# DDFV METHODS FOR THE EULER EQUATIONS

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## Introduction

- Hyperbolic system of conservation laws in 2D

$$\partial_t W + \partial_x f(W) + \partial_y g(W) = 0 \qquad (1)$$

$W : \mathbb{R}^2 \times \mathbb{R}^+ \to \Omega \subset \mathbb{R}^d$: unknown state vector
$f, g : \Omega \to \mathbb{R}^d$: flux functions

- **Example:** the 2D Euler equations

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \; f(W) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}, \; g(W) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix},$$

where $\rho$ is the density, $(u, v)$ the velocity, $E$ the total energy and $p$ the pressure given by the perfect gas law

$$p = (\gamma - 1)\left(E - \frac{\rho}{2}\left(u^2 + v^2\right)\right)$$

- $\Omega$ convex set of physical states. In the Euler case:

$$\Omega = \left\{ W \in \mathbb{R}^4; \rho > 0, (u,v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2+v^2\right) > 0 \right\}$$

- **Objectif:** derive a numerical scheme
  - → Second order accurate
  - → $\Omega$-preserving
  - → Unstructured meshes
  - → CFL restriction

## 1. MUSCL schemes

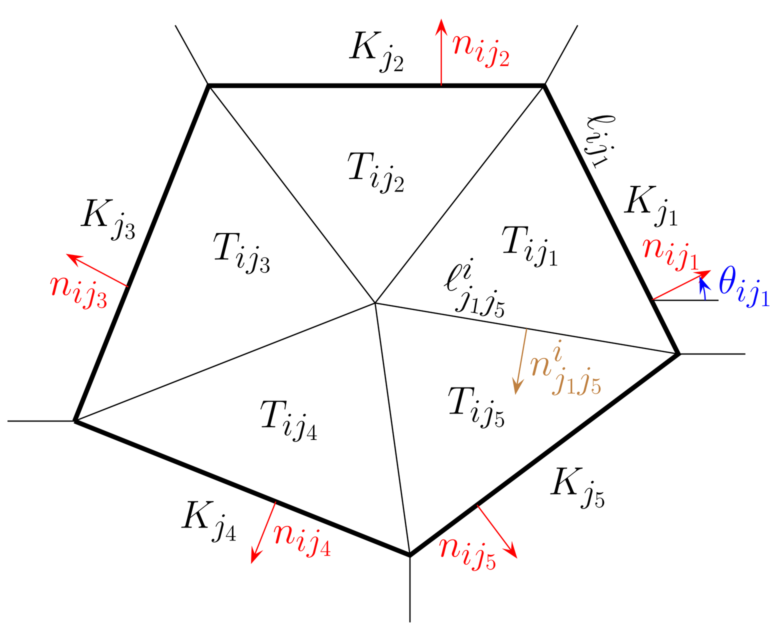


**Fig. 1:** A cell $K_i$

- **First-order scheme on the cell $K_i$**

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_i^n, W_j^n, \theta_{ij}\right) \qquad (2)$$

– $\phi$ 1D Godunov-type flux: under the CFL condition $\frac{\Delta t}{\delta} \max\{\lambda^{\pm}(W_L, W_R, \theta)\} \le \frac{1}{2}$, we have

$$\phi(W_L, W_R, \theta) = h_\theta(W_L) + \frac{\delta}{2\Delta t} W_L - \frac{1}{\Delta t} \int_{-\frac{\delta}{2}}^{0} \widetilde{W}_\theta\left(\frac{x}{\Delta t}, W_L, W_R\right) dx$$

▷ $h_\theta = \cos\theta f + \sin\theta g$: flux in the direction $\theta$
▷ $\widetilde{W}_\theta$ approximate Riemann solver **valued in $\Omega$**

– Consistency: $\phi(W, W, \theta) = h_\theta(W)$
– Conservation: $\phi(W_L, W_R, \theta) = -\phi(W_R, W_L, \theta + \pi)$

- **Second-order scheme on the cell $K_i$**

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_{ij}, W_{ji}, n_{ij}\right) \qquad (3)$$

$W_{ij}$ and $W_{ji}$ second-order approximations at the interface between $K_i$ and $K_j$ (see Fig. 2)
→ **How to compute $W_{ij}$ ?**

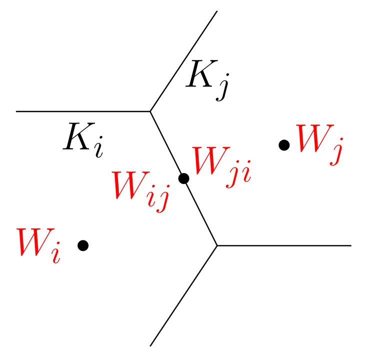


**Fig. 2:** States $W_{ij}$ and $W_{ji}$

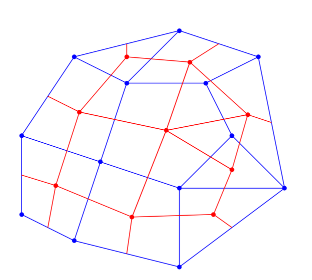

**Fig. 3:** Primal mesh and dual mesh

- **DDFV meshes**
We write a MUSCL scheme on two meshes: a primal mesh and its dual mesh (see Fig. 3)

## 2. Robustness and CFL restriction

- **Theorem 1: Robustness of the first-order scheme**
If the following hypothesis are satisfied
(i) $W_i^n \in \Omega$, $\forall i \in \mathbb{Z}$
(ii) We have the CFL condition

$$\Delta t \frac{\mathcal{P}_i}{|K_i|} \max_{j \in \nu(i)} \left\{ \lambda^{\pm}(W_i^n, W_j^n, \theta_{ij}) \right\} \le 1, \forall i$$

with $\mathcal{P}_i$ the perimeter of the cell $K_i$

Then the states $W_i^{n+1}$ evoluted by the first-order scheme (2) remain in $\Omega$.

- **Theorem 2: Robustness of the MUSCL scheme**
If the following hypothesis are satisfied
(i) The initial states $W_i^n$ are in $\Omega$
(ii) The reconstructed states $W_{ij}$ are in $\Omega$
(iii) The reconstruction satisfies the conservation property

$$\sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} W_{ij} = W_i^n \qquad (4)$$

(iv) We have the CFL condition $\forall i \in \mathbb{Z}$

$$\Delta t \max_{j \in \nu(i)} \frac{\mathcal{P}_{ij}}{|T_{ij}|} \max_{k \in \nu(i,j)} \left\{ \lambda^{\pm}(W_{ij}, W_{ji}, \theta_{ij}), \lambda^{\pm}(W_{ij}, W_{ik}, \theta^i_{jk}) \right\} \le 1$$

Then the states $W_i^{n+1}$ evoluted by the MUSCL scheme (3) remain in $\Omega$.

## 3. Reconstruction procedure

- **Computation of the states at the vertices**
  – The states at a vertex of the dual mesh is exactly the state at the center of the associated primal cell
  – We approximate the state at a vertex of the primal mesh by the state at the center of the associated dual cell
- **Local notations on cell $K$:**

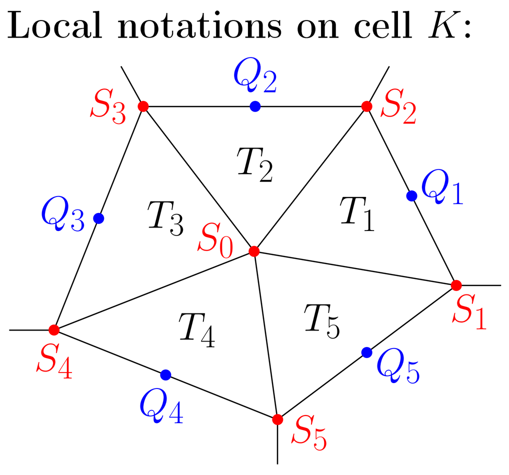


**Fig. 4:** Geometry of the cell $K$

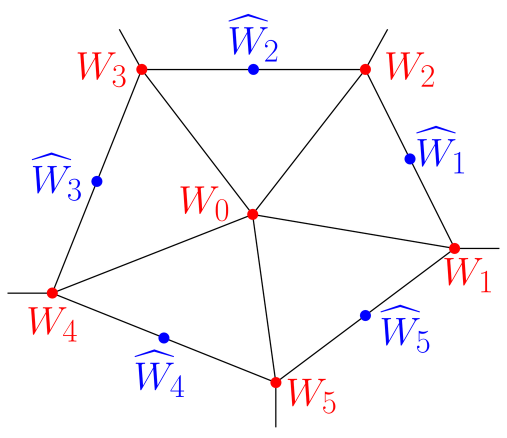


**Fig. 5:** Known states and reconstructed states

$S_0$, mass center → $W_0$, known state
$S_j$, vertex → $W_j$, known state
$Q_j$, midpoint of the edge → $\widehat{W}_j$, to be reconstructed in $\Omega$

- **Reconstruction procedure**

1. **Gradient reconstruction**
$\overline{W} : K \to \mathbb{R}^d$: continuous function, piecewise linear on each triangle $T_j$ and such that $\overline{W}(S_j) = W_j$, $j \in \nu(i)$

2. **Projection**
For $1 \le k \le d$, we define

$$E_k(\nu) = \int_K \left| \overline{W}_k(X) - [(W_0)_k + \nu \cdot (X - S_0)] \right|^2 dX,$$

where the subscript $k$ denotes the $k$-th component.
Let $\mu \in \mathbb{R}^d$ be the vector whose $k$-th component is the solution of

$$E_k(\mu_k) = \min_{\nu \in \mathbb{R}^2} E_k(\nu).$$

We define $\widetilde{W}_\mu(X) : K \to \mathbb{R}^d$ the linear function whose $k$-th component is $(W_0)_k + \mu_k \cdot (X - S_0)$.

3. **Limitation of the slope $\mu$**
We restrict $\Omega$ to a close set $\Omega_\epsilon$. In the Euler case,

$$\Omega_\epsilon = \left\{ W \in \mathbb{R}^4; \rho \ge \epsilon, (u,v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2+v^2\right) \ge \epsilon \right\}.$$

We define the optimal slope limiter by

$$\alpha = \max \left\{ t \in [0,1], \widetilde{W}_{t\mu}(Q_j) \in \Omega_\epsilon, \forall j \in \nu(i) \right\}.$$

4. Finally, the reconstructed states are given by $\widehat{W}_j = \widetilde{W}_{\alpha\mu}(Q_j)$.

Limitation procedure $\Rightarrow \widehat{W} j \in \Omega$
$\overline{W}(S_0) = W_0 \Rightarrow \sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} \widehat{W}_j = W_0$

⇒ **The DDFV-MUSCL scheme is robust**

## 5. Numerical tests

**Meshes**

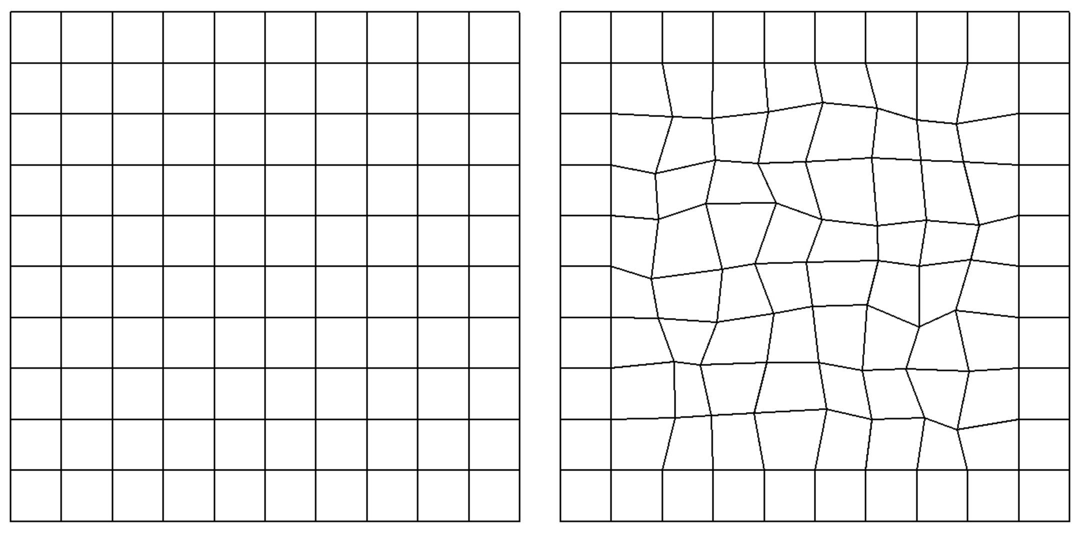

**Fig. 6:** $10 \times 10$ *square mesh (left) and* $10 \times 10$ *quadrilateral mesh (right)*

**Case 1 : four shocks**

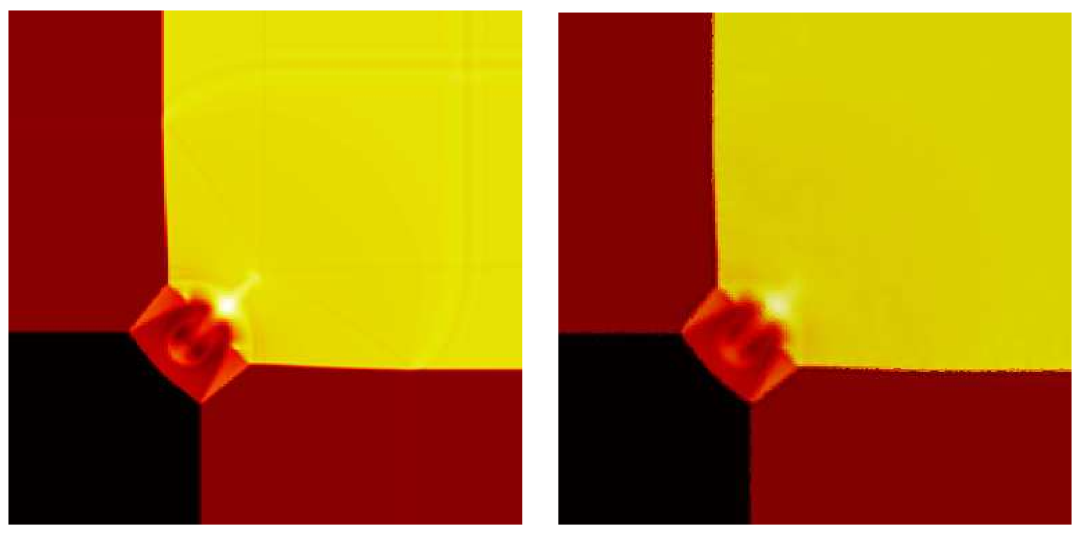

**Fig. 7:** $200 \times 200$ *square mesh (left) and* $200 \times 200$ *quadrilateral mesh (right)*

**Case 2 : four contact discontinuities**

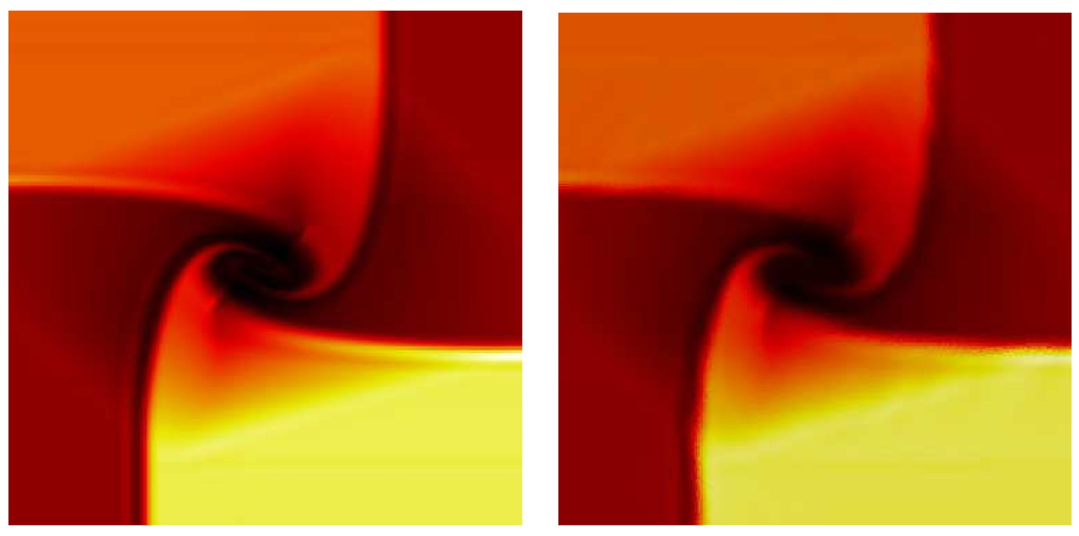

**Fig. 8:** $200 \times 200$ *square mesh (left) and* $200 \times 200$ *quadrilateral mesh (right)*

## Perspectives

- Allow non-conservative reconstructions, i.e. which don't satisfy (4)
- Optimization of the CFL condition in the robustness theorem for the MUSCL scheme
- Better approximation of the value at the vertices of the primal mesh, especially in the case of very distorded meshes (see Fig. 9)

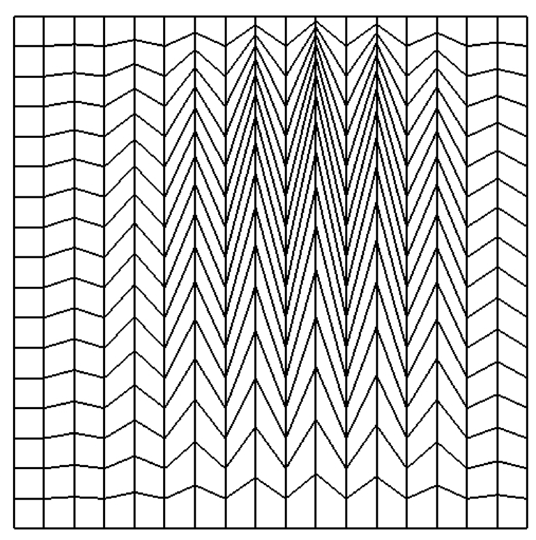

**Fig. 9:** *Distorded mesh*

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