

Total mass of the right-hand side :

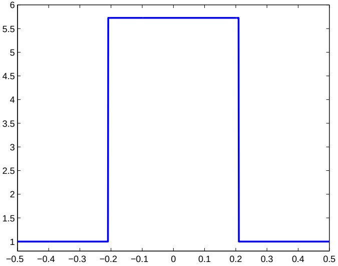
$$I^{\Delta} = \Delta x \sum_{i,n} (P_i^n - \eta(w_i^n))$$

1–rarefaction



$I^{\Delta} \overset{?}{\rightarrow} 0$

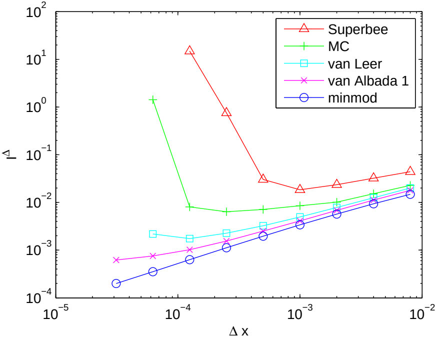
Double shock



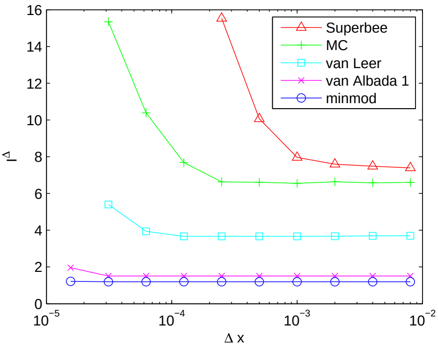
$I^{\Delta} \overset{?}{\rightarrow} c > 0$

Numerical results obtained with a first-order time scheme

1-rarefaction



Double shock



1-rarefaction : what superbee does

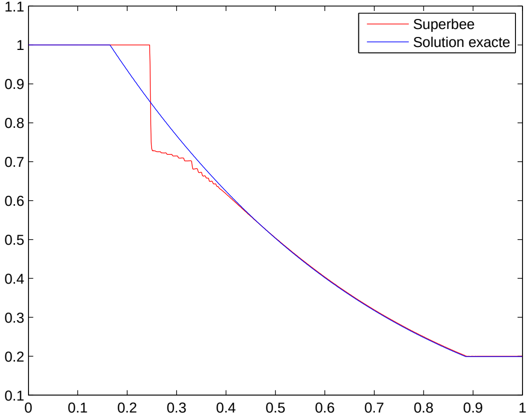


FIGURE: Solution given by the superbee limiter with 1000 cells

1-rarefaction : what superbee does

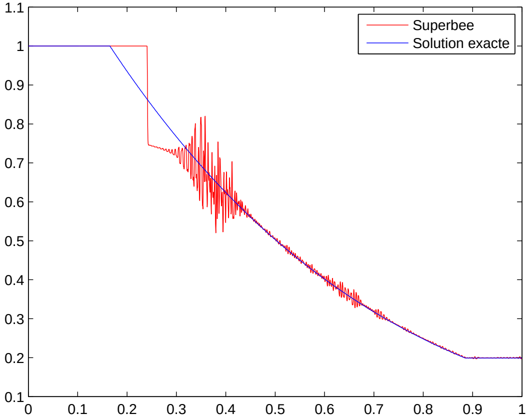
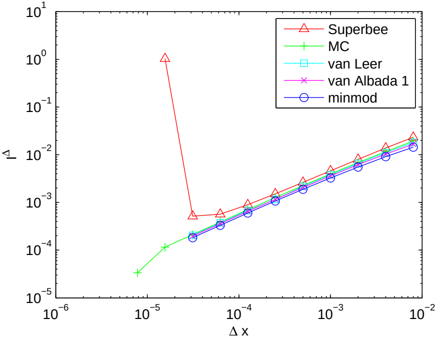


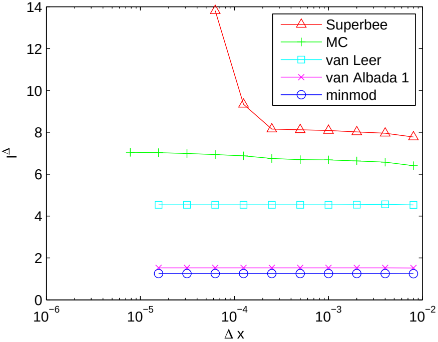
FIGURE: Solution given by the superbee limiter with 2000 cells

Numerical results obtained with a second-order time scheme

1-rarefaction



Double shock



Conclusion

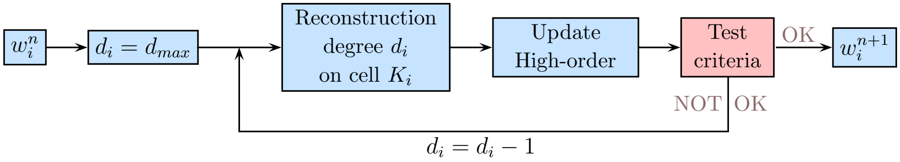
- Numerical results confirm the Hou-le Floch conjecture : when the scheme converges, the measure δ seems to be concentrated on the curves of discontinuity of w.
- This does not imply that the limit is not entropic, but the usual DEI are not relevant to apply the Lax-Wendroff theorem.
- We have to enforce the stronger DEI

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{G_{i+1/2}^n - G_{i-1/2}^n}{\Delta x} \leq 0.$$

- We suggest to extend the *a posteriori* methods (MOOD) introduced in [Clain, Diot & Loubère '11].

The MOOD framework

- High-order reconstruction procedure without any limitation
- List of constraints $\mathcal{A}$ that the approximation has to fulfill
- First-order scheme that satisfies the $\mathcal{A}$–criteria



$\mathcal{A}$–criteria in the original method

- Positivity preserving
- Discrete maximum principle

$\rightarrow$ We want to use the strong DEI as a criterion.

Problem : There are an infinity of entropies.

The family of entropies for the Euler equations

Lemma

The entropy pairs $(\eta, \mathcal{G})$ *of the Euler system rewrite*

$$\eta = \rho\psi(r), \quad \mathcal{G} = \rho\psi(r)u,$$

where $r = -\frac{p^{1/\gamma}}{\rho}$ *and* ψ *is a smooth increasing convex function.*

We consider the scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right),$$

where $w_i^n = (\rho_i^n, \rho_i^n u_i^n, E_i^n)^T$ and $F_{i+1/2} = (F^{\rho}_{i+1/2}, F^{\rho u}_{i+1/2}, F^{E}_{i+1/2})^T$.

We introduce $r^n_{i+1/2} = \begin{cases} r^n_{i+1} & \text{if } F^\rho_{i+1/2} < 0 \\ r^n_i & \text{if } F^\rho_{i+1/2} > 0 \end{cases}.$

Theorem

Assume the scheme preserves Ω. Assume the scheme satisfies the specific discrete entropy inequality

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2} r^n_{i+1/2} - F^\rho_{i-1/2} r^n_{i-1/2}\right).$$

Assume the additional CFL like condition

$$\frac{\Delta t}{\Delta x}\left(\max\left(0, F^\rho_{i+1/2}\right) - \min\left(0, F^\rho_{i-1/2}\right)\right) \leq \rho_i^n.$$

Then the scheme is entropy preserving : for all smooth increasing convex function ψ, we have

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n \psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2}\psi(r^n_{i+1/2}) - F^\rho_{i-1/2}\psi(r^n_{i-1/2})\right).$$

Proof of the Theorem (1)

Using the upwind definition of $r^n_{i+1/2}$, the specific DEI writes

$$r_i^{n+1} \leq \frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n,$$

where we have set

$$a = \frac{\Delta t}{2\Delta x}\left(F^{\rho}_{i-1/2} + \left|F^{\rho}_{i-1/2}\right|\right),$$

$$b = \rho_i^n - \frac{\Delta t}{2\Delta x}\left(F^{\rho}_{i+1/2} + \left|F^{\rho}_{i+1/2}\right| - F^{\rho}_{i-1/2} + \left|F^{\rho}_{i-1/2}\right|\right),$$

$$c = \frac{\Delta t}{2\Delta x}\left(\left|F^{\rho}_{i+1/2}\right| - F^{\rho}_{i+1/2}\right).$$

- We have $a + b + c = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^{\rho}_{i+1/2} - F^{\rho}_{i-1/2}\right) = \rho_i^{n+1}$.
- $a \geq 0, \ c \geq 0$
- $b \geq 0$ thanks to the CFL like condition

$\Rightarrow r_i^{n+1}$ is less than a convex combination of r^n_{i-1}, r^n_i and r^n_{i+1}.

Proof of the Theorem (2)

- We consider an entropy pair $(\rho\psi(r), \rho\psi(r)u)$ with ψ a smooth increasing convex function.
- ψ is increasing :

$$\psi\left(r_i^{n+1}\right) \leq \psi\left(\frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n\right)$$

- Jensen inequality (ψ is convex) :

$$\psi\left(r_i^{n+1}\right) \leq \frac{a}{\rho_i^{n+1}} \psi\left(r_{i-1}^n\right) + \frac{b}{\rho_i^{n+1}} \psi\left(r_i^n\right) + \frac{c}{\rho_i^{n+1}} \psi\left(r_{i+1}^n\right)$$

- Replacing a, b and c by their value, we get

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n \psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(F^\rho_{i+1/2}\psi^n_{i+1/2} - F^\rho_{i-1/2}\psi_{i-1/2}\right),$$

with $\psi^n_{i+1/2} = \begin{cases} \psi\left(r_{i+1}^n\right) & \text{if} \quad F^\rho_{i+1/2} < 0 \\ \psi\left(r_i^n\right) & \text{if} \quad F^\rho_{i+1/2} > 0 \end{cases}.$

First-order scheme

We consider a first-order scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^n, w_{i+1}^n\right) - F\left(w_{i-1}^n, w_i^n\right)\right).$$

For a time step restricted according to the CFL condition

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left|\lambda^{\pm}\left(w_i^n, w_{i+1}^n\right)\right| \leq \frac{1}{2},$$

the first-order scheme is assumed to satisfy :

- Robustness : $\forall i \in \mathbb{Z}, \quad w_i^n \in \Omega \quad \Rightarrow \quad \forall i \in \mathbb{Z}, \quad w_i^{n+1} \in \Omega$
- Stability :

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\Big(F^{\rho}\left(w_i^n, w_{i+1}^n\right) r_{i+1/2}^n - F^{\rho}\left(w_{i-1}^n, w_i^n\right) r_{i-1/2}^n\Big).$$

Example : the HLLC/Suliciu relaxation scheme

Reconstruction procedure

- We consider high-order reconstructed states $w_i^{n,\pm}$ on the cell K_i at the interfaces $x_{i\pm 1/2}$.
- These reconstructed states can be obtained by any reconstruction procedure (MUSCL, ENO/WENO, PPM...).
- Assumptions :
  - The reconstruction is Ω-preserving : $w_i^{n,\pm} \in \Omega$;
  - The reconstruction is conservative :

$$w_i^n = \frac{1}{2}\left(w_i^{n,-} + w_i^{n,+}\right).$$

The e-MOOD algorithm

1. Reconstruction step : For all $i \in \mathbb{Z}$, we evaluate high-order reconstructed states $w_i^{n,\pm}$ located at the interfaces $x_{i\pm 1/2}$.
2. Evolution step : We compute a candidate solution as follows :

$$w_i^{n+1,\star} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(w_i^{n,+}, w_{i+1}^{n,-}\right) - F\left(w_{i-1}^{n,+}, w_i^{n,-}\right)\right).$$

3. A posteriori limitation step : We have the following alternative :
   - if for all $i \in \mathbb{Z}$, we have

$$\rho^{n+1,\star} r_i^{n+1,\star} \leq \rho_i^n r(w_i^n) - \frac{\Delta t}{\Delta x}\Big(F^\rho\left(w_i^{n,+}, w_{i+1}^{n,-}\right) r_{i+1/2}^n - F^\rho\left(w_{i-1}^{n,+}, w_i^{n,-}\right) r_{i-1/2}^n\Big), \quad (1)$$

     then the solution is valid and the updated solution at time $t^n + \Delta t$ is defined by $w_i^{n+1} = w_i^{n+1,\star}$;
   - otherwise, for all $i \in \mathbb{Z}$ such that (1) is not satisfied, we set $w_i^{n,\pm} = w_i^n$ and we go back to step 2.

Theorem

Assume the time step Δt is chosen in order to satisfy the two following CFL like conditions :

$$\frac{\Delta t}{\Delta x} \max_{i\in\mathbb{Z}} \left(\left| \lambda^{\pm}\left(w_i^{n,+}, w_{i+1}^{n,-}\right) \right|, \left| \lambda^{\pm}\left(w_i^{n,-}, w_i^{n,+}\right) \right| \right) \leq \frac{1}{4},$$

$$\frac{\Delta t}{\Delta x} \left(\max\left(0, F^{\rho}_{i+1/2}\right) - \min\left(0, F^{\rho}_{i-1/2}\right) \right) \leq \rho_i^n.$$

Then the updated states w_i^{n+1}, given by the e-MOOD scheme, belong to Ω. Moreover, for all smooth increasing convex function ψ, the e-MOOD scheme satisfies

$$\frac{1}{\Delta t}\left(\rho_i^{n+1}\psi(r_i^{n+1}) - \rho_i^n \psi\left(r_i^n\right)\right) + \frac{1}{\Delta x}\left(F^{\rho}\left(w_i^{n,+}, w_{i+1}^{n,-}\right)\psi(r^n_{i+1/2}) - F^{\rho}\left(w_{i-1}^{n,+}, w_i^{n,-}\right)\psi(r^n_{i-1/2})\right) \leq 0.$$

The e-MOOD scheme is thus entropy preserving.

Numerical results : smooth solution

Computational domain $[0,1]$ with periodic boundary conditions
Initial data $u_0(x) = 1$, $p_0(x) = 1$, $\rho_0(x) = 1 + \chi_{[0.2,0.8]}(x) \exp\left(\frac{(x-0.5)^2}{(x-0.2)(x-0.8)}\right)$

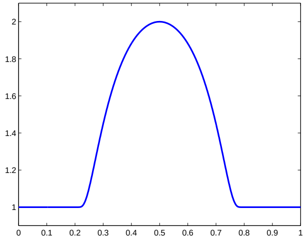
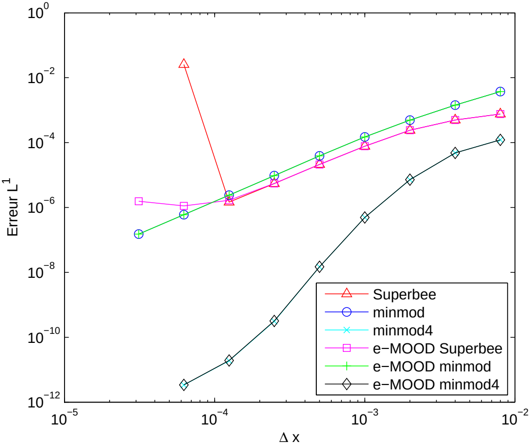


FIGURE: Initial density

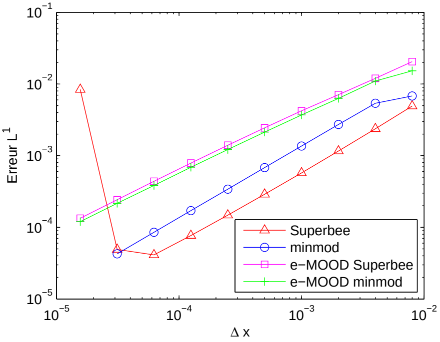


Numerical results obtained with a second-order time scheme

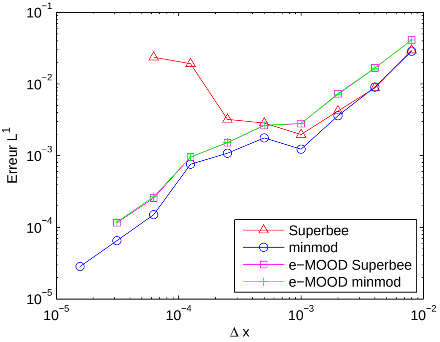
L^1 error :

$$\sum_i \left| \rho_i^N - \rho_{ex}(x_i, T) \right|$$

1-rarefaction



Double shock



Conclusions

- Theoretical and numerical study of entropy stability of classical high-order schemes :
  - OK for high-order time schemes
  - Not OK for high-order space schemes
- **e-MOOD scheme** : high-order entropy preserving scheme for the Euler equations in 1D

Perspectives

- Extension of the e-MOOD method to 2D
- Extension of the e-MOOD scheme to other systems
  - Euler with general pressure law
  - Shallow-water system : first-order entropy preserving scheme ?

Thank you for your attention !